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THE AXIOM OF SPHERES IN RIEMANNIAN GEOMETRY

DOMINIC S. LEUNG & KATSUMI NOMIZU

In his book on Riemannian geometry [1] Elie Cartan defined the axiom of r-planes as follows. A Riemannian manifold M of dimension $n \geq 3$ satisfies the *axiom of r-planes*, where r is a fixed integer $2 \leq r < n$, if for each point p of M and any r-dimensional subspace S of the tangent space $T_p(M)$ there exists an r-dimensional totally geodesic submanifold V containing p such that $T_p(V) = S$. He proved that if M satisfies the axiom of r-planes for some r, then M has constant sectional curvature [1, § 211].

We propose

Axiom of r-spheres. *For each point p of M and any r-dimensional subspace S of $T_p(M)$, there exists an r-dimensional umbilical submanifold V with parallel mean curvature vector field such that $p \in V$ and $T_p(V) = S$.*

We shall prove

Theorem. *If a Riemannian manifold M of dimension $n \geq 3$ satisfies the axiom of r-spheres for some r, $2 \leq r < n$, then M has constant sectional curvature.*

The special case where $r = n - 1$ was proved by J. A. Schouten (see [3, p. 180]). In this case the condition of parallel mean curvature vector field simply means constancy of the mean curvature.

1. Preliminaries

Let M be a Riemannian manifold of class C^∞, and let V be a submanifold. The Riemannian connections of M and V are denoted by ∇ and ∇', respectively, whereas the normal connection (in the normal bundle of V in M) is denoted by $\nabla^\perp$. The second fundamental form α is defined by

$$\nabla_X Y = \nabla'_X Y + \alpha(X, Y) ,$$

where X and Y are vector fields tangent to V. On the other hand, for any vector field ξ normal to V, the tensor field A_ξ of type (1,1) on V is given by

$$\nabla_X \xi = -A_\xi(X) + \nabla^\perp_X \xi ,$$

where X is a vector field tangent to V. We have

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$$g(\alpha(X, Y), \xi) = g(A_\xi X, Y)$$

for X and Y tangent to V and ξ normal to V, where g is the Riemannian metric on M.

Among the fundamental facts we recall the following equation of Codazzi (which is essentially equivalent to that given in [2, p. 25]):

(*) *For X and Y tangent to V and ξ normal to V, the tangential component of $R(X, Y)\xi$ is equal to*

$$(\nabla'_Y A_\xi)(X) - (\nabla'_X A_\xi)(Y) + A_{\nabla_X^\perp \xi}(Y) - A_{\nabla_Y^\perp \xi}(X) .$$

Here R is the curvature tensor of M.

The mean curvature vector field η of V in M is defined by the relation

$$\operatorname{trace} A_\xi / r = g(\xi, \eta) \qquad \text{for all } \xi \text{ normal to } V ,$$

where $r = \dim V$. We say that η is *parallel* (with respect to the normal connection) if $\nabla^\perp \eta = 0$.

We say that V is *umbilical* in M if

$$\alpha(X, Y) = g(X, Y)\eta \qquad \text{for all } X \text{ and } Y \text{ tangent to } V .$$

Equivalently, V is umbilical in M if

$$A_\xi = g(\xi, \eta)I \qquad \text{for all } \xi \text{ normal to } V ,$$

where I is the identity transformation. An umbilical submanifold is totally geodesic if and only if η vanishes on V.

A word of explanation may be in order. If M is a space of constant sectional curvature, then an umbilical submanifold V has parallel mean curvature vector field. V is also contained in a totally geodesic submanifold of M of one higher dimension. When M is one of the standard models of spaces of constant sectional curvature, that is, R^n, S^n and H^n, one can thus determine all connected, complete umbilical submanifolds.

2. Proof of theorem

To prove that M has constant sectional curvature we use

Lemma [1, § 212]. *If $g(R(X, Y)Z, X) = 0$ whenever X, Y and Z are three orthonormal tangent vectors of M, then M has constant sectional curvature.*

For the sake of completeness we give a simple proof of this lemma. For X, Y, and Z orthonormal, let

$$Y' = (Y + Z)/\sqrt{2} \quad \text{and} \quad Z' = (Y - Z)/\sqrt{2} .$$

Since X, Y' and Z' are again orthonormal, we have

$$g(R(X, Y')Z', X) = 0 ,$$

from which we get

$$g(R(X, Y)Y, X) = g(R(X, Z)Z, X) .$$

This means that the sectional cnrvature for the 2-plane $X \wedge Y$ is equal to that of the 2-plane $X \wedge Z$. It is easily seen that all the 2-planes (at each point) have the same sectional curvature. By Schur's theorem, M is a space of constant sectional curvature ($\dim M \geq 3$).

Now, in order to prove the theorem, let X, Y and Z be three orthonormal vectors in $T_p(M)$, where p is an arbitrary point of M, and let S be an r-dimensional subspace of $T_p(M)$ containing X and Y and normal to Z. By the axiom there exists an r-dimensional umbilical submanifold V with parallel mean curvature vector field η such that $p \in V$ and $T_p(V) = S$. Let U be a normal neighborhood of p in V. For each point $q \in U$, let ξ_q be the normal vector at q to V which is parallel to Z with respect to the normal connection $\nabla^{\perp}$ along the geodesic from p to q in U. Along each geodesic we have $g(\xi, \eta) =$ constant, say, λ, so that $A_\xi = \lambda I$ at every point of U. Thus

$$\nabla'_X A_\xi = \nabla'_Y A_\xi = 0 \qquad \text{at } p .$$

We have also

$$\nabla^{\perp}_X \xi = \nabla^{\perp}_Y \xi = 0 \qquad \text{at } p .$$

Now the equation of Codazzi (*) implies that the tangential component (namely, the S-component) of $R(X, Y)Z$ is 0. In particular, $g(R(X, Y)Z, X) = 0$. By the lemma we conclude that M has constant sectional curvature.

We wish to conclude with the following remark. If we drop in the axiom of spheres the requirement that V has parallel mean curvature vector field, then this weaker axiom for $n \geq 4$ and $r = n - 1$ implies that M is conformally flat (see [3, p. 180]). It is easy to extend this result to the case $3 \leq r < n$.

References

[1] E. Cartan, *Leçons sur la géométrie des espaces de Riemann,* Gauthier-Villars, Paris, 1946.
[2] S. Kobayashi & K. Nomizu, *Foundations of differential geometry,* Vol. II, Wiley-Interscience, New York, 1969.
[3] J. A. Schouten, *Der Ricci-Kalkül,* Springer, Berlin, 1924.

BROWN UNIVERSITY