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THE AXELROD MODEL FOR THE DISSEMINATION OF CULTURE REVISITED

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This article is concerned with the Axelrod model, a stochastic process which similarly to the voter model includes social influence, but unlike the voter model also accounts for homophily. Each vertex of the network of interactions is characterized by a set of F cultural features, each of which can assume q states. Pairs of adjacent vertices interact at a rate proportional to the number of features they share, which results in the interacting pair having one more cultural feature in common. The Axelrod model has been extensively studied during the past ten years, based on numerical simulations and simple mean-field treatments, while there is a total lack of analytical results for the spatial model itself. Simulation results for the one-dimensional system led physicists to formulate the following conjectures. When the number of features F and the number of states q both equal two, or when the number of features exceeds the number of states, the system converges to a monocultural equilibrium in the sense that the number of cultural domains rescaled by the population size converges to zero as the population goes to infinity. In contrast, when the number of states exceeds the number of features, the system freezes in a highly fragmented configuration in which the ultimate number of cultural domains scales like the population size. In this article, we prove analytically for the one-dimensional system convergence to a monocultural equilibrium in terms of clustering when $F = q = 2$, as well as fixation to a highly fragmented configuration when the number of states is sufficiently larger than the number of features. Our first result also implies clustering of the one-dimensional constrained voter model.

1. Introduction.

Opinion and cultural dynamics are driven by social influence, the tendency of individuals to become more similar when they interact, which is the basic mechanism of the voter model introduced independently by Clifford and Sudbury [4] and Holley and Liggett [8]. Social influence alone usually drives the system to a monocultural equilibrium, whereas differences between individuals and groups persist in the real world. In his seminal paper [2], political scientist Robert Axelrod explains the diversity of cultures as a consequence of homophily, which is the tendency to interact more frequently with individuals which are more

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similar. In his model, actors are characterized by a finite number of cultural features. In Axelrod's own words, the more similar an actor is to a neighbor, the more likely that actor will adopt one of the neighbor's traits. The network of interactions is a finite connected graph G with vertex set V and edge set E, where each vertex x is characterized by a vector $X(x)$ of F cultural features, each of which assuming q possible states,

$$X(x) = (X^1(x), \dots, X^F(x)),$$

where

$$X^i(x) \in \{1, 2, \dots, q\} \qquad \text{for } i = 1, 2, \dots, F.$$

At each time step, a vertex x is picked uniformly at random from the vertex set along with one of its neighbors y. Then, with a probability equal to the fraction of features x and y have in common, one of the features for which states are different (if any) is selected, and the state of vertex x is set equal to the state of vertex y for this cultural feature. Otherwise nothing happens. In order to describe more generally the Axelrod dynamics on both finite and infinite graphs, we assume that the system evolves in continuous-time with each pair of adjacent vertices interacting at rate one, which causes one of the two vertices chosen uniformly at random to mimic the other vertex in the case of an update. This induces a continuous-time Markov process whose state at time t is a function X_t that maps the vertex set of the graph into the set of cultures $\{1, 2, \dots, q\}^F$, and whose dynamics are described by the generator Ω_{ax} defined on the set of cylinder functions by

$$\Omega_{\mathrm{ax}} f(X) = \sum_{x \in V} \sum_{x \sim y} \sum_{i=1}^{F} \frac{1}{2F} \left[\frac{F(x, y)}{1 - F(x, y)} \right] \mathbb{1}\{X^i(x) \neq X^i(y)\} [f(X^i_{y \to x}) - f(X)],$$

where $x \sim y$ means that x and y are connected by an edge,

$$X^i_{y \to x}(x) = (X^1(x), \dots, X^{i-1}(x), X^i(y), X^{i+1}(x), \dots, X^F(x))$$

and $X^i_{y \to x}(z) = X(z)$ for all $z \neq x$, and

$$F(x, y) = \frac{1}{F} \sum_{i=1}^{F} \mathbb{1}\{X^i(x) = X^i(y)\}$$

denotes the fraction of cultural features vertices x and y have in common. To explain the expression of the Markov generator of the Axelrod model, we note that

$$\frac{1}{2F} \left[\frac{F(x, y)}{1 - F(x, y)} \right] = F(x, y) \times \frac{1}{F(1 - F(x, y))} \times \frac{1}{2},$$

which represents the fraction of features both vertices have in common, which is the rate at which the vertices interact, times the reciprocal of the number of features for which both vertices disagree, which is the probability that any of these features is the one chosen to be updated, times the probability one half that vertex x rather than vertex y is chosen to be updated.

The two-feature, two-state Axelrod model is also closely related to the constrained voter model introduced by Vázquez et al. [11] which identifies two of the cultures with no common feature to be a centrist opinion, and the other two cultures to be a leftist and a rightist opinions. This results in a stochastic process somewhat similar to the voter model except that leftists and rightists are too incompatible to interact. Thinking of leftist as $-$ state, centrist as 0 state and rightist as $+$ state, the constrained voter model is the Markov process whose state at time t is a function Z_t that maps the vertex set of the graph into the opinion set $\{-1, 0, +1\}$, and whose dynamics are described by the Markov generator Ω_{cv} defined on the set of cylinder functions by

$$\Omega_{\mathrm{cv}} f(Z) = \frac{1}{2} \sum_{x \in V} \sum_{x \sim y} \sum_{\varepsilon=-1}^{1} \mathbb{1}\{Z(y) = \varepsilon\} \mathbb{1}\{Z(x) \neq -\varepsilon\} [f(Z_{x,\varepsilon}) - f(Z)],$$

where $Z_{x,\varepsilon}$ is the configuration defined by

$$Z_{x,\varepsilon}(z) = \varepsilon \qquad \text{if } z = x \quad \text{and} \quad Z_{x,\varepsilon}(z) = Z(z) \qquad \text{if } z \neq x.$$

In order to understand these opinion and cultural dynamics, we study analytically the number and mean size of the cultural domains at equilibrium. The number $N(t)$ of cultural domains at time t is the number of connected components of the graph obtained by removing all the edges that connect two vertices that do not share the same culture at time t, while the mean size $S(t)$ is defined as the mean number of vertices per connected component. Note that when the dynamics take place on a finite connected graph, the mean size of the cultural domains is also equal to the total number of vertices divided by the number of cultural domains.

The constrained voter model and Axelrod model have been extensively studied over the past ten years by social scientists as well as statistical physicists based on numerical simulations and simple mean field treatments, while there is a total lack of analytical results for the spatial models. We refer the reader to Sections III.B and IV.A of Castellano et al. [3] for a review, and references therein for more details about numerical results. Because spatial simulations are usually difficult to interpret, there is a need for rigorous analytical results, and this article is intended to provide analytical proofs of important conjectures suggested by the simulations, which also gives insight into the mechanisms that promote convergence to either a monocultural equilibrium or, on the contrary, a highly fragmented configuration where cultural domains are uniformly bounded.

Convergence to a monocultural equilibrium. Letting θ denote the initial density of centrists in the one-dimensional constrained voter model, the mean-field analysis in [11] suggests that the average domain length at equilibrium is

$$\lim_{t \to \infty} E(S(t)) \sim N^{2\psi(\theta)} \qquad \text{where } \psi(\theta) = -\frac{1}{8} + \frac{2}{\pi^2} \left[\cos^{-1}\left(\frac{1 - 2\theta}{\sqrt{2}} \right) \right]^2,$$

when the length N of the system is large. Vázquez et al. [11] also showed that these predictions agree with their numerical simulations when the initial density of centrists is small enough, as indicated by their Figure 5, from which they conclude that, for small θ, the system ends up with high probability to a frozen mixture of leftists and rightists. Their simulations, however, also suggest that a typical final frozen state is characterized by two spatial scales, with few cultural domains covering macroscopically large fractions of the universe and a number of small domains. The presence of two spatial scales also holds for the two-feature two-state Axelrod model. Even though it was not the conclusion of Vázquez et al. [11], this somewhat suggests convergence to a monocultural equilibrium in which the number of cultural domains does not scale like the population size, that is, the cardinality of the vertex set. Our first result shows that both the constrained voter model and the two-feature two-state Axelrod model on the one-dimensional infinite lattice indeed converge to a monocultural equilibrium in terms of a clustering similar to that of the one-dimensional voter model. This clustering indicates that the only stationary distributions are the ones supported on the set of configurations in which all vertices share the same culture.

THEOREM 1. *Starting from a translation invariant product measure in which each culture/opinion occurs with positive probability, the one-dimensional two-feature two-state Axelrod model and the one-dimensional constrained voter model cluster, that is,*

$$\lim_{t\to\infty} P\big(X_t(x) \neq X_t(y)\big) = \lim_{t\to\infty} P\big(Z_t(x) \neq Z_t(y)\big) = 0 \qquad \text{for all } x, y \in \mathbb{Z}.$$

The apparent contradiction between our analytical result and the numerical results of [11] is due to the fact that, when taking place on a finite connected graph, the dynamics of the Axelrod model and constrained voter model may drive the system to a culturally fragmented frozen configuration even though they promote convergence to a monocultural equilibrium, which again reveals the difficulty in interpreting spatial simulations and the need for analytical results.

Fixation to a fragmented configuration. For the one-dimensional Axelrod model with an arbitrary number of features and states per feature, Vilone et al. [13] have predicted through the analysis of mean-field approximation supported by simulation results that convergence to a monocultural equilibrium occurs when $F > q$ whereas fixation to a highly fragmented configuration occurs when $F < q$. Our second result establishes partly the latter in the sense that the expected number of cultural domains at equilibrium on a path-like graph scales like the length of the graph for an infinite proper subset of the parameter region $F < q$.

THEOREM 2. *Assume that $G = \{0, 1, \ldots, N\}$ and $F < q$. Then, starting from a translation invariant product measure in which each culture occurs with the same* *probability* q^{-F},

$$N^{-1} \lim_{t\to\infty} E(N(t)) \geq \left(1-\frac{1}{q}\right)^F + \frac{F}{q-F}\left(\left(1-\frac{1}{q}\right)^F - \left(1-\frac{1}{q}\right)\right).$$

Note that the lower bound for the expected number of cultural domains also gives an upper bound for the expected length of the cultural domains since the expected number of domains times their expected length is equal to the cardinal of the vertex set. Note also that the theorem does not a priori exclude clustering of the system on the infinite one-dimensional lattice, but it strongly suggests the latter since it gives upper bounds for the expected length of the cultural domains which are uniform in the size of the network of interactions. Finally, we point out that, even though our estimate holds for all $F < q$, the theorem does not fully prove the conjecture of [13] since the lower bound can be negative: to make the coexistence region more explicit when both parameters are large, we fix $F/q = x$ and notice that the lower bound in Theorem 2 converges to

$$\left(\frac{1}{x}-1\right)^{-1}\left(\frac{e^{-x}}{x}-1\right) \qquad \text{as } F \to \infty.$$

The equation $e^{-x} = x$ has a unique positive solution given by $x_0 \approx 0.567$, which indicates that the one-dimensional Axelrod model coexists in the sense that the expected number of cultural domains scales like the population size when $F < x_0 \times q$ and F is large. For explicit conditions for coexistence when the parameters are small, we refer the reader to Table 1 which gives some values of the upper bound for the expected length of the cultural domains at equilibrium.

The intuition behind Theorems 1 and 2 that appears in our proofs can be interpreted in terms of active versus frozen boundaries between adjacent cultural domains. Here, we call an active boundary the boundary between two cultural domains with at least one feature in common. Even though the infinite system keeps

TABLE 1
Upper bounds for the expected length of the cultural domains at equilibrium

	$q=4$	$q=8$	$q=12$	$q=16$	$q=20$	$q=24$	$q=28$	$q=32$	$q=36$
$F=2$	2.6667	1.3714	1.2121	1.1487	1.1146	1.0932	1.0785	1.0679	1.0597
$F=3$	neg.	1.8286	1.3861	1.2535	1.1890	1.1508	1.1255	1.1074	1.0940
$F=4$	–	3.3629	1.6645	1.3938	1.2810	1.2188	1.1792	1.1519	1.1318
$F=5$	–	neg.	2.1989	1.5943	1.3985	1.3007	1.2417	1.2022	1.1738
$F=6$	–	neg.	3.7048	1.9091	1.5552	1.4017	1.3154	1.2599	1.2211
$F=7$	–	neg.	45.641	2.4851	1.7767	1.5304	1.4040	1.3268	1.2746
$F=8$	–	–	neg.	3.9072	2.1170	1.7007	1.5132	1.4058	1.3360
$F=9$	–	–	neg.	13.637	2.7127	1.9385	1.6514	1.5005	1.4071

evolving indefinitely, Theorem 2 indicates that for reasonably large values of the number of states q the incompatibility between adjacent vertices prevents a positive fraction of boundaries to ever become active, that is, a positive fraction of the boundaries frozen initially stay frozen at any time. In contrast, the result of Theorem 1 is symptomatic of a large activity of the system in the sense that each vertex changes its culture infinitely often which results in the destruction of the frozen boundaries thus in the presence of cultural domains that keep growing indefinitely. Cultural dynamics including two features and two states per feature taking place on finite graphs operate similarly by promoting convergence to a monocultural equilibrium. However, due to the finiteness of the network of interactions, the system may fixate before reaching a total consensus in which case the final frozen configuration is characterized by two spatial scales: cultural domains whose length scales like the size of the system and domains which are uniformly bounded, as observed in Vázquez et al. [11]. The rest of the article is devoted to the proofs.

2. Proof of Theorem 1. Note first that the constrained voter model is obtained from the two-feature two-state Axelrod model by identifying two cultures without common features with the centrist opinion, and each of the other two cultures with the leftist and rightist opinions, respectively. Therefore the mean cluster size is stochastically larger for the constrained voter model than the Axelrod model, so it suffices to prove the result for the latter. To study the probability of a consensus when $F = q = 2$, but also the expected number of cultural domains at equilibrium when $F < q$ in the next section, the idea is to analyze the evolution of the agreements along the edges rather than the actual opinion at each vertex. The network can be viewed as a weighted graph where each edge is assigned a weight that counts the number of features its endpoints have in common. We call $e = \{x, x+1\} \in E$ an edge with weight j at time t, or simply a j-edge at time t, whenever

$$\bar{X}_t(e) = \sum_{i=1}^{F} \mathbb{1}\{X_t^i(x) = X_t^i(x+1)\} = j.$$

The key to proving Theorem 1 is to observe that, when $F = 2$, clustering of the Axelrod model is equivalent to almost sure extinction of the 1-edges and the 0-edges. The former follows from the clustering of a certain voter model coupled with the Axelrod model, while the latter follows from the combination of the clustering and the site recurrence property of the voter model that we define below. Before going into the details of the proof, we start by collecting important results about the connection between the Axelrod model, the voter model and coalescing random walks.

The first ingredient is to observe, as pointed out by Vázquez and Redner [12], that one recovers the voter model from the two-feature two-state Axelrod model by identifying cultures that have no feature in common (see Figure 1 for simulation

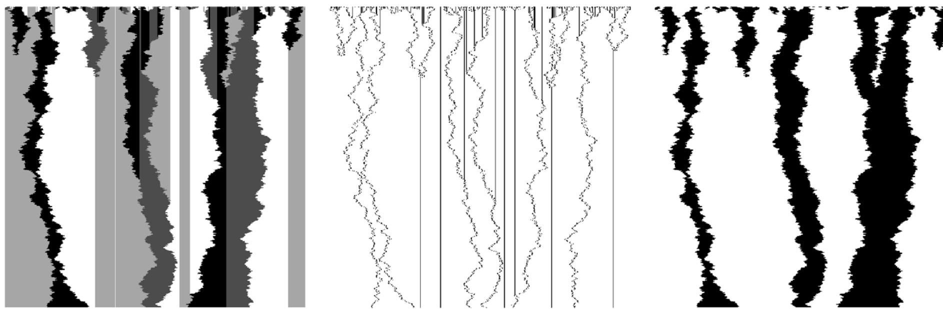

FIG. 1. *Coupling between the Axelrod model and the voter model.*

pictures of the coupled Axelrod model and voter model). Indeed, when $F = q = 2$, we have

$$\Omega_{\mathrm{ax}} f(X) = \frac{1}{4} \sum_{x \in \mathbb{Z}} \sum_{x \sim y} \sum_{i \neq j} \mathbb{1}\{X^i(x) \neq X^i(y)\} \mathbb{1}\{X^j(x) = X^j(y)\} \times [f(X^i_{y \to x}) - f(X)].$$

Therefore, letting $Y(x) = |X^1(x) - X^2(x)|$ for all $x \in \mathbb{Z}$ and noticing that

$$\{Y(x) \neq Y(y)\} = \bigcup_{i \neq j} \{X^i(x) \neq X^i(y)\} \cap \{X^j(x) = X^j(y)\},$$

we obtain that $\{Y_t : t \geq 0\}$ is the Markov process with generator

$$\Omega_{\mathrm{vm}} f(Y) = \frac{1}{4} \sum_{x \in \mathbb{Z}} \sum_{x \sim y} \mathbb{1}\{Y(x) \neq Y(y)\}[f(Y_{y \to x}) - f(Y)],$$

where $Y_{y \to x}$ is the configuration defined by

$$Y_{y \to x}(z) = Y(y) \qquad \text{if } z = x \quad \text{and} \quad Y_{y \to x}(z) = Y(z) \qquad \text{if } z \neq x.$$

This indicates that $\{Y_t : t \geq 0\}$ is a time change of the voter model run at rate $1/2$, but since we are only interested in the limiting distribution of the Axelrod model, we shall for simplicity speed up time by a factor two in order to get the usual voter model run at rate 1.

The voter model can be constructed graphically using an idea of Harris [7], which also allows us to exhibit a duality relationship between the voter model and coalescing random walks. This construction is now standard, so we only give a brief description. To each vertex $x \in \mathbb{Z}$, we attach a Poisson process with parameter one. Then, at the arrival times of this process, we choose one of the two neighbors $x \pm 1$ uniformly at random, then draw an arrow from vertex $x \pm 1$ to vertex x and put a δ at x to indicate that x updates its opinion by mimicking $x \pm 1$. The connection between the voter model and coalescing random walks appears when keeping track of the ancestry of each vertex going backward in time which also defines the so-called dual process. We say that there is a dual path from (x, T) to $(y, T - s)$ if there are sequences of times and vertices

$$s_0 = T - s < s_1 < \cdots < s_{n+1} = T \quad \text{and} \quad x_0 = y, x_1, \dots, x_n = x$$

such that the following two conditions hold:

(1) for $i = 1, 2, \dots, n$, there is an arrow from x_{i-1} to x_i at time s_i;
(2) for $i = 0, 1, \dots, n$, the vertical segment $\{x_i\} \times (s_i, s_{i+1})$ does not contain any δ's.

Then, for $A \subset \mathbb{Z}$ finite, the dual process starting at (A, T) is the set-valued process

$$\hat{Y}_s(A, T) = \{y \in \mathbb{Z}\text{: there is a dual path from } (x, T) \text{ to } (y, T - s) \text{ for some } x \in A\}.$$

The dual process is naturally defined only for dual times $0 \le s \le T$. However, it is convenient to assume that the Poisson processes in the graphical representation are also defined for negative times so that the dual process can be defined for all $s \ge 0$. The reason for introducing the dual process is that it allows one to deduce the state of the process at the current time from the configuration at earlier times based on the duality relationship

$$Y_t(x) = Y_{t-s}(\hat{Y}_s(x, t)) = Y_0(\hat{Y}_t(x, t)) \qquad \text{for all } s \in (0, t).$$

Moreover, it can be seen from the graphical representation that the dual process evolves according to a system of simple symmetric coalescing random walks that jump at rate 1, so questions about the voter model can be answered by looking at this system of coalescing random walks.

The first step, as previously mentioned, is to establish extinction of the 1-edges, which is directly related to the clustering of the one-dimensional voter model.

LEMMA 3. *There is almost sure extinction of the* 1-*edges*, *that is*,

$$\lim_{t\to\infty} P(\bar{X}_t(e) = 1) = 0 \qquad \text{for all } e \in E.$$

PROOF. Since the one-dimensional voter model clusters [4, 8], we have

$$\begin{aligned}
&\lim_{t\to\infty} P(\bar{X}_t(\{x, x+1\}) = 1) \\
&\quad = \lim_{t\to\infty} P(X_t^1(x) \neq X_t^1(x+1) \text{ and } X_t^2(x) = X_t^2(x+1)) \\
&\qquad + \lim_{t\to\infty} P(X_t^1(x) = X_t^1(x+1) \text{ and } X_t^2(x) \neq X_t^2(x+1)) \\
&\quad = \lim_{t\to\infty} P(Y_t(x) \neq Y_t(x+1)) = 0
\end{aligned}$$

for every vertex $x \in \mathbb{Z}$, which proves extinction of the 1-edges. In words, the 1-edges in the Axelrod model correspond to the interfaces of the underlying voter model, which evolve according to a system of annihilating random walks that, because of clustering, goes extinct. □

The second step is to prove that there is almost sure extinction of the 0-edges, which follows from the combination of clustering and the following property:

$$P\big(Y_t(x) \neq Y_s(x) \text{ for some } t > s\big) = 1 \qquad \text{for all } x \in \mathbb{Z} \text{ and } s > 0$$

that we shall call site recurrence of the one-dimensional voter model. The terminology is motivated by the article of Erdős and Ney [6] who conjectured that, given a system of discrete-time annihilating random walks starting with one particle at each site except the origin, the probability that the origin is visited infinitely often is one, property that Arratia [1] called later site recurrence. The continuous-time version of their conjecture has been proved by Schwartz [10] based on the connection with the one-dimensional voter model whose interfaces precisely evolve according to a system of annihilating random walks. In particular, the idea of her proof is to show that the one-dimensional voter model starting from a particular deterministic configuration is site recurrent in the sense defined above. Her result easily extends to the process starting from more general random configurations, but we give a somewhat shorter proof in Lemma 4 below. Note that the site recurrence of the voter model in higher dimensions directly follows from the law of large numbers for the occupation time of the process established in Cox and Griffeath [5] which does not hold in one dimension.

LEMMA 4. *The one-dimensional voter model is site recurrent*, *that is*,

$$P\big(Y_t(x) \neq Y_s(x) \text{ for some } t > s\big) = 1 \qquad \text{for all } x \in \mathbb{Z} \text{ and } s > 0.$$

PROOF. The key is to observe that, for all $y \in \mathbb{Z}$, the process that keeps track of the number of vertices at time t that descend from y; namely

$$M_t(y) := \operatorname{card}\{z \in \mathbb{Z}: \text{there is a dual path from } (z, t) \text{ to } (y, 0)\}$$

is a martingale absorbed at state 0. Note that, though stated for the one-dimensional voter model only, this property holds in any spatial dimension. Since in addition $M_t(y)$ is an integer-valued process, a straightforward application of the martingale convergence theorem implies that it converges almost surely to its absorbing state. Now, let $x \in \mathbb{Z}$ and $s > 0$, and define

$$\Phi(s) := \inf\{t > 0 : M_t(y) = 0\} \qquad \text{where } y := \hat{Y}_s(x, s).$$

Since there is a dual path from (x, s) to $(y, 0)$, this stopping time is larger than s, but almost sure convergence of $M_t(y)$ to zero implies that time $\Phi(s)$ is almost surely finite. In addition,

$$\text{there is no dual path from } (x, \Phi(s)) \text{ to } (y, 0)$$

from which it follows that the spin at $(x, \Phi(s))$ and the spin at (x, s) originate from different vertices at time 0 and thus are independent since the initial configuration is distributed according to a product measure. This holds for all $s > 0$ hence defining recursively $s_0 = s$ and

$$s_{i+1} := \Phi(s_i) = \inf\{t > 0 : M_t(y) = 0\} \qquad \text{where } y := \hat{Y}_{s_i}(x, s_i)$$

for all integers $i \geq 0$ induces an increasing sequence of stopping times which are all almost surely finite. Moreover, the collection of spins at (x, s_i) are independent, determined from the spins of different vertices at time 0. It is straightforward to deduce that

$$P\big(Y_s(x) := Y_{s_0}(x) = Y_{s_1}(x) = \cdots = Y_{s_i}(x)\big) \to 0 \qquad \text{as } i \to \infty$$

since each type occurs initially with positive probability. The lemma follows. □

LEMMA 5. *There is almost sure extinction of the* 0-*edges*, *that is*,

$$\lim_{t\to\infty} P\big(\bar{X}_t(e) = 0\big) = 0 \qquad \text{for all } e \in E.$$

PROOF. First of all, we observe that, since the initial configuration, as well as the evolution rules of the process, are translation invariant in space, the probability of edge e being a 0-edge at time t does not depend on the specific choice of e. This key property is implicitly used repeatedly in the proof of the lemma. Now, we let $0 < s < t < \infty$, and partition the set of 0-edges at time t into the subset Ω_- of those edges that have been lately updated by time s and the subset Ω_+ of those edges that have been lately updated after time s, namely

$$\Omega_- = \{e \in E : \bar{X}_u(e) = 0 \text{ for all } u \in (s, t)\},$$
$$\Omega_+ = \{e \in E : \bar{X}_t(e) = 0 \text{ and } \bar{X}_u(e) = 1 \text{ for some } u \in (s, t)\}.$$

Note that an update at vertex x in the Axelrod model corresponds to the simultaneous update of the pair of edges incident to this vertex. In addition, the culture at x flips at a positive rate if and only if x has exactly one feature in common with one of its two nearest neighbors, that is, if and only if at least one of both edges incident to x is a 1-edge. Also, since only one feature changes at a time, when an edge pair is updated, the weight of each edges varies by exactly one unit. In particular, accounting for symmetry, there are only four possible transitions of the edge pairs:

$$(1, 0) \to (2, 1), \qquad (1, 1) \to (2, 0), \qquad (1, 1) \to (2, 2), \qquad (1, 2) \to (2, 1).$$

It follows that the probability of e being a 1-edge is nonincreasing (this can be seen from the fact that the 1-edges evolve according to a system of annihilating random walks) and that a 0-edge can only result from the annihilation of two 1-edges. This, together with Lemma 3 which claims extinction of the 1-edges, implies that, for all $\varepsilon > 0$, there exists s large such that

$$2 \times P(e \in \Omega_+) \le P(\bar{X}_s(e) = 1) \le \varepsilon.$$

Time s being fixed, Lemma 4 implies the existence of $t > s$ such that

$$\begin{aligned} P(e = \{x, x+1\} \in \Omega_-) &= P(X_u^i(x) \neq X_u^i(x+1) \text{ for all } u \in (s,t) \text{ and } i = 1, 2) \\ &\le P(Y_u(x) = Y_u(x+1) \text{ for all } u \in (s,t)) \\ &\le P(Y_u(x) = Y_s(x) \text{ for all } u \in (s,t)) \le \varepsilon. \end{aligned}$$

Combining the previous two estimates, we obtain that, for all $\varepsilon > 0$ small, there exists a large but finite time $s > 0$ and a large but finite time $t > s$ such that

$$P(\bar{X}_t(e) = 0) = P(e \in \Omega_+) + P(e \in \Omega_-) \le 2\varepsilon,$$

which establishes extinction of the 0-edges. □

Having established that both sets of 0-edges and 1-edges go extinct, the proof of Theorem 1 is now straightforward, and follows the lines of Lemma 3. While the latter shows that clustering of the voter model implies extinction of the 1-edges, the last step is to prove that, conversely, extinction of type 0 and type 1 edges implies clustering of the Axelrod model. Fix $x < y$, and let

$$z_0 = x < z_1 < \cdots < z_k = y \qquad \text{with } k = |x - y|.$$

Denote by $e_i = \{z_i, z_{i+1}\}$ the edge connecting vertex z_i and vertex z_{i+1}. Then, extinction of both the 1-edges and the 0-edges given, respectively, by Lemmas 3 and 5 implies that

$$\begin{aligned} &\lim_{t\to\infty} P(X_t(x) \neq X_t(y)) \\ &\quad \le \lim_{t\to\infty} P(X_t(z_i) \neq X_t(z_{i+1}) \text{ for some } i = 0, 1, \ldots, k-1) \\ &\quad \le \lim_{t\to\infty} \sum_{i=0}^{k-1} P(\bar{X}_t(e_i) = 0 \text{ or } \bar{X}_t(e_i) = 1) = 0. \end{aligned}$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2. This section is devoted to the proof of Theorem 2 which again relies on the analysis of the agreements along the edges rather than the actual opinion at each vertex. Before going into the details of the proof, we briefly introduce its main steps. First, it is noted in Lemma 6 that the process on a finite graph reaches almost surely one of its absorbing states which consist of the configurations in which each edge has either weight zero or weight F. The ultimate number of cultural domains on a path-like graph is roughly equal to the ultimate number of edges with weight zero, so the strategy is to bound from below the number of such edges. To do so, we introduce

$$W(t) := \sum_{j=0}^{F} j W_j(t) \qquad \text{where } W_j(t) := \operatorname{card}\{e \in E : \bar{X}_t(e) = j\}$$

and where $\bar{X}_t(e)$ is defined as in the previous section. In words, $W(t)$ keeps track of the total number of agreements in the system. Let y be a vertex with degree 2, and let x and z be its two nearest neighbors. Then, we observe that if the individual at vertex y updates its culture at time t by mimicking the ith feature of vertex x, then we have the following alternatives:

(0) $W(t) - W(t-) = 0$ whenever $X_{t-}^i(y) = X_{t-}^i(z)$.
(1) $W(t) - W(t-) = 1$ whenever $X_{t-}^i(x) \neq X_{t-}^i(z)$ and $X_{t-}^i(y) \neq X_{t-}^i(z)$.
(2) $W(t) - W(t-) = 2$ whenever $X_{t-}^i(x) = X_{t-}^i(z)$.

The first key step is to prove that if the ith feature of vertex y differs from the ith feature of both of its neighbors (cases 1 and 2 above), then the ith feature of vertex x and the ith feature of vertex z are independent, which is established in Lemma 7. It follows that

$$P\big(W(t) - W(t-) = 2 | W(t) \neq W(t-)\big) \leq (q-1)^{-1}$$

as stated in Lemma 8. Note that two nearest neighbors, say y and z, cannot interact as long as they are connected by an edge with weight zero; therefore edges with weight zero can possibly change their state only when two agreements emerge simultaneously (case 2 above). This, together with the previous inequality, leads us to consider the following urn problem which is also based on an idea initially introduced in [9]. There are $F + 1$ boxes labeled from box 0 to box F containing all together a total of N balls, which corresponds to the number of edges. The game starts with as many balls in box j as there are edges with weight j in the Axelrod model at time 0, and evolves in discrete time according to the following stochastic rules:

(1) At each time step, we move a ball from box j to box $j + 1$ where (if it exists) box j is chosen uniformly at random from the set of nonempty inner boxes.

(2) In case a ball has indeed been moved, and box 0 is nonempty, we move an additional ball from box 0 to box 1 only with probability $(q-1)^{-1}$.

(3) The game halts when all the inner boxes are empty.

Here, inner boxes refer to boxes $1, 2, \ldots, F-1$. Using coupling arguments and the inequality established in Lemma 8, we prove that the expected number of balls in box 0 when the game halts is larger than the expected number of edges with weight zero in the Axelrod model when the system gets trapped. This is done in Lemma 9. Finally, the lower bound of Theorem 2 is proved to be a lower bound for the expected number of balls in box 0 when the game halts, and thus a lower bound for the expected value of the ultimate number of domains, in Lemmas 10 and 11. The key to keeping track of the number of balls in box 0 is to divide the game into rounds and paint the balls with two different colors at the beginning of each round.

LEMMA 6. *We have* $\lim_{t\to\infty} N(t) - W_0(t) = 1$.

PROOF. The number $N(t)$ of cultural domains at time t is the number of connected components of the graph obtained by removing all edges whose weight at time t differs from F. In the case of a finite tree, this results in a forest whose number of connected components is equal to the number of edges removed plus one, from which it follows that

$$N(t) = W_0(t) + W_1(t) + \cdots + W_{F-1}(t) + 1.$$

Since the Axelrod model on a finite graph converges to one of its absorbing states and that each absorbing state is characterized by all the edges having weight either zero or F, we also have

$$\lim_{t\to\infty} W_j(t) = 0 \qquad \text{for } j = 1, 2, \dots, F-1.$$

The result follows. $\square$

LEMMA 7. *Let* $0 \le x < y < z \le N$ *and fix* $i \in \{1, 2, \dots, F\}$. *Then*

$$P\big(X_t^i(x) = X_t^i(z) | X_t^i(x) \neq X_t^i(y) \text{ and } X_t^i(y) \neq X_t^i(z)\big) = (q-1)^{-1}.$$

PROOF. The idea is that, given that the pairs $\{x, y\}$ and $\{y, z\}$ disagree on their ith feature, the states of this feature for x and z must originate from different vertices at time 0. To make this argument rigorous, we first construct the Axelrod model graphically from collections of independent random variables: for each oriented edge $e = (u, v)$ and $n \ge 1$:

(1) $T_n(e)$ is the nth arrival time of a Poisson process with rate $1/2$;
(2) $U_n(e)$ is the discrete random variable uniformly distributed over $\{1, 2, \dots, F\}$;
(3) $V_n(e) = \{V_{n,k}(e) : k \ge 1\}$ is an infinite sequence of independent random variables which are uniformly distributed over the set of features $\{1, 2, \dots, F\}$.

The process starting from any initial configuration is constructed inductively as follows. Assume that the process has been constructed up to time $t-$ where $t = T_n(e)$, and let

$$I_{t-}(e) = I_{t-}((u, v)) = \{i : X_{t-}^i(u) = X_{t-}^i(v)\}$$

and

$$J_{t-}(e) = \{i : X^i_{t-}(u) \neq X^i_{t-}(v)\}.$$

In words, the sets $I_{t-}(e)$ and $J_{t-}(e)$ are the sets of features for which the vertices u and v agree and disagree, respectively. Then we have the following alternative:

(1) if $U_n(e) \in I_{t-}(e)$ and $J_{t-}(e) \neq \varnothing$, then we draw an arrow from u to v at time t;
(2) if $U_n(e) \in J_{t-}(e)$ or $J_{t-}(e) = \varnothing$, then we do nothing.

Thinking of an arrow oriented from u to v as representing an interaction that causes vertex v to mimic one of the features of u, and noticing that

$$P\big(U_n(e) \in I_{t-}(e)\big) = \frac{1}{F}\,\mathrm{card}(I_{t-}(e)) = \frac{1}{F}\sum_{i=1}^{F} \mathbb{1}\{X^i_{t-}(u) = X^i_{t-}(v)\},$$

this indicates that, as required, adjacent vertices with different cultures interact at a rate proportional to the number of features they share. To complete the construction, the last step is to use the random variables $V_{n,k}(e)$ to determine the feature vertex v mimics on the event that 1 above occurs. To choose this feature uniformly at random from $J_{t-}(e)$, as required, we let

$$m = \inf\{k \geq 1 : V_{n,k}(e) \in J_{t-}(e)\} \quad \text{and} \quad i = V_{n,m}(e)$$

and label the arrow with an i to indicate that the ith feature of v at time t is set equal to the ith feature of vertex u. Since the network of interactions is finite, the times of the Poisson events can be ordered; therefore the Axelrod model can be constructed going forward in time using the previous rules. However, an idea of Harris [7] allows us as well to construct the process on infinite lattices in the same manner. Given an initial configuration in which features are independent and uniformly distributed, and the collections of independent random variables introduced above, we draw the arrows along with their label up to time t following the rules previously described. Then we say that there is an i-lineage from (u, t) to $(w, t - s)$ if there are

$$s_0 = t - s < s_1 < \cdots < s_{n+1} = t \quad \text{and} \quad u_0 = w, u_1, \ldots, u_n = u$$

such that the following two conditions hold:

(1) for $j = 1, 2, \ldots, n$, there is an i-arrow from u_{j-1} to u_j at time s_j;
(2) for $j = 0, 1, \ldots, n$, the segment $\{u_j\} \times (s_j, s_{j+1})$ does not contain any tips of i-arrow.

Note that for all $s \in (0, t)$, there exists a unique vertex w such that (1) and (2) hold. We define the process that keeps track of the unique i-lineage starting at (u, t) by letting

$$\hat{X}^i_s(u, t) = \{w\text{: there is an } i\text{-lineage from } (u, t) \text{ to } (w, t - s)\},$$

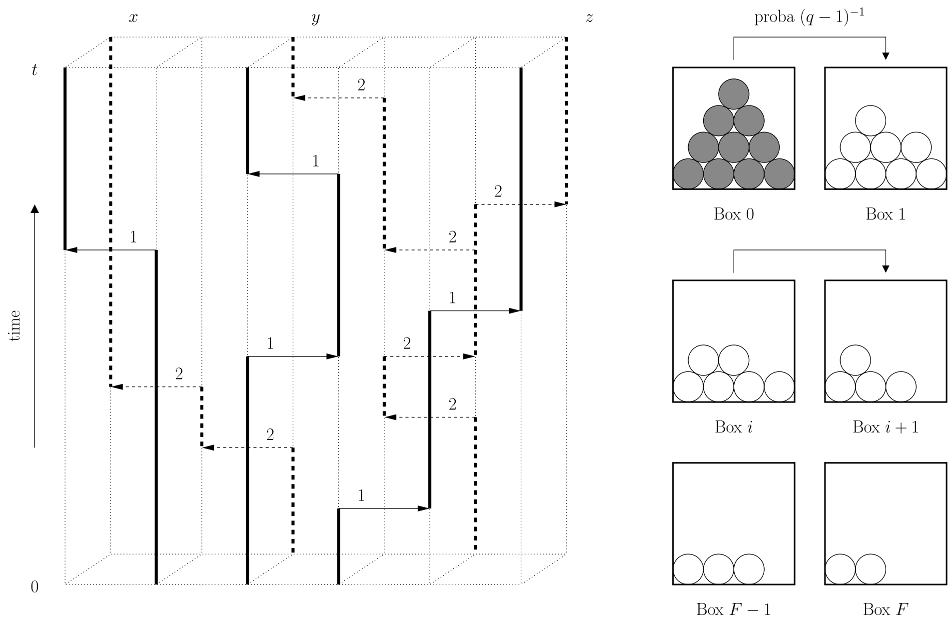

FIG. 2. *Pictures related to the proof of Theorem 2—lineages in the Axelrod model on the left-hand side, and schematic illustration of a single step evolution of the urn problem on the right-hand side.*

and refer the reader to Figure 2 for an illustration. Although i-lineages in the Axelrod model are somewhat reminiscent of dual paths in the voter model, the latter can be constructed from the graphical representation regardless of the initial configuration, whereas the construction of the former also depends on the initial configuration, since one needs to construct the process forward in time up to time t in order to identify the labels on the arrows. In particular, the culture of a given space–time point cannot be determined from the initial configuration by simply going backward in time along the graphical representation. However, by construction, one has

$$X_t^i(u) = X_{t-s}^i(\hat{X}_s^i(u,t)) = X_0^i(\hat{X}_t^i(u,t)) \qquad \text{for all } 0 \le s \le t$$

from which it follows that

$$X_t^i(y) \notin \{X_t^i(x), X_t^i(z)\} \quad \text{implies} \quad \hat{X}_t^i(y,t) \notin \{\hat{X}_t^i(x,t), \hat{X}_t^i(z,t)\}.$$

Moreover, after the random labels on the arrows have been determined by constructing the process up to time t, the system of i-lineages is constructed from the set of i-arrows in the same manner as the system of dual paths in the voter model. Due, in addition, to the presence of one-dimensional nearest neighbor interactions, it is straightforward to deduce that, regardless of the random configuration of the i-arrows, the dynamics preserve the order of the i-lineages in the sense that

$$\hat{X}_s^i(x,t) \le \hat{X}_s^i(y,t) \le \hat{X}_s^i(z,t) \qquad \text{for all } 0 \le s \le t.$$

Combining the previous two properties, we deduce that

$$X_t^i(y) \notin \{X_t^i(x), X_t^i(z)\} \quad \text{implies} \quad \hat{X}_t^i(x,t) < \hat{X}_t^i(y,t) < \hat{X}_t^i(z,t),$$

indicating that the ith feature of vertex x and the ith feature of vertex z at time t are determined by the initial ith features of two different vertices. Since these features are independent and uniformly distributed at time 0, we obtain that, for all $q_1, q_2 \neq X_t^i(y)$,

$$\begin{aligned} &P\big(X_t^i(x) = q_1 \text{ and } X_t^i(z) = q_2 | X_t^i(x) \neq X_t^i(y) \text{ and } X_t^i(y) \neq X_t^i(z)\big) \\ &\quad = P\big(X_0^i(\hat{X}_t^i(x,t)) = q_1 \text{ and } X_0^i(\hat{X}_t^i(z,t)) = q_2 | \hat{X}_t^i(x,t) \neq \hat{X}_t^i(z,t) \\ &\qquad\qquad \text{and } X_t^i(x) \neq X_t^i(y) \text{ and } X_t^i(y) \neq X_t^i(z)\big) \\ &\quad = (q-1)^{-2}. \end{aligned}$$

The lemma follows by summing over all the $q-1$ possible values of $q_1 = q_2$. $\square$

LEMMA 8. *We have* $P(W(t) - W(t-) = 2 | W(t) \neq W(t-)) \leq (q-1)^{-1}$.

PROOF. First, we observe that there exist $n \geq 1$ and $e = (x, y)$ such that $t = T_n(e)$ on the conditional event that the total number of agreements increases at time t. Then we construct the process up to time $t-$ using the collection of independent random variables and following the rules introduced in Lemma 7 in order to identify the set

$$J_{t-}(e) = \{1 \leq i \leq F : X_{t-}^i(x) \neq X_{t-}^i(y)\},$$

which is nonempty on the event that $W(t) \neq W(t-)$. Note that if vertex y is at the boundary of the system, then only the weight of edge e is updated at time t; therefore

$$P\big(W(t) - W(t-) = 2 | W(t) \neq W(t-)\big) = 0 \qquad \text{whenever } y = 0 \text{ or } y = N.$$

To deal with the nontrivial case when vertex y has degree 2, observe that

$$\begin{aligned} \{W(t) - W(t-) = 2\} \cap \{V_{n,m}(e) = i\} &= \{X_{t-}^i(x) = X_{t-}^i(z)\} \cap \{V_{n,m}(e) = i\}, \\ \{W(t) \neq W(t-)\} \cap \{V_{n,m}(e) = i\} &= \{X_{t-}^i(y) \neq X_{t-}^i(z)\} \cap \{V_{n,m}(e) = i\}, \end{aligned}$$

where vertex z is the unique nearest neighbor of y different from x and where m is defined in the construction of the process given in Lemma 7. Since the random variables $V_{n,k}$, $k \geq 1$, are independent of the configuration of the system at time $t-$, we have in addition that

$$\begin{aligned} &P\big(X_{t-}^i(x) = X_{t-}^i(z) | X_{t-}^i(y) \neq X_{t-}^i(z) \text{ and } V_{n,m}(e) = i\big) \\ &\quad = P\big(X_{t-}^i(x) = X_{t-}^i(z) | X_{t-}^i(y) \neq X_{t-}^i(z) \text{ and } i \in J_{t-}(e)\big). \end{aligned}$$

Combining the previous two properties, we deduce that

$$\begin{aligned}
&P\big(W(t) - W(t-) = 2 | W(t) \neq W(t-) \text{ and } V_{n,m}(e) = i\big) \\
&\quad = P\big(X_{t-}^i(x) = X_{t-}^i(z) | W(t) \neq W(t-) \text{ and } V_{n,m}(e) = i\big) \\
&\quad = P\big(X_{t-}^i(x) = X_{t-}^i(z) | X_{t-}^i(y) \neq X_{t-}^i(z) \text{ and } V_{n,m}(e) = i\big) \\
&\quad = P\big(X_{t-}^i(x) = X_{t-}^i(z) | X_{t-}^i(y) \neq X_{t-}^i(z) \text{ and } i \in J_{t-}(e)\big) \\
&\quad = P\big(X_{t-}^i(x) = X_{t-}^i(z) | X_{t-}^i(x) \neq X_{t-}^i(y) \text{ and } X_{t-}^i(y) \neq X_{t-}^i(z)\big) \\
&\quad = (q-1)^{-1},
\end{aligned}$$

where the last equality follows from Lemma 7. This completes the proof. □

As previously mentioned, to find a lower bound for the expected value of the ultimate number of edges with weight zero, we need to compare the number of such edges with the ultimate number of balls in box 0 for the game described at the beginning of this section. To do so, the first step is to couple the Axelrod dynamics with another urn problem that evolves in continuous-time. As previously, we start with as many balls in box j as there are edges with weight j in the Axelrod model at time 0, but the evolution is now coupled with the cultural dynamics as follows:

(0) if $W(t) - W(t-) = 0$, we do nothing;

(1) if $W(t) - W(t-) = 1$, we move a ball from box j to box $j+1$ where (if it exists), box j is chosen uniformly at random from the set of nonempty inner boxes;

(2) if $W(t) - W(t-) = 2$, we repeat the same as in 1 above, and, in case a ball has indeed been moved and box 0 is nonempty, we move another ball from box 0 to box 1.

The game halts when the Axelrod model hits an absorbing state, that is, when all the edges have either weight zero or weight F. Let $\mathfrak{B}_j(t)$ denote the number of balls in box j at time t. The next lemma indicates that, at any time, the number of balls in box 0 is smaller than the number of edges with weight zero in the Axelrod model.

LEMMA 9. *For all* $t \geq 0$, *we have* $\mathfrak{B}_0(t) \leq W_0(t)$.

PROOF. The intuition behind the result is that a ball is removed from box 0 if and only if two agreements emerge simultaneously in the Axelrod model, whereas the latter is only a necessary condition for an edge with weight zero to change its weight. To make this argument rigorous, we introduce the key variable $\bar{B}(t)$ that represents the number of steps required after time t to move all the balls to box F, excluding the ones which are in box 0 at time t, along with its analog for the Axelrod model that we denote by $\bar{W}(t)$. More precisely, we introduce

$$\bar{B}(t) := \sum_{j=1}^{F} (F - j)\mathfrak{B}_j(t) \quad \text{and} \quad \bar{W}(t) := \sum_{j=1}^{F} (F - j) W_j(t).$$

Then, the idea is to prove by induction that, as long as box 0 is nonempty (note that once it is empty the result is trivial), we have

$$\mathfrak{B}_0(t) \le W_0(t) \quad \text{and} \quad \bar{B}(t) \ge \bar{W}(t).$$

The two inequalities to be proved are obviously true at time 0 since initially there are as many balls in box j as there are edges with weight j. Assume that they are true at time $t-$ and that a culture is updated at time t. Since

$$\bar{W}(t) = F \times \big(N - W_0(t)\big) - \sum_{j=1}^{F} j W_j(t) = F \times \big(N - W_0(t)\big) - W(t),$$

and a weight jumps from 0 to 1 at time t only if $W(t) - W(t-) = 2$, we have:

(1) Assume that $W(t) - W(t-) = 0$. Then

$$\begin{aligned} \mathfrak{B}_0(t) &= \mathfrak{B}_0(t-) \le W_0(t-) = W_0(t), \\ \bar{B}(t) &= \bar{B}(t-) \ge \bar{W}(t-) = F \times \big(N - W_0(t)\big) - W(t) = \bar{W}(t). \end{aligned}$$

(2) Assume that $W(t) - W(t-) = 1$. Then $\bar{W}(t-) > 0$ and so $\bar{B}(t-) > 0$ by assumption. In particular, one of the inner boxes is nonempty, which implies that a ball is indeed moved from some box j to box $j + 1$. It follows that

$$\begin{aligned} \mathfrak{B}_0(t) &= \mathfrak{B}_0(t-) \le W_0(t-) = W_0(t), \\ \bar{B}(t) &= \bar{B}(t-) - 1 \ge \bar{W}(t-) - 1 = F \times \big(N - W_0(t-)\big) - W(t-) - 1 \\ &= F \times \big(N - W_0(t)\big) - W(t) = \bar{W}(t). \end{aligned}$$

(3) Assume that $W(t) - W(t-) = 2$. In case box 0 is empty at time $t-$, the result is trivial. Otherwise, using as previously that $\bar{B}(t-) \ge \bar{W}(t-) > 0$, we obtain

$$\begin{aligned} \mathfrak{B}_0(t) &= \mathfrak{B}_0(t-) - 1 \le W_0(t-) - 1 \le W_0(t), \\ \bar{B}(t) &= \bar{B}(t-) - 1 + (F - 1) \ge \bar{W}(t-) - 1 + (F - 1) \\ &= F \times \big(N - W_0(t-)\big) - W(t) + F \\ &\ge F \times \big(N - W_0(t) - 1\big) - W(t) + F = \bar{W}(t). \end{aligned}$$

This completes the proof. $\square$

LEMMA 10. *For all $j = 0, 1, \ldots, F$, we have*

$$E(W_j(0)) = N p_j := N \binom{F}{j} q^{-j} (1 - q^{-1})^{F-j}.$$

PROOF. Since initially nearest neighbors agree on each of their features with probability q^{-1} and that all features are independent, the probability that a given edge is a j-edge is

$$P(X = j) \qquad \text{where } X \sim \text{Binomial}(F, q^{-1}).$$

Since the graph contains N edges, the result follows. $\square$

LEMMA 11. *Assume that $F < q$. Then*

$$N^{-1} \lim_{t \to \infty} E(W_0(t)) \geq \left(1 - \frac{1}{q}\right)^F + \frac{F}{q - F}\left(\left(1 - \frac{1}{q}\right)^F - \left(1 - \frac{1}{q}\right)\right).$$

PROOF. In view of Lemma 9, it suffices to prove the result for $\mathfrak{B}_0(t)$ instead of $W_0(t)$. First, we consider the discrete-time game introduced at the beginning of this section. To count the number of balls more easily, we divide the evolution of the latter into rounds as follows.

Round 1. We paint in black all the balls in box 0 and in white all the other balls and, at each time step, move a white ball from box j to box $j + 1$ where (if it exists) box j is chosen uniformly at random from the set of inner boxes that contain at least one white ball. In case a white ball has indeed been moved, and box 0 is nonempty, we move a black ball from box 0 to box 1 with probability $(q - 1)^{-1}$. The round halts when all the white balls are in box F.

Round 2. Note that, at the end of round 1, all the boxes are empty but boxes 0 and 1 that contain only black balls, and box F that contains only white balls. We paint in white all the balls in box 1 after which the game evolves as described in round 1.

Any other round is defined starting from the final configuration of the previous round in the same way as round 2 is defined starting from the final configuration of round 1, and the game halts when all the balls are either in box 0 or box F. We refer the reader to the right-hand side of Figure 2 for a schematic illustration of a single step evolution. Note that, letting $\mathfrak{A}_j(t)$ denote the number of balls in box j at step t for this game, it follows from Lemma 8 that

$$\lim_{t \to \infty} E(\mathfrak{A}_0(t)) \leq \lim_{t \to \infty} E(\mathfrak{B}_0(t)),$$

whenever

$$\mathfrak{A}_j(0) = \mathfrak{B}_j(0) \qquad \text{for } j = 0, 1, \ldots, F;$$

therefore it suffices to bound from below the limit on the left-hand side. Let T_k denote the time at which round k halts. Since $F - j$ steps are required to move a white ball from box j to box F, and all the white balls are either in box 1 or box F at the beginning of round $k \geq 2$,

$$T_1 = \sum_{j=1}^{F} (F - j)\mathfrak{A}_j(0) \quad \text{and} \quad T_{k+1} = T_k + (F - 1)\mathfrak{A}_1(T_k).$$

The expression of time T_1 together with Lemma 10 implies that

$$\begin{aligned} E(T_1) &= \sum_{j=1}^{F} (F-j) E(\mathfrak{A}_j(0)) = \sum_{j=1}^{F} (F-j) N p_j \\ &= \sum_{j=0}^{F} (F-j) N p_j - N F p_0 \\ &= N F (1 - p_0) - N \sum_{j=0}^{F} j p_j \\ &= N F \biggl(1 - \biggl(1 - \frac{1}{q}\biggr)^F - \frac{1}{q}\biggr). \end{aligned}$$

In other respects, in view of the expression of time T_{k+1}, and since at each step a black ball is moved from box 0 to box 1 with probability $(q-1)^{-1}$, we also have

$$E(\mathfrak{A}_1(T_{k+1})) = \biggl(\frac{F-1}{q-1}\biggr) E(\mathfrak{A}_1(T_k)) = \biggl(\frac{F-1}{q-1}\biggr)^k E(\mathfrak{A}_1(T_1)).$$

Combining the previous two equations, we obtain

$$\begin{aligned} E(\mathfrak{A}_1(T_{k+1})) &= \biggl(\frac{F-1}{q-1}\biggr)^k \frac{1}{q-1} E(T_1) \\ &= N \biggl(\frac{F}{q-1}\biggr) \biggl(\frac{F-1}{q-1}\biggr)^k \biggl(1 - \biggl(1 - \frac{1}{q}\biggr)^F - \frac{1}{q}\biggr). \end{aligned}$$

Finally, using again Lemma 10 and some basic algebra, we deduce that

$$\begin{aligned} &N^{-1} \lim_{t\to\infty} E(W_0(t)) \\ &\quad \geq N^{-1} \lim_{t\to\infty} E(\mathfrak{A}_0(t)) \geq N^{-1} \Biggl(E(\mathfrak{A}_0(0)) - \sum_{k=1}^{\infty} E(\mathfrak{A}_1(T_k))\Biggr) \\ &\quad \geq \biggl(1 - \frac{1}{q}\biggr)^F - \sum_{k=0}^{\infty} \biggl(\frac{F}{q-1}\biggr)\biggl(\frac{F-1}{q-1}\biggr)^k \biggl(1 - \biggl(1 - \frac{1}{q}\biggr)^F - \frac{1}{q}\biggr) \\ &\quad \geq \biggl(1 - \frac{1}{q}\biggr)^F + \frac{F}{q-F} \biggl(\biggl(1 - \frac{1}{q}\biggr)^F - \biggl(1 - \frac{1}{q}\biggr)\biggr). \end{aligned}$$

This completes the proof. $\square$

Theorem 2 directly follows from the combination of Lemmas 6 and 11.

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