J. Math. Soc. Japan
Vol. 37, No. 2, 1985

The automorphism group of Leech lattice and elliptic modular functions

Dedicated to Professor Hirosi Nagao on his 60th birthday

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(Received May 28, 1984)

Introduction.

As usual, we denote by $\cdot 0$ the automorphism group of Leech lattice which is an even unimodular lattice in 24-dimensional Euclidean space [**1**]. So $\cdot 0$ has a natural 24-dimensional representation ρ_0 over the rational number field. In this paper, Frame shapes of conjugacy classes of $\cdot 0$ with respect to ρ_0, the list of which is given in Table I of Appendix, will play a central role. For the definition of Frame shape, see § 1.2.

Let $\mathfrak{F}$ be the set of all elliptic modular functions $f(z)$ satisfying the following conditions:

(1) $f(z)$ is a modular function with respect to a discrete subgroup Γ of $SL(2, \boldsymbol{R})$ containing $\Gamma_0(N)$ for some integer N (i.e. $f\left(\frac{az+b}{cz+d}\right)=f(z)$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and meromorphic on the upper half plane and at all cusps of Γ),

(2) the genus of Γ is zero and $f(z)$ is a generator of a function field for Γ,

(3) $f(z)$ has a Fourier expansion of the form $f(z)=1/q+\sum_{n=0}^{\infty} a_n q^n$ $(q=e^{2\pi iz})$.

Now the main result of this paper is to show that *various "transformations"* (cf. § 1.1) *of Frame shapes of* $\cdot 0$ *yield functions of* $\mathfrak{F}$ (Th. 3.2, 3.4, 3.5 and Table II～IV in Appendix). Furthermore, an application of this result is as follows: Let G be a finite group which has a d-dimensional representation ρ over the rational number field where d is a divisor of 24. For each of such many (not all) pairs (G, ρ), we can construct a mapping from G to $\mathfrak{F}$

$$G \ni \sigma \longmapsto j_\sigma(z) \in \mathfrak{F}$$

such that all coefficients $a_k(\sigma)$ $(k \geqq 1)$ of a Fourier expansion $j_\sigma(z)=1/q+\sum_{k=0}^{\infty} a_k(\sigma) q^k$ are generalized characters of G (Th. 4.6, 4.8 and 4.10). Such a mapping is called a *moonshine* of G. A moonshine of Fischer-Griess's Monster is constructed in a remarkable paper of Conway-Norton [**2**] and other examples of moonshines can be found in Queen [**10**] and Koike [**4**]. Constructions of moonshines in this paper are rather elementary compared with those of Conway-Norton-Queen. For examples of pairs (G, ρ) which does not yield a moonshine, we refer readers to Remark 4.4.

The author is very grateful to Prof. Masao Koike for many valuable discussions and suggestions.

NOTATIONS.

$\boldsymbol{Z}$ the ring of rational integers

$\boldsymbol{Q}$ the field of rational numbers

$\boldsymbol{R}$ the field of real numbers

$SL(2, \boldsymbol{Z})=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \boldsymbol{Z} \text{ and } ad-bc=1\right\}$

$SL(2, \boldsymbol{R})=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \boldsymbol{R} \text{ and } ad-bc=1\right\}$

$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z}) \,\middle|\, c \equiv 0 \bmod N\right\}$

$\langle\cdots\rangle$=a group generated by $\cdots$.

For the notations of conjugacy classes of $\cdot 0$, see the first paragraph of Appendix.

§ 1. Generalized permutations and Frame shapes.

1.1. A symbol

$$\prod_t t^{r_t} = 1^{r_1} 2^{r_2} \cdots \qquad (r_t \in \boldsymbol{Z})$$

is called a *generalized permutation* if $r_t \neq 0$ for only a finite number of positive integers t. For a generalized permutation $\pi=\prod_t t^{r_t}$, set

$$\deg \pi = \sum_t t r_t ,$$

$$\operatorname{sgn} \pi = \prod_t (-1)^{(t-1) r_t} .$$

Obviously $\deg \pi$ and $\operatorname{sgn} \pi$ are generalizations of degree and sign of a permutation on a finite set.

Now we will define some transformations of a generalized permutation. Let r be a positive integer and $\pi=\prod_t t^{r_t}$ be a generalized permutation. Then define

$$\pi/r = \prod_t (rt)^{r_t/r}, \qquad \text{where } r \mid r_t \text{ for any } t,$$

$$\pi \circ r = \prod_t t^{r_t/(r+1)} (rt)^{r_t/(r+1)}, \qquad \text{where } r+1 \mid r_t \text{ for any } t,$$

$$\pi \circ (r/1) = \prod_t (rt)^{r_t/(r-1)} t^{-r_t/(r-1)}, \qquad \text{where } r-1 \mid r_t \text{ for any } t.$$

These are called the *r-th harmonic, the r-transformation and the* $(r/1)$*-transformation* of π respectively. All of these transformations have the same degree as π.

We note that (2/1)-transformation can be defined for all generalized permutations.

Let $\eta(z)$ be Dedekind eta-function:

$$\eta(z)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q=e^{2\pi i z}.$$

For a generalized permutation $\pi=\prod_t t^{r_t}$, we put

$$\eta_\pi(z)=\prod_t \eta(tz)^{r_t}. \tag{1.1}$$

The meaning of the transformations defined above consists in considering the transformation of functions $\eta_\pi(z)$:

$$\eta_\pi(z)\longmapsto \eta_{\pi^*}(z) \qquad \pi^*=\pi/r,\ \ \pi\circ(r/1) \ \text{ or } \ \pi\circ r.$$

1.2. Let G be a finite group and

$$G\ni\sigma\longmapsto\rho(\sigma)\in GL(d, \boldsymbol{Q})$$

be a d-dimensional representation of G over the rational number field $\boldsymbol{Q}$. Then we will assign to every element (or every conjugacy class) of G a generalized permutation of degree d as follows. The characteristic polynomial $\det(xI_d-\rho(\sigma))$ (I_d=the identity matrix of degree d) of $\rho(\sigma)(\sigma\in G)$ can be written in the form

$$\prod_t (x^t-1)^{r_t} \qquad (r_t\in\boldsymbol{Z})$$

where t ranges over all positive integers dividing the order of G. Then a generalized permutation $\prod_t t^{r_t}$ of degree d is called *Frame shape* of an element σ with respect to the representation ρ. We also refer to Frame shape of a conjugacy class of G (w.r.t. ρ), as two conjugate elements of G yield the same Frame shape.

REMARK 1.1. If a representation ρ is a permutation representation of G (i.e. every $\rho(\sigma)$ is a permutation matrix), the Frame shape of σ w.r.t. ρ is just a cycle decomposition of a permutation corresponding to $\rho(\sigma)$. Thus Frame shape can be regarded as a generalization of a cycle decomposition of an element in a permutation group.

REMARK 1.2. If G has no subgroup of index 2 and so $\det\rho(\sigma)=1$ for all $\sigma\in G$, we have $\operatorname{sgn}\pi=1$ for all Frame shapes π of conjugacy classes of G, because $\det\rho(\sigma)=\operatorname{sgn}\pi$.

REMARK 1.3. A generalized permutation is not always a Frame shape. For example, a generalized permutation $1.2^{-2}4$ is not a Frame shape, as $(x-1)(x^4-1)/(x^2-1)^2$ is not a polynomial.

§ 2. A class of elliptic modular functions.

2.1. As in the introduction, let $\mathfrak{F}$ be the set of all elliptic modular functions $f(z)$ having the following properties:

(1) $f(z)$ is a modular function with respect to a discrete subgroup Γ of $SL(2, \boldsymbol{R})$ containing some $\Gamma_0(N)$,

(2) the genus of Γ is zero and $f(z)$ is a generator of a function field for Γ and

(3) $f(z)$ has a Fourier expansion of the form $f(z)=1/q+\sum_{n=0}^{\infty} a_n q^n$ $(q=e^{2\pi i z})$.

For simplicity, we call Γ in (1) and (2) a group for $f(z)$ and also $f(z)$ a Hauptmodule for Γ. Clearly the well known modular invariant $j(z)$ belongs to $\mathfrak{F}$ and $\Gamma_0(1)=SL(2, \boldsymbol{Z})$ is a group for $j(z)$. Other examples of $f(z)\in\mathfrak{F}$ and a group for $f(z)$ can be found in Table 3 of [**2**] which is very useful in this paper. In these examples, a group for $f(z)\in\mathfrak{F}$ is the one obtained by adjoining to $\Gamma_0(N)$ some of its Atkin-Lehner's involutions W_Q

$$W_{Q,N}=W_Q=\begin{pmatrix} aQ & b \\ cN & dQ \end{pmatrix} \qquad a, b, c, d\in \boldsymbol{Z}$$

where $Q\|N$, i.e. Q is a divisor of N with $(Q, N/Q)=1$ and $\det W_Q=Q$. As in [**2**] and [**10**], we use the notations

$$N+Q_1, Q_2, \cdots, \qquad N-, \qquad N+$$

which denote

$$\langle \Gamma_0(N), W_{Q_1}, W_{Q_2}, \cdots\rangle, \qquad \Gamma_0(N), \qquad \langle\Gamma_0(N), W_Q \mid Q\|N\rangle$$

respectively.

LEMMA 2.1. *Let $\eta_\pi(z)$ be a function defined by* (1.1) *for a generalized permutation π. Assume that*

(1) $\deg\pi=-24$,

(2) *$\eta_\pi(z)$ is a modular function w.r.t. a discrete subgroup Γ of $SL(2, \boldsymbol{R})$ containing some $\Gamma_0(N)$,*

(3) *$\Gamma_{i\infty}=\{M(i\infty)=i\infty \mid M\in\Gamma\}$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,*

(4) *$z=i\infty$ is the unique pole of $\eta_\pi(z)$ among all inequivalent cusps of Γ. Then $\eta_\pi(z)\in\mathfrak{F}$ and Γ is a group for $\eta_\pi(z)$.*

PROOF. The condition (1) means that $\eta_\pi(z)$ has a Fourier expansion of the form $1/q+\sum_{n=0}^{\infty} a_n q^n$ and the condition (3) shows that q can be taken as a local parameter of $\eta_\pi(z)$ at $z=i\infty$ and so $z=i\infty$ is a pole of $\eta_\pi(z)$ of order 1. Let $\mathfrak{R}$ be a Riemann surface corresponding to Γ, i.e. $\mathfrak{R}=\Gamma\backslash\mathfrak{H}^*$ where $\mathfrak{H}^*$ is a union of the upper half plane and the set of all cusps of Γ. Since $\eta(z)$ has no poles on the upper half plane, the condition (4) means that $z=i\infty$ is the unique pole of $\eta_\pi(z)$ on $\mathfrak{R}$ and so $\eta_\pi(z)$ gives an isomorphism from $\mathfrak{R}$ onto the Riemann sphere. Thus the genus of $\mathfrak{R}$ is zero and $\eta_\pi(z)$ is a generator of a function field of $\mathfrak{R}$. This completes the proof of Lemma 2.1.

2.2. Here we mention the well known transformation formula of Dedekind eta-function:

$$\eta\left(\frac{az+b}{cz+d}\right)=v(M)(cz+d)^{1/2}\eta(z) \qquad \text{for } M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2, \boldsymbol{Z}) \tag{2.1}$$

where $v(M)^{24}=1$. An explicit formula of $v(M)$ was given by Petersson [**6**; Th. 2 of Chap. 4]:

$$v(M)=\begin{cases} \left(\dfrac{d}{c}\right)^* \exp\left\{\dfrac{\pi i}{12}[(a+d)c-bd(c^2-1)-3c]\right\} & \text{if } c \text{ is odd} \\ \left(\dfrac{c}{d}\right)_* \exp\left\{\dfrac{\pi i}{12}[(a+d)c-bd(c^2-1)+3d-3-3cd]\right\} & \text{if } c \text{ is even} \end{cases} \tag{2.2}$$

where, by using Jacobi symbol $\left(\dfrac{n}{m}\right)$, we put

$$\left(\frac{c}{d}\right)^*=\left(\frac{c}{|d|}\right) \text{ and } \left(\frac{c}{d}\right)_*=\left(\frac{c}{|d|}\right)(-1)^e, \qquad e=\frac{\operatorname{sgn} c-1}{2}\cdot\frac{\operatorname{sgn} d-1}{2},$$

$$\left(\frac{0}{\pm 1}\right)^*=1, \qquad \left(\frac{0}{1}\right)_*=1 \text{ and } \left(\frac{0}{-1}\right)_*=-1.$$

Now we give some formulas which are useful for our calculations of $\eta_\pi(z)$:

$$\eta\left(z+\frac{1}{2}\right)=e^{\pi i/24}\eta(2z)^3/\eta(z)\eta(4z). \tag{2.3}$$

(2.4) If $2|N_0|N$ and $(2N_0, N)=N_0$,

$$\eta(2N_0z/(Nz+1))=v(M)\,(Nz+1)^{1/2}\eta((N_0z+1)/2),$$

$$M=\begin{pmatrix} 2 & -1 \\ N/N_0 & (N_0-N)/2N_0 \end{pmatrix}\in SL(2, \boldsymbol{Z}).$$

$$\eta\left(z+\frac{1}{3}\right)\eta\left(z+\frac{2}{3}\right)=e^{\pi i/12}\eta(3z)^4/\eta(z)\eta(9z). \tag{2.5}$$

(2.6) If $3|N_0|N$, $(3N_0, N)=N_0$ and $N/N_0\equiv\varepsilon \bmod 3$ $(\varepsilon=\pm 1)$,

$$\eta(3N_0z/(Nz+1))=v(M)(Nz+1)^{1/2}\eta((N_0z+\varepsilon)/3),$$

$$M=\begin{pmatrix} 3 & -\varepsilon \\ N/N_0 & (N_0-N\varepsilon)/3 \end{pmatrix}\in SL(2, \boldsymbol{Z}).$$

(2.7) If $W_Q=\begin{pmatrix} aQ & b \\ cN & dQ \end{pmatrix}$ is an Atkin-Lehner's involution of $\Gamma_0(N)$,

$$\eta(tW_Q(z))=v(M)(Q,t)^{-1/2}(cNz+dQ)^{1/2}\eta((Qt/(Q,t)^2)z),$$

$$\text{where } M=\begin{pmatrix} a(Q,t) & bt/(Q,t) \\ cN(Q,t)/Qt & dQ/(Q,t) \end{pmatrix}\in SL(2,\boldsymbol{Z}).$$

(2.3) and (2.5) are obtained by direct computations. (2.4), (2.6) and (2.7) follow from (2.1).

LEMMA 2.2 (M. Newmann [**9**; Th. 1]). *Let* $\pi=\prod_{t|N}t^{r_t}$ *be a generalized permutation and* $\eta_\pi(z)=\prod_{t|N}\eta(tz)^{r_t}$, *where* t *ranges over all positive divisors of some integer* N. *Assume that*

(0) $$\sum_t r_t=0,$$

(1) $$\sum_t r_t t\equiv 0 \mod 24,$$

(2) $$\sum_t r_t N/t\equiv 0 \mod 24$$

(3) *the number* $\prod_{t|N}t^{r_t}$ *is a rational square.*

Then $\eta_\pi(z)$ *is a modular function* w.r.t. $\Gamma_0(N)$.

A proof of Lemma 2.2 can be done by using (2.1) and (2.2).

LEMMA 2.3. *Let* $\pi=\prod_t t^{r_t}$ *be a generalized permutation of degree* 24 *and* $r>1$ *be an integer with* $r|r_t$ *for any* t. *Assume that*

(1) $$\eta_\pi(z)^{-1}\in\mathcal{F},$$

(2) $\prod_t t^{r_t/r}$ *is a rational square.*

Then $\eta_{\pi/r}(z)^{-1}\in\mathcal{F}$, *where* π/r *is the* r*-th harmonic of* π.

PROOF. Let $f(z)=\eta_\pi(z)^{-1}$ and $g(z)=\eta_{\pi/r}(z)^{-1}$. Then we have

(*) $$g(z)=f(rz)^{1/r}.$$

If Γ is a group for $f(z)\in\mathcal{F}$, $f(rz)$ is a modular function w.r.t. $\Gamma_1=\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}^{-1}\Gamma\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$ and $[\boldsymbol{C}(g(z)):\boldsymbol{C}(f(rz))]=r$. Also we have, by (*),

(**) $$g(Mz)=\delta g(z) \quad (\delta^r=1) \qquad \text{for any } M\in\Gamma_1$$

and, in particular,

(***) $$g\left(z+\frac{1}{r}\right)=e^{-2\pi i/r}g(z) \qquad \text{for } \begin{pmatrix} 1 & 1/r \\ 0 & 1 \end{pmatrix}\in\Gamma_1.$$

Now let $\Gamma_2=\{M\in\Gamma_1 | g(Mz)=g(z)\}$. Then, by (**) and (***), we must have $[\Gamma_1:\Gamma_2]=r$ and so $\boldsymbol{C}(g(z))$ is a function field for Γ_2. By the assumption (2) and

Lemma 2.2, Γ_2 contains some $\Gamma_0(N)$. Thus $g(z)\in\mathfrak{F}$. This completes the proof of Lemma 2.3.

LEMMA 2.4. *For a generalized permutation* $\pi=\prod_t t^{r_t}$ *and* $T=\begin{pmatrix}1 & 1/2\\ 0 & 1\end{pmatrix}\in SL(2, \boldsymbol{R})$, *define*

$$\pi\circ T=\prod_{t:\ \text{even}} t^{r_t}\prod_{t:\ \text{odd}}(2t)^{3r_t}t^{-r_t}(4t)^{-r_t}.$$

If $\eta_\pi(z)\in\mathfrak{F}$, *we have* $\mathfrak{F}\ni\eta_{\pi\circ T}(z)=-\eta_\pi(Tz)$.

PROOF. If $\eta_\pi(z)$ is a modular function w.r.t. Γ, $\eta_{\pi\circ T}(z)$ is a modular function w.r.t. $T^{-1}\Gamma T$. From this we get $\mathfrak{F}\ni\eta_{\pi\circ T}(z)$ if $\eta_\pi(z)\in\mathfrak{F}$. The equality $\eta_{\pi\circ T}(z)=-\eta_\pi(Tz)$ follows from (2.3), q.e.d.

§ 3. Frame shapes of conjugacy classes of $\cdot 0$.

3.1. The automorphism group $\cdot 0$ of Leech lattice has a natural 24-dimensional representation ρ_0 over $\boldsymbol{Q}$. It is not difficult to compute Frame shape of every conjugacy class of $\cdot 0$ by using the character values of the representation ρ_0 and the power mapping of conjugacy classes of $\cdot 0$ [**11**; Table 1]. The list of Frame shapes of conjugacy classes of $\cdot 0$ is given in Table I of Appendix. The following observation of the list may be useful (cf. [**2**; p. 315]):

THEOREM 3.1. *Let* $\pi=\prod_t t^{r_t}$ *be a Frame shape of* $\cdot 0$. *Then* $\sum_t r_t$ *is even and if* $\sum_t r_t=0$, *we have* $\eta_\pi(z)^{-1}\in\mathfrak{F}$ (*=a class of elliptic modular functions defined in* § 2).

PROOF. By inspection of Table I of Appendix, $\sum_t r_t$ is even. It can be seen from Table 3 of [2] that, if $\sum_t r_t=0$, $\eta_\pi(z)^{-1}\in\mathfrak{F}$.

REMARK 3.1. Let $\sum_t r_t=2k\neq 0$ and N be a product of L.C.M. and G.C.D. of $\{t\,|\,r_t\neq 0\}$. Then $\eta_\pi(z)$ is a cusp form or an Eisenstein series of level N and weight k with some character, according as $r_t\geqq 0$ for any t or not (cf. [**3**], [**5**] and [**8**]).

The following theorem is one of the main results of this paper:

THEOREM 3.2. *Let* $\pi=\prod_t t^{r_t}$ *be a Frame shape of* $\cdot 0$ *and* $r>1$ *be an integer with* $r-1\,|\,24$ *and* $r-1\,|\,r_t$ *for any* t. *Then we have*

(3.1) $\eta_{\pi\circ(r/1)}(z)^{-1}\in\mathfrak{F}$ ($\pi\circ(r/1)=(r/1)$-*transformation of* π (§ 1.1)),

except for the following cases:

r	*classes*			
3	$\pm 4C$,	$4F$,	$8D$,	$12M$
5	$2C$,	$4B$,	$6I$,	$8B$
7	$4F$			
13	$2C$			

In these exceptional cases, we have $\mathrm{sgn}(\pi^{1/(r-1)})=-1$, *where* $\pi^{1/(r-1)}=\prod_t t^{r_t/(r-1)}$.

REMARK 3.2. The case $r=2$ of Th. 3.2 is a part of a theorem of Conway-Norton-Queen [**2**], [**10**] which says that a mapping

$$\cdot 0 \ni \sigma \longmapsto \eta_{\sigma\circ(2/1)}(z)^{-1}=q^{-1}\prod_{n=0}^{\infty}\prod_{i=1}^{24}(1-\varepsilon_i(\sigma)q^{2n-1})$$

is a moonshine of $\cdot 0$, where $\varepsilon_i(\sigma)$ $(1\leqq i\leqq 24)$ are eigenvalues of $\rho_0(\sigma)$. In Table II of Appendix, we will give $\sigma\circ(2/1)$ and a group for $\eta_{\sigma\circ(2/1)}(z)^{-1}$ for each σ. This table can be also found in Queen [**10**; Table 1], but, in Queen's table, some conjugacy classes of $\cdot 0$ are missing and, in our table, a group for $\eta_{\sigma\circ(2/1)}(z)^{-1}$ $(\sigma\in\cdot 0)$ is described more explicitly than in Queen's table.

REMARK 3.3. Notations being as in Th. 3.2, let G be a finite group with no subgroup of index 2 and ρ be a $24/(r-1)$-dimensional representation over $\boldsymbol{Q}$. If $\mathrm{sgn}(\pi^{1/(r-1)})=-1$, $\pi^{1/(r-1)}$ is not a Frame shape of G w.r.t. ρ, because $\mathrm{sgn}(\pi^{1/(r-1)})=-1$ means that the determinant of a linear transformation with Frame shape $\pi^{1/(r-1)}$ is -1.

PROOF OF THEOREM 3.2. This is done by using Table I of Appendix and examining (3.1) in case by case for each conjugacy class of $\cdot 0$. Here we will give a proof of the case $r=3$. (Also the case $r=2$ (cf. Remark 3.2) can be dealt with quite similarly, and other cases $r>3$ are rather easy to be examined.)

First of all, we see from Table I of Appendix that, if π is a Frame shape of $\cdot 0$, $\pi\circ(3/1)$ is

(1) a Frame shape of $\cdot 0$,

(2) for some r, the r-th harmonic of a Frame shape $\prod_t t^{r_t}$ of $\cdot 0$ such that $\sum_t r_t=0$ and $\prod_t t^{r_t/r}$ is a rational square, or

(3) one of the following generalized permutations:

classes of π	$\pi\circ(3/1)$	
$\pm 3A$	$9^6 1^6/3^{12}$,	$2^6 3^{12} 18^6/1^6 6^{12} 9^6$
$\pm 4C$	$3^2 6.\,12^2/1^2 2.\,4^2$,	$1^2 6^3 12^2/2^3 3^2 4^2$
$4F$	$12^3/4^3$	
$\pm 6A$	$1^2 2^2 9^2 18^2/3^4 6^4$,	$2^4 3^4 18^4/1^2 6^8 9^2$
$6B$	$4^3 6^6 36^3/2^3 12^6 18^3$	
$8D$	$4.\,24^2/8^2 12$	
$\pm 12A$	$3^4 4^2 36^2/1^2 9^2 12^4$,	$1^2 4^2 6^4 9^2 36^2/2^2 3^4 12^4 18^2$
$12B$	$4^2 6^2 36^2/2.\,12^4 18$	
$12C$	$2.\,4.\,18.\,36/6^2 12^2$	
$12M$	$36/12$	
$\pm 15B$	$1.\,5.\,9.\,45/3^2 15^2$,	$2.\,3^2 10^2 15.\,18.\,90/1.\,5.\,6^2 9.\,30^2 45$
$\pm 21A$	$3^2 7.\,63/1.\,3.\,21^2$,	$1.\,6^2 9.\,14.\,21^2 126/2.\,3^2 7.\,18.\,42^2 63$
$24A$	$6^2 8.\,72/2.\,18.\,24^2$	

If we have the case (1) or (2), we can conclude from Th. 3.1 and Lemma 2.3 that $\eta_{\pi\circ(3/1)}(z)^{-1}\in\mathcal{F}$. So suppose we have the case (3).

Classes $\pm 4C$, $4F$, $8D$, $12M$; These classes are exceptional ones in Th. 3.2 and then we have $\mathrm{sgn}(\pi^{1/(r-1)})=-1$.

Classes $\pm 6A$; We see from Table 3 of [**2**] that $\pi\circ(3/1)$ is a Hauptmodule for 18+ or 18+9.

Classes $12B$ or $12C$; By Table 3 of [**2**], $\pi\circ(3/1)$ is the 2nd-harmonic of a Hauptmodule for 18+9 or 18+. Then (3.1) follows from Lemma 2.3.

Classes $+3A$, $-12A$ or $+15B$; $\pi\circ(3/1)$ is a Hauptmodule for 9+, 36+ or 45+ respectively by Table 3 of [**2**].

Now conjugacy classes $-3A$, $6B$, $+12A$, $-15B$, $\pm 21A$ and $24A$ remain to be examined. Since $\pi\circ(3/1)$ for $6B$ or $24A$ is the 2nd-harmonic of $-3A$ or $+12A$ respectively, it is sufficient to see (Lemma 2.3) that, for five classes $-3A$, $+12A$, $-5A$ and $\pm 21A$, $\eta_{\pi\circ(3/1)}(z)^{-1}\in\mathcal{F}$. Among these classes, we will prove (3.1) for the class $-3A$, as other classes can be also dealt with quite similarly.

Let π be the Frame shape of the class $-3A$ and let

$$f(z)=\eta_{\pi\circ(3/1)}(z)^{-1},$$

$$M=W_2\begin{pmatrix}1&0\\6&1\end{pmatrix}\begin{pmatrix}1&-1/3\\0&1\end{pmatrix}\quad\text{and}$$

$$\Gamma=\langle\Gamma_0(18),\ M\rangle,$$

where $W_2=\begin{pmatrix}18&1\\18&2\end{pmatrix}$ is an Atkin-Lehner's involution of $\Gamma_0(18)$ (cf. § 2.2). Firstly it follows from Lemma 2.2 that $f(z)$ is a modular function w.r.t. $\Gamma_0(18)$ and then we see that, by using (2.2), (2.4), (2.6) and (2.7), $f(Mz)=f(z)$. Thus $f(z)$ is a modular function w.r.t. Γ. Now we will apply Lemma 2.1 to show that $f(z)\in\mathcal{F}$ and Γ is a group for $f(z)$. A representative of inequivalent cusps of $\Gamma_0(18)$ is

$$0,\ 1/2,\ \pm 1/3,\ \pm 1/6,\ 1/9,\ 1/18$$

and these cusps are divided into two classes under Γ:

$$0\sim -1/6\sim 1/6\sim 1/9\quad\text{and}\quad 1/3\sim -1/3\sim 1/2\sim 1/18\sim i\infty.$$

Since, by (2.7), $f(W_{18}(z))=c/f(z)$ (c=constant and $W_{18}=\begin{pmatrix}0&-1\\18&0\end{pmatrix}$, an Atkin-Lehner's involution of $\Gamma_0(18)$), we have $f(1/18)=f(i\infty)=\infty$ and $f(0)=0$. Thus $z=i\infty$ is the unique pole of $f(z)$ among inequivalent cusps of Γ. Then Lemma 2.1 yields that $f(z)\in\mathcal{F}$ and Γ is a group for $f(z)$. This completes the proof of Th. 3.2.

In Table III of Appendix, we will give the list of $\pi\circ(3/1)$ and groups for

$\eta_{\pi\circ(3/1)}(z)^{-1}\in\mathfrak{F}$. Also we will give the list of $\pi\circ(r/1)$ $(r>3)$ in Table IV, together with the list of $\pi\circ s$, s-transformations of π.

3.2. In this paragraph, we give theorems analogous to Th. 3.2 for r-th harmonics and r-transformations. For that purpose, it is convenient to introduce the "*ghost classes*" of $\cdot 0$. We call the following generalized permutations "ghost classes" of $\cdot 0$:

Name of classes	Frame shapes
$\pm 9Z$	$3^2 9^2$, $6^2 18^2/3^2 9^2$
$16Z$	8.16
$18Z$	6.18
$\pm 25Z$	$25/1$, $1.50/2.25$

We refer to [2], [3] and [8] for these classes. In [2], the class $-25Z$ is denoted by $50Z$.

It is easy to see the following

THEOREM 3.3. *For all Frame shapes π of the ghost classes of $\cdot 0$, we have $\eta_{\pi\circ(2/1)}(z)^{-1}\in\mathfrak{F}$. Also we have $\eta_{\pi\circ(3/1)}(z)^{-1}\in\mathfrak{F}$ for Frame shapes of the classes $\pm 9Z$.*

THEOREM 3.4. *Let $s>1$ be an integer with $s+1|24$. For each Frame shape $\pi=\prod_t t^{r_t}$ of $\cdot 0$ with $s+1|r_t$ for any t, the s-transformation $\pi\circ s$ of π is a Frame shape of $\cdot 0$ or that of a ghost class of $\cdot 0$ except for the following cases:*

s	*classes*	
2	$4E$,	$12F$
3	$2B$,	$6H$
5	$4E$	
11	$2B$	

For these exceptional classes, $\pi\circ s$ is an r-th harmonic of a Frame shape of $\cdot 0$ for some $r|24$ and also we have $\eta_{\pi\circ s}(z)^{-1}\in\mathfrak{F}$ and $\eta_{\pi\circ s\circ(2/1)}(z)^{-1}\in\mathfrak{F}$.

The proof of this theorem is done just by inspection of Table I of Appendix and we don't need any facts from the theory of elliptic modular functions other than Lemma 2.3 (cf. Table IV of Appendix). Finally we have the following theorem for r-th harmonics of Frame shapes of $\cdot 0$:

THEOREM 3.5. *Let $r>1$ be an integer and $\pi=\prod_t t^{r_t}$ be a Frame shape of $\cdot 0$ with $r|r_t$ for any t.*

(1) *Let $\sum_t r_t=0$. If $\prod_t t^{r_t/r}$ is a rational square, we have $\eta_{\pi/r}(z)^{-1}\in\mathfrak{F}$.*

(2) *Let $\sum_t r_t\neq 0$. Then one of the following holds:*

(i) *π/r is a Frame shape of a conjugacy class or ghost class of $\cdot 0$,*

(ii) *$(\sum_t r_t)/r$ is odd, or*

(iii) *$r=4$ and $\pi=1^8 4^8/2^8$, a Frame shape of the class $-4A$.*

Also the proof of this theorem is obtained by inspection of Table I of Appendix and Lemma 2.3.

§4. Some examples of moonshines for finite groups.

4.1. In this paragraph, we collect some lemmas on group characters.

LEMMA 4.1. *Let G be a finite group and $\sigma\mapsto\rho(\sigma)$ $(\sigma\in G)$ be an n-dimensional representation of G over the complex number field. Let $\varepsilon_1(\sigma), \varepsilon_2(\sigma), \cdots, \varepsilon_n(\sigma)$ be eigenvalues of $\rho(\sigma)$. Define functions $a_k(\sigma)$ $(1\leqq k<\infty)$ and $b_k(\sigma)$ $(1\leqq k\leqq n)$ as follows:*

$$\{(1-\varepsilon_1(\sigma)q)\cdots(1-\varepsilon_n(\sigma)q)\}^{-1}=\sum_{k=0}^{\infty}a_k(\sigma)q^k\,,$$

$$(1+\varepsilon_1(\sigma)q)\cdots(1+\varepsilon_n(\sigma)q)=\sum_{k=0}^{n}b_k(\sigma)q^k$$

where q is a variable. Then $a_k(\sigma)$ and $b_k(\sigma)$ are characters of G.

PROOF. It can be easily seen that, if V is a representation space of G, $a_k(\sigma)$ (resp. $b_k(\sigma)$) is a character of the representation which ρ induces on the space of symmetric (resp. anti-symmetric) tensors of V of degree k, q.e.d.

LEMMA 4.2. *Let G be a finite group and $\sigma\mapsto\rho(\sigma)$ $(\sigma\in G)$ be a representation of G over $\boldsymbol{Q}$. For an element σ of G with Frame shape $\prod_t t^{r_t}$, define a function $\chi_k(\sigma)$ as follows:*

$$\eta_\sigma(z)^{-1}=q^{-m/24}\Big(\sum_{k=0}^{\infty}\chi_k(\sigma)q^k\Big),$$

where $m=\sum_t r_t t$ is the degree of ρ and $\eta_\sigma(z)$ is a function defined by (1.1). *Then $\chi_k(\sigma)$ is an ordinary character of G.*

PROOF. Let

$$\xi(q)=(\prod_t(1-q^t)^{r_t})^{-1}=\sum_k a_k(\sigma)q^k\,.$$

Then, by Lemma 4.1, $a_k(\sigma)$ is an ordinary character of G, as we have clearly

$$\prod_t(1-q^t)^{r_t}=(1-\varepsilon_1(\sigma)q)\cdots(1-\varepsilon_m(\sigma)q)$$

where $\varepsilon_1(\sigma), \cdots, \varepsilon_m(\sigma)$ are eigenvalues of $\rho(\sigma)$. Then it follows from $\eta_\sigma(z)^{-1}=\prod_{n=1}^{\infty}\xi(q^n)$ that the coefficients $\chi_k(\sigma)$ of $\eta_\sigma(z)^{-1}$ are ordinary characters, q.e.d.

LEMMA 4.3. *Let $\mathfrak{S}_n$ be the symmetric group of degree n and $\xi(q)=\sum_{k=0}^{\infty}c_kq^k$ $(c_0=1)$ be a formal power series of q with non-negative integral coefficients, i.e. $0\leqq c_k\in\boldsymbol{Z}$ $(k=1, 2, 3, \cdots)$. For an element σ of $\mathfrak{S}_n$ with a cycle decomposition $\prod_t t^{r_t}$, define functions $d_k(\sigma)$ as follows:*

$$\prod_t\xi(q^t)^{r_t}=\sum_{k=0}^{\infty}d_k(\sigma)q^k\,.$$

Then $d_k(\sigma)$ *are characters of* $\mathfrak{S}_n$.

PROOF. We may assume that $\xi(q)$ is a polynomial, as $d_k(\sigma)$ $(1\leqq k\leqq n)$ are clearly the same as the ones which are obtained from a polynomial $\sum_{k=0}^{n} c_k q^k$. Thus we may assume $\xi(q)=1+c_1q+\cdots+c_nq^n$. Let

$$m=c_1+c_2+\cdots+c_n$$

and $x_0, x_1, \cdots, x_m$ be $m+1$ independent variables. For $\sigma=\prod_t t^{r_t}\in\mathfrak{S}_n$, define a function $\chi_{k_0k_1\cdots k_m}(\sigma)$ on $\mathfrak{S}_n$ as follows:

$$\prod_t (x_0^t+\cdots+x_m^t)^{r_t}=\sum_{k_0+\cdots+k_m=n}\chi_{k_0k_1\cdots k_m}(\sigma)x_0^{k_0}x_1^{k_1}\cdots x_m^{k_m}.$$

It is well known [**5**; § 5.2] that $\chi_{k_0\cdots k_m}(\sigma)$ is a character of $\mathfrak{S}_n$ which is induced from the principal character of the subgroup $\mathfrak{S}_{k_0}\times\mathfrak{S}_{k_1}\times\cdots\times\mathfrak{S}_{k_m}$ of $\mathfrak{S}_n$. Now, by putting,

$$x_0=1,\qquad x_1=\cdots=x_{c_1}=q,$$
$$x_{c_1+1}=\cdots=x_{c_1+c_2}=q^2,$$
$$\cdots$$
$$x_{c_1+\cdots+c_{n-1}}=\cdots=x_m=q^n,$$

we see that $d_k(\sigma)$ is a linear combination of several induced characters $\chi_{k_0\cdots k_m}(\sigma)$ with non-negative coefficients. Thus $d_k(\sigma)$ is a character of $\mathfrak{S}_n$, q.e.d.

LEMMA 4.4. *Let* $d>1$ *be an integer and let*

$$\eta(2z)\eta(dz)/\eta(z)\eta(2dz)=q^{(1-d)/24}(1+\sum_k c_kq^k).$$

Then we have $c_k\geqq 0$ $(1\leqq k<\infty)$.

PROOF. If $k\leqq 2dn$, the c_k are the same as the coefficients of q-expansion of

$$\prod_{h=1}^{dn}(1-q^{2h})\prod_{h=1}^{2n}(1-q^{dh})\Big/\prod_{h=1}^{2dn}(1-q^h)\prod_{h=1}^{n}(1-q^{2dh})$$
$$=\Big(\prod_{h=1}^{n}(1-q^{d(2h-1)})/(1-q^{2h-1})\Big)\prod_{h=n}^{dn-1}(1-q^{2h+1})^{-1}.$$

Clearly all coefficients of q-expansion of the right hand side are non-negative.

4.2. Let G be a finite group and $\mathfrak{F}$ be a class of elliptic modular functions defined in the introduction. A mapping from G to $\mathfrak{F}$

$$G\ni\sigma\longmapsto j_\sigma(z)\in\mathfrak{F}$$

is said to satisfy *Moonshine condition* or simply to be a *moonshine* if every coefficient $a_k(\sigma)$ $(k\geqq 1)$ of a Fourier expansion of $j_\sigma(z)=1/q+\sum_{k=0}^{\infty}a_k(\sigma)q^k$ is a generalized character of G. Furthermore, if every coefficient $a_k(\sigma)$ $(k\geqq 1)$ of $j_\sigma(z)$ is an ordinary character of G, this moonshine is called *proper*.

REMARK 4.1. In the above moonshine condition, the constant term $a_0(\sigma)$ of a Fourier expansion of $j_\sigma(z)$ need not necessarily to be any generalized character. But, in all moonshines which appear in this paper, $a_0(\sigma)$ will be also a generalized character.

Let $d>1$ be a divisor of 24 and $\sigma\mapsto\rho(\sigma)$ $(\sigma\in G)$ be a d-dimensional representation of G over $\boldsymbol{Q}$.

LEMMA 4.5. *Let G, ρ and d be as above and $\pi=\prod_h h^{d_h}$ be a generalized permutation of degree $24/d$. For every element σ of G with Frame shape $\sigma=\prod_t t^{r_t}$, we put*

$$j_\sigma^\pi(z)=\prod_t(\prod_h \eta(htz)^{d_h})^{-r_t}.$$

Then $j_\sigma^\pi(z)$ has a Fourier expansion of the form

$$(*)\qquad j_\sigma^\pi(z)=\frac{1}{q}+\sum_{k=0}^{\infty}a_k(\sigma)q^k$$

and $a_k(\sigma)$ $(k=0, 1, 2, \cdots)$ are generalized characters of G.

PROOF. It is clear that $j_\sigma^\pi(z)$ has a Fourier expansion of the form (*). For each h, the coefficients of $\prod_t \eta(htz)^{r_t}$ are generalized characters of G by Lemma 4.2. From this, the second statement follows, q. e. d.

Now we ask when we have $j_\sigma^\pi(z)\in\mathcal{F}$, i. e. a mapping $\sigma\mapsto j_\sigma^\pi(z)$ is a moonshine.

THEOREM 4.6. *Let G, ρ and d be as above. Assume that,*

($\sharp$) *for any element σ of G with a Frame shape $\prod_t t^{r_t}$, a generalized permutation $\prod_t t^{24r_t/d}$ of degree 24 is a Frame shape of $\cdot 0$.*

(1) *Let $d^+=1+(24/d)$. Then a mapping*

$$G\ni\sigma\longmapsto j_{d,\sigma}^+(z)=(\prod_t(\eta(d^+tz)/\eta(tz))^{r_t})^{-1}$$

is a moonshine of G, where $\prod_t t^{r_t}$ is a Frame shape of σ.

(2) *Let $d^-=(24/d)-1$. Then a mapping*

$$G\ni\sigma\longmapsto j_{d,\sigma}^-(z)=(\prod_t(\eta(2tz)\eta(2d^-tz)/\eta(tz)\eta(d^-tz))^{r_t})^{-1}$$

is also a moonshine of G, where $\prod_t t^{r_t}$ is a Frame shape of σ.

PROOF. (1) Let π^+ be a generalized permutation $d^+/1$. Then we have $j_{d,\sigma}^+(z)=j_\sigma^{\pi^+}(z)$ in the notation of Lemma 4.5. Thus the coefficients of a Fourier expansion of $j_{d,\sigma}^+(z)$ are generalized characters by Lemma 4.5. On the other hand, we have

$$j_{d,\sigma}^+(z)=\eta_{\sigma'\circ(d^+/1)}(z)^{-1}$$

where $\sigma'=\prod_t t^{24r_t/d}$ and $\sigma'\circ(d^+/1)$ is a $(d^+/1)$-transformation of σ'. Then it follows from ($\sharp$) and Th. 3.2 that $j_{d,\sigma}^+(z)\in\mathcal{F}$.

(2) Let π^- be a generalized permutation $(2.2d^-)/(1.d^-)$. Then we have, in

the notation of Lemma 4.5,

$$j^{-}_{d,\sigma}(z)=j^{\pi^{-}}_{\sigma}(z),$$

and also

$$j^{-}_{d,\sigma}(z)=\eta_{\sigma'\circ d^{-}\circ(2/1)}(z)^{-1}.$$

Then (2) follows from Lemma 4.5, (♯) and Th. 3.3, q.e.d.

REMARK 4.2. Moonshines in Th. 4.6 are not proper, i.e. Fourier coefficients of $j^{+}_{d,\sigma}(z)$ and $j^{-}_{d,\sigma}(z)$ are not necessarily ordinary characters. Some examples of proper moonshines will be given in the next paragraph § 4.3.

Let p be a prime with $p+1|24$ and $d>1$ be an integer with $d|24$. Many d-dimensional rational representations of $SL(2,p)$ satisfy the condition (♯) of Th. 4.6. In the following, we will give examples of such representations and Frame shapes w.r.t. them for $p=5, 7$.

$SL(2,5)$

$\pm 1A$	$2A$	$\pm 3A$	$\pm 5A$	
1^4	2^2	1.3	$5/1$	absolutely irreducible
1^6	$1^2 2^2$	3^2	1.5	a permutation representation
1^6	$2^4/1^2$	3^2	1.5	absolutely irreducible
$\{1^4$	$4^2/2^2$	1.3	$5/1$	absolutely irreducible
$2^4/1^4$		$2.6/1.3$	$1.10/2.5$	
$\{1^4$	$4^2/2^2$	$3^2/1^2$	$5/1$	not absolutely irreducible
$2^4/1^4$		$1^2 6^2/2^2 3^2$	$1.10/2.5$	
$\{1^6$	$4^3/2^3$	3^2	1.5	not absolutely irreducible
$2^6/1^6$		$6^2/3^2$	$2.10/1.5$	

$SL(2,7)$

$\pm 1A$	$2A$	$\pm 3AB$	$4AB$	$\pm 7AB$	
1^6	$1^2 2^2$	3^2	2.4	$7/1$	absolutely irreducible
1^6	$2^4/1^2$	3^2	$1^2 4^2/2^2$	$7/1$	not absolutely irreducible
1^8	2^4	$1^2 3^2$	4^2	1.7	permutation representation
1^8	2^4	$3^3/1$	4^2	1.7	absolutely irreducible
$\{1^8$	$4^4/2^4$	$3^3/1$	$8^2/4^2$	1.7	absolutely irreducible
$2^8/1^8$		$1.6^3/2.3^3$		$2.14/1.7$	
$\{1^8$	$4^4/2^4$	$1^2 3^2$	$8^2/4^2$	1.7	not absolutely irreducible
$2^8/1^8$		$2^2 6^2/1^2/3^2$		$2.14/1.7$	
$\{1^{12}$	$8^3/4^3$	3^4	$4^6/2^6$	$7^2/1^2$	not absolutely irreducible
$2^{12}/1^{12}$		$6^4/3^4$		$1^2 14^2/2^2 7^2$	

Notations: For $SL(2,p)\ni\sigma$, if homomorphic image of σ in $PSL(2,p)$ is of order n, conjugate class of σ is denoted by nA or $\pm nA$, according as σ and $-\sigma$ are conjugate in $SL(2,p)$ or not. And nAB (resp. $\pm nAB$) expresses that there exist two conjugate classes nA and nB (resp. $\pm nA$ and $\pm nB$) of $PSL(2,p)$ of order n with the same Frame shapes.

REMARK 4.3. Let ρ be one of representations of $SL(2, 5)$ of degree 6 in the above table. Then if σ is a class of order 5 and so its Frame shape is 1.5, we have $j^{+}_{6,\sigma}(z)=\eta(z)/\eta(25z)$ which is a ghost element of Monster's moonshine. Similarly another ghost element $\eta(2z)\eta(25z)/\eta(z)\eta(50z)$ of Monster's moonshine [2] also appears in the moonshines $\sigma\mapsto j^{-}_{4,\sigma}(z)$ which are obtained from any one of 4-dimensional representations of $SL(2, 5)$.

REMARK 4.4. $SL(2, 9)$ has representations of degree 4 and 6 with the following Frame shapes:

$\pm 1A$	$2A$	$\pm 3A$	$\pm 3B$	$4A$	$\pm 5A$
1^4	$4^2/2^2$	1.3	$3^2/1^2$	$8/4$	$5/1$
$2^4/1^4$		$2.6/1.3$	$1^2 6^2/2^2 3^2$		$1.10/2.5$
1^6	$1^2 2^2$	$1^3 3$	3^2	2.4	1.5

The one of degree 6 is a natural permutation representation of $PSL(2, 9)\simeq\mathfrak{A}_6$ (=the Alternating group of degree 6). The representation of degree 4 satisfies the condition (♯) of Th. 4.6, while the one of degree 6 does not, as $(1^3 3)^4=1^{12}3^4$ is not a Frame shape of $\cdot 0$. It is easy to see that, if $\sigma=(1^3 3)^4$, $\eta_{\sigma\circ(5/1)}(z)^{-1}$ does not satisfy the second condition in the definition of $\mathcal{F}$. Thus the representation of degree 6 does not yield a moonshine. Similarly the permutation representation of Mathieu group M_{12} of degree 12 also does not yield a moonshine. In fact, there are elements of M_{12} with cycle decompositions $1^4 4^2$ and $1^2 2.8$. These permutations do not satisfy (♯) of Th. 4.6 and it can be shown that, if σ is one of these permutations, $\eta_{\sigma^2\circ(3/1)}(z)^{-1}\notin F$, where $\sigma^2=1^8 4^4$ or $1^4 2^2 8^2$.

4.3. In this paragraph, we give some examples of proper moonshines for Mathieu group M_{24} and $PSL(2, p)$ $(p+1|24)$.

LEMMA 4.7. *Let $\sigma=\prod_t t^{r_t}$ be a cycle decomposition (=Frame shape) of an element σ of M_{24} w.r.t. the natural permutation representation of M_{24}. Then (2/1)-transformation of σ is a Frame shape of $\cdot 0$. More explicitly, we have the following table:*

σ	1^8	$1^8 2^8$	2^{12}	$1^6 3^6$	3^8	$1^4 2^2 4^4$	$2^4 4^4$	4^6	$1^4 5^4$	$1^2 2^2 3^2 6^2$	6^4	$1^3 7^3$
$\sigma\circ(2/1)$	$-2A$	$4A$	$2B$	$-3A$	$-3D$	$8C$	$8A$	$4E$	$-5B$	$12E$	$6H$	$-7B$

σ	$1^2 2.4.8^2$	$2^2 10^2$	$1^2 11^2$	$2.4.6.12$	12^2	$1.2.7.14$	$1.3.5.15$	3.21	1.23
$\sigma\circ(2/1)$	$16B$	$10C$	$-11B$	$24C$	$12L$	$28B$	$-30B$	$-21C$	$-23AB$

where the second line denotes conjugacy classes of $\cdot 0$ with Frame shapes $\sigma\circ(2/1)$.

PROOF. This can be seen immediately from Table I of Appendix.

THEOREM 4.8. *For an element σ of M_{24} with a cycle decomposition $\prod_t t^{r_t}$, we put*

$$j_\sigma(z)=\prod_t (\eta(2tz)^2/\eta(tz)\eta(4tz))^{r_t}.$$

Then a mapping

$$M_{24}\ni\sigma\longmapsto j_\sigma(z)$$

is a proper moonshine of M_{24}.

PROOF. Let $\sigma'=\sigma\circ(2/1)\circ(2/1)$. Then we have

$$\sigma'=\prod_t t^{r_t}(4t)^{r_t}(2t)^{-2r_t}$$

and

$$j_\sigma(z)=\eta_{\sigma'}(z)^{-1}.$$

Then it follows from Lemma 4.7 and Th. 3.2 that $j_\sigma(z)\in\mathcal{F}$. Furthermore, we see from Lemma 4.3 and 4.4 that this moonshine is proper, q.e.d.

LEMMA 4.9. *Let p be a prime with $p+1|24$ and ρ_p be a permutation representation of $PSL(2, p)$ of degree $p+1$ on a projective line over F_p, a finite field of p elements. For an element σ of $PSL(2, p)$, let $\sigma=\prod_t t^{r_t}$ be a Frame shape of σ w.r.t. ρ_p. Then a generalized permutation $\prod_t t^{24r_t/(p+1)}$ of degree 24 is a Frame shape of M_{24}.*

PROOF. It is easy to check this for each $p=2, 3, 5, 7, 11$ and 23. See the table in §4.2 for $p=5, 7$.

THEOREM 4.10. *Notations being as in Lemma 4.9, we put, for $PSL(2, p)\ni\sigma$,*

$$j_{p,\sigma}(z)=\prod_t (\eta(2tz)\eta(dtz)/\eta(2dtz)\eta(tz))^{r_t}$$

where $d=24/(p+1)+1$ and $\prod_t t^{r_t}$ is a Frame shape of σ. Then a mapping

$$\sigma\longmapsto j_{p,\sigma}(z)$$

is a proper moonshine of $PSL(2, p)$.

PROOF. Let $\sigma'=\prod_t t^{24r_t/(p+1)}$. Then we have

$$\sigma'\circ(d/1)\circ(2/1)=\prod_t (2dt)^{r_t}t^{r_t}(2t)^{-r_t}(dt)^{-r_t}$$

and so

$$j_{p,\sigma}(z)=\eta_{\sigma'\circ(d/1)\circ(2/1)}(z)^{-1}.$$

By Lemma 4.9, σ' is a Frame shape of M_{24} and then, by Lemma 4.7 and Th. 3.2, $j_{p,\sigma}(z)\in\mathcal{F}$. By Lemma 4.3 and 4.4, this moonshine is proper, q.e.d.

REMARK 4.5. $j_\sigma(z)$ and $j_{23,\sigma}(z)$ being in Th. 4.8 and 4.10 respectively, we have $j_\sigma(z)=j_{23,\sigma}(z)$. Since $PSL(2, 23)$ is a subgroup of M_{24} and the embedding is unique up to conjugation, a moonshine $\sigma\mapsto j_{23,\sigma}(z)$ of $PSL(2, 23)$ is a restriction of a moonshine $\sigma\mapsto j_\sigma(z)$ of M_{24}.

4.4. Here we will make two remarks on Th. 4.6, 4.8 and 4.10. In these theorems, we constructed moonshine of a finite group G by using a representation of G of degree d satisfying the condition (♯) of Th. 4.6 and one of transformations $d^+/1$, $d^-\circ(2/1)$ and $(2/1)\circ(2/1)$ (in this case, $d=24$) of "degree" $24/d$. These are, however, not all transformations we can use to construct moonshines. In fact, for $d=4$ or 6, we can also use the following transformations of "degree" $24/d$:

$$d=4 \qquad 5\circ(2/1)\circ(2/1), \qquad (3/1)\circ(4/1)$$

$$d=6 \qquad 3\circ(2/1)\circ(2/1), \qquad (3/1)\circ(3/1).$$

But these transformations do not always yield a moonshine. For example, for a representation of $SL(2, 5)$ or $SL(2, 7)$ in which a Frame shape $2^4/1^2$ appears (§ 4.2), a transformation $3\circ(2/1)\circ(2/1)$ does not yield a moonshine, because we can easily see that $\pi=(2^{16}/1^8)\circ3\circ(2/1)\circ(2/1)=2^8 6^8 8^4 24^4/1^2 3^2 4^{10} 12^{10}$ but $\eta_\pi(z)^{-1}\notin\mathcal{F}$.

The second remark is that $j_\sigma(z)$ in Th. 4.8 can be related to some even lattice.

Let V be a 24-dimensional vector space over $\boldsymbol{Q}$ and e_i $(1\leqq i\leqq 24)$ be a basis of V. Furthermore let (u, v) $(u, v\in V)$ be an inner product of V with $(e_i, e_j)=2\delta_{ij}$. Set $L=\sum_{i=1}^{24}\boldsymbol{Z}e_i\in V$. Then L is an even lattice of V on which the Mathieu group M_{24} acts in such a way that $e_i^\sigma=e_{\sigma(i)}$ $(\sigma\in M_{24})$. For each $\sigma\in M_{24}$, we put

$$L_\sigma=\{v\in L\,|\,v^\sigma=v\}$$

and

$$\Theta_\sigma(z)=\sum_{v\in L_\sigma} e^{\pi i(v,v)z} \qquad (\Theta\text{-series of } L_\sigma).$$

THEOREM 4.11. *$j_\sigma(z)$ being as in Th. 4.8, we have*

$$(*) \qquad \Theta_\sigma(z)=j_\sigma(z)^2\eta_\sigma(2z)$$

where $\prod_t t^{r_t}$ is a cycle decomposition of σ and $\eta_\sigma(z)=\prod_t\eta(tz)^{r_t}$.

PROOF. Let $\theta(z)=\sum_{x\in Z}e^{2\pi i x^2 z}$. It is easy to see that $\Theta_\sigma(z)=\prod_t\theta(tz)^{r_t}$. Then (*) follows from the identity $\theta(z)=\eta(2z)^5/\eta(z)^2\eta(4z)^2$, q. e. d.

Appendix. Table I∼IV.

For conjugacy classes of ·0, we use the following notations in § 3∼§ 4 and Table I∼IV. The heading column nA, nB, ⋯ of Table I are the Atlas names of conjugacy classes of the Conway's simple group ·1 (=the factor group of ·0 by its center $\langle\pm1\rangle$) [**11**; Table 1], i.e. conjugacy classes of ·1 of order n are named nA, nB, ⋯ in descending order of their centralizer sizes.

Case (1). If the inverse image in ·0 of a class nX $(X=A, B, \cdots)$ of ·1 is a conjugacy class of ·0, this class is also denoted by nX.

Case (2). If the inverse image in $\cdot 0$ of a class nX of $\cdot 1$ consists of two conjugacy classes of $\cdot 0$, these classes are denoted by $+nX$ and $-nX$.

Table I; In case (1), Frame shape of a class nX is written after the heading column and in case (2), firstly Frame shape of $+nX$ and then that of $-nX$ are written. For a Frame shape $\pi=\prod_t t^{r_t}$ with $\sum_t r_t=0$, a group for $\eta_\pi(z)^{-1}$ is given in parenthesis after the Frame shape by using notations of Table 2 and 3 of [2] (cf. also § 2.1 of this paper).

Table II$\sim$IV; Let π be a Frame shape of $\cdot 0$. If $\pi\circ(r/1)$ (resp. $\pi\circ s$) is a Frame shape of a conjugacy class nX ($+nX$ or $-nX$) of $\cdot 0$, $\pi\circ(r/1)$ (resp. $\pi\circ s$) is also denoted by nX (nX or $-nX$). Note that $+$ of $+nX$ is omitted. And if $\pi\circ(r/1)$ (resp. $\pi\circ s$) is the m-th harmonic of a Frame shape of a class nX ($+nX$ or $-nX$), $\pi\circ(r/1)$ (resp. $\pi\circ s$) is denoted by nX/m (nX/m or $-nX/m$).

Some of $(r/1)$-transformations are expressed by generalized permutations with symbols (?). These are exceptional classes in Th. 3.2.

In Table II and III, groups for $\eta_{\pi\circ(2/1)}(z)^{-1}$ or $\eta_{\pi\circ(3/1)}(z)^{-1}$ are given in parenthesis after $\pi\circ(2/1)$ or $\pi\circ(3/1)$. Then the following notations are used:

$$T=\begin{pmatrix}1 & 1/2\\ 0 & 1\end{pmatrix},\quad S=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix},\quad T_n=\begin{pmatrix}1 & 0\\ n & 1\end{pmatrix},$$

$$\left(\frac{1}{h}n\right)=\begin{pmatrix}1 & 1/h\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ n & 1\end{pmatrix},\quad \left(n\frac{1}{h}\right)=\begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}\begin{pmatrix}1 & 1/h\\ 0 & 1\end{pmatrix},$$

W_Q=an Atkin-Lehner's involution of $\Gamma_0(N)$ for some N.

A notation like "$N+Q$, $\left(h\frac{1}{n}\right)$, ..." denotes $\langle \Gamma_0(N), W_Q, \begin{pmatrix}1 & 1/h\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ n & 1\end{pmatrix}, \cdots\rangle$.

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Table I. Frame shapes of conjugacy classes of $\cdot 0$.

1A	1^{24},		$2^{24}/1^{24}$ (2-)
2A	$1^{8}2^{8}$,		$2^{16}/1^{8}$
2B		$4^{12}/2^{12}$ (4/2-)	
2C		2^{12}	
3A	$3^{12}/1^{12}$ (3-),		$1^{12}6^{12}/2^{12}3^{12}$ (6+6)
3B	$1^{6}3^{6}$,		$2^{6}6^{6}/1^{6}3^{6}$ (6+3)
3C	$3^{9}/1^{3}$,		$1^{3}6^{9}/2^{3}3^{9}$ (6-)
3D	3^{8},		$6^{8}/3^{8}$ (6/3-)
4A	$4^{8}/1^{8}$ (4-),		$1^{8}4^{8}/2^{8}$
4B		$4^{8}/2^{4}$	
4C	$1^{4}2^{2}4^{4}$,		$2^{6}4^{4}/1^{4}$
4D		$2^{4}4^{4}$	
4E		$8^{6}/4^{6}$ (8/4-)	
4F		4^{6}	
5A	$5^{6}/1^{6}$ (5-),		$1^{6}10^{6}/2^{6}5^{6}$ (10+10)
5B	$1^{4}5^{4}$,		$2^{4}10^{4}/1^{4}5^{4}$ (10+5)
5C	$5^{5}/1$,		$1.10^{5}/2.5^{5}$ (10-)
6A	$3^{4}6^{4}/1^{4}2^{4}$ (6+2),		$1^{4}6^{8}/2^{8}3^{4}$ (6-)
6B		$2^{6}12^{6}/4^{6}6^{6}$ (12/2+6)	
6C	$1^{4}2.6^{5}/3^{4}$,		$2^{5}3^{4}6/1^{4}$
6D	$2.6^{5}/1^{5}3$ (6-),		$1^{5}3.6^{4}/2^{4}$
6E	$1^{2}2^{2}3^{2}6^{2}$,		$2^{4}6^{4}/1^{2}3^{2}$
6F	$3^{3}6^{3}/1.2$,		$1.6^{6}/2^{2}3^{3}$
6G		$2^{3}6^{3}$	
6H		$12^{4}/6^{4}$ (12/6-)	
6I		6^{4}	
7A	$7^{4}/1^{4}$ (7-),		$1^{4}14^{4}/2^{4}7^{4}$ (14+14)
7B	$1^{3}7^{3}$,		$2^{3}14^{3}/1^{3}7^{3}$ (14+7)
8A		$8^{4}/2^{4}$ (8/2-)	
8B		$2^{4}8^{4}/4^{4}$	
8C	$2^{2}8^{4}/1^{4}4^{2}$ (8-),		$1^{4}8^{4}/2^{2}4^{2}$
8D		$8^{4}/4^{2}$	
8E	$1^{2}2.4.8^{2}$,		$2^{3}4.8^{2}/1^{2}$
8F		$4^{2}8^{2}$	

Table I (continued)

9A	$9^3/1^3$ (9-),		$1^3 18^3/2^3 9^3$ (18+18)
9B	$9^3/3$,		$3.18^3/6.9^3$ (18-)
9C	$1^3 9^3/3^2$,		$2^3 3^2 18^3/1^3 6^2 9^3$ (18+9)
10A	$5^2 10^2/1^2 2^2$ (10+2),		$1^2 10^4/2^4 5^2$ (10-)
10B		$2^3 20^3/4^3 10^3$ (20/2+10)	
10C		$4^2 20^2/2^2 10^2$ (20/2+5)	
10D	$1^2 2.10^3/5^2$,		$2^3 5^2 10/1^2$
10E	$2.10^3/1^3 5$ (10-),		$1^3 5.10^2/2^2$
10F		$2^2 10^2$	
11A	$1^2 11^2$,		$2^2 22^2/1^2 11^2$ (22+11)
12A	$1^4 12^4/3^4 4^4$ (12+12),		$2^4 3^4 12^4/1^4 4^4 6^4$ (12+4)
12B		$2^2 12^4/4^4 6^2$ (12-)	
12C		$6^2 12^2/2^2 4^2$ (12/2+2)	
12D	$1.12^3/3^3 4$ (12-),		$2.3^3 12^3/1.4.6^3$
12E	$4^2 12^2/1^2 3^2$ (12+3),		$1^2 3^2 4^2 12^2/2^2 6^2$
12F		$4^3 24^3/8^3 12^3$ (24/4+6)	
12G		$4^2 12^2/2.6$	
12H	$2^3 6.12^2/1.3.4^2$,		$1.2^2 3.12^2/4^2$
12I	$1^2 4.6^2 12/3^2$,		$2^2 3^2 4.12/1^2$
12J		2.4.6.12	
12K	$2^2 3.12^3/1^3 4.6^2$ (12-),		$1^3 12^3/2.3.4.6$
12L		$24^2 12^2$ (24/12-)	
12M		12^2	
13A	$13^2/1^2$ (13-),		$1^2 26^2/2^2 13^2$ (26+26)
14A		$2^2 28^2/4^2 14^2$ (28/2+14)	
14B	1.2.7.14,		$2^2 14^2/1.7$
15A	$1^3 15^3/3^3 5^3$ (15+15),		$2^3 3^3 5^3 30^3/1^3 6^3 10^3 15^3$ (30+6,10)
15B	$3^2 15^2/1^2 5^2$ (15+5),		$1^2 5^2 6^2 30^2/2^2 3^2 10^2 15^2$ (30+5,6)
15C	$15^2 3^2$ (15/3-),		$3^2 30^2/6^2 15^2$ (30/3+10)
15D	1.3.5.15,		2.6.10.30/1.3.5.15 (30+3,5)
15E	$1^2 15^2/3.5$,		$2^2 3.5.30^2/1^2 6.10.15^2$ (30+15)
16A		$2^2 16^2/4.8$	
16B	$2.16^2/1^2 8$ (16-),		$1^2 16^2/2.8$
18A	9.18/1.2 (18+2),		$1.18^2/2^2 9$ (18-)
18B	$2.3.18^2/1^2 6.9$ (18-),		$1^2 9.18/2.3$
18C	$1.2.18^2/6.9$,		$2^2 9.18/1.6$

Table I (continued)

20A	$1^2 20^2/4^2 5^2$ (20+20),	$2^2 5^2 20^2/1^2 4^2 10^2$ (20+4)
20B	4.20	
20C	1.2.10.20/4.5,	$2^2 5.20/1.4$
21A	$1^2 21^2/3^2 7^2$ (21+21),	$2^2 3^2 7^2 42^2/1^2 6^2 14^2 21^2$ (42+6,14)
21B	7.21/1.3 (21+3),	1.3.14.42/2.6.7.21 (42+3,14)
21C	3.21	6.42/3.21 (42/3+7)
22A	2.22	
23A	1.23,	2.46/1.23 (46+23)
23B	1.23,	2.46/1.23 (46+23)
24A	$2^2 24^2/6^2 8^2$ (24/2+12)	
24B	$1^2 4.6.24^2/2.3^2 8^2 12$ (24+24),	$2.3^2 4.24^2/1^2 6.8^2 12$ (24+8)
24C	8.24/2.6 (24/2+3)	
24D	12.24/4.8 (24/4+2)	
24E	2.6.8.24/4.12	
24F	1.4.6.24/3.8,	2.3.4.24/1.8
26A	2.52/4.26 (52/2+26)	
28A	4.28/1.7 (28+7),	1.4.7.28/2.14
28B	4.56/8.28 (56/4+14)	
30A	1.2.15.30/3.5.6.10 (30+2,15),	$2^2 3.5.30^2/1.6^2 10^2 15$ (30+15)
30B	2.10.12.60/4.6.20.30 (60/2+5,6)	
30C	6.60/12.30 (60/6+10)	
30D	1.6.10.15/3.5,	2.3.5.30/1.15
30E	2.30/3.5 (30+15),	2.3.5.30/6.10
33A	3.33/1.11 (33+11),	1.6.11.66/2.3.22.33 (66+6,11)
35A	1.35/5.7 (35+35),	2.5.7.70/1.10.14.35 (70+10,14)
36A	1.36/4.9 (36+36),	2.9.36/1.4.18 (36+4)
39A	1.39/3.13 (39+39),	2.3.13.78/1.6.26.39 (78+6,26)
39B	1.39/3.13 (39+39),	2.3.13.78/1.6.26.39 (78+6,26)
40A	2.40/8.10 (40/2+20)	
42A	4.6.14.84/2.12.28.42 (84/2+6,14)	
60A	3.4.5.60/1.12.15.20 (60+12,15),	1.4.6.10.15.60/2.3.5.12.20.30 (60+4,15)

Table II. (2/1)-transformations of Frame shapes of ·0.

1A	−1A (2−),	$1^{24}4^{24}/2^{48}$ (4+)
2A	4A (4−),	$1^{8}4^{16}/2^{24}$ (4−)
2B	−1A∘(2/1)/2 $(16+16,(\frac{1}{2}8))$	
2C	2B $(8+(\frac{1}{2}4))$	
3A	−3A (6+6),	$2^{24}3^{12}12^{12}/1^{12}4^{12}6^{24}$ $((6+6)^{T})$
3B	−3B (6+3),	$1^{6}3^{6}4^{6}12^{6}/2^{12}6^{12}$ (12+)
3C	−3C (6−),	$2^{6}3^{9}12^{9}/1^{3}4^{3}6^{18}$ (12+4)
3D	−3D $(18+9,(-\frac{1}{3}6))$,	−1A∘(2/1)/3 $(36+4,9,(\frac{1}{3}12))$
4A	$1^{8}8^{8}/2^{8}4^{8}$ (8+),	$2^{16}8^{8}/1^{8}4^{16}$ $(8+8^{T})$
4B	−2A∘(2/1)/2 (8−)	
4C	8C (8−),	$1^{4}4^{2}8^{4}/2^{10}$ (8−)
4D	8A $(16+(\frac{1}{2}8))$	
4E	−1A∘(2/1)/4 $(64+64,(\frac{1}{4}16))$	
4F	4E $(32+(-\frac{1}{4}8),(\frac{1}{2}16))$	
5A	−5A (10+10),	$2^{12}5^{6}20^{6}/1^{6}4^{6}10^{12}$ $((10+10)^{T})$
5B	−5B (10+5),	$1^{4}4^{4}5^{4}20^{4}/2^{8}10^{8}$ (20+)
5C	−5C (10−),	$2^{2}5^{5}20^{5}/1.4.10^{10}$ (20+4)
6A	12A (12+12),	$2^{4}3^{4}12^{4}/1^{4}4^{4}6^{4}$ $(12+12^{T})$
6B	−3A∘(2/1)/2 $(24+24,(\frac{1}{2}12))^{ST12}$	
6C	$2^{3}3^{4}4.12^{5}/1^{4}6^{9}$ $(12+12^{TW4})$,	$1^{4}4^{5}6^{3}12/2^{9}3^{4}$ $(12+12^{TW4})$
6D	$1^{5}3.4.12^{5}/2^{6}6^{6}$ (12+12),	$2^{9}12^{4}/1^{5}3.4^{4}6^{3}$ $(12+12^{T})$
6E	12E (12+3),	$1^{2}3^{2}4^{4}12^{4}/2^{6}6^{6}$ (12+3)
6F	12D (12−),	$2^{3}3^{3}12^{6}/1.4^{2}6^{9}$ (12−)
6G	−3B/2 $(24+3,(\frac{1}{2}12))$	
6H	−1A∘(2/1)/6 $(144+144,(\frac{1}{6}24),(\frac{1}{3}48))$	
6I	6H $(72+9,(-\frac{1}{6}12),(-\frac{1}{3}24))$	
7A	−7A (14+14),	$2^{8}7^{4}28^{4}/1^{4}4^{4}14^{8}$ $((14+14)^{T})$
7B	−7B (14+7),	$1^{3}4^{3}7^{3}28^{3}/2^{6}14^{6}$ (28+)
8A	4A∘(2/1)/2 $(32+32,(\frac{1}{2}16))$	
8B	−4A∘(2/1)/2 $(32+32^{T8},(\frac{1}{2}16))$	
8C	$1^{4}4^{4}16^{4}/2^{6}8^{6}$ (16+),	$2^{6}16^{4}/1^{4}8^{6}$ $(16+16^{T})$
8D	−2A∘(2/1)/2 (16−)	
8E	16B (16−),	$1^{2}4^{2}16^{2}/2^{5}8$ (16−)
8F	4A/4 $(64+(-\frac{1}{4}16))$	
9A	−9A (18+18),	$2^{6}9^{3}36^{3}/1^{3}4^{3}18^{6}$ $((18+18)^{T})$
9B	−9B (18−),	$6^{3}9^{3}36^{3}/3.12.18^{6}$ (36+4)
9C	−9C (18+9),	$1^{3}4^{3}6^{4}9^{3}36^{3}/2^{6}3^{2}12^{2}18^{6}$ (36+)

Table II (continued)

10A	20A (20+20),	$2^6 5^2 20^4/1^2 4^4 10^6$ $(20+20^T)$
10B	$-5A\circ(2/1)/2$ $((40+40,(\frac{1}{2}20))^{ST20})$	
10C	$-5B\circ(2/1)/2$ $(80+5,16,(\frac{1}{2}40))$	
10D	$2.4.5^2 20^3/1^2 10^5$ $(20+20^{TW4})$,	$1^2 4^3 10.20/2^5 5^2$ $(20+20^{TW4})$
10E	$1^3 4.5.20^3/2^4 10^4$ (20+20),	$2^5 20^2/1^3 4^2 5.10$ $(20+20^T)$
10F	10C $(40+5,(\frac{1}{2}20))$	
11A	−11A (22+11),	$1^2 4^2 11^2 44^2/2^4 22^4$ (44+)
12A	$2^4 3^4 4^4 24^4/1^4 6^4 8^4 12^4$ $(24+8,W_3(\frac{1}{2}12))$,	$1^4 4^8 6^8 24^4/2^8 3^4 8^4 12^8$ $(24+24,W_3(\frac{1}{2}12))$
12B	$-6A\circ(2/1)/2$ $(24+W_3T_{12})$	
12C	24A $(48+48,(\frac{1}{2}24))$	
12D	$2.3^3 4.24^3/1.6^3 8.12^3$ (24+8),	$1.4^2 6^6 24^3/2^2 3^3 8.12^6$ $(24+8^T)$
12E	$1^2 3^2 8^2 24^2/2^2 4^2 6^2 12^2$ (24+),	$2^4 6^4 8^2 24^2/1^2 3^2 4^4 12^4$ $(24+3^T,8^T)$
12F	$-3A\circ(2/1)/4$ $(96+96,(\frac{1}{4}24),(\frac{1}{2}48))$	
12G	$-6E\circ(2/1)/2$ $(24+W_3T)$	
12H	$1.3.4^5 24^2/2^4 6^2 8^2 12$ $(24+W_3T_{12})$,	$4^4 6.24^2/1.2.3.8^2 12^2$ $(24+W_3T_{12})$
12I	$2^2 8.3^2 12.24/1^2 4.6^4$ $(24+W_3T)$,	$1^2 4.8.6^2 24/2^4 3^2 12$ $(24+W_3T)$
12J	24C $(48+3,(\frac{1}{2}24))$	
12K	$1^3 4^3 6^3 24^3/2^5 3.8.12^5$ (24+24),	$2^4 3.24^3/1^3 8.12^4$ $(24+24^T)$
12L	$-1A\circ(2/1)/12$ $(576+(\frac{1}{6}96),(\frac{1}{3}192),(\frac{1}{4}144),576;\ Z_2\text{x}Z_2\text{x}D_8)$	
12M	12L $(288+(-\frac{1}{12}24),-\frac{1}{6}48),(-\frac{1}{4}72),(-\frac{1}{2}144)$	
13A	−13A (26+26),	$2^4 13^2 52^2/1^2 4^2 26^4$ $((26+26)^T)$
14A	$-7A\circ(2/1)/2$ $((56+56,(\frac{1}{2}/2))^{ST28})$	
14B	28A (28+7),	$1.4^2 7.28^2/2^3 14^3$ (28+7)
15A	−15A (30+6,10),	$1^3 4^3 6^6 10^6 15^3 60^3/2^6 3^3 5^3 12^3 20^3 30^6$ $((30+6,10)^T)$
15B	−15B (30+5,6),	$2^4 3^2 10^4 12^2 15^2 60^2/1^2 4^2 5^2 6^4 20^2 30^4$ $((30+5,6)^T)$
15C	−15C $(90+9,10,(-\frac{1}{3}30))$,	$-5A\circ(2/1)/3$ $((90+9,10,(-\frac{1}{3}30))^T)$
15D	−15D (30+3,5),	$1.3.4.5.12.15.20.60/2^2 6^2 10^2 30^2$ (60+)
15E	−15E (30+15),	$1^2 4^2 6^2 10^2 15^2 60^2/2^4 3.5.12.20.30^4$ $((30+15)^T)$
16A	$-8C\circ(2/1)/2$ $(64+(\frac{1}{2}32),64^T 16)$	
16B	$1^2 4.8.32^2/2^3 16^3$ (32+),	$2^3 8.32^2/1^2 4.16^3$ $(32+32^T)$
18A	36A (36+36),	$2^3 9.36^2/1.4^2 18^3$ $(36+36^T)$
18B	$1^2 4.6^2 9.36^2/2^3 3.12.18^3$ (36+36),	$2^3 3.36/1^2 4.6.9$ $(36+36^T)$
18C	$4.6.9.36^2/1.12.18^3$ $(36+36^{W4T})$,	$1.4^2 6.36/2^3 9.12$ $(36+36^{W4T})$
20A	$2^2 4^2 5^2 40^2/1^2 8^2 10^2 20^2$ $(40+40,W_5(\frac{1}{2}20))$,	$1^2 4^4 10^4 40^2/2^4 5^2 8^2 20^4$ $(40+8,W_5(\frac{1}{2}20))$
20B	$-5B/4$ $(160+5,(-\frac{1}{4}40),(\frac{1}{2}80))$	
20C	$4^2 5.40/1.8.10^2$ $(40+W_5(\frac{1}{2}20))$,	$1.4^3 10.40/2^3 5.8.20$ $(40+W_5(\frac{1}{2}20))$

Table II (continued)

21A	$-21A\ (42+6,14)$,	$1^2 4^2 6^4 14^4 21^2 84^2/2^4 3^2 7^2 12^2 28^2 42^4\ ((42+6,14)^T)$
21B	$-21B\ (42+3,14)$,	$2^2 6^2 7.21.28.84/1.3.4.12.14^2 42^2\ ((42+3,14)^T)$
21C	$-21C\ (126+7,9,(-\frac{1}{3}42))$,	$3.12.21.84/6^2 42^2\ (252+4,9,7,(\frac{1}{3}84))$
22A	$-11A/2\ (88+11,(\frac{1}{2}44))$	
23AB	$-23A\ (46+23)$,	$1.4.23.92/2^2 46^2\ (92+)$
24A	$12A\circ(2/1)/2\ (96+32,96^{T24},(\frac{1}{2}48))$	
24B	$2^3 3^2 8^3 12^2 48^2/1^2 4^2 6^3 16^2 24^3\ (48+48,16^T)$,	$1^2 6^3 8^3 48^2/2^3 3^2 16^2 24^3\ (48+16,48^T)$
24C	$12E\circ(2/1)/2\ (96+3,32,(\frac{1}{2}48))$	
24D	$12A/4\ (192+192,(\frac{1}{4}48);\ D_8)$	
24E	$-12E\circ(2/1)/2\ (96+3,32^{T24},(\frac{1}{2}48))$	
24F	$2.3.8^2 12.48/1.4.6^2 16.24\ (48+W_3(\frac{1}{2}24))$,	$1.6.8^2 48/2^2 3.16.24\ (48+W_3(\frac{1}{2}24))$
26A	$-13A\circ(2/1)/2\ ((104+104,(\frac{1}{2}52))^{ST52})$	
28A	$1.7.8.56/2.4.14.28\ (56+)$,	$2^2 8.14^2 56/1.4^2 7.28^2\ (56+7^T,8^T)$
28B	$-7A\circ(2/1)/4\ ((224+224,(\frac{1}{4}56),(\frac{1}{2}112))^{ST112})$	
30A	$60A\ (60+12,15)$,	$1.4^4 6^3 10^3 15.60^2/2^3 3.5.12^2 20^2 30^3\ (60+12^T,15^T)$
30B	$-15B\circ(2/1)/2\ ((120+5,24,(\frac{1}{2}60))^{ST60})$	
30C	$-5A\circ(2/1)/6\ ((360+360,(\frac{1}{6}60),(\frac{1}{3}120),(\frac{1}{2}180))^{ST60})$	
30D	$3.4.5.60/2.6.10.30\ (60+12,15)$,	$4.6^2 10^2 60/2.3.5.12.20.30\ (60+12^T,15^T)$
30E	$2.3.5.12.20.30/1.6^2 10^2 15\ (60+15,TW_5)$,	$1.4.6.10.15.60/2^2 3.5.30^2\ (60+15,TW_5)$
33A	$-33A\ (66+6,11)$,	$2^2 3.12.22^2 33.136/1.4.6^2 11.44.66^2\ ((66+6,11)^T)$
35A	$-35A\ (70+10,14)$,	$1.4.10^2 14^2 35.140/2^2 5.7.20.28.70^2\ ((70+10,14)^T)$
36A	$2.4.9.72/1.8.18.36\ (72+72,W_9(\frac{1}{2}36))$,	$1.4^2 18^2 72/2^2 8.9.36^2\ (72+8,W_9(\frac{1}{2}36))$
39AB	$-39A\ (78+6,26)$,	$1.4.6^2 26^2 39.156/2^2 3.12.13.52.78^2\ ((78+6,26)^T)$
40A	$20A\circ(2/1)/2\ (160+32,160^{T40},(\frac{1}{2}40))$	
42A	$-21A\circ(2/1)/2\ ((168+21,56,(\frac{1}{2}84))^{ST84})$	
60A	$1.6.8.10.15.20.48.120/2.3.4.5.24.30.40.60\ (120+15,120,W_3(\frac{1}{2}60))$,	$2^2 3.5.8.12.20^2 30^2 48.120/1.4^2 6^2 10^2 15.24.40.60^2 (120+15,24,W_3(\frac{1}{2}60))$

Table III. (3/1)-transformations of Frame shapes of ·0.

1A	$3A\ (3-)$,	$-3A\ (6+6)$
2A	$6A\ (6+2)$,	$-6A\ (6-)$
2B	$6B\ (12/2+6)$	
2C	$3A/2\ (6/2-)$	
3A	$1^6 9^6/3^{12}\ (9+)$,	$2^6 3^{12} 18^6/1^6 6^{12} 9^6\ (18+W_2(6(-\frac{1}{3})))$
3B	$9A\ (9-)$,	$-9A\ (18+18)$
3D	$3A/3\ (9/3-)$,	$-3A/3\ (18/3+6)$

Table III (continued)

4A	12A (12+12), −12A (12+4)
4B	12B (12−)
4C	$3^2 6.12^2/1^2 2.4^2$ (?), $1^2 6^3 12^2/2^3 3^2 4^2$ (?)
4D	12C (12/2+2)
4E	12F (24/4+6)
4F	$12^3/4^3$ (?)
5A	15A (15+15), −15A (30+6,10)
5B	15B (15+5), −15B (30+5,6)
6A	$1^2 2^2 9^2 18^2/3^4 6^4$ (18+), $2^4 3^4 18^4/1^2 6^8 9^2$ (18+9)
6B	−3A∘(3/1)/2 $(72+(\frac{1}{2}36), W_{72}(-\frac{1}{3}24); D_8)$
6E	18A (18+2), −18A (18−)
6H	−3A/6 (36/6+6)
6I	3A/6 (18/6−)
7A	21A (21+21), −21A (42+6,14)
8A	24A (24/2+12)
8B	−12A/2 (24/2+4)
8C	24B (24+24), −24B (24+8)
8D	$4.24^2/8^2 12$ (?)
8F	24D (24/4+2)
10A	30A (30+2,15), −30A (30+15)
10C	30B (60/2+5,6)
10F	15B (30/2+5)
11A	33A (33+11), −33A (66+6,11)
12A	$3^4 4^2 36^2/1^2 9^2 12^4$ $(36+9, W_{36}(12(\frac{1}{3})))$, $1^2 4^2 6^4 9^2 36^2/2^2 3^4 12^4 18^2$ (36+)
12B	(−6A)∘(2/1)/2 (36/2+9)
12C	6A∘(2/1)/2 (36/2+)
12E	36A (36+36), −36A (36+4)
12L	−3A/12 (72/12+6)
12M	36/12 (?)
13A	39AB (39+39), −39AB (78+6,26)
14A	42A (84/2+6,14)
15B	$1.5.9.45/3^2 15^2$ (45+), $2.3^2 10.15^2 18.90/1.5.6^2 9.30^2 45$ $(90+5, W_2(30(\frac{1}{3}); D_8)$
15C	15A/3 (45/3+15), −15A/3 (90/3+6,10)
20A	60A (60+12,15), −60A (60+4,15)
21A	$3^2 7.63/1.9.21^2$ $(63+9, W_7(21(\frac{1}{3})))$, $1.6^2 9.14.21^2 126/2.3^2 7.18.42^2 63$ $(126+9, 126, W_7(42(\frac{1}{3})))$
24A	12A∘(3/1)/2 $(144+9, W_{16}(48(\frac{1}{3})), (\frac{1}{2}36); Z_2 \times Z_2 \times Z_2)$

Table IV. Other transformations of Frame shapes of ·0.

	2-transf.	(4/1)-transf.
1A	2A, 4A	4A, 4A∘(2/1)
2B	8A	4A∘(2/1)/2
2C	4D	8A
3A	6A, 12A	12A, 12A∘(2/1)
3B	6E, 12E	12E, 12E∘(2/1)
3C	6F, 12D	12D, 12D∘(2/1)
4E	4A/4	4A∘(2/1)
4F	8F	4A/4
5A	10A, 20A	20A, 20A∘(2/1)
6B	24A	12A∘(2/1)/2
6G	12J	24C
7B	14B, 28A	28A, 28A∘(2/1)
9A	18A, 36A	36A, 36A∘(2/1)
10B	40A	20A∘(2/1)/2
12F	12A/4	12A∘(2/1)/4
15A	30A, 60A	60A, 60A∘(2/1)

	3-transf.	(5/1)-transf.
1A	3B, -3B	5A, -5A
2A	6E, -6E	10A, -10A
2B	-3B/2	10B
2C	6G	$10^3/2^3$ (?)
3A	9A, -9A	15A, -15A
3D	9Z, -9Z	15C, -15C
4A	12E, -12E	20A, -20A
4B	12G	$2.20^2/4^2 10$ (?)
4D	12J	10A/2
5B	15D, -15D	25Z, -25Z
6A	18A, -18A	30A, -30A
6H	-3B/6	-5A/6
6I	18Z	30/6 (?)
7A	21B, -21B	35A, -35A
8A	24C	20A/2
8B	24E	4.10.40/2.8.20 (?)
12A	36A, -36A	60A, -60A

	5-transf.	(7/1)-transf.
1A	5B, -5B	7A, -7A
2B	10C	14A
2C	10F	7A/2
3A	15B, -15B	21A, -21A
3B	15D, -15D	21B, -21B
4E	-5B/4	28B
4F	20B	28/4 (?)
5A	25Z, -25Z	35A, -35A
6B	30B	42A

	7-transf.	(9/1)-transf.
1A	7B, -7B	9A, -9A
2A	14B, -14B	18A, -18A
3D	21C, -21C	9A/3, -9A/3
4A	28A, -28A	36A, -36A

	11-transf.	(13/1)-transf.
1A	11A, -11A	13A, -13A
2B	-11A/2	26A
2C	22A	26/2 (?)
3A	33A, -33A	39A, -39A

	23-transf.	(25/1)-transf.
1A	23A, -23A	25Z, -25Z

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