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The asymptotic nature of the analytic spread — Source link

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The asymptotic nature of the analytic spread

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In (3), corollary, p. 373) Burch gives the following inequality for the analytic spread $l(I)$ of an ideal I of a noetherian local ring (R, m):

$$l(I) \leqslant \dim (R) - \min_n \operatorname{depth} (R/I^n). \tag{0}$$

In this paper we shall improve this by showing that the number min depth (R/I^n) may be replaced by the asymptotic value of depth (R/I^n) for large n (which exists) (see Section (2)). By its definition (see (6), def. 3)) the analytic spread is of asymptotic nature, i.e. depends on the modules $I^n/mI^n = U_n$ only for large n. We shall prove a stronger result, Section (4), which also shows the asymptotic nature of $l(I)$. This result might be interesting for itself, particularly as it is not of local nature. Once Section (4) is proved and once we know that depth (R/I^n) is asymptotically constant (which turns out to be an easy consequence of (1), (1)), our improved inequality is easily established: Indeed, replacing R by R/xR where x is regular with respect to almost all modules R/I^n, we perform a change which affects only finitely many of the modules U_n (see Section (8)).

In Section (12) we give a result which relates the depths of the R-modules I^{n-1}/I^n and R/I^n. Namely, if R is Cohen–Macaulay, and if $ht(I) > 0$, these two values coincide asymptotically.

As an application we shall give another proof (and slight improvement) of a criterion for local complete intersections, due to Vogel–Achilles (7), which, in its turn, improves a corresponding result of Cowsik and Nori (4), p. 219.

We intend to make our results as general as possible, in particular to show that the arguments we use are not confined to local rings. So, unlike Burch, we shall not use completions or Nakayama's lemma. On the other hand, to keep the paper self-contained, we reproduce some arguments which may be known to the reader familiar with the subject.

Let R be a noetherian ring, and let $I, J \subseteq R$ be ideals. Then by the Rees ring of I, we mean the graded R algebra

$$\tilde{R} = \bigoplus_{n \geqq 0} I^n.$$

Now for each R-module M we may form the graded $\tilde{R}$-modules

$$\tilde{M} = \bigoplus_{n \geqq 0} I^n M$$

and

$$M^* = \bigoplus_{n \geqq 0} I^n M / J I^n M = \tilde{M} / J \tilde{M}.$$

Next define the *analytic spread of I at J with respect to M* as the Krull dimension of the $\tilde{R}$-annihilator of M^*, i.e.

$$l_J(I, M) = \dim_{\tilde{R}}(M^*).$$

(1) *Remark.* If (R, J) is local, $l_J(I, R)$ is the analytic spread $l(I)$ of I, introduced by Northcott and Rees (6). If N is a finitely generated R-module, let the J-grade of N, $\mathrm{gr}_J(N)$, be the maximal length of N-sequences in J. Now we are ready to state:

(2) THEOREM. *Let M be an R-module of finite type. Then*

(i) $\mathrm{gr}_J(M/I^nM)$ *takes a constant value* $g_J(I, M)$ *for large n.*

(ii) *If* $I \subseteq J$, *we have the inequality* $l_J(I, M) \leqslant \dim_R(M) - g_J(I, M)$.

(3) *Remark.* If we take (R, J) local and $M = R$, (ii) obviously gives the announced improvement of Burch's inequality (0). Let us prove (i) by induction on the inferior limit

$$h_M = \liminf_{n\to\infty} \mathrm{gr}_J(M/I^nM).$$

If $h_M = 0$, we have

$$J \subseteq \bigcup_{p \in A(n)} p$$

for infinitely many n, where $A(n) = \mathrm{Ass}(M/I^nM)$. But by [1, (1)] $A(n)$ is stable for large n, and so we see that $\mathrm{gr}_J(M/I^nM) = 0$ for sufficiently large n.

If $h_M > 0$, use the same stability argument to show that there is an $x \in J$ which lies outside $\bigcup_{p \in A(n)} p$ for sufficiently large n. Then we clearly have, with $\bar{M} = M/xM$,

$$h_{\bar{M}} < h_M; \quad \mathrm{gr}_J(\bar{M}/I^n\bar{M}) = \mathrm{gr}_J(M/I^nM) - 1 \quad (n \gg 0),$$

so (i) follows by induction.

Now we prove (ii) by induction on $g_J(I, M) = c$. So let $c = 0$. Then, obviously,

$$\mathrm{ann}_{\tilde{R}}(\tilde{M}) = \bigoplus_{n \geqq 0} \mathrm{ann}_R(M) \cap I^n,$$

which shows that $\tilde{R}/\mathrm{ann}_{\tilde{R}}(\tilde{M})$ is isomorphic to the Rees-ring $\tilde{\bar{R}}$ of $I\bar{R}$ where

$$\bar{R} = R/\mathrm{ann}_R(M).$$

So we get

$$l_J(I, M) \leqslant \dim(\tilde{\bar{R}}/J\tilde{\bar{R}}).$$

But on the other hand, it is easy to see that

$$\dim(\tilde{\bar{R}}) \leqslant \begin{cases} \dim(\bar{R}), & \text{if} \quad \dim(\tilde{\bar{R}}/I\tilde{\bar{R}}) = \dim(\tilde{\bar{R}}) \\ \dim(\bar{R}) + 1, & \text{otherwise.} \end{cases}$$

As $\dim(\bar{R}) = \dim_R(M)$ we are done in the first case. Note that, in the second case, clearly

$$\dim(\tilde{\bar{R}}/J\tilde{\bar{R}}) < \dim(\tilde{\bar{R}});$$

hence $\dim(\tilde{\bar{R}}/J\tilde{\bar{R}}) \leqslant \dim_R(M)$.

If $c > 0$, we need more information on M^*. To formulate our results, let us introduce the following notations. If N is any graded $\tilde{R}$-module, let $(N)_n$ be the R-module of its n-forms, and let $(N)_{\geqq n}$ be the R-submodule of $\tilde{M}$ given by

$$(N)_{\geqq n} = \bigoplus_{m \geqq n} (N)_m.$$

We claim

(4) For all n we have

$$(\sqrt{\{\mathrm{ann}_{\tilde{R}}(M^*)\}})_{\geq 1} = (\sqrt{\{\mathrm{ann}_{\tilde{R}}((M^*)_{\geq n})\}})_{\geq 1}.$$

The inclusion $\subseteq$ is clear. So let $a \in R_h = I^h$ $(h > 0)$ be a form which annihilates $(M^*)_m$ for some m. As, over $\tilde{R}$, M^* is generated by its 0-forms, it suffices to show that a^s annihilates $(M^*)_0$ for an appropriate s.

By our hypothesis we have that

(5) $$aI^mM \subseteq JI^{m+h}M,$$

and we have to find an s such that

(6) $$a^sM \subseteq JI^{sh}M.$$

Note that (5) induces that $a^sI^mM \subseteq J^sI^{m+sh}M \subseteq I^{m+sh}JM$ for all s. Let $t < s$. Then, obviously, $a^sM \subseteq I^{sh}M \subseteq I^{th}JM$. Now, by an Artin–Rees argument, we have

$$(O: I)_{I^{th}JM} = 0$$

for sufficiently large values of t [see (1), (5)]. (If A, B are sub-modules of an R-module C, and $L \subseteq R$ is an ideal, $(A:L)_B$ is defined as $\{b \in B \mid Lb \subseteq A\}$.) Then by [(1), (4)] we find an $s > t$ such that for each $n \geqslant 0$

$$(I^{n+sh}JM: I)_{I^{th}JM} = (I^{n+(s-t)h}(I^{th}JM): I)_{I^{th}JM} = I^{n+(s-t)h-1}(I^{th}JM) = I^{n-1+sh}JM.$$

Applying this repeatedly, we get finally that

$$a^sM \subseteq (I^{m+sh}JM: I^m)_{I^{th}JM} \subseteq I^{sh}JM.$$

Note the following obvious fact.

(7) (i) $\mathrm{ann}_{\tilde{R}}(M^*)_0 = \mathrm{ann}_R(M/JM)$, consequently,
(ii) if $x \in J$, $\mathrm{ann}_{\tilde{R}}((M/xM)^*)_0 = \mathrm{ann}_{\tilde{R}}(M^*)_0$.

Now we complete the proof of (2). As $c = g_J(I, M) > 0$ (by our assumption) the asymptotic stability of $A(n)$ shows that there is an $x \in J$ which is regular with respect to M/I^nM for all sufficiently large n. Now there is a canonical homomorphism of graded R-modules

$$M^* \xrightarrow{\phi} (M/xM)^*,$$

given in degree n by the canonical maps

$$(M^*)_n = I^nM/JI^nM \xrightarrow{\phi_n} I^n(M/xM)/JI^n(M/xM) = (M/xM)^*_n.$$

We claim that ϕ is a quasiisomorphism, i.e. that

(8) ϕ_n is an isomorphism for all large n. Indeed, we have the canonical isomorphisms

$$\begin{aligned}(M/xM)^*_n &\simeq (I^nM + xM)/(JI^nM + xM) \simeq I^nM/(JI^nM + xM) \cap I^nM \\ &= I^nM/(JI^nM + xM \cap I^nM),\end{aligned}$$

and it suffices to show that $xM \cap I^nM \subseteq JI^nM$ for large values of n. As $x \in J$, this means proving that $(I^nM:x)_M = I^nM$ for sufficiently large n. But this is clear by our choice of x.

(8), (7) and (4) now induce

(9) $$l_J(I, M) = l_J(I, M/xM).$$

On the other hand, we have, obviously,

(10) $$g_J(I, M) = g_J(I, M/xM) + 1.$$

As $I \subseteq J$, it is clear that any minimal prime divisor of M containing J would belong to $A(n)$ for all sufficiently large n. This shows that x moreover may be chosen outside of all minimal prime divisors of M, hence that

(11) $$\dim(M) = \dim(M/xM) + 1.$$

But, as by induction, we have

$$l_J(I, M/xM) \leqslant \dim(M/xM) - g_J(I, M/xM);$$

(9)–(11) give the inequality (2(ii)).

Now we want to compare the asymptotic behaviour of $\operatorname{gr}_J(M/I^nM)$ to that of $\operatorname{gr}_J(I^{n-1}M/I^nM)$.

(12) PROPOSITION. *If M is finitely generated,*

(i) $\operatorname{gr}_J(I^{n-1}M/I^nM)$ *takes a constant value* $\bar{g}_J(I, M)$ *for all sufficiently large n, which satisfies the inequality* $g_J(I, M) \leqslant \bar{g}_J(I, M)$.

(ii) *If* $I \subseteq J$, $\operatorname{gr}_I(M) > 0$, *and* $\operatorname{gr}_J(M) = \operatorname{ht}(J/\operatorname{ann}(M))$, *we have equality in* (i).

Proof. (i) To show that $\operatorname{gr}_J(I^{n-1}M/I^nM)$ asymptotically stabilizes, we could apply the same argument as in the proof of (2(ii)), if we knew that

$$B(n) = \operatorname{Ass}(I^{n-1}M/I^nM)$$

is asymptotically stable. But this is clear, as $B(n)$ is increasing for large n ((1), proof (1)), and as $B(n) \subseteq A(n)$. This latter relation then also immediately proves the inequality.

(ii) As $\operatorname{gr}_I(M) > 0$ we have that $A(n)$ and $B(n)$ take the same asymptotic value A^* by [(1), (7)]. This proves the equality in case $\bar{g} = \bar{g}_J(I, M) \leqslant 1$. Let $y \in I \cap \operatorname{reg}(M)$. If $g > 1$ there is an $x \in J \cap \operatorname{reg}(M) \cap \operatorname{reg}(M/yM)$, but outside of $\bigcup_{p \in A^*} p$, as otherwise $\bar{g} \leqslant \operatorname{ht}(J/\operatorname{ann}(M)) = \operatorname{gr}_J(M) \leqslant 1$. Thus we have $y \in \operatorname{reg}(M/xM)$; hence

$$\operatorname{gr}_I(M/xM) > 0,$$

and $g_J(I, M/xM) = g_J(I, M) - 1$, $\bar{g}_J(I, M/xM) = \bar{g}_J(I, M) - 1$. Finally it holds that $\operatorname{ht}(J/\operatorname{ann}(M/xM)) < \operatorname{ht}(J/\operatorname{ann}(M))$, and so, by induction, applied to M/xM, we get the equality.

(13) *Remark.* The example [(1), (8)] shows that $g_J(I, M)$ and $\bar{g}_J(I, M)$ do not coincide in general. If, on the other hand, $R = M$ is a Cohen–Macaulay ring, and if $\operatorname{ht}(I) > 0$, they do.

Now we want to give the announced application.

(14) COROLLARY (cf. Cowsik–Nori (4), p. 219), Vogel–Achilles (7), Waldi (8)).

Let (R, m) be a local Cohen–Macaulay ring, and let $I \subseteq R$ be an ideal of height $h > 0$.

Assume that IR_p is generated by h elements for each minimal prime P of I. Then the following statements are equivalent:

(i) *I^{n-1}/I^n is a Cohen–Macaulay module over R/I for infinitely many n.*

(ii) *R/I^n is a Cohen–Macaulay ring for infinitely many n.*

(iii) *I is generated by h elements (hence a complete intersection).*

Proof. (i) ⇒ (ii) is an immediate consequence of (12).

(iii) ⇒ (i). (iii) implies that R/I is C.M. and that all the R/I-modules I^{n-1}/I^n are free. So it remains to prove (ii) ⇒ (iii). But this implication is clear by (2(ii)), which induces

$$l(I) = l_m(I, R) \leqslant \dim(R) - \dim(R/I) = h,$$

and the following result essentially due to Cowsik–Nori (4).

(15) Let R and I be as in (14). Then I is generated by h elements if and only if $l(I) \leqslant h$.

(16) *Remark.* Cowsik and Nori give a slightly more special form of (15). The above statement is given in (5), p. 179.

Finally, note that Section (14) generalizes a part of (2), (6·5), where only the case $h = \dim(R) - 1$ was treated.

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