The asymptotic manifold of a perturbed linear differential equation as determined by the comparison principle (*).

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Abstract. - *The asymptotic properties of the solutions of a linear homogeneous system of differential equations determine, under suitable restrictions, the asymptotic properties of a set of solutions of a nonlinear perturbation of this linear equation. The comparison principle is used here to generate an asymptotic manifold of the perturbed equation. The majorant function that is used in connection with the comparison technique is usually assumed to be nondecreasing in the dependent variable. However, properties of the asymptotic manifold are discussed here under the opposite monotonicity assumption, namely, that the majorant function is nonincreasing in the dependent variable. This type of majorant function arises, for example, in certain gravitation problems. The main result on the structure of asymptotic manifolds which have an asymptotic uniformity is that solutions close to the manifold are either in the manifold or do not exist in the future.*

I. - Introduction.

Consider the linear system of differential equations

$$dy/dt = A(t)y \tag{1}$$

and the nonlinear perturbation of this linear equation

$$dx/dt = A(t)x + f(t, x). \tag{2}$$

In these equations, we will require that $A(t)$ is a real valued continuous $n \times n$ matrix defined on $J = [0, \infty)$; $f(t, x)$ is a continuous function from $J \times U$, where $U = R^n - \{0\}$, to R^n. The symbol $\| \cdot \|$ will designate some convenient vector or matrix norm. The fundamental matrix of (1) that is equal to the $n \times n$ identity matrix at $t = t_0$ will be denoted by $Y(t)$. In order to have a measurement of the size of $Y(t)$, we will require that there exists a nonsingular continuous $n \times n$ matrix $\triangle = \triangle(t)$ defined on J, a continuous positive scalar valued function $\alpha = \alpha(t)$ defined on J, and a constant k, $k \geq \|I_n\|$, (where I_n is the $n \times n$ identity matrix) such that

$$\|\nabla(t)\, Y(t)\| \leq k\alpha(t), \quad t \in J; \tag{3}$$

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and

$$\| Y^{-1}(t)\Delta^{-1}(t)\| \leq \alpha^{-1}(t), \qquad t \in J; \tag{4}$$

Under suitable hypotheses, the asymptotic properties of the solutions of the linear system of differential equations (1) are transfered to certain of the solutions of the nonlinear system of differential equations (2). A technique that has often been used to effect this transfer of asymptotic behavior is the comparison principle. The comparison principle requires a majorant of the function f. We will assume that the majorant is given by the inequality

$$\| Y^{-1}(t)f(t, x)\| \leq w(t, \|\Delta(t)x\|\alpha^{-1}(t)), \qquad t \in J, \qquad x \in U. \tag{5}$$

In (5), $w(t, r)$ is nonnegative, continuous on $J \times (J - \{0\})$, and nonincreasing in r, $r > 0$, for each $t \in J$.

Perturbation problems which involve this particular class of majorant functions have received very little attention in the literature. In fact, most research articles in this area consider majorant functions that are nondecreasing in r for each fixed $t \in J$; (for example, see the references, [1]-[6]). This remains true despite fundamental applications (for example, gravitational problems) where the differential equations involved contain a decreasing function of the dependent variable. However, there is one result in a fundamental paper on asymptotic behavior by J. K. HALE and N. ONUCHIC [3] where the nth order scalar equation considered has the nonlinear portion of the differential equation majorized by a nonincreasing function of the dependent variable (see Corollary 2, [3, p. 72]).

Some of the results of this article are closely related to results of F. BRAUER and J. S. W. WONG [2]. In [2], the majorant function $w = w(t, r)$ in (5) is assumed to be nondecreasing in r for each fixed t in J. Subject to this hypothesis, asymptotic correspondences between the solutions of (1) and (2) are obtained for arbitrarily small initial positions. Under our assumption (5), we obtain a dual result that solutions of (1) and (2) with arbitrarily large initial positions have a prescribed asymptotic behavior. These results indicate that (under certain conditions) a manifold of solutions of the nonlinear differential equation (2) is generated by the linear equation (1). This manifold and a certain openness property (perturbable) was considered by TOROSHELIDZE in [6] for some scalar differential equations. A more formal development of the properties of this manifold was given in a more general setting by J.W. HEIDEL and the author in [5] (see also [4]). As is indicated below, the nonincreasing hypothesis in (5) leads to an asymptotic manifold that is essentially closed-in-itself.

We now state the form of the comparison principle that we will need.

LEMMA 1. - (Viswanatham, [7, Theorem 2]). *Let $w = w(t, r)$ be continuous and nonincreasing in r for each fixed t in the region R defined by $|t - t_0| \leq a$, $|r - \eta| \leq b$ where a and b are positive real numbers. Suppose that $\varphi = \varphi(t)$ is continuous for $|t - t_0| \leq a$ and satisfies the inequality*

$$\varphi(t) \geq \eta - \int_{t_0}^{t} w(s, \varphi(s))ds$$

there. Then, $\varphi(t) \geq \rho(t)$ for $t_0 \leq t \leq t_0 + \alpha$ where $\alpha = \min[a, b/M]$, $M > 0$ and $|w(t, r)| \leq M$ on R, and $\rho = \rho(t)$ is the minimal solution of

$$dr/dt = -w(t, r) \tag{6}$$

through the point (t_0, η).

II. - The asymptotic manifold of the perturbed linear equation.

We find it convenient to view the above perturbation problem in terms of an asymptotic manifold of solutions.

DEFINITION 1. - *The asymptotic manifold $S = S(\Delta, \alpha)$ of* (2) *generated by* (1) is the set of all solutions $x = x(t)$ of equation (2) that satisfy the order relation

$$\|\Delta(t)(x(t) - y(t))\| = o(\alpha(t)), \qquad (t \to \infty), \tag{7}$$

for some solution $y = y(t)$ of equation (1).

The first result below demonstrates that, under certain hypotheses, the asymptotic manifold of (2) generated by (1) is nonempty. A related result for nondecreasing majorants may be found in [2, Theorem 1].

THEOREM 1. - *Let the conditions* (3), (4), *and* (5) *be satisfied. Suppose that the initial value problem*

$$dr/dt = -w(t, r), \; r(t_0) = r_0 > 0 \tag{8}$$

has a minimal solution $\rho = \rho(t)$ that is positive on $J_0 = [t_0, \infty)$. Then, any solution $x = x(t; t_0, x_0)$ of (2) *that initially satisfies the inequality*

$$\|x_0\| \geq r_0 \tag{9}$$

is in $S(\Delta, \alpha)$. *Furthermore, the solution* $y = y(t)$ *of* (1) *that corresponds to* x *by the asymptotic relation* (7) *is unique. Also, if in addition to satisfying* (9), $\|x_0\|$ *satisfies the inequality*, $\|x_0\| > r_0 - \rho_\infty$, $\rho_\infty = \lim_{t\to\infty}\rho(t)$, *then* y *is nonzero.*

PROOF. – By using the above integral inequality (Lemma 1) we will find a lower bound for $\|\Delta(t)x(t;\, t_0^1,\, x_0^1)\|\alpha^{-1}(t)$ provided that inequality (9) holds. First, we note that condition (4) implies that

$$\|\Delta(t)Y(t)\| \geq \|I_n\|\alpha(t) \tag{10}$$

where I_n is the $n \times n$ identity matrix. The inequalities (3) and (10) mean that the pair (Δ, α) is a "good" measurement of the fundamental matrix of (1).

Using the variation of parameters formula, we have

$$Y^{-1}(t)x(t;\, t_0,\, x_0) = x_0 + \int_{t_0}^{t} Y^{-1}(s)f(s,\, x(s;\, t_0,\, x_0))ds. \tag{11}$$

From the equations (10) and (11) we obtain

$$\begin{aligned}
&\|x_0\| - \int_{t_0}^{t} w(s,\, \|\Delta(s)x(s;\, t_0,\, x_0)\|\alpha^{-1}(s))ds \\
&\leq \|x_0\| - \int_{t_0}^{t} \|Y^{-1}(s)f(s,\, x(s;\, t_0,\, x_0))\|ds \\
&\leq \|Y^{-1}(t)x(t;\, t_0,\, x_0)\| \\
&\leq \alpha^{-1}(t)\|\Delta(t)x(t;\, t_0,\, x_0)\|.
\end{aligned}$$

Since (9) is valid, an application of Lemma 1 establishes the inequality

$$\|\Delta(t)x(t;\, t_0,\, x_0)\| \geq \rho(t)\alpha(t) > 0, \qquad t \geq t_0; \tag{12}$$

hence, $x(t;\, t_0,\, t_0)$ exists for $t \geq t_0$.

The next portion of the proof proceeds much as in [2]; the only changes that need to be made are due to the differences in the monotonicity hypotheses. To find a solution y of (1) satisfying (7) we consider the expression

$$x(t_0) + \int_{t_0}^{t} Y^{-1}(s)f(s,\, x(s;\, t_0,\, x_0))ds.$$

Using conditions (5) and (12), we obtain

$$\int_{t_0}^{t} \| Y^{-1}(s) f(s,\ x(s;\ t_0,\ x_0)) \| ds \leq r_0 - \rho(t), \qquad t \geq t_0 .$$

Therefore, the limit

$$\lim_{t\to\infty} \int_{t_0}^{t} Y^{-1}(s) f(s,\ x(s;\ t_0,\ x_0)) ds =$$

$$\int_{t_0}^{\infty} Y^{-1}(s) f(s,\ x(s;\ t_0,\ x_0)) ds$$

exists; hence, let

$$c = x_0 + \int_{t_0}^{\infty} Y^{-1}(s) f(s,\ x(s;\ t_0,\ x_0)) ds . \tag{13}$$

Using the equations (11) and (13), we can write

$$\Delta(t) x(t;\ t_0,\ x_0) = \Delta(t) Y(t) c - \Delta(t) Y(t) \int_{t}^{\infty} Y^{-1}(s) f(s,\ x(s;\ t_0,\ x_0)) ds.$$

Using condition (3) and the above equation, the asymptotic relationship (7) with $y(t) = Y(t)c$ is easily seen to be satisfied.

The conclusion of the theorem concerning the uniqueness of the correspondence $y \to x$ as determined by (7), follows from inequality (10). For if $y_1(t) = Y(t)c_1$ and $y_2(t) = Y(t)c_2$, $c_1 \neq c_2$, both satisfy the asymptotic relationship (7) for the same solution x of (2), then

$$\| \Delta(t)(y_1(t) - y_2(t)) \| = o(\alpha(t)), \qquad (t \to \infty).$$

Since $c_1 \neq c_2$, an application of (10) leads to a contradiction.

It remains to show that $y = y(t) = Y(t)c$ is nonzero provided $\|x_0\|$ is sufficiently large. To accomplish this, we note that

$$\left\| x_0 + \int_{t_0}^{t} Y^{-1}(s) f(s,\ x(s;\ t_0,\ x_0)) ds \right\|$$

$$\geq \|x_0\| + \rho(t) - r_0, \qquad t \geq t_0 .$$

If $\|x_0\| > r_0 - \rho_\infty$ and (9) is satisfied, then $\|c\| > 0$; and, hence, y is nonzero. This completes the proof of Theorem 1.

REMARK 1. - The inequality (9), $\|x_0\| \geq r_0$, is a weaker hypothesis than assuming that $\|\Delta(t_0)x_0\| \geq r_0\alpha(t_0)$. The stronger hypothesis has been imposed previously (see [2]) in analogous situations.

As mentioned in the proof of Theorem 1, conditions (3) and (4) require (in some sense) that the function $\Delta(t)\alpha^{-1}(t)$ be a good measurement of the fundamental matrix $Y(t)$ of (1). Some obvious choices of Δ that are useful in certain instances are $\Delta = I_n$ and $\Delta = Y^{-1}$. To demonstrate that functions other than these will also yield asymptotic results, we consider the second order nonlinear scalar equation

$$u'' + t^{-1}(t+1)^{-2}a(t)[(u - tu')^2 + u'^2]^{-1} = 0$$

where $a(t) \in C[1, \infty) \cap L^1[1, \infty)$. Using $u = x_1$, $u' = x_2$, we write the above equation as a system in the form (2) with

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$f(t, x) = \begin{pmatrix} 0 \\ -a(t)/t(t+1)^2[(x_1 - tx_2)^2 + x_2^2] \end{pmatrix}.$$

The fundamental matrix Y of the associated linear equation such that $Y(1) = I_2$ is

$$Y(t) = \begin{pmatrix} 1 & t-1 \\ 0 & 1 \end{pmatrix}.$$

For the purposes of this example, let $\|\cdot\|$ designate the sum of the absolute values of the components of a vector or matrix. With Δ and α chosen as

$$\Delta(t) = \begin{pmatrix} t^{-1} & -1 \\ 0 & t^{-1} \end{pmatrix}, \; \alpha(t) = 3^{-1}(t+1)^{-1},$$

it is easy to verify that (3) and (4) are satisfied with $k = 18$.

Since

$$\|Y^{-1}(t)f(t, x)\| = \frac{|a(t)|}{(t+1)^2[(x_1 - tx_2)^2 + x_2^2]},$$

the function w in (5) may taken as

$$w(t, r) = 18\,|a(t)|\,r^{-2},$$

where

$$r = \|\Delta(t)x\|\alpha^{-1}(t) = 3(t+1)[\,|x_1 - tx_2| + |x_2|\,].$$

Since $a \in L^1[1, \infty)$, the initial value problem (8) has a solution that remains positive provided r_0 is sufficiently large. Hence, the hypotheses of Theorem 1 have been verified for this example.

The hypothesis that the majorant function is nondecreasing in r presents some difficulties in a study of properties of the asymptotic manifold that do not occur when the majorant is nondecreasing in r. These difficulties might be alleviated by the development of the theory of differential and integral inequalities for such majorant functions. However, it seems to be very desirable to have an example of a differential equation of the from (6), with the property that given any initial data (t_0, r_0), $t_0 \geq 0$, $r_0 > 0$, there exists a minimal solution $r(t; t_0, r_0)$ of the initial value problem (8) that remains positive on $[t_0, \infty)$. The existence of such an example is not readily established. For instance, if a differential equation with variables separable is considered, under very natural assumptions, the above property cannot be demonstrated.

REMARK 2. - Let (*i*) $g(r)$ be positive, nondecreasing on $(0, \infty)$ and continuous on $[0, \infty)$ with $g(0) = 0$; and, (*ii*) $a(t)$ be continuous, positive, and in L^1 on $J = [0, \infty)$. Then, the equation $g(r)dr/dt = -a(t)$ has a solution that tends to zero as $t \to \infty$. Furthermcre, the equation has solutions that remain positive (that is, exist) only on a finite interval. These conclusions are a consequence of the following statements.

If $G r) = \int_0^r g(s)ds$, then $\lim_{t\to\infty} r(t;\, t_0,\, r_0) = 0$ provided $G(r_0) = \int_{t_0}^{\infty} a(s)ds$.

If $r_0 > 0$ and $G(r_0) < \int_{t_0}^{\infty} a(s)ds$ then $\lim_{t\to T^-} r(t;\, t_0,\, r_0) = 0$ for some T, $t_0 < T < \infty$.

The hypothesis $a \in L^1[0, \infty)$ of (*ii*) is a necessary condition for a solution to have a limit. In fact, for the more general equation (6), it is easy to show that if the initial value problem (8) has a solution $r = r(t;\, t_0,\, r_0)$ with $\lim_{t\to\infty} r(t;\, t_0,\, r_0) = r_\infty \geq 0$, then $\int^{\infty} w(s, \lambda)ds < \infty$ for all $\lambda > r_\infty$.

The above remarks point out the difficulties that arise when one attempts to proceed as was done in the case of a nondecreasing majorant, see [5]. We will continue our investigation by utilizing another property of asymptotic manifolds.

A positive, continuous, scalar valued function $\eta = \eta(t)$ defined on some interval $[T, \infty)$ satisfying $\lim_{t\to\infty} \eta(t) = 0$ will be called a ***null function.***

DEFINITION 2. - A subset A of the asymptotic manifold $S = S(\Delta, \alpha)$ of (2) generated by (1) is *locally asymptotically uniform* if given any solution $x_1 = x(t;\ t_1,\ x_1)$ of (2) in A, there exists a $\delta = \delta(x_1) > 0$ and a null function $\eta_{x_1} = \eta_{x_1}(t)$ defined on $[T, \infty)$ for some $T \geq t_1$, such that whenever $x(t;\ t_1,\ x_2)$ is in A with $\|x_1 - x_2\| < \delta$, then

$$\|\Delta(t)(x(t:\ t_1,\ x_2) - y_2(t))\| \leq \eta_{x_1}(t)\alpha(t), \qquad t \geq T$$

where $y_2 = y_2(t)$ denotes the solution of (1) that corresponds to $x(t;\ t_1,\ x_2)$ by (7).

DEFINITION 3. - A subset A of $S(\Delta, \alpha)$ is *asymptotically uniform* if there exists a null function $\eta = \eta(t)$ defined on $[T, \infty)$ for some $T \geq 0$, such that

$$\|\Delta(t)(x(t) - y(t))\| \leq \eta(t)\alpha(t), \qquad t \geq T$$

for all solutions $x = x(t)$ of (2) in A. The function y is the solution of (1) that corresponds to x by (7).

The next result shows that the comparison principle leads to asymptotic uniformity for certain subsets of $S(\Delta,\ \alpha)$.

THEOREM 2. - *Let the hypotheses of Theorem 1 be satisfied. Suppose that $I = [T_0,\ T_1]$ is a compact subinterval of $J_0 = [t_0,\ \infty)$. Then, the subset $A_\rho(I)$ of $S(\Delta,\ \alpha)$ which consists of all solutions $x = x(t)$ of (2) in $S(\Delta, \alpha)$ that satisfy the inequality*

$$\|Y^{-1}(t_1)x(t_0)\| \geq \rho(t_1). \tag{14}$$

for some t_1 in I, is asymptotically uniform.

PROOF. - Let x be in $A_\rho(I)$; then, since $A_\rho(I) \subseteq S(\Delta,\ \alpha)$, the argument used in the proof of Theorem 1 leads to the equation

$$\begin{aligned} &\Delta(t)(x(t) - y(t)) \\ &= -\Delta(t)Y(t)\int_t^\infty Y^{-1}(s)f(s,\ x(s))ds \end{aligned} \tag{15}$$

for some solution y of (1). The hypothesis (14) and Lemma 1 imply that for any x in $A_\rho(I)$

$$\|\Delta(t)x(t)\| \geq \rho(t)\alpha(t), \qquad t \geq T_1. \tag{16}$$

Equation (15) and the inequality (16) lead to

$$\|\Delta(t)(x(t)-y(t))\|\alpha^{-1}(t)$$

$$\leq k\int_t^{\infty} w(s, \|\Delta(s)x(s)\|\alpha^{-1}(s))ds$$

$$\leq k[\rho(t)-\rho_\infty], \qquad t\geq T_1 .$$

The right side of the above inequality yields a uniform null function, whose domain is $[T_1, \infty)$, that satisfies Definition 3 for the class $A_\rho(I)$. This completes the proof of Theorem 2.

COROLLARY. - *Let the hypotheses of Theorem* 1 *be satisfied and suppose that initial value problems of* (2) *have unique solutions. The class* A_ρ *of all solutions* $x=x(t)$ *of* (2) *that are in* $S(\Delta, \alpha)$ *and satisfy the inequality*

$$\|Y^{-1}(t_1^*)x(t_1^*)\|>\rho(t_1^*) \tag{17}$$

for some $t_1^*\in J_0$ *is locally asymptotically uniform.*

PROOF. - Let $x=x(t;\, t_1, x_1)$ be in A_ρ; then, the inequality (17) is satisfied for some $t_1^*\in J_0$. Since solutions of (2) depend continuously upon their initial data, there exists a $\delta>0$ such that if x_2 satisfies the inequality $\|x_1-x_2\|<\delta$ then

$$\|Y^{-1}(t_2)x(t_2;\, t_1, x_2)\|\geq\rho(t_2)$$

for t_2 in some compact subinterval I, of J_0, that contains t_1^*. Theorem 2 implies that $A_\rho(I)$ is asymptotically uniform. Hence, A_ρ is locally asmptotically uniform.

Our next result shows that limits of solutions in the asymptotic manifold are either in the asymptotic manifold or have a finite escape time.

THEOREM 3. - *Let the hypotheses of Theorem* 1 *be satisfied. Suppose that the asymptotic manifold,* $S=S(\Delta, \alpha)$, *of* (12) *generated by* (1) *is locally asymptotically uniform. If* $\{x_n\}_{n=1}^{\infty}$ *is any sequence of initial positions of solutions* $x_n(t;\, t_n, x_n)$ *that are in* $S(\Delta, \alpha)$ *with* $\lim_{n\to\infty}(t_n, x_n)=(t_*, x_*)$ *where* $\|x_*\|>0$, *and* $t_*\geq t_0$, *then either*

(a) *there exists a* T *where* $t_*<T<\infty$ *such that* $\lim_{t\to T^-}\|x(t;\, t_*, x_*)\|=0$; *or*

(b) *the solution* $x(t:\, t_*, x_*)$ *is in the asymptotic manifold* $S(\Delta, \alpha)$ *of* (2) *generated by* (1).

PROOF. - If the solution $x = x(t;\ t_*,\ x_*)$ does not exist for all $t \geq t_*$ and (a) does not occur then there exists a T_1, $t_* < T_1 < \infty$ such that $\lim_{t \to T_1^-} \| x(t,\ t_*,\ x^*) \| = \infty$. In particular, there is a t_1^*, $t_* \leq t_1^* < T_1$ so that

$$\| Y^{-1}(t_1^*) x(t_1^*;\ t_*,\ x_*) \| \geq \rho(t_1^*).$$

Lemma 1 implies that

$$\| \Delta(t) x(t;\ t_*,\ x_*) \| \geq \rho(t) \alpha(t)$$

for all $t \geq t_1^*$ for which $x(t;\ t_*,\ x_*)$ exists.

However, from the variation of parameters equation, we obtain

$$\begin{aligned}
&\| \Delta(t) x(t;\ t_*,\ x_*) \| \alpha^{-1}(t) \\
&\leq k\alpha^{-1}(t_1^*) \| \Delta(t_1^*) x(t_1^*;\ t_*,\ x_*) \| \\
&+ k \int_{t_1^*}^{t} \| Y^{-1}(s) f(s,\ x(s;\ t_*,\ x_*) \| \, ds \\
&\quad \leq k\alpha^{-1}(t_1^*) \| \Delta(t_1^*) x(t_1^*;\ t_*,\ x_*) \| \\
&+ k \int_{t_1^*}^{t} w(s,\ \| \Delta(s) x(s;\ t_*,\ x_*) \| \alpha^{-1}(s) ds \\
&\leq k\alpha^{-1}(t_1^*) \| \Delta(t_1^*) x(t_1^*;\ t_*,\ x_*) \| \\
&+ k \int_{t_1^*}^{t} w(s,\ \rho(s)) ds, \qquad t \geq t_1^*.
\end{aligned}$$

This yields a contradiction and shows that if $x(t;\ t_*,\ x_*)$ has a finite escape time T, then condition (a) must hold at T.

Next, the case where the solution $x(t;\ t_*,\ x_*)$ exists for all $t \geq t_*$ will be considered. In this instance we will show that condition (b) holds; that is, $x(t;\ t_*,\ x_*)$ is in $S(\Delta,\ \alpha)$. Since $x_n(t;\ t_n,\ x_n)$ is in $S(\Delta,\ \alpha)$ is locally asymptotically uniform, there exists a vector function $z(t)$, where $\lim_{t \to \infty} \| z(t) \| = 0$, which satisfies the equation

$$\Delta(t) y_n(t) = \Delta(t) x_n(t;\ t_n,\ x_n) + z(t) \alpha(t), \tag{18}$$

for $t \geq T$ and $n = 1,\ 2,\ \ldots$ The uniform convergence of the sequence of solutions $\{x_n(t;\, t_n,\, x_n)\}$ to the solution $x(t;\, t_*,\, x_*)$ on compact subintervals implies that $\lim_{n\to\infty} x_n(T;\, t_n,\, x_n) = x(T;\, t_*,\, x_*)$.

We note that if solutions to initial value problems of (2) are not unique, then a choice of a subsequence of $\{x_n(t;\, t_n,\, x_n)\}$ might be necessary to guarantee the uniform convergence that is desired. This subsequence will be designated, as the original sequence was, by $\{x_n(t;\, t_n,\, x_n)\}$. In (18), let $y_n(t) = Y(t)c_n$; then, we obtain

$$c_n = Y^{-1}(T)x_n(T;\, t_n,\, x_n) + Y^{-1}(T)\Delta^{-1}(T)z(T)\alpha(T). \tag{19}$$

The equation (19) implies that $\lim_{n\to\infty} c_n$ exists; let $c = \lim_{n\to\infty} c_n$. The solution $y(t) = X(t)c$ of (1) is the asymptotic correspondent (under (7)) of $x(t;\, t_*,\, x_*)$. To verify this statement we note that the limit

$$\lim_{t\to\infty} \|\Delta(t)(x_n(t;\, t_n,\, x_n) - y_n(t))\|\alpha^{-1}(t) = 0$$

is uniform in n since $S(\Delta,\, \alpha)$ is locally asymptotically uniform. Also, for each $t \geq T$,

$$\lim_{n\to\infty} \|\Delta(t)(x_n(t;\, t_n,\, x_n) - y_n(t))\|\alpha^{-1}(t)$$

$$= \|\Delta(t)(x(t;\, t_*,\, x_*) - y(t))\|\alpha^{-1}(t).$$

An application of the MOORE–OSGOOD Theorem leads to

$$\lim_{t\to\infty} \|\Delta(t)(x(t;\, t_*,\, x_*) - y(t))\|\alpha^{-1}(t) = 0\,;$$

that is, $x(t;\, t_*,\, x_*)$ is in the asymptotic manifold $S(\Delta,\, \alpha)$ of (2) generated by (1). This completes the proof of Theorem 4.

REMARK 3. – The conclusion (a) of Theorem 3 may not be neglected. The following example demonstrates that limits of solutions in the asymptotic manifold need not exist in the future. Consider the differential equation

$$\frac{dr}{dt} = \begin{cases} -1 & 0 \leq t \leq 1, \quad r > 0 \\ t-2 & 1 < t \leq 1, \quad r > 0\,; \\ 0 & 2 < t, \quad\quad r > 0. \end{cases} \tag{20}$$

The solutions of (20) are given by

$$r(t)=\begin{cases} -t+c_0 & 0\le t\le 1, \quad r>0\\ \dfrac{t^2}{2}-2t+c_1 & 1<t\le 2, \quad r>0\\ c_2 & 2<t. \quad r>0,\end{cases}$$

where c_i, $i=1, 2$, are constants.

The separatrix solution $r_*=r_*(t)$ occurs when the constants are chosen as $c_0=\frac{3}{2}$ and $c_1=2$; we note that r_* exists on $[0, 2)$. Any sequence of solutions, $\{r_n(t;\ t_0,\ r_n)\}_{n=1}^{\infty}$, of (20) such that $r_n(t_0;\ t_0,\ r_n)\downarrow r_*(t_0)\,(n\to\infty)$ for $0\le t_2<2$ has the property that $r_n(t;\ t_0,\ r_n)$ is in the asymptotic manifold that is obtained by considering (20) as a perturbation of the equation $dv/dt=0$. However, $\lim_{t\to 2^-}(r_*(t)=0$; that is, r_* has a finite escape time.

Next, we consider the necessity of assuming that the manifold $S(\Delta,\ \alpha)$ is locally asymptotically uniform. We will restrict our remarks to scalar comparison equation of the form (6).

THEOREM 4. - *Let $w=w(t,\ r)$ satisfy the condition* (5) *and let $dr/dt=-w(t,\ r)$ have unique solutions to the initial value problem. Furthermore, suppose that the set $S(1,\ 1)$, which consists of all positive solutions $r=r(t)$ of* (6) *that satisfy the condition $\lim_{t\to\infty} r(t)=c$ for some constant c, is closed in the sense that condition* (b) *of Theorem 3 always holds then $S(1,\ 1)$ is locally asymptotically uniform.*

PROOF. - Let $r_0=r_0(t)$ be in $S(1,\ 1)$. Suppose that $\delta>0$ is chosen so that

$$r_0(t_1)-\delta>\varepsilon>0$$

where ε is some positive number and for some $t_1\ge 1$.

Let r_* denote the infimum of all initial positions r_1 where $r_1>r_0(t_1)-\delta$ and the solution $r(t;\ t_1,\ r_1)$ of (6) is in $S(1,\ 1)$. Since $S(1,\ 1)$ is closed, the solution $r_*(t)=r(t;\ t_1,\ r_*)$ is also in $S(1,\ 1)$. By uniqueness of solutions to initial value problems,

$$r(t;\ t_1,\ r_1)\ge r_*(t;\ t_1,\ r_*)$$

for any solution $r(t;\ t_1,\ r_1)$ in $S(1,\ 1)$ where $r_1\ge r_*$. Therefore,

$$r(t;\ t_1,\ r_1)-c_1=\int_t^\infty w(s,\ r(s;\ t_1,\ r_1))ds$$

$$\leq \int_{t}^{\infty} w(s, r_*(s; t_1, r_*))ds$$

$$= r_*(t; t_1, r_*) - c_* .$$

where $\lim_{t\to\infty} r_*(t; t_1, r_*) = c_*$. Since the function $\eta(t) = r_*(t; t_1, r_*) - c_*$ is a null function, $S(1, 1)$ is locally asymptotically uniform.

II. - The asymptotic manifold of the linear equation as determined by the nonlinear equation.

The next result is a dual result to Theorem 1. The duality is concerned with the interchange of the roles of the equations (1) and (2). The analogue for nondecreasing majorant functions may be found in [5].

THEOREM 5. - *Let the hypotheses of Theorem 1 be satisfied. Then, corresponding to any solution $y = y(t) = Y(t)c$ of (1) with $\|c\|$ sufficiently large there exists a solution $x = x(t)$ of (2) so that the asymptotic relationship (7) is satisfied.*

REMARK 4. - The hypotheses imply that $\lim_{t\to\infty}\rho(t) = \rho_\infty$ exists. The proof of the theorem will show that "$\|c\|$ sufficiently large" means $\|c\| > \rho_\infty$.

PROOF OF THEOREM 5. - It is easy to see that

$$\int^{\infty} w(t, \lambda)dt < \infty \tag{21}$$

for all $\lambda > \rho_\infty$. Let $\|c\| > \rho_\infty$ and choose λ_0 satisfying the inequality $\|c\| > \lambda_0 > \rho_\infty$. Define $\gamma = \|c\| - \lambda_0 > 0$ and consider the set

$F = \{u;\ u(t) = \Delta(t)\alpha^{-1}(t)x(t)$ where x is continuous on J and $\lambda_0 \leq \|u\| \leq \leq k(\|c\| + \gamma)\}$.

Using (21), we suppose that t_0 is sufficiently large so that

$$\int_{t_0}^{\infty} w(s, \lambda_0)ds < \gamma .$$

The transformation T defined on F by

$$Tu(t) = \frac{\Delta(t)Y(t)c}{\alpha(t)} \tag{22}$$

$$-\frac{\Delta(t)Y(t)}{\alpha(t)}\int_t^\infty Y^{-1}(s)f(s,\ \Delta^{-1}(s)\alpha(s)u(s))ds$$

maps F into itself since using (5) and (2), we obtain the lower bound

$$\begin{aligned}\lambda_0\alpha^{-1}(t) &\leq \alpha^{-1}(t)\left[\|c\| - \int_t^\infty w(s,\ \lambda_0)ds\right]\\ &\leq \alpha^{-1}(t)\left[\|c\| - \int_t^\infty \|Y^{-1}(s)f(s,\ \Delta^{-1}(s)\alpha(s)u(s))\|ds\right]\\ &\leq \|Y^{-1}(t)\Delta^{-1}(t)Tu(t)\|\\ &\leq \alpha^{-1}(t)\|Tu(t)\|.\end{aligned}$$

The upper bound on $\|Tu\|$ is obtained from (22) as follows:

$$\begin{aligned}\|Tu(t)\| &\leq k\|c\| + k\int_t^\infty \|Y^{-1}(s)f(s,\ \Delta^{-1}(s)u(s))\|ds\\ &\leq k\left[\|c\| + \int_t^\infty w(s,\ \lambda_0)ds\right]\\ &\leq k[\|c\| + \gamma].\end{aligned}$$

These two inequalities show that $TF \subset F$.

The operator T is continuous and the functions in the image set TF are equicontinuous and bounded at every point of J_0. The details which are used to verify this statement are similar to those found in the proof of Theorem 2 of [5]. A modification that is required in the details of [5] is that the positive lower bound λ_0 in the definition of the class F is used because of the nonincreasing nature of the function w.

The SCHAUDER–TYCHONOFF Theorem now implies that there exists a $u \in F$ such that $u(t) = Tu(t)$; that is, there exists a solution $x = x(t)$ of the equation

$$x(t) = Y(t)c - Y(t)\int_t^\infty Y^{-1}(s)f(s,\ x(s))ds.$$

From this equation, it follows that $x(t)$ is a solution of equation (2) which possesses the asymptotic behavior (7). This completes the proof of the theorem.

REMARK 5. - Motivated by the Corollary to Theorem 5 of [5], which is valid for nondecreasing majorants, it might be expected that (under certain hypotheses) the asymptotic manifold of the linear equation as generated by equation (2) consists of all of the solutions of (1).

An obvious condition that one could impose upon the comparison equation is that there exists a minimal solution $\rho = \rho(t)$ to an initial value problem (8) which satisfies $\lim_{t\to\infty}\rho(t) = 0$. Under this assumption, Theorem 5 shows that the linear asymptotic manifold contains all solutions of the form $y(t) = Y(t)c$ where c is in $R^n - \{0\}$. Because of our required domain of definition of f, this is the best that we can do, as the following simple example demonstrates. Consider the scalar equations

$$\frac{dx}{dt} = 0 \qquad t \in J, \qquad |x| > 0. \tag{25}$$

In the comparison equation, take $w(t, r) = a(t)r^{-1}$, where $a(t)$ is continuous and positive on J and $\int_0^\infty a(t)dt < \infty$. The comparison equation has a solution that tends to zero as t tends to infinity. However, (25) has no solution to correspond to $y \equiv 0$ under the asymptotic correspondence (7).

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