The asymptotic optimality of the LPT rule

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by

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THE ASYMPTOTIC OPTIMALITY OF THE LPT RULE

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Abstract

For the problem of minimizing makespan on parallel machines of different
speed, the behaviour of list scheduling rules is subjected to a probabilistic
analysis under the assumption that the processing requirements of the jobs are
independent, identically distributed nonnegative random variables. Under mild
conditions on the probability distribution, we obtain strong asymptotic
optimality results for arbitrary list scheduling and even stronger ones for
the LPT (Longest Processing Time) rule, in which the jobs are assigned to the
machines in order of nonincreasing processing requirements.

Keywords: scheduling, parallel machines, list scheduling, LPT rule,
probabilistic analysis, asymptotic optimality.

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1. Introduction

One of the fundamental problems in scheduling is the minimization of makespan on parallel identical machines. In this problem n jobs have to be distributed among m machines so as to minimize the time needed to process to them. We shall denote the processing requirement of the j-th job by p_j ($j=1,\ldots,n$). If the set of jobs assigned to the i-th machine is denoted by M_i ($i=1,\ldots,m$), then the time required by that machine to process all its jobs is equal to $Z_{i,n} \triangleq \Sigma_{j \in M_i} p_j$ and the objective is to minimize the makespan

$$Z_n^{(m)} \triangleq \max_i \{Z_{i,n}\}.$$

This problem is well known to be NP-hard if $m \geq 2$ [11]. This motivates the design and analysis of heuristic methods that with moderate computational effort produce a value $Z_n^{(m)}(\text{HEUR})$ which is reasonably close to the optimal value $Z_n^{(m)}(\text{OPT})$. Among these heuristics, particular attention has been paid to list scheduling rules (LS), in which jobs are assigned successively to the first available machine in the order in which they appear on a predetermined priority list. Indeed, one of the oldest worst case results in scheduling theory [12] is concerned with the behaviour of such rules; it states that

$$\frac{Z_n^{(m)}(\text{LS})}{Z_n^{(m)}(\text{OPT})} \leq 2 - \frac{1}{m}. \tag{1}$$

The examples for which this worst-case bound is actually achieved suggest that a better bound should be obtainable if the jobs appear in the list in order of nonincreasing p_j. And indeed, for this LPT (Longest Processing Time) rule, it is shown in [13] that

$$\frac{Z_n^{(m)}(\text{LPT})}{Z_n^{(m)}(\text{OPT})} \leq \frac{4}{3} - \frac{1}{3m}. \tag{2}$$

Such worst-case results, however, are inherently pessimistic and do not necessarily provide much information about the performance of the heuristic in practice. To carry out a rigorous study of the latter phenomenon, it is necessary to specify a probability distribution over the class of problem instances and study the relation between the random variables $\underline{Z}_n^{(m)}(\text{HEUR})$ and $\underline{Z}_n^{(m)}(\text{OPT})$. The common way to arrive at such a probability distribution is to assume that the processing requirements p_j are independent, identically

distributed nonnegative random variables, generated from some given probability distribution.

The initial probabilistic analyses of the LPT rule strengthened the intuition that it is a reasonable heuristic for this scheduling model. For instance, under the assumption that the p_j are uniformly distributed on $[0,1]$ it is known that [4]

$$\frac{EZ_n^{(m)}(LPT)}{EZ_n^{(m)}(OPT)} = 1 + O\left(\frac{m^2}{n^2}\right),\tag{3}$$

so that the heuristic is asymptotically relatively optimal in expectation:

$$\lim_{n\to\infty} \frac{EZ_n^{(m)}(LPT)}{EZ_n^{(m)}(OPT)} = 1\tag{4}$$

In addition, the absolute difference $Z_n^{(m)}(LPT) - Z_n^{(m)}(OPT)$ has been studied under the assumption that the p_j have finite first moment Ep; it is known to be bounded by an a.s. fixed valued random variable [14]. Below, these results are extended in various ways.

To start with, we shall extend the underlying scheduling model to allow for uniform rather than identical machines: the i-th machine has speed s_i and $Z_{i,n}$ is redefined as $(\Sigma_{j\in M_i} p_j)/s_i$. The extension of list scheduling rules to this new situation is straightforward.

In Section 2, we consider the LPT rule for this model. We assume that the density function of the processing requirements is strictly positive in a neighbourhood of 0 and show that, if Ep is finite, the LPT rule is asymptotically absolutely optimal almost surely:

$$\lim_{n\to\infty} (Z_n^{(m)}(LPT) - Z_n^{(m)}(OPT)) = 0 \;(a.s.).\tag{5}$$

We also show that, if Ep^2 is finite, the LPT rule is asymptotically absolutely optimal in expectation

$$\lim_{n\to\infty} (EZ_n^{(m)}(LPT) - EZ_n^{(m)}(OPT)) = 0.\tag{6}$$

These results represent strong forms of asymptotic optimality for the LPT

rule. In Section 3, we consider the speed at which convergence to absolute optimality occurs. For almost sure convergence (5) we show that if the p_j are generated from a _uniform distribution_ or a _negative exponential distribution_, then the speed of convergence is almost surely proportional to $(\log n)/n$. For convergence in expectation, we show that if the p_j are uniformly distributed, then

$$EZ_{-n}^{(m)}(LPT) - EZ_{-n}^{(m)}(OPT) = 0 \ (\frac{m^2}{n}), \tag{7}$$

thus generalizing the result in [4] (cf. (3)).

In Section 4, we show how similar techniques yield comparable (but not surprisingly somewhat weaker) results for arbitrary list scheduling rules. For the case of _identical machines_ ($s_i=1$ for all i) and under the assumption that Ep^3 is finite, we show that

$$EZ_{-n}^{(m)}(LS) - EZ_{-n}^{(m)}(OPT) \leq (1 - \frac{1}{m}) \ Ep + (1 - \frac{1}{2m}) \ \frac{\sigma^2(p)}{Ep} \ . \tag{8}$$

We also indicate how this result can be extended to the case of arbitrary uniform machines. Finally, in Section 6, we show how the results for the LPT rule can be applied to yield speed of convergence results for certain _hierarchical scheduling heuristics_ [6]. We also provide some concluding remarks and topics for further research.

2. The LPT rule for uniform machines

Let us assume that the machines are numbered in such a way that $s_1 \geq s_2 \geq \ldots \geq s_m$. A formal description of _arbitrary list scheduling_ can be given as follows. If the _partial sums_ $Z_{i,n-1}(i=1,\ldots,m)$ are ranked in non-decreasing order:

$$Z_{n-1}^{(1)} \leq Z_{n-1}^{(2)} \leq \ldots \leq Z_{n-1}^{(m)}, \tag{9}$$

then the n-th job is assigned to machine k such that

$$Z_{k,n-1} = Z_{n-1}^{(1)} = \min_i \ \{Z_{i,n-1}\}, \tag{10}$$

so that

$$z_n^{(m)} = \max \{z_{n-1}^{(m)}, \ z_{n-1}^{(1)} + \frac{P_n}{s_k}\} \tag{11}$$

As in [14], much of our analysis will focus on the difference between the largest and the average partial sum:

$$D_n \triangleq z_n^{(m)} - \frac{1}{m} \sum_{i=1}^{m} z_n^{(i)} =$$

$$= z_n^{(m)} - \frac{1}{m} \sum_{i=1}^{m} Z_{i,n} \geq 0. \tag{12}$$

Applying (11), we obtain the following recurrence

$$D_n = z_n^{(m)} - \frac{1}{m} \sum_{i=1}^{m} Z_{i,n} =$$

$$= \max \{z_{n-1}^{(m)} - \frac{1}{m} \sum_{i=1}^{m} Z_{i,n},$$

$$z_{n-1}^{(1)} + \frac{P_n}{s_k} - \frac{1}{m} \sum_{i=1}^{m} Z_{i,n}\} =$$

$$= \max \{z_{n-1}^{(m)} - \frac{1}{m} \sum_{i=1}^{m} Z_{i,n-1} - \frac{P_n}{ms_k},$$

$$z_{n-1}^{(1)} - \frac{1}{m} \sum_{i=1}^{m} Z_{i,n-1} + \frac{(m-1)P_n}{ms_k}\} \leq$$

$$\leq \max \{D_{n-1} - \frac{P_n}{ms_1}, \frac{(m-1)P_n}{ms_m}\} \tag{13}$$

By iteration, we obtain

$$D_n \leq \max \{D_1 - \sum_{j=2}^{n} \frac{P_j}{ms_1}, \frac{(m-1)P_2}{ms_m} - \sum_{j=3}^{n} \frac{P_j}{ms_1}, \ldots, \frac{(m-1)P_n}{ms_m}\} \tag{14}$$

Since, by definition

$$D_1 = \frac{p_1}{s_1} - \frac{1}{m}\frac{p_1}{s_1} =$$

$$= \frac{(m-1)p_1}{ms_1} \leq$$

$$\leq \frac{(m-1)p_1}{ms_m}, \tag{15}$$

we obtain that

$$D_n \leq \frac{1}{ms_1} \max_{1 \leq k \leq n} \{\alpha p_k - \textstyle\sum_{j=k}^{n} p_j\} \tag{16}$$

with

$$\alpha \triangleq \frac{(m-1)s_1}{s_m} + 1. \tag{17}$$

The above inequality (16) holds for arbitrary list scheduling rules. In the case of the LPT rule, we know that in addition

$$p_1 \geq p_2 \geq \cdots \geq p_n \tag{18}$$

Hence, if $p^{(j)}$ are the order statistics of $\{p_1,\ldots,p_n\}$, with

$$p^{(1)} \leq p^{(2)} \leq \cdots \leq p^{(n)}, \tag{19}$$

we have that in this case

$$D_n(LPT) \leq \frac{1}{ms_1} \max_{1 \leq k \leq n} \{\alpha p^{(k)} - \textstyle\sum_{j=1}^{k} p^{(j)}\}. \tag{20}$$

Now, let us assume that the p_j are i.i.d. nonnegative random variables with distribution function F, whose density function is strictly positive on $(0,\tilde{\epsilon})$ for some $\tilde{\epsilon} > 0$.

Theorem 1. If the expected processing time $E\underline{p}$ is finite, then

$$\lim_{n\to\infty} \underline{D}_n(\text{LPT}) = 0 \quad (\text{a.s.}) \tag{21}$$

Proof. For a certain $\epsilon \in (0,\tilde{\epsilon})$ to be chosen later, we give separate consideration in (20) to values $k \in \{1,\ldots,[\epsilon n]\}$ ([x] is the <u>integer rounddown</u> of x) and $k \in \{[\epsilon n] + 1,\ldots,n\}$. Clearly,

$$\max_{1 \leq k \leq [\epsilon n]} \{\alpha\underline{p}^{(k)} - \sum_{j=1}^{k} \underline{p}^{(j)}\} \leq \alpha\underline{p}^{([\epsilon n])} \tag{22}$$

and

$$\max_{[\epsilon n]+1 \leq k \leq n} \{\alpha\underline{p}^{(k)} - \sum_{j=1}^{k} \underline{p}^{(j)}\} \leq$$

$$\leq \alpha\underline{p}^{(n)} - \sum_{j=1}^{[\epsilon n]}\underline{p}^{(j)}. \tag{23}$$

Hence,

$$\underline{D}_n(\text{LPT}) \leq \frac{1}{m s_1} \max\{\alpha\underline{p}^{([\epsilon n])}, \alpha\underline{p}^{(n)} - \sum_{j=1}^{[\epsilon n]}\underline{p}^{(j)}, 0\}, \tag{24}$$

and we shall prove that the right hand can be made arbitrarily small almost surely.

Define

$$\underline{\xi}_{\epsilon,n} \triangleq \inf \{x : |\{k : \underline{p}_k \leq x\}| \geq \epsilon n\}. \tag{25}$$

Obviously,

$$\underline{p}^{([\epsilon n])} \leq \underline{\xi}_{\epsilon,n} \leq \underline{p}^{([\epsilon n]+1)} \tag{26}$$

Now consider the interval $(0,F(\tilde{\epsilon}))$. This interval is not empty since F has been assumed to be strictly increasing on $(0,\tilde{\epsilon})$. It follows that for all $y \in (0,F(\tilde{\epsilon}))$ [7, p.75]

$$\lim_{n\to\infty} \underline{\xi}_{y,n} = \xi_y \quad (\text{a.s.}) \tag{27}$$

with $F(\xi_y) = y$.

Obviously,

$$\lim_{y \downarrow 0} \xi_y = 0. \tag{28}$$

Thus, for every $\delta > 0$, ε can be chosen in such a way that $\xi_\varepsilon \in (0, \tilde{\varepsilon})$ and

$$\lim \sup_{n \to \infty} \frac{1}{ms_1} \alpha\underline{p}^{([\varepsilon n])} \leq \delta \quad \text{(a.s.)} \tag{29}$$

For this particular choice of ε, we shall show that $(\sum_{j=1}^{[\varepsilon n]} \underline{p}^{(j)})/n$ converges to a positive constant almost surely. Since $E\underline{p} < \infty$ implies that [2, p.212] $\lim_{n \to \infty} \underline{p}^{(n)}/n = 0$ (a.s.), we then know that

$$\lim_{n \to \infty} \alpha\underline{p}^{(n)} - \sum_{j=1}^{[\varepsilon n]} \underline{p}^{(j)} = -\infty \quad \text{(a.s.)} \tag{30}$$

and hence, together with (29), the desired result follows immediately.

We first observe that

$$\frac{1}{n} \sum_{j=1}^{[\varepsilon n]} \underline{p}^{(j)} = \int_0^{\underline{p}^{([\varepsilon n])}} x \, d\underline{F}_n(x) \tag{31}$$

where $\underline{F}_n(x)$ is the _empirical distribution function_. Now, (31) can be rewritten as

$$\int_0^{F(\underline{p}^{([\varepsilon n])})} F^+(y) \, d\underline{U}_n(y) = \int_0^{\underline{U}^{([\varepsilon n])}} F^+(y) \, d\underline{U}_n(y), \tag{32}$$

where $\underline{U}^{(j)}$ are the order statistics of n random variables that are uniformly distributed on $[0,1]$.

Through partial integration, we obtain

$$\left| \int_0^{\underline{U}^{([\varepsilon n])}} F^+(y) \, d\underline{U}_n(y) - \int_0^\varepsilon F^+(y) \, dy \right| \leq$$

$$\leq \mid \frac{[\varepsilon n]}{n} F^+(\underline{U}^{([\varepsilon n])}) - \varepsilon F^+(\varepsilon) \mid +$$

$$+ \mid \int_0^\varepsilon (y - \underline{U}_n(y)) dF^+(y) \mid +$$

$$+ \mid \int_{\underline{U}([\varepsilon n])}^\varepsilon \underline{U}_n(y) dF^+(\overset{+}{y}) \mid \qquad (33)$$

We claim that all the three terms on the right hand side of (33) converge to 0 almost surely. Indeed, for the first term this is implied by the specific choice of ε ($\varepsilon < F(\tilde{\varepsilon})$) and the continuity of F on $(0,\tilde{\varepsilon})$. For the second term, this is implied by the Glivenko-Cantelli lemma [2, p. 232]. And for the third term, this follows from the fact that the term is bounded from above by $\mid F^+(\varepsilon) - F^+(\underline{U}^{([\varepsilon n])}) \mid$.

Hence, we have shown that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{[\varepsilon n]} \underline{p}^{(j)} = \int_0^\varepsilon F^+(y) dy \quad (a.s.), \qquad (34)$$

which completes the proof.

Theorem 1 will now be seen to imply that the LPT rule is <u>asymptotically absolutely optimal almost surely</u>, this confirming a conjecture in [14].

<u>Corollary 1</u>. If $E\underline{p}$ is finite, then

$$\lim_{n\to\infty} (\underline{Z}_n^{(m)}(LPT) - \underline{Z}_n^{(m)}(OPT)) = 0 \quad (a.s.) \qquad (35)$$

<u>Proof</u>. Theorem 1 implies that

$$\lim_{n\to\infty} (\underline{Z}_n^{(m)}(LPT) - \frac{1}{m} \sum_{i=1}^m \underline{Z}_n^{(i)}(LPT)) = 0. \qquad (36)$$

From

$$0 \leq \underline{Z}_n^{(m)}(LPT) - \underline{Z}_n^{(i)}(LPT) = \frac{m}{i}(\frac{i}{m} \underline{Z}_n^{(m)}(LPT) - \frac{i}{m} \underline{Z}_n^{(i)}(LPT)) \qquad (37)$$

it follows that $0 \leq \underline{Z}^{(m)}(LPT) - \underline{Z}^{(i)}(LPT) \leq$

$$0 \leq \frac{m}{i} (\frac{i}{m} \underline{Z}_n^{(m)}(LPT) - \frac{1}{m} \sum_{j=1}^i \underline{Z}_n^{(j)}(LPT)) \leq$$

$$\leq \frac{m}{i} \left(\underline{z}_n^{(m)}(LPT) - \frac{1}{m} \sum_{j=i+1}^m \underline{z}_n^{(j)}(LPT) - \frac{1}{m} \sum_{j=1}^i \underline{z}_n^{(j)}(LPT) \right)$$

$$= \frac{m}{i} \underline{D}_n(LPT) \text{ for every } i \in \{1, \ldots, m\} \tag{38}$$

and hence

$$\lim_{n \to \infty} \left(\underline{z}_n^{(m)}(LPT) - \underline{z}_n^{(i)}(LPT) \right) = 0 \quad (a.s.) \tag{39}$$

for every $i \in \{1, \ldots, m\}$.

Now, by summing (39) over all i, we obtain that

$$\lim_{n \to \infty} \left(\left(\sum_{i=1}^m s_i \right) \underline{z}_n^{(m)}(LPT) - \sum_{j=1}^n \underline{p}_j \right) = 0 \quad (a.s.). \tag{40}$$

Since, trivially,

$$Z_n(OPT) \geq \frac{\sum_{j=1}^n \underline{p}_j}{\sum_{i=1}^m s_i}, \tag{41}$$

this leads to the desired result.

We can use the upper bound (23) on $\underline{D}_n(LPT)$ in a similar manner to show that the LPT rule is <u>asymptotically absolutely optimal in expectation</u> under a somewhat stronger condition on the distribution of the $\underline{p}_j$.

<u>Theorem 2</u>. If $E\underline{p}^2$ is finite, then

$$\lim_{n \to \infty} E\underline{D}_n(LPT) = 0. \tag{42}$$

<u>Proof</u>. Starting from (23), we obtain upper bounds for the expected value of the terms on the right hand side.
First, we derive an upper bound on $E\underline{p}^{([\epsilon n])}$. As in (27), let $\xi_{(1+\beta)\epsilon}$ satisfy

$$F(\xi_{(1+\beta)\epsilon}) = (1+\beta)\epsilon \tag{43}$$

for some $\beta > 0$. Then

$$E\underline{p}^{([\epsilon n])} = \int_0^\infty (1 - Pr\{\underline{p}^{([\epsilon n])} \leq x\})dx =$$

$$= \int_0^{\xi_{(1+\beta)\epsilon}}(1 - Pr\{\underline{p}^{([\epsilon n])} \leq x\})dx +$$

$$+ \int_{\xi_{(1+\beta)\epsilon}}^\infty (1 - Pr\{\underline{p}^{([\epsilon n])} \leq x\})dx \leq$$

$$\leq \xi_{(1+\beta)\epsilon} + \int_{\xi_{(1+\beta)\epsilon}}^\infty (1 - Pr\{\underline{p}^{([\epsilon n])} \leq x\})dx \leq$$

$$\leq \xi_{(1+\beta)\epsilon} + n(1 - Pr\{\underline{p}^{([\epsilon n])} \leq \xi_{(1+\beta)\epsilon}\}) +$$

$$+ \int_n^\infty (1 - Pr\{\underline{p}^{([\epsilon n])} \leq x\})dx. \tag{44}$$

The first term can be made arbitrarily small for every $\beta \epsilon (0,1]$ (cf. (28)).
The second term is equal to

$$n \sum_{j=0}^{[\epsilon n]-1} \binom{n}{j} F(\xi_{(1+\beta)\epsilon})^j (1 - F(\xi_{(1+\beta)\epsilon}))^{n-j} =$$

$$= n \sum_{j=0}^{[\epsilon n]-1} \binom{n}{j}((1 + \beta)\epsilon)^j (1 - (1 + \beta)\epsilon)^{n-j}$$

$$\leq n\, e^{-2(\beta\epsilon)^2 n} \tag{45}$$

(cf. [9]). Similarly, the third term is equal to

$$\sum_{j=0}^{[\epsilon n]-1} \binom{n}{j} \int_n^\infty F(x)^j (1 - F(x))^{n-j} dx \leq$$

$$\leq \sum_{j=0}^{[\epsilon n]} \binom{n}{j} \int_n^\infty (\frac{1}{2x})^{n-j} dx \leq$$

$$\leq \frac{1}{2^{n-[\epsilon n]}} \sum_{j=0}^{[\epsilon n]} \binom{n}{j} \int_n^\infty x^{j-n} dx \leq$$

$$\leq \frac{n}{2^{n-[\epsilon n]}} \sum_{j=0}^{[\epsilon n]} \binom{n}{j} (\frac{1}{n})^{n-j} \leq$$

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$$\leq \frac{n}{2^{n-[\varepsilon n]}} \, (1 + \frac{1}{n})^n \tag{46}$$

where the first inequality in (46) is implied by the fact that $E\underline{p}$ is finite and hence $\lim_{x \to \infty} x(1-F(x)) = 0$. Obviously, both (45) and (46) converge exponentially to 0.

We next consider $E \max\{\alpha\underline{p}^{(n)} - \sum_{j=1}^{[\varepsilon n]} \underline{p}^{(j)}, 0\}$ and bound it by conditioning on $\alpha\underline{p}^{(n)}$ being greater or smaller than δn respectively, where

$$\delta \stackrel{\Delta}{=} \min\{\varepsilon^2, \int_0^{\varepsilon/2} F^+(z)dz, \, \big(\int_0^{\varepsilon/2} F^+(z)dz\big)^2\} \tag{47}$$

We bound the expectation, conditioned on $\alpha\underline{p}^{(n)} \geq \delta n$, by

$$\int_{\delta n}^{\infty} x \, d(\Pr\{\alpha\underline{p}^{(n)} \leq x\}) =$$

$$= \delta n\big(1 - F(\frac{\delta n}{\alpha})^n\big) + \int_{\delta n}^{\infty} \big(1 - F(\frac{x}{\alpha})^n\big)dx \leq$$

$$\leq \delta n^2\big(1 - F(\frac{\delta n}{\alpha})\big) + n \int_{\delta n}^{\infty} \big(1 - F(\frac{x}{\alpha})\big)dx. \tag{48}$$

Both these final terms converge to 0, since $E\underline{p}^2 < \infty$ implies that $\lim_{x \to \infty} x^2(1 - F(x)) = 0$ and $\lim_{x \to \infty} x \int_x^{\infty} (1 - F(z))dz = 0$.

The term conditioned on $\alpha\underline{p}^{(n)} < \delta n$ is bounded by

$$\delta n \, \Pr\{\sum_{j=1}^{[\varepsilon n]}\underline{p}^{(j)} \leq \alpha\underline{p}^{(n)} < \delta n\} \leq$$

$$\leq \delta n \, \Pr\{\sum_{j=1}^{[\varepsilon n]} \underline{p}^{(j)} < \delta n\}. \tag{49}$$

Similar to (32), we observe that

$$\Pr\{\sum_{j=1}^{[\varepsilon n]} \underline{p}^{(j)} < \delta n\} =$$

$$\Pr\{\sum_{j=1}^{[\varepsilon n]} F^+(\underline{U}^{(j)}) < \delta n\} =$$

$$= \int_0^1 \Pr\{\sum_{j=1}^{[\varepsilon n]} F^+(\underline{U}_j(y)) < \delta n\} d(\Pr\{\underline{U}^{([\varepsilon n]+1)} \leq y\})$$

$$\leq \Pr\{\underline{U}^{([\varepsilon n]+1)} \leq \frac{\varepsilon}{2}\} +$$

$$+ \int_{\varepsilon/2}^{1} \Pr\{\frac{1}{[\varepsilon n]} \sum_{j=1}^{[\varepsilon n]} (F^+(\underline{U}_j(y)) - \frac{1}{y} \int_0^y F^+(z)dz)$$

$$< \frac{\delta}{\varepsilon} - \frac{1}{y} \int_0^y F^+(z)dz\} d(\Pr\{\underline{U}^{([\varepsilon n]+1)} \leq y\}) \quad (50)$$

where $\underline{U}_j(y)$ are independently uniformly distributed on $[0,y]$ and where we have conditioned on the value of the $([\varepsilon n]+1)$-th uniform order statistic $\underline{U}^{([\varepsilon n]+1)}$ ([1, p. 103]).

The first term on the right hand side of (50) corresponds to the tail of a binomial distribution, converging exponentially to 0.

We bound the term within the remaining integral by observing that, for every $y \in [\varepsilon/2,1]$,

$$\frac{\delta}{\varepsilon} - \frac{1}{y} \int_0^y F^+(z)dz \leq$$

$$\leq \frac{\delta}{\varepsilon} - \frac{2}{\varepsilon} \int_0^{\varepsilon/2} F^+(z)dz \leq$$

$$\leq -\frac{1}{\varepsilon} \int_0^{\varepsilon/2} F^+(z)dz \quad (51)$$

where we have used (47) and the (easily verified) fact that $1/y \int_0^y F^+(z)dz$ is nondecreasing in y. It follows from Chebyshev's inequality that the probability within the integral is bounded by

$$\frac{\sigma^2(F^+(\underline{U}_j(y)))\varepsilon^2}{[\varepsilon n](\int_0^{\varepsilon/2} F^+(z)dz)^2} \quad (52)$$

so that the last term in (50) multiplied by δn itself can be bounded by

$$\frac{\delta\varepsilon}{(\int_0^{\varepsilon/2} F^+(z)dz)^2} \int_{\varepsilon/2}^1 \sigma^2(F^+(\underline{U}_j(y)))d(\Pr\{\underline{U}^{([\varepsilon n]+1)} \leq y\}) \leq \varepsilon M, \quad (53)$$

where M is the uniform upperbound on $\sigma^2(F^+(\underline{U}_j(y)))$ for $y \in [0,1]$.

Collecting the various upper bounds, we conclude that

$$\lim \sup_{n \to \infty} \underline{ED}_n(\text{LPT}) \leq \xi_{(1+\beta)\epsilon} + \epsilon M \tag{54}$$

and the theorem follows by letting ϵ go to 0.

Corollary 2. If $E\underline{p}^2$ is finite then

$$\lim_{n \to \infty} (E\underline{Z}_n^{(m)}(\text{LPT}) - E\underline{Z}_n^{(m)}(\text{OPT})) = 0 \tag{55}$$

Proof: The proof is similar to that of Corollary 1.

Corollaries 1 and 2 confirm the excellent asymptotic properties of the LPT rule. The usefulness of such asymptotic results is, however, much enhanced by some insight into the speed at which convergence occurs. This forms the subject of the next section.

3. Speed of convergence results

In this section, we first analyse the speed of convergence to absolute almost sure optimality of the LPT rule for two special cases:

(i) the $\underline{p}_j$ are uniformly distributed on $[0,1]$;

(ii) the $\underline{p}_j$ are exponentially distributed with parameter λ.

For both cases, we obtain the same result.

Theorem 3. If the $\underline{p}_j$ are uniformly or exponentially distributed, then

$$\lim \sup_{n \to \infty} \frac{n}{\log n} (\underline{Z}_n^{(m)}(\text{LPT}) - \underline{Z}_n^{(m)}(\text{OPT})) < \infty \quad (\text{a.s.}) \tag{56}$$

Proof. We first consider the uniform case. Here, we know that [1, p. 107] $(\underline{p}^{(1)}, p^{(2)}, \ldots, \underline{p}^{(n)})$ is distributed as

$$\left(\frac{q_1}{q_{n+1}} \, , \, \frac{q_2}{q_{n+1}} \, , \ldots, \, \frac{q_n}{q_{n+1}} \right) \tag{57}$$

with $\underline{q}_j = \Sigma_{\ell=1}^j \, \underline{r}_\ell$, and $\underline{r}_\ell$ independent exponentially distributed random variables ($\ell=1,\ldots,n+1$) with parameter $\lambda = 1$. From (20), we conclude that in this case

$$\underline{D}_n(\text{LPT}) \overset{d}{\leq} \frac{1}{ms_1 \underline{q}_{n+1}} \max_{1 \leq k \leq n} \{ \alpha \underline{q}_k - \Sigma_{j=1}^k \, \underline{q}_j \}. \tag{58}$$

We now consider two possibilities. First, if $k \leq \alpha$, then

$$\alpha \underline{q}_k - \Sigma_{j=1}^k \, \underline{q}_j = \alpha \, \Sigma_{\ell=1}^k \, \underline{r}_\ell - \Sigma_{j=1}^k \, \Sigma_{\ell=1}^j \, \underline{r}_\ell \leq$$

$$\leq (\alpha-1) \, \Sigma_{\ell=1}^k \, \underline{r}_\ell \leq$$

$$\leq (\alpha-1)\alpha \, \max_{1 < \ell \leq n} \{ \underline{r}_\ell \}. \tag{59}$$

Secondly, if $k > \alpha$, then also

$$\alpha \underline{q}_k - \Sigma_{j=1}^k \, \underline{q}_j \leq (\alpha-1) \, \Sigma_{\ell=1+k-\alpha}^k \, \underline{r}_\ell \leq$$

$$\leq (\alpha-1)\alpha \, \max_{1 < \ell \leq n} \{ \underline{r}_\ell \}. \tag{60}$$

Hence,

$$\underline{D}_n(\text{LPT}) \overset{d}{\leq} \frac{(\alpha-1)\alpha \, \max_{1 < \ell \leq n} \{ \underline{r}_\ell \}}{ms_1 \, \underline{q}_{n+1}} \tag{61}$$

and since [2, p. 224]

$$\lim_{n \to \infty} \frac{\max_{1 < \ell \leq n} \{ \underline{r}_\ell \}}{\log n} = 1 \ (\text{a.s.}), \tag{62}$$

the strong law of large numbers applied to $\underline{q}_{n+1}$ yields that $\underline{D}_n$ converges to 0 as fast as $(\log n)/n$. Hence (cf. (36)-(41)), so does $\underline{z}_n^{(m)}(\text{LPT}) - \underline{z}_n^{(m)}(\text{OPT})$.

Next, we consider the exponential case, where we may as well assume that $\lambda = 1$. Here, we know [3, p. 18] that $\underline{p}^{(j)} - \underline{p}^{(j-1)}$ is distributed as $\underline{r}_j/(n-j+1)(j=1,\ldots,n)$ with $p^{(o)} = 0$ and $\underline{r}_j$ as defined above. Thus, (20) implies that in this case

$$\underline{D}_n(\text{LPT}) \overset{d}{\leq} \frac{1}{ms_1} \max_{1 \leq k \leq n} \{\alpha \sum_{\ell=1}^{k} \frac{\underline{r}_\ell}{n-\ell+1} - \sum_{j=1}^{k} \sum_{\ell=1}^{j} \frac{\underline{r}_\ell}{n-\ell+1}$$

$$= \frac{1}{ms_1} \max_{1 \leq k \leq n} \{\sum_{\ell=1}^{k} \frac{\alpha-k+\ell-1}{n-\ell+1} \underline{r}_\ell\} \tag{63}$$

In the proof of Theorem 1 we have seen that, if $k \in \{[\epsilon n] + 1, \ldots, n\}$, then

$$\lim_{n \to \infty} \{\alpha \underline{p}^{(k)} - \sum_{j=1}^{k} \underline{p}^{(j)}\} = -\infty \quad (\text{a.s.}) \tag{64}$$

(cf. (22), (30)). Thus, we only need consider the cases that $k \leq \alpha$ and that $\alpha + 1 \leq k \leq [\epsilon n]$. In the former we have that

$$\alpha \underline{p}^{(k)} - \sum_{j=1}^{k} \underline{p}^{(j)} \overset{d}{\leq} \frac{\alpha(\alpha-1) \max_{1 \leq \ell \leq n} \{\underline{r}_\ell\}}{n-\alpha+1}, \tag{65}$$

in the latter we have by a similar argument that

$$\alpha \underline{p}^{(k)} - \sum_{j=1}^{k} \underline{p}^{(j)} \overset{d}{\leq} \frac{\alpha(\alpha-1) \max_{1 \leq \ell \leq n} \{\underline{r}_\ell\}}{[(1-\epsilon)n]}. \tag{66}$$

The remaining part of the proof is as above. □

We now consider the speed of convergence to absolute optimality in expectation for the LPT rule, and restrict ourselves to the case that the $\underline{p}_j$ are uniformly distributed on $[0,1]$.

Theorem 4. If the $\underline{p}_j$ are uniformly distributed, then

(i) in the case of identical machines

$$E\underline{Z}_n^{(m)}(\text{LPT}) - E\underline{Z}_n^{(m)}(\text{OPT}) = O(\frac{m}{n}) \tag{67}$$

(ii) in the case of uniform machines

$$E\underline{Z}_n^{(m)}(\text{LPT}) - E\underline{Z}_n^{(m)}(\text{OPT}) = O(\frac{\alpha^2}{n}) \tag{68}$$

Proof: Starting from (58), we observe that

$$E \max_{1 \leq k \leq n} \{ \alpha \underline{q}_k - \sum_{j=1}^{k} \underline{q}_j \} \leq$$

$$\leq E \max_{1 \leq k \leq [3\alpha]} \{ \alpha \underline{q}_k - \sum_{j=1}^{k} \underline{q}_j, \ 0 \} +$$

$$+ \ E \max_{[3\alpha]+1 \leq k \leq n} \{ \alpha \underline{q}_k - \sum_{j=1}^{k} \underline{q}_j, \ 0 \} \leq$$

$$\leq E \alpha \underline{q}_{[3\alpha]} + E \max_{[3\alpha]+1 \leq k \leq n} \{ \alpha \underline{q}_k - \sum_{j=1}^{k} \underline{q}_j, \ 0 \}$$

$$= O(\alpha^2) + E \max_{[3\alpha]+1 \leq k \leq n} \{ \alpha \underline{q}_k - \sum_{j=1}^{k} \underline{q}_j, \ 0 \} \qquad (69)$$

We define $\underline{S}_k \triangleq \sum_{\ell=1}^{k} (\alpha - k + \ell - 1) \underline{r}_\ell$ and rewrite the second term in (69) as follows:

$$E \max_{[3\alpha]+1 \leq k \leq n} \{ \alpha \underline{q}_k - \sum_{j=1}^{k} \underline{q}_j, \ 0 \} \leq$$

$$\leq \sum_{k=[3\alpha]+1}^{n} \int_0^\infty \mathrm{Pr}\{ \alpha \underline{q}_k - \sum_{j=1}^{k} \underline{q}_j \geq x \} dx =$$

$$= \sum_{k=[3\alpha]+1}^{n} \int_0^\infty \mathrm{Pr}\{ \underline{S}_k \geq x \} \ dx =$$

$$= \sum_{k=[3\alpha]+1}^{n} \int_0^\infty \mathrm{Pr}\{ \underline{S}_k - E\underline{S}_k \geq x - \frac{k(2\alpha-1-k)}{2} \} \ dx \qquad (70)$$

Through Chebyshev's inequality, we can choose any $p \in \mathbb{N}$ and bound the probability within the integral by $E((\underline{S}_k - E\underline{S}_k)^{2p} \cdot (x - \frac{1}{2}k(2\alpha-1-k))^{-2p}$. Thus, for every $p \in \mathbb{N}$, (70) is bounded from above by

$$\sum_{k=[3\alpha]+1}^{n} E((\underline{S}_k - E\underline{S}_k)^{2p}) \int_0^\infty \left(x - \frac{k(2\alpha-1-k)}{2} \right)^{-2p} dx$$

$$= O(\sum_{k=[3\alpha]+1}^{n} E((\underline{S}_k - E\underline{S}_k)^{2p}) \ ((k+1-2\alpha)k)^{-2p+1}). \qquad (71)$$

We now apply the Marczinkiewicz-Zygmund inequality [18, p. 41] to $E((\underline{S}_k - E\underline{S}_k)^{2p})$:

$$E((\underline{S}_k - E\underline{S}_k)^{2p}) \leq A k^{p-1} \sum_{\ell=1}^{k} (\alpha - k + \ell - 1)^{2p} E((\underline{r}_\ell - 1)^{2p}) \qquad (72)$$

with A depending only on p. Hence, $E((\underline{S}_k - E\underline{S}_k)^{2p}) = O(k^{p-1}(k-\alpha)^{2p+1})$ and by substitution we find that (70) is finally bounded by $O(\sum_{k=[3\alpha]+1}^{n} k^{-p+2})$, which is $O(1)$ if $p=4$. Combining this with (69) and by conditioning on the events $\{\underline{q}_{n+1} > \frac{1}{2}(n+1)\}$ and $\{\underline{q}_n < \frac{1}{2}(n+1)\}$ respectively, we easily verify that

$$\underline{ED}_n(LPT) = O(\alpha^2/mn).$$

□

This proves (i), i.e., the case in which $\alpha = m$. For (ii), we see from (37) that $\underline{EZ}_n^{(m)}(LPT) - \underline{EZ}_n^{(i)}(LPT) = O(\alpha^2/in)$ for every $i \in \{1,\ldots,m\}$

Since Theorem 4 trivially implies that

$$\frac{\underline{EZ}_n^{(m)}(LPT)}{\underline{EZ}_n^{(m)}(OPT)} = 1 + O(\frac{m^2}{n^2}), \tag{73}$$

this result generalizes the bound in [4] (cf. (3)).

4. A bound for arbitrary list scheduling rules

As pointed out in Section 2, the basic result

$$D_n \le \frac{1}{ms_1} \max_{1 \le k \le n} \{\alpha p_k - \sum_{j=k}^n p_j\} \tag{74}$$

holds for arbitrary list scheduling rules (LS). We can use it to derive the following bound on the expected absolute difference between the result produced by such rules and the optimal value.

Theorem 5. If $E\underline{p}^3$ is finite and $s_i = 1$ for all i, then

$$\underline{EZ}_n^{(m)}(LS) - \underline{EZ}_n^{(m)}(OPT) \le (1 - \frac{1}{m}) E\underline{p} + (1 - \frac{1}{2m}) \frac{\sigma^2(\underline{p})}{E\underline{p}} \tag{75}$$

Proof. Since $s_i = 1$ for all i, we have that (cf. (17))

$$\underline{D}_n(LS) \le \max_{1 \le k \le n} \{\underline{p}_k - \frac{1}{m} \sum_{j=k}^n \underline{p}_j\} \tag{76}$$

If we denote the right hand side of (76) by $\underline{V}_n(m)$, then we obtain after renumbering

$$\underline{V}_n(m) \stackrel{d}{=} \max_{1 \le k \le n} \{\underline{p}_k - \frac{1}{m} \sum_{j=1}^k \underline{p}_j\} \tag{77}$$

Thus, $\Pr\{\underline{V}_n(m) \leq x\}$ is a nonincreasing sequence for fixed x. Hence, $E\underline{V}_n(m) \leq E\underline{V}_{n+1}(m)$ and since

$$\lim_{n \to \infty} (\underline{p}_n - \frac{1}{m} \sum_{j=1}^{n} \underline{p}_j) = -\infty \quad (a.s.) \tag{78}$$

this implies that $\underline{V}_n(m)$ converges in distribution to an a.s. finite valued nonnegative random variable $\underline{V}(m)$ that satisfies the following recurrence (cf. (13)):

$$\underline{V}(m) = \max \left\{ \underline{V}(m) - \frac{\underline{p}_\infty}{m}, \frac{(m-1)\underline{p}_\infty}{m} \right\} \tag{79}$$

where $\underline{p}_\infty$ does not depend on $\underline{V}(m)$. Hence,

$$\underline{V}(m) = \max \{ \underline{V}(m) - \underline{p}_\infty, 0 \} + \frac{(m-1)\underline{p}_\infty}{m}. \tag{80}$$

By applying a technique used first by Kingman [5,16], it is easy to show that this implies that, if $E\underline{V}(m)^2$ is finite, then

$$E\underline{V}(m) \leq (1 - \frac{1}{m})E\underline{p}_\infty + (1 - \frac{1}{2m}) \frac{\sigma^2(\underline{p}_\infty)}{E\underline{p}_\infty}$$

$$= (1 - \frac{1}{2m}) \frac{E\underline{p}_\infty^2}{E\underline{p}_\infty} - \frac{1}{2m} E\underline{p}_\infty \tag{81}$$

where $\underline{p}_\infty$ is distributed as the p_j (cf. Appendix A)

Since $E\underline{D}_n(LS) \leq E\underline{V}_n(m) \leq E\underline{V}(m)$ and

$$z_n^{(m)}(LS) - z_n^{(m)}(OPT) \leq$$

$$\leq z_n^{(m)}(LS) - \frac{1}{m} \sum_{j=1}^{n} p_j =$$

$$= D_n(LS), \tag{82}$$

all that remains to be done is to verify that $E\underline{V}(m)^2$ is indeed finite. For this proof, we refer to Appendix B.

□

We note that Theorem 5 implies that, if $E\underline{p}^3$ is finite, then

$$\frac{EZ_{\underline{n}}^{(m)}(LS)}{EZ_{\underline{n}}^{(m)}(OPT)} = 1 + \frac{m-1}{n} + \frac{2m-1}{2n} \frac{\sigma^2(\underline{p}_\infty)}{(E\underline{p}_\infty)^2}, \quad (83)$$

so that under this assumption arbitrary list scheduling is asymptotically relatively optimal in expectation; the speed of convergence is $O(m/n)$.

It is also possible to extend the above analysis to the case of uniform machines. One finds that

$$E\underline{V}(m) \leq (1 - \frac{1}{m}) \frac{E\underline{p}_\infty}{s_m} + (1 - \frac{1}{m}(1 - \frac{s_m}{2s_1})) \frac{\sigma^2(\underline{p}_\infty)}{s_m E\underline{p}_\infty} \quad (84)$$

and hence

$$EZ_{\underline{n}}^{(m)}(LS) - EZ_{\underline{n}}^{(m)}(OPT) \leq \frac{E\underline{p}_\infty}{s_m}(m-1) + \frac{\sigma^2(\underline{p}_\infty)}{s_m E\underline{p}_\infty}(m-1+\frac{s_m}{2s_1}) \quad (85)$$

with similar conslusions to be drawn as in the identical machine case; we leave the details to the reader.

5. Conluding remarks

The analysis in the previous sections confirms that the LPT rule requires only slightly more work than arbitrary list scheduling and yet has very strong properties of asymptotic optimality.

This insight can be fruitfully applied whenever asymptotic results that were obtained for arbitrary list scheduling have to be improved. An example of such a situation occurs in hierarchical, two-stage scheduling problems, where in the first stage m identical machines have to be acquired at cost c each, subject to probabilistic information about the n jobs that have to be scheduled on these machines in the second stage so as to minimize makespan. The objective is to find the value m(OPT) such that

$$\underline{C}(OPT) = cm(OPT) + \underline{z}_{\underline{n}}^{(m(OPT))}(OPT) \quad (86)$$

is minimal in expectation.

In the heuristic method proposed in [6] to solve this problem, m is chosen so
as to minimize the expected value of a lower bound on the objective function,
given by

$$\underline{LB}(m) = cm + \frac{\sum_{j=1}^{n} \underline{P}_j}{m} ,$$ (87)

i.e.,

$$m(H) \in \{ [(\frac{nEp}{c})^{\frac{1}{2}}], \langle (\frac{nEp}{c})^{\frac{1}{2}} \rangle \}$$ (88)

($\langle x \rangle$ is the integer roundup of x). In the second stage, the jobs are scheduled
on the m(H) machines by some list scheduling rule.

It is easy to see that the value $\underline{C}(H)$ produced by this heuristic satisfies

$$\underline{C}(H) = \underline{LB}(m(H)) + \left(\underline{z}_n^{(m(H))}(LS) - \frac{\sum_{j=1}^{n} \underline{P}_j}{m(H)} \right)$$ (89)

where, of course, the second term is equal to $\underline{D}_n(LS)$ for m = m(H).

Hence, if we replace arbitrary list scheduling by the LPT rule,

$$E\underline{C}(H) = E\underline{LB}(m(H)) + E\underline{D}_n(LPT) \leq$$

$$\leq E\underline{C}(OPT) + E\underline{D}_n(LPT).$$ (90)

If $E\underline{p}^3$ is finite, then one can prove as in Section 2 that
$\lim_{n \to \infty} E\underline{D}_n(LPT) = 0$. Hence, since $E\underline{C}(OPT) = O(\sqrt{n})$, we obtain the following
strengthened version of the asymptotic relative optimality result in
expectation from [17]

$$\frac{E\underline{C}(H)}{E\underline{C}(OPT)} = 1 + o(\frac{1}{\sqrt{n}})$$ (91)

The other asymptotic optimality results for this model from [6] can be
strengthened in a similar manner. For instance, it is possible to obtain speed

of convergence results for the rate at which $\underline{C}(H)/\underline{C}(OPT)$ converges to 1 almost surely, by applying the law of the iterated logarithm so as to obtain

$$\lim \sup_{n\to\infty} \frac{\underline{C}(H)}{\underline{C}(OPT)} = 1 + O\left(\left(\frac{\log \log n}{n}\right)^{\frac{1}{2}}\right) \quad (a.s.). \tag{92}$$

As a final remark, we note that the results on the LPT rule are all based on replacing $\underline{z}_n^{(m)}(OPT)$ by $(\sum_{j=1}^n \underline{p}_j)/(\sum_{i=1}^m s_i)$; as such they show that this approximation is asymptotically accurate almost surely. Indeed, for the uniform case, our results show that the difference between the LPT result and this value converges to 0 as $(\log n)/n$, whereas it can be shown that the difference between the true optimal value and its approximation converges _exponentially_ to 0 [15]. It is of interest to note that a heuristic was recently proposed [15] for the case that $s_i = 1$ for all i for which the absolute error converges as $n^{-\log n}$; it is tempting to conjecture that this result is the strongest possible one for a polynomial time heuristic.

Acknowledgement

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Appendix A

Define $X^+ \triangleq \max\{X,0\}$ and $X^- \triangleq \max\{-X,0\}$. Since $E\underline{V}(m) < \infty$ (see Appendix B) we find from (80) that

$$E\underline{V}(m) - E\underline{p}_\infty =$$

$$= E(\underline{V}(m) - \underline{p}_\infty)^+ - E(\underline{V}(m) - \underline{p}_\infty)^-$$

$$= E\underline{V}(m) - \frac{m-1}{m} E\underline{p}_\infty - E(\underline{V}(m) - \underline{p}_\infty)^- \tag{A.1}$$

so that $E(\underline{V}(m) - \underline{p}_\infty)^- = E\underline{p}_\infty/m$.

Because $\underline{V}(m)$ and $\underline{p}_\infty$ are independent, we also have that (use (80) again)

$$\sigma^2(\underline{V}(m)) + \sigma^2(\underline{p}_\infty) = \sigma^2(\underline{V}(m) - \underline{p}_\infty) =$$

$$= \sigma^2((\underline{V}(m) - \underline{p}_\infty)^+) + \sigma^2((\underline{V}(m) - \underline{p}_\infty)^-) +$$

$$+ 2E(\underline{V}(m) - \underline{p}_\infty)^+ E(\underline{V}(m) - \underline{p}_\infty)^-$$

$$\geq \sigma^2(\underline{V}(m)) + (\frac{m-1}{m})^2 \sigma^2(\underline{p}_\infty) +$$

$$+ 2 \frac{E\underline{p}_\infty}{m} (E\underline{V}(m) - \frac{m-1}{m} E\underline{p}_\infty). \tag{A.2}$$

Since $\sigma^2(\underline{V}(m)) < \infty$ (see Appendix B), this implies that

$$\frac{2m-1}{m^2} \sigma^2(\underline{p}_\infty) \geq 2 \frac{E\underline{p}_\infty}{m} (E\underline{V}(m) - \frac{m-1}{m} E\underline{p}_\infty) \tag{A.3}$$

which in turn implies (81).

Appendix B

It is sufficient to prove that $E\underline{V}_n^2(m)$ is uniformly bounded (use the fact that $\underline{V}_n(m)$ converges to $\underline{V}(m)$ in distribution and [19, p. 164]), i.e. that

$$\int_0^\infty x \ \Pr\{\underline{V}_n(m) \geq x\} \ dx \tag{A.4}$$

is uniformly bounded.

By definition of $\underline{V}_n(m)$,

$$\int_0^\infty x \ \Pr\{\underline{V}_n(m) \geq x\}dx \leq \sum_{k=1}^n \int_0^\infty x \ \Pr\{\underline{p}_k - \frac{1}{m}\sum_{j=1}^k \underline{p}_j \geq x\}dx \tag{A.5}$$

and so by conditioning on $\underline{p}_k$ for every $k=1,\ldots,n$, we get

$$\int_0^\infty x \ \Pr\{\underline{V}_n(m) \geq x\}dx \leq \sum_{k=2}^n \int_0^\infty \int_x^\infty \Pr\{\sum_{j=1}^{k-1} \underline{p}_j \leq m(y-x)\} \ F(dy)dx$$

$$+ \int_0^\infty x(1-F(x))dx = \int_0^\infty \int_x^\infty \sum_{k=0}^{n-1} F^{k*}(m(y-x)) \ F(dy)dx \tag{A.6}$$

where F^{k*} denotes the k-fold convolution of F for $k=1,2,\ldots$ and F^{0*} the distribution of a random variable degenerate in 0.

Hence for every $n \in \mathbb{N}$

$$\int_0^\infty x \ \Pr\{\underline{V}_n(m) \geq x\}dx \leq \int_0^\infty \int_x^\infty U(m(y-x)) \ F(dy)dx \tag{A.7}$$

where $U(x) \triangleq \sum_{k=0}^\infty F^{k*}(x)$ is the well-known renewal function ([1], [10]). It is easy to derive, using the renewal equation, that $t = \int_{0-}^t F_1(t-y)U(dy)$ with $F_1(t) \triangleq \int_0^t (1-F(z))dz$. Hence $t \geq \int_0^{t/2} F_1(t-y)U(dy) \geq F_1(t/2)U(t/2)$ or equivalently

$$\frac{U(t)F_1(t)}{t} \leq 2. \tag{A.8}$$

Using this observation, we find for every $n \in \mathbb{N}$

$$\int_0^\infty x \; \Pr\{\underline{V}_n(m) \geq x\}dx \leq 2m \int_0^\infty x \int_x^\infty \frac{y-x}{F_1(y-x)} F(dy)dx. \qquad (A.9)$$

Since $\lim_{x \downarrow 0} F_1(x)/x = 1$ and $F_1(x)$ is a strictly increasing function, we can find a constant M such that for every $x \geq 0$

$$\int_x^\infty \frac{(y-x)}{F_1(y-x)} F(dy) \leq M \; [\int_x^\infty (y-x)F(dy) + F(x+1) - F(x)] \qquad (A.10)$$

and so by (A.9) with $\mu \stackrel{\Delta}{=} F_1(\infty) < \infty$

$$\int_0^\infty x \; \Pr\{\underline{V}_n(m) \geq x\}dx \leq$$

$$\leq 2mM \int_0^\infty x \int_x^\infty (y-x)F(dy)(dx) + 2mM \int_0^\infty x \; (F(x+1) - F(x))dx \leq$$

$$\leq 2mM \int_0^\infty x \; (\mu - F_1(x))dx + 2mM \int_0^\infty x(1 - F(x))dx \qquad (A.11)$$

Using $E\underline{p}^3 \leq \infty$, we can easily derive that the upperbound in (A.11) is finite and hence $\int_0^\infty x \; \Pr\{\underline{V}_n(m) \geq x\}dx$ is bounded by a uniform constant.

$\square$