# The attracting centre of a continuous self-map of the interval 

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(Received 15 August 1986 and revised 19 February 1987)


#### Abstract

Let $f$ denote a continuous map of a compact interval $I$ to itself. A point $x \in I$ is called a $\gamma$-limit point of $f$ if it is both an $\omega$-limit point and an $\alpha$-limit point of some point $y \in I$. Let $\Gamma$ denote the set of $\gamma$-limit points. In the present paper, we show that (1) $\bar{P}-\Gamma$ is either empty or countably infinite, where $\bar{P}$ denotes the closure of the set $P$ of periodic points, (2) $x \in I$ is a $\gamma$-limit point if and only if there exist $y_{1}$ and $y_{2}$ in $I$ such that $x$ is an $\omega$-limit point of $y_{1}$, and $y_{1}$ is an $\omega$-limit point of $y_{2}$, and if and only if there exists a sequence $y_{1}, y_{2}, \ldots$ of points in $I$ such that $x$ is an $\omega$-limit point of $y_{1}$, and $y_{i}$ is an $\omega$-limit point of $y_{i+1}$ for every $i \geq 1$, and (3) the period of each periodic point of $f$ is a power of 2 if and only if every $\gamma$-limit point is recurrent.


## 1. Introduction

Throughout this paper $f$ will be a continuous map of the interval $I=[0,1]$ to itself, $P$ the set of periodic points of $f, R$ the set of recurrent points of $f$, and $\Omega$ the set of nonwandering points of $f$.

For a subset $Y$ of $I$, define $\Lambda(Y)=\bigcup_{x \in Y} \omega(x)$, where $\omega(x)$ is the set of $\omega$-limit points of $x$. Let $\Lambda^{1}=\Lambda(I)$ and for any $n>1$, inductively define $\Lambda^{n}=\Lambda\left(\Lambda^{n-1}\right)$. Obviously, $\Lambda^{1} \supset \Lambda^{2} \supset \Lambda^{3} \supset \cdots$. The set $\Lambda^{\infty}=\bigcap_{n=1}^{\infty} \Lambda^{n}$ will be called the attracting centre of $f$.

We will say that a point $y$ is a $\gamma$-limit point of $x \in I$ if $y \in \omega(x) \cap \alpha(x)$, where $\alpha(x)$ is the set of $\alpha$-limit points of $x$. Let $\gamma(x)=\omega(x) \cap \alpha(x)$ and $\Gamma=\bigcup_{x \in I} \gamma(x)$.
$\operatorname{In}[8]$ the author investigated the set $\Omega-\bar{P}$ and showed that it is always countable. In this paper we show

Theorem 1. Suppose that $f$ is a continuous map of the interval $I$. Then
(1) $\Omega-\Gamma$ is countable.
(2) $\Lambda^{1}-\Gamma$ and $\bar{P}-\Gamma$ are either empty or countably infinite.
A. N. Sharkovskii [6] has shown that $\Lambda^{1}$ is closed and hence that $\bar{P} \subset \Lambda^{1}$. L. Block and E. Coven [1] have shown that $\omega(x)$ is an infinite minimal set for any $x \in \Lambda^{1}-\bar{P}$. It follows that $\Lambda^{2} \subset \bar{P}$, because each minimal set is contained in $R$ and $\bar{R}=\bar{P}$ (see [7], for example). In [1] and [2], one can find examples in which $\Lambda^{1} \neq \bar{P}$. Therefore,
$\Lambda^{1}=\Lambda^{2}$ does not hold in general. However, we will prove
Theorem 2. Suppose that $f$ is a continuous map of the interval I. Then

$$
=\Lambda^{\infty}=\cdots=\Lambda^{3}=\Lambda^{2}=\Lambda(\bar{P})=\Lambda(\Omega)=\Gamma .
$$

In particular, $\Gamma$ is the attracting centre of $f$.
Remark 1. Theorem 2 shows that the following conditions are equivalent.
(1) $y \in I$ is an $\omega$-limit point of a nonwandering point of $f$.
(2) $y \in I$ is an $\omega$-limit point of a point in the closure of the set of periodic points of $f$.
(3) There is a point $x \in I$ such that $y \in I$ is both an $\omega$-limit point and an $\alpha$-limit point of $x$.
(4) For each $n \geq 2$, there are $n$ points $x_{1}, x_{2}, \ldots, x_{n} \in I$ such that $y$ is an $\omega$-limit point of $x_{1}, x_{1}$ is an $\omega$-limit point of $x_{2}, \ldots$, and $x_{n-1}$ is an $\omega$-limit point of $x_{n}$.
(5) There is a sequence $x_{1}, x_{2}, \ldots$ of points in $I$ such that $y$ is an $\omega$-limit point of $x_{1}$, and $x_{i}$ is an $\omega$-limit point of $x_{i+1}$ for every $i \geq 1$.

The continuous maps of the interval $I$ into itself can be divided into two disjoint classes, determined by whether or not the period of each periodic point is a power of 2. It has been shown that maps in different classes have quite different dynamical properties. (See [3] for a survey, and [9] and [10] for some new results.) In the following theorem it will be shown that the class to which an interval map belongs is determined by whether or not $\Gamma-R$ is empty.

Theorem 3. Suppose that $f$ is a continuous map of the interval I to itself. Then the following conditions are equivalent.
(1) The period of each periodic point of $f$ is a power of 2 .
(2) Every $\gamma$-limit point of $f$ is recurrent (i.e. $\Gamma=R$ ).

## 2. Preliminaries

Recall that $f$ is a continuous map of the interval $I=[0,1]$ to itself. Let $x \in I$.
A point $y \in I$ is called an $\omega$-limit point of $x$ if there exist $n_{i} \rightarrow \infty$ such that $f^{n \prime}(x) \rightarrow y$. Let $\omega(x)$ denote the set of $\omega$-limit points of $x$. We will use the symbols $\omega_{+}(x)$ (resp. $\left.\omega_{-}(x)\right)$ to denote the set of all points $y$ such that there exist $n_{i} \rightarrow \infty$ such that $f^{n_{i}}(x) \rightarrow y$ and $y<f^{n_{i}}(x)$ (resp. $\left.f^{n_{i}}(x)<y\right)$ for every $i>0$. Clearly, $y \in \omega_{+}(x)\left(r e s p . ~ y \in \omega_{-}(x)\right)$ if and only if there exist $n_{i} \rightarrow \infty$ such that $f^{n_{i}}(x) \rightarrow y$ and $y<\cdots<f^{n_{2}}(x)<f^{n_{1}}(x)$ (resp. $\left.f^{n_{1}}(x)<f^{n_{2}}(x)<\cdots<y\right)$. It is clear that if $x \notin P$, then $\omega(x)=\omega_{+}(x) \cup \omega_{-}(x)$. Define $\Lambda_{+}=\bigcup_{x \in I} \omega_{+}(x)$ and $\Lambda_{-}=\bigcup_{x \in I} \omega_{-}(x)$.

A point $y \in I$ is called an $\alpha$-limit point of $x$ if there exist $n_{i} \rightarrow \infty$ and $x_{i} \rightarrow y$ such that $f^{n_{i}}\left(x_{i}\right)=x$ for every $i>0$. We will use the symbols $\alpha_{+}(x)$ (resp. $\left.\alpha_{-}(x)\right)$ to denote the set of all points $y$ such that there exist $n_{i} \rightarrow \infty$ and $x_{i} \rightarrow y$ such that $f^{n_{i}}\left(x_{i}\right)=x$ and $y<x_{i}\left(\right.$ resp. $\left.x_{i}<y\right)$ for every $i>0$. It is clear that if $x \notin P$, then $\alpha(x)=\alpha_{+}(x) \cup$ $\alpha_{-}(x)$.

A point is called a $\gamma$-limit point of $x$ if it is both an $\omega$-limit point of $x$ and an $\alpha$-limit point of $x$. The symbol $\gamma(x)$ denotes the set of $\gamma$-limit points of $x$ and
$\Gamma=\bigcup_{x \in I} \gamma(x)$. Define $\gamma_{+}(x)=\omega_{+}(x) \cap \alpha_{+}(x)$ and $\gamma_{-}(x)=\omega_{-}(x) \cap \alpha_{-}(x)$. Then $\Gamma_{+}=\bigcup_{x \in I} \gamma_{+}(x)$ and $\Gamma_{-}=\bigcup_{x \in I} \gamma_{-}(x)$.

The forward orbit $O_{P}(x)$ of $x \in I$ is the set $\left\{f(x), f^{2}(x), \ldots\right\}$ and the reverse orbit $O_{N}(x)$ of $x \in I$ is the set $\bigcup_{n=1}^{\infty} f^{-n}(x)$.
Let $Y$ be a subset of $I$. $\bar{Y}$ denotes the closure of $Y$ as usual. A point $y \in I$ is called a right-sided (resp. left-sided) accumulation point of $Y$ if for any $\varepsilon>0$, $(y, y+\varepsilon) \cap Y \neq \varnothing($ resp. $(y-\varepsilon, y) \cap Y \neq \varnothing)$. The right-sided closure $\bar{Y}_{+}$(resp., the left-sided closure $\bar{Y}_{-}$) is the union of $Y$ and the set of right-sided (resp. left-sided) accumulation points of $Y$. A point which is both a right-sided and a left-sided accumulation point of $Y$ is called a two-sided accumulation point of $Y$. It is easy to see that $\bar{Y}=\bar{Y}_{+} \cup \bar{Y}_{-}$.

We need the following known results.
Proposition A [5]. $x \in \Omega$ if and only if $x \in \alpha(x)$.
An interval (i.e. a connected subset of the real line) $J \subset I$ is said to be of positive (resp. negative) type if there exist $x^{\prime} \in J$ and $n^{\prime}>0$ such that $f^{\prime \prime}(x) \in J$, and for any $x \in J$ and any $n>0, x<f^{n}(x)\left(\right.$ resp. $\left.f^{n}(x)<x\right)$ provided $f^{n}(x) \in J$. An interval $J \subset I$ is said to be of free type if $f^{n}(x) \notin J$ for any $x \in J$ and any $n>0$.

Proposition B [4], [7]. If $J \subset I$ is an interval such that $J \cap P=\varnothing$, then one and only one of the following conditions holds:
(1) $J$ is of positive type;
(2) $J$ is of negative type;
(3) $J$ is of free type.

The following proposition is a slightly stronger version of a theorem of Sharkovskii [6]. (See also [4].)

Proposition C. $\bar{P}_{+}-P \subset \Lambda_{+}$and $\bar{P}_{-}-P \subset \Lambda_{-}$.
Proof. Let $x \in \bar{P}_{-}-P$. Choose a sequence $z_{1}, z_{2}, \ldots$ of periodic points of $f$ such that $z_{i} \rightarrow x$ and $z_{1}<z_{2}<\cdots<x$. Let $p_{i}$ denote the period of $z_{i}$ with respect to $f$.

Fix $i>0$, and let $g=f^{p_{i}}$. Then

$$
K=L_{i} \cup g\left(L_{i}\right) \cup g^{2}\left(L_{i}\right) \cup \ldots
$$

is an interval, where $L_{i}=\left[z_{i}, x\right]$. Let $n_{j}$ denote the period of $z_{j}$ with respect to $g$. For $k=1,2$, or 3 , suppose a subsequence of $g^{n_{j+1}-k}\left(z_{j+1}\right), g^{n_{j+2}-k}\left(z_{j+2}\right), \ldots$ converges to $u_{k} \in \bar{K}$. It is clear that $g^{k}\left(u_{k}\right)=x$. If $u_{k^{\prime}}=u_{k^{\prime \prime}}$ for some $k^{\prime}$ and $k^{\prime \prime}$ with $k^{\prime}<k^{\prime \prime}$, then

$$
g^{k^{\prime \prime}-k^{\prime}}(x)=g^{k^{\prime \prime}}\left(u_{k^{\prime}}\right)=g^{k^{\prime \prime}}\left(u_{k^{\prime \prime}}\right)=x,
$$

and so $x$ is periodic. Thus $u_{1}, u_{2}$, and $u_{3}$ are distinct points, and $u_{\vec{k}} \in K$, where $u_{\vec{k}}$ is the one which lies between the other two. Choose $v_{i} \in L_{i}$ and $\tilde{m}_{i}>0$ so that $u_{\tilde{k}}=g^{\dot{m}_{i}}\left(v_{i}\right)$. Let $m_{i}=\tilde{k}+\tilde{m}_{i}$. Then $g^{m_{i}}\left(v_{i}\right)=x$.

Summarizing, for each $i>0$, we have $v_{i} \in L_{i}$ and $m_{i}>0$ such that $f^{m_{i} p_{i}}\left(v_{i}\right)=x$.
Let $q_{i}=p_{i} m_{i}$. Since $f^{q_{i}}\left(z_{i}\right)=z_{i}$ and $f^{q_{i}}\left(v_{i}\right)=x$, it follows that $f^{q_{i}}\left(L_{i}\right) \supset L_{i}$. Let $F_{0}=I$ and inductively define $F_{n}=F_{n-1} \cap f^{t_{n}}\left(L_{n}\right), n>0$, where $t_{n}=\sum_{i=1}^{n} q_{i}$. Obviously, for any $n \geq 0, F_{n}$ is closed. Note that $F_{n} \neq \varnothing$ for any $n \geq 0$. On the other hand, it is
clear that $F_{0} \supset F_{1} \supset F_{2} \supset \cdots$. Hence, $\bigcap_{n=1}^{\infty} F_{n} \neq \varnothing$. Let $y \in \bigcap_{n=1}^{\infty} F_{n}$. Then $f^{t_{n}}(y) \in L_{n}$ for any $n>0$. Therefore $f^{t_{n}}(y) \rightarrow x$, and since $x \notin P, f^{t_{n}}(y)<x$ for every $n>0$. Thus $x \in \omega_{-}(y)$, completing the proof of the proposition.

Proposition D [9]. The following conditions are equivalent.
(1) The period of each periodic point is a power of 2.
(2) The set $\bar{P}-R$ is countable.

Proposition E [10]. If the period of each periodic point is a power of 2, then for any $x \in \bar{P}-P$, any $n \geq 0$, and any odd integer $m>0$, between $x$ and $f^{m 2^{n}}(x)$ there is a periodic point with period $2^{n}$, and there is no periodic point with period $2^{n^{\prime}}$ for every $n^{\prime}<n$.

## 3. Proof of theorem 1

Lemma 1. If $y \in \Omega$, then
(1) $\omega_{+}(y)=\gamma_{+}(y)$ and $\omega_{-}(y)=\gamma_{-}(y)$,
(2) $\omega(y)=\gamma(y)$.

Therefore $\Gamma \supset \Lambda(\Omega)$.
Proof. (1) Without loss of generality, we prove only that $\omega_{+}(y)=\gamma_{+}(y)$. Let $x \in \omega_{+}(y)$. There exist $n_{i} \rightarrow \infty$ such that $f^{n_{i}}(y) \rightarrow x$ and $x<f^{n_{i}}(y)$ for every $i>0$. It follows from Proposition A that $y \in \alpha(y)$. It is easy to see that $f^{n}(y) \in \alpha(y)$ for any $i>0$. Hence, it follows immediately that $x \in \alpha_{+}(y)$, and so $x \in \gamma_{+}(y)$. This shows that $\omega_{+}(y) \subset$ $\gamma_{+}(y)$. On the other hand, it is trivial that $\omega_{+}(y) \supset \gamma_{+}(y)$.
(2) If $y \in P$, it is clear that $\omega(y)=\gamma(y)$. If $y \notin P$, then $\omega(y)=\omega_{+}(y) \cup \omega_{-}(y)=$ $\gamma_{+}(y) \cup \gamma_{-}(y) \subset \gamma(y)$, and hence $\omega(y)=\gamma(y)$.
Lemmia 2. For $y \in I$,
(1) $\overline{\left.O_{N}(y)\right)_{+}}=O_{N}(y) \cup \alpha_{+}(y)$, and
(2) $\overline{\left(O_{N}(y)\right)_{-}}=O_{N}(y) \cup \alpha_{-}(y)$.

Proof. Without loss of generality, we prove only (1). Obviously, $\overline{\left(O_{N}(y)\right)_{+}} \supset O_{N}(y) \cup$ $\alpha_{+}(y)$. On the other hand, if $x$ is a right-sided accumulation point of $O_{N}(y)$, then we may choose a sequence $v_{i} \rightarrow x$ of points in $O_{N}(y)$ such that $x<v_{i}$ for every $i \geq 1$. Let $m_{i}>0$ be such that $f^{m_{i}}\left(v_{i}\right)=y$. If the sequence $m_{i}$ has a constant subsequence $m_{i(j)}=m$, then $f^{m}\left(v_{i(j)}\right)=y$ and $f^{m}(x)=y$, i.e. $x \in O_{N}(y)$. If the sequence $m_{i}$ has a subsequence $m_{i(j)} \rightarrow \infty$, then $x \in \alpha_{+}(y)$ may be shown by verifying that the sequence $m_{i(j)}$ and the sequence $v_{i(j)}$ satisfy the conditions of the definition of $\alpha_{+}(y)$. Therefore, $\overline{\left(O_{N}(y)\right)_{+}} \subset O_{N}(y) \cup \alpha_{+}(y)$.

Lemma 3. For $y \in I$,
(1) $\omega_{+}(y) \cap \overline{\left(O_{N}(y)\right)_{+}}=\gamma_{+}(y)$, and
(2) $\omega_{-}(y) \cap \overline{\left(O_{N}(y)\right)_{-}}=\gamma_{-}(y)$.

Proof. Without loss of generality, we prove only (1). By lemma 2,

$$
\omega_{+}(y) \cap{\overline{\left(O_{N}(y)\right)_{+}}}=\left(\omega_{+}(y) \cap O_{N}(y)\right) \cup \gamma_{+}(y)
$$

It is trivial that (1) holds if $\omega_{+}(y) \cap O_{N}(y)=\varnothing$. If $\omega_{+}(y) \cap O_{N}(y) \neq \varnothing$, choose a
point $x$ in this set, and let $m>0$ be such that $f^{m}(x)=y$. Since $x \in \omega_{+}(y)$, we know that $y \in \omega_{+}(y)$. Hence $y$ is recurrent, and so nonwandering. By lemma $1, \omega_{+}(y)=$ $\gamma_{+}(y)$, and hence (1) follows.

Lemma 4. (1) If $x \in \Lambda_{+}$, and if for every $\varepsilon>0$ there exist $v \in(x, x+\varepsilon)$ and $m>0$ such that $f^{m}(v) \in(x, x+\varepsilon)$ and $f^{m}(x)>x$, then $x \in \Gamma_{+}$.
(2) If $x \in \Lambda_{-}$, and if for every $\varepsilon>0$ there exist $v \in(x-\varepsilon, x)$ and $m>0$ such that $f^{m}(v) \in(x-\varepsilon, x)$ and $f^{m}(x)<x$, then $x \in \Gamma_{-}$.
Proof. Without loss of generality, we prove only (1). Let $x \in \Lambda_{+}$and let $y$ be a point such that $x \in \omega_{+}(y)$. There exist $n_{i} \rightarrow \infty$ such that $f^{n_{i}}(y) \rightarrow x$ and $x<f^{n_{i}}(y)$ for every $i>0$. For each $i>0$, choose $v_{i} \in\left(x, f^{n_{i}}(y)\right)$ and $m_{i}>0$ such that $f^{m_{i}}\left(v_{i}\right) \in\left(x, f^{n_{i}}(y)\right)$ and $f^{m_{i}}(x)>x$. If the sequence $m_{i}$ has a subsequence $m_{i(j)}$ such that $f^{m_{i(j)}}(x) \rightarrow x$, then $x \in R$, and so $x \in \omega_{+}(x)=\gamma_{+}(x)$ by lemma 1 . If no such subsequence exists, then there exists $i^{\prime}>0$ such that $f^{m_{i}}(x)>f^{n_{i}}(y)$ for every $i>0$. Choose $N>0$ such that $f^{m_{i}}\left(v_{i}\right) \in\left(x, f^{n_{i}}(y)\right)$ whenever $i \geq N$. Let $z=f^{n_{i}}(y)$. Since $z \in f^{m_{i}}\left(\left(x, f^{n_{i}}(y)\right)\right.$ for every $i \geq N$, we know that $x \in \overline{\left(O_{N}(z)\right)_{+}}$. Clearly, $\omega_{+}(y)=\omega_{+}(z)$. Therefore $x \in$ $\omega_{+}(z) \cap{\overline{\left(O_{N}(z)\right)}}_{+}=\gamma_{+}(z)$ by lemma 3.

Proposition 1. Suppose that $f$ is a continuous map of the interval I. Let $D$ denote a connected component of $I-P$, and let a denote the left end point of $D, b$ the right end point of $D$. Then
(1) If $D$ is of positive (resp. negative) type, then $b \in \Gamma_{+} \cup P\left(\right.$ resp. $\left.a \in \Gamma_{-} \cup P\right)$.
(2) If $D$ is of free type, then either $a \in \Gamma_{-} \cup P$ or $b \in \Gamma_{+} \cup P$.

Proof. (1) Suppose $D$ is of positive type. If $b \in P$, there is nothing to prove. Assume then that $b \notin P$. In this case, $b \in D$. Since an interval of positive type is not a singleton, we see that $b \in \bar{P}_{+}-P$. It follows from Proposition C that $b \in \omega_{+}(y)$ for some $y \in I$.

Since $D$ is of positive type, we may choose $d \in D$ and $k>0$ such that $f^{k}(d) \in D$ and $d<f^{k}(d)$. Since $D$ contains no periodic points, it follows that $b<f^{k}(b)$.

We verify that $b$ satisfies the hypotheses of lemma 4(1). Let $\varepsilon>0$. Choose a periodic point $u \in\left(b, f^{k}(b)\right) \cap(x, x+\varepsilon)$, and let $m$ denote the period of $u$. Since $b \in \bar{P}$, it follows that $f^{m}(b) \notin D$. If $f^{m}(b) \leq a$, then $\left.f^{m}(b, u)\right) \supset[d, b]$, and hence $f^{m+k}([d, b]) \supset[d, b]$, by the fact that $f^{k}([d, b]) \supset(b, u)$. Therefore there is a periodic point in $[d, b]$, which is contained in $D$, a contradiction. Thus the only possibility is that $f^{m}(b)>b$. We have verified that the hypotheses of lemma 4(1) are satisfied by $b$, and hence $b \in \Gamma_{+}$.
(2) Suppose $D$ is of free type. If either $a$ or $b$ is periodic, there is nothing to prove. Assume then that neither $a$ nor $b$ is periodic. It follows that $a, b \in D$ and $D$ is closed. We divide our discussion into three cases.
Case I. $a=0$. In this case $D=[0, b]$ and $b \in \bar{P}_{+}-P$. By proposition C, we know that $a \in \Lambda_{+}$. Let $\varepsilon>0$. Choose a periodic point $v \in(b, b+\varepsilon)$ and let the period of $v$ be $m$. Then $f^{m}(v)=v \in(b, b+\varepsilon)$ and $f^{m}(b)>b$, because $f^{m}(0)>0$. Therefore it follows from lemma 4 that $b \in \Gamma_{+}$.

Case II. $b=1$. In this case, an argument similar to the one used in case I leads us to the fact that $a \in \Gamma_{\ldots}$.

Case III. $0<a \leq b<1$. In this case, $a \in \bar{P}_{-}-P$ and $b \in \bar{P}_{+}-P$. By proposition C, $a \in \Lambda_{-}$and $b \in \Lambda_{+}$. To prove that either $a \in \Gamma_{-}$or $b \in \Gamma_{+}$, we show that $b \in \Gamma_{+}$under the assumption that $a \notin \Gamma_{-}$. First, there exists $\delta^{\prime}>0$ such that $O_{P}(a) \cap\left(a-\delta^{\prime}, a\right)=\varnothing$. (If not, $a \in \omega_{-}(a), a$ is a recurrent point, and hence $a \in \gamma_{-}(a)$ by lemma 1, contradicting our assumption.) Then $O_{P}(a) \cap\left(a-\delta^{\prime}, b\right]=\varnothing$ because $D$ is of free type. Second, it follows from lemma 4 that we may choose $\delta>0$ with $\delta^{\prime}>\delta>0$ such that whenever $m>0$ with $f^{m}(u) \in(a-\delta, a)$ for some $u \in(a-\delta, a)$, then $f^{m}(a)>a$.

We verify that $b$ satisfies the hypotheses of lemma 4(1). We have shown that $b \in \Lambda_{+}$. Let $\varepsilon>0$. Choose a periodic point $u \in(a-\delta, a)$ and let $p$ be its period, choose a periodic point $v \in(b, b+\varepsilon)$ and let $q$ be its period. Clearly, $f^{p q}(v)=v \in$ $(b, b+\varepsilon)$. On the other hand, since $f^{p q}(u)=u \in(a-\delta, a)$, it follows that $f^{p q}(a)>a$, and hence that $f^{p q}(b)>b$. Therefore $b \in \Gamma_{+}$.

Corollary 1. Suppose that $f$ is a continuous map of the interval I. Then
(1) $\min \bar{P} \in \Gamma_{+} \cup P$ and $\max \bar{P} \in \Gamma_{-} \cup P$. In particular, $\min \bar{P}$ and $\max \bar{P}$ are $\gamma$-limit points.
(2) No end point of $I$ is in $\Omega-\Gamma$.
(3) If a two-sided accumulation point of periodic points is not periodic, then it is either in $\Gamma_{+}$or in $\Gamma_{-}$. Therefore $\bar{P}_{+} \cap \bar{P}_{-} \subset \Gamma$.

Proof. (1) If $\min \bar{P}$ is not periodic, then $[0, \min \bar{P}]$ is a connected component of $I-P$ which is not of negative type. If $[0, \min \bar{P}]$ is of positive type, then its right end point $\min P$ is in $\Gamma_{+} \cup P$ by proposition 1. In the case that $[0, \min \bar{P}]$ is of free type, it follows from proposition 1 that $\min \bar{P} \in \Gamma_{+} \cup P$, because $0 \notin \Gamma_{-}$.

By a similar argument, we also see that $\max \bar{P} \in \Gamma_{-} \cup P$.
(2) If $0 \in \Omega$, then $0 \in \bar{P}$ by lemma 2.7 in [5], and hence it follows by (1) of this corollary that $0 \in \Gamma$. Similarly, if $1 \in \Omega$, then $1 \in \Gamma$.
(3) Let $x$ be a two-sided accumulation point of $P$. If $x$ is not periodic, then the singleton $\{x\}$ is a connected component of $I-P$ which is of free type, and hence its unique end point $x$ is either in $\Gamma_{+}$or in $\Gamma_{-}$, by proposition 1 .
Proof of theorem 1. (1) For any subset $Y$ of $I$, each point of $\bar{Y}-\left(\bar{Y}_{+} \cap \bar{Y}_{-}\right)$is an end point of a connected component of $I-\bar{Y}$. Since $I-\bar{Y}$ has only countably many connected components, $\bar{Y}-\left(\bar{Y}_{+} \cap \bar{Y}_{-}\right)$is a countable set. It follows that $\bar{P}-$ ( $\bar{P}_{+} \cap \bar{P}_{-}$) is countable. By corollary $1, \bar{P}-\Gamma$ is also countable. It follows from [8] that $\Omega-\bar{P}$ is countable, and so is $\Omega-\Gamma$.
(2) We claim that if $Y$ is a strictly invariant subset of $I$, and $Z$ an invariant subset of $I$ containing $P$, then $Y-Z$ is either empty or infinite. To show this, note that if $Y-Z \neq \varnothing$, then we may choose by induction a sequence $y_{1}, y_{2}, \ldots$ such that $y_{n} \in Y-Z$ and $y_{n}=f\left(y_{n+1}\right)$ for every $n \geq 1$. Since there is no periodic point in $Y-Z$, the points $y_{1}, y_{2}, \ldots$ are pairwise distinct, and hence $Y-Z$ is infinite. The proof of the claim is complete.

It follows from the claim above that $\Lambda^{1}-\Gamma$ and $\bar{P}-\Gamma$ are either empty or infinite, because $\Lambda^{1}$ and $\bar{P}$ are strictly invariant and $\Gamma$ is invariant. On the other hand, $\Lambda^{1}-\Gamma$ and $\bar{P}-\Gamma$, as subsets of the countable set $\Omega-\Gamma$, are countable.

## 4. Proof of theorem 2

Lemma 5. $\Omega \supset \Lambda^{1} \supset \bar{P} \supset \Gamma$.
Proof. The inclusion $\Omega \supset \Lambda^{1}$ is obvious, and the inclusion $\Lambda^{1} \supset \bar{P}$ is an immediate consequence of the theorem of Sharkovskii mentioned in § 1. (See also proposition C.)

It remains to prove that $\bar{P} \supset \Gamma$. To do this, assume that $x \in \Gamma-\bar{P}$. Let $y$ be such that $x \in \omega(y) \cap \alpha(y)$, and $D$ be the connected component of $I-\bar{P}$ containing $x$. Clearly $D$ is not of free type. By proposition B, we may assume, without loss of generality, that $D$ is of positive type. Since $x \in \omega(y)$, there exist $n_{i} \rightarrow \infty$ such that $f^{n}(y) \rightarrow x$. Let $i$ be an integer such that $f^{n_{i}}(y) \in D$. Since $D$ is of positive type, we have that $f^{n_{1}}(y)<x$. Since $x \in \alpha(y)$, there exists $u \in D \cap\left(f^{n_{1}}(y), 1\right]$ such that $f^{m}(u)=$ $y$ for some $m>0$. Then $f^{m+n_{i}}(u)=f^{n_{i}}(y)$. Hence $D$ fails to be of positive type, a contradiction.

Lemma 6. For $y \in I$,
(1) $\omega_{+}(y) \cap \alpha_{-}(y) \subset \Gamma_{+} \cup P$, and
(2) $\omega_{-}(y) \cap \alpha_{+}(y) \subset \Gamma_{-} \cup P$.

Proof. Without loss of generality, we prove only (1). Let $x \in \omega_{+}(y) \cap \alpha_{-}(y)$. If $x \notin \Gamma_{+}$, then it follows from lemma 3 and lemma 4 that there exists $\varepsilon>0$ such that
(a) $(x, x+\varepsilon) \cap O_{N}(y)=\varnothing$, and
(b) if $m>0$ and $f^{m}(u) \in(x, x+\varepsilon)$ for some $u \in(x, x+\varepsilon)$, then $f^{m}(x) \leq x$.

Since $x \in \omega_{+}(y)$, we may choose $m$ and $n$ with $m>n>0$ such that $f^{m}(y), f^{n}(y) \in$ $(x, x+\varepsilon)$. Then by condition (b), $f^{m-n}(x) \leq x$. On the other hand, it follows from the condition (a) that $f^{m-n}(x) \geq x$. For if $f^{m-n}(x)<x$, then $f^{m-n}((x, x+\varepsilon)) \supset$ ( $\left.f^{m-n}(x), x\right)$, which contains points of $O_{N}(y)$ because $x \in \alpha_{-}(y)$. Therefore $x=$ $f^{m-n}(x)$ and $x$ is a periodic point.

Proposition 2. Suppose that $f$ is a continuous map of the interval I. Then $\Gamma=\Gamma_{+} \cup$ $\Gamma_{-} \cup P$.
Proof. Obviously, $\Gamma \supset \Gamma_{+} \cup \Gamma_{-} \cup P$. On the other hand, it is easily seen that $\omega(y)$ is a periodic orbit if $y$ is periodic. If $y$ is not periodic, then it follows that $\omega(y)=$ $\omega_{+}(y) \cup \omega_{-}(y)$ and $\alpha(y)=\alpha_{+}(y) \cup \alpha_{-}(y)$. Therefore

$$
\begin{aligned}
\gamma(y) & =\gamma_{+}(y) \cup \gamma_{-}(y) \cup\left(\omega_{+}(y) \cap \alpha_{-}(y)\right) \cup\left(\omega_{-}(y) \cap \alpha_{+}(y)\right) \\
& \subset \Gamma_{+} \cup \Gamma_{-} \cup P
\end{aligned}
$$

by lemma 6. Thus $\Gamma=\bigcup_{y \in I} \gamma(y) \subset \Gamma_{+} \cup \Gamma_{-} \cup P$.
Lemma 7. Let $y \in I-P$, and for each $n \geq 0$, let $C_{n}$ denote the connected component of $I-O_{N}(y)$ containing $f^{n}(y)$. Then
(1) $f^{n}\left(\bar{C}_{0}\right) \subset \bar{C}_{n}$, and
(2) $\bar{C}_{0} \cap \Gamma \neq \varnothing$.

Proof. (1) $f^{n}\left(C_{0}\right)$ is a connected subset of $I-O_{N}(y)$ containing $f^{n}(y)$. Therefore $f^{n}\left(C_{0}\right) \subset C_{n}$, and so $f^{n}\left(\bar{C}_{0}\right) \subset \bar{C}_{n}$.
(2) If $\bar{C}_{0} \cap P \neq \varnothing$, there is nothing to prove. Assume then that $\bar{C}_{0} \cap P=\varnothing$. In this case, $\bar{C}_{0}$ is contained in some connected component $D$ of $I-P$. Let $\bar{C}_{0}=[a, b]$, and let $a^{\prime}$ denote the left end point of $D, b^{\prime}$ the right end point of $D$.

We claim that $\bar{C}_{0}$ and $D$ have at least one end point in common. If not, then $a^{\prime}<a \leq y \leq b<b^{\prime}$. Since $\left(a^{\prime}, a\right] \cap O_{N}(y) \neq \varnothing$ and $\left(b, b^{\prime}\right] \cap O_{N}(y) \neq \varnothing, D$ is not of positive type, of negative type, or of free type. This contradicts proposition B.

If the end points of $\bar{C}_{0}$ and of $D$ coincide, then $\bar{C}_{0} \cap \Gamma \neq \varnothing$ because at least one of the endpoints of $D$ is in $\Gamma$ by proposition 1 .

Assume then, without loss of generality, that $a^{\prime}<a$. In this case, it follows that $D$ is of positive type, because $\left(a^{\prime}, a\right] \cap O_{N}(y) \neq \varnothing$. Then the right end point $b=b^{\prime}$ of $D$ is in $\Gamma_{+} \cup P$.

The proof is complete.
Proof of theorem 2. Lemma 1 shows that $\Gamma \supset \Lambda(\Omega)$, and the inclusions $\Lambda(\Omega) \supset \Lambda^{2} \supset$ $\Lambda(\bar{P}) \supset \Lambda(\Gamma)$ are immediate consequences of lemma 5 .

We prove that $\Lambda(\Gamma) \supset \Gamma$ as follows. Let $x \in \Gamma$. Obviously $x \in \Lambda(\Gamma)$ if $x$ is periodic. Assume then that $x \notin P$. Then either $x \in \Gamma_{+}$or $x \in \Gamma_{-}$by proposition 2 . Assume, without loss of generality, that $x \in \Gamma_{+}$. Let $y$ be such that $x \in \gamma_{+}(y)$, and $C_{n}$ the connected component of $I-O_{N}(y)$ containing $f^{n}(y)$. Since $x \in \gamma_{+}(y)$, there exist $n_{i} \rightarrow \infty$ such that $f^{n_{i}}(y) \rightarrow x, x<\cdots<f^{n_{2}}(y)<f^{n_{1}}(y)$, and for every $i>0$ the interval $\left(f^{n_{i+1}}(y), f^{n_{i}}(y)\right)$ contains at least two distinct points of $O_{N}(y)$. Then the intervals $\bar{C}_{n_{1}}, \bar{C}_{n_{2}}, \ldots$ are pairwise disjoint and $L_{i} \rightarrow 0$, where $L_{i}$ denotes the length of $\bar{C}_{n_{i}}$. By lemma $7(2)$, there exists $u \in \bar{C}_{0} \cap \Gamma$, and it follows from lemma $7(1)$ that $\mid f^{n_{1}}(u)-$ $f^{n_{i}}(y) \mid \leq L_{i}$. Therefore $f^{n_{i}}(u) \rightarrow x$, and so $x \in \Lambda(\Gamma)$.

Up to now, we have shown that

$$
\Gamma=\Lambda(\Omega)=\Lambda^{2}=\Lambda(\bar{P})=\Lambda(\Gamma) .
$$

Then it follows by induction that for every $n \geq 2$,

$$
\Lambda^{n+1}=\Lambda\left(\Lambda^{n}\right)=\Lambda(\Gamma)=\Gamma
$$

The proof of theorem 2 is completed.

## 5. Proof of theorem 3

$(1) \Rightarrow(2)$. Suppose that condition (1) holds. Let $x \in \Gamma$. We show that $x \in R$ as follows. Obviously, $x \in R$ if $x$ is periodic. Assume then that $x \notin P$.
$\Gamma=\Lambda(\bar{P})$ by theorem 2 , so there exists $y \in \bar{P}$ such that $x \in \omega(y)$. Since $x$ is not periodic, $y$ is not either. Therefore $x \in \omega_{+}(y) \cup \omega_{-}(y)$. We may assume, without loss of generality, that $x \in \omega_{+}(y)$. Then there exist $n_{i} \rightarrow \infty$ such that $f^{n_{i}}(y) \rightarrow x$ and $x<f^{n_{i}}(y)$ for every $i>0$.

Given $\varepsilon>0$, let $p>0$ be the least integer such that there is a periodic point in ( $x, x+\varepsilon$ ) with period $2^{p}$, and let $u$ be the smallest periodic point in $(x, x+\varepsilon)$ with period $2^{p}$. Choose $j>0$ such that $x<f^{n}(y)<u$. Let $q>0$ be the least integer such that there is a periodic point in $\left(x, f^{n}(y)\right)$ with period $2^{q}$, and $v$ a periodic point in $\left(x, f^{n_{j}}(y)\right)$ with period $2^{q}$. Clearly, $q>p$. Then, let $i^{\prime}>0$ be such that $n_{i}>n_{j}$ and $f^{n}(y) \in(x, v)$ whenever $i \geq i^{\prime}$.

Fix $i \geq i^{\prime}$. Let $n_{i}-n_{j}=m_{i} 2^{t_{i}}$, where $t_{i}>0$ and $m_{i}>0$ is odd. We claim that $t_{i}=q$. For if $t_{i}<q$, then by proposition E , there would be a periodic point between $f^{n}(y)$ and $f^{n_{i}}(y)\left(=f^{m_{i} 2^{t_{1}}}\left(f^{n_{i}}(y)\right)\right)$ with period $2^{t^{i}}$, contradicting the definition of $q$, and if $t_{i}>q$, then by the same proposition, there would be no periodic point with period $2^{q}$ between the two points above.

Therefore if we write $n_{i}-n_{j}+2^{q}=\tilde{m}_{i} 2^{\tilde{i}_{i}}$, where $\tilde{t}_{i}>0$ and $\tilde{m}_{i}>0$ is odd, then $\tilde{t}_{i}>q$. By proposition E , between $f^{n_{\mathrm{i}}+2^{q}}(y)$ and $f^{n_{j}}(y)$ there is no periodic point with period either $2^{p}$ or $2^{q}$. Thus, $f^{n_{i}+2^{q}}(y) \in(v, u)$.

Since $f^{n_{i}+2^{q}}(y) \rightarrow f^{2^{9}}(x)$, we have that $f^{2^{q}}(x) \in(x, x+\varepsilon)$. This shows that $x \in R$ and the proof of the implication (1) $\Rightarrow(2)$ is complete.
(2) $\Rightarrow(1)$. If $\Gamma=R$, then $\bar{P}-R$ is countable by theorem 1 . Therefore, it follows from proposition $D$ that (1) holds.
Acknowledgments. The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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