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THE AXISYMMETRIC BRANCHING BEHAVIOR OF COMPLETE SPHERICAL SHELLS*

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Abstract. The purpose of this paper is to describe the axisymmetric branching behavior of complete spherical shells subjected to external pressure. By means of an asymptotic integration technique (based on the smallness of the ratio of the shell thickness to the shell radius) applied directly to a differential equation formulation, we are able to continue the solution branches from the immediate vicinity of the bifurcation points, where the solution has the functional form predicted by the linear buckling theory, to the region where the solution consists of either one or two "dimples" with the remainder of the shell remaining nearly spherical. The analysis deals with a novel aspect of bifurcation theory involving "closely spaced" eigenvalues.

1. Introduction. The field of solid mechanics is a rich source of nonlinear stability problems, one of the most important of these being the buckling of a complete spherical shell under uniform pressure. The classical or linear buckling theory yields an eigenvalue problem with the eigenvalue parameter proportional to the pressure. The spectrum is discrete and the lowest eigenvalue is usually called the buckling load. Several authors have analyzed the axisymmetric branching of solutions of a nonlinear theory for loads near the eigenvalues of the linearized theory [1]. Recently it has been shown rigorously that the standard perturbation expansion in powers of α (where α measures the relative deviation of the loading parameter from the branch or bifurcation point) is indeed asymptotic [2]. However, Koiter [3] made the very important discovery that this expansion has an extremely small region of validity ($\alpha = o(\delta)$ as $\delta \to 0$ where the parameter δ measures the ratio of the thickness to the radius of the undeformed shell).

Now it is well known that an initial postbuckling analysis can provide useful information regarding the imperfection-sensitivity of an elastic structure. The fact that the standard expansion breaks down outside the immediate vicinity of the branch point means that it is of limited value in assessing the imperfection-sensitivity of complete spherical shells. Thus it is important for practical as well as theoretical reasons to have a means of extending the branching solutions beyond the region of validity of the standard perturbation expansions. To this end Koiter [3] performed a refined asymptotic analysis based on

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the general theory of [4]. However, because of very serious analytical complications associated with expressing the solution in terms of a series of the linear eigenfunctions, Koiter was forced to assume a restrictive functional form for the leading order behavior of the branching solution. As a consequence, it does not appear that his refined analysis provides the desired extension. The asymptotic solutions which he constructs do not agree with the deflection patterns obtained by numerical calculations [5]. In the present work, we develop a technique which allows us to handle this singular branching problem.

In [5] Bauer, Reiss and Keller used a shooting technique to compute numerically the solution branches for complete spherical shells. By means of this very powerful procedure they were able to continue the solution branches for large deformations. It is of particular mathematical interest that the nontrivial solution branches are all connected. However, it should be pointed out that the numerical scheme employed in [5] becomes less efficient for small values of δ and that the results presented in [5] are for fairly thick shells. On the other hand, the analytical procedure which we develop is based on the asymptotic limit of $\delta \to 0$; so, in this sense, our approach complements that presented in [5].

To be more precise, the aim of the present work is to establish the axisymmetric branching behavior of complete spherical shells under uniform pressure by means of an asymptotic integration technique applied directly to a differential equation formulation. We take as the governing equations the set of equations derived by Reissner [6] for the axisymmetrical deformations of shells of revolution. By means of our asymptotic integration technique, we are able to follow the solution branches from the immediate neighborhood of the branch point where the solution has the functional form predicted by the linear buckling theory (a large number of waves directed inward and outward and extending over the whole shell surface) to the advanced state in which the buckled shape becomes primarily an inward dimpling at the poles of the shell. The latter deflection state is the one usually observed in experiments [7] and it has served as the starting point for a number of investigations (see [3] for a review). Moreover, we are able to show the existence of stable upward-branching solutions (see Sec. 2 for a definition) from odd eigenvalues for all sufficiently small values of δ. The latter result has special physical significance as it indicates that the shell can deform continuously into a nonspherical shape without jumping. Such solutions were observed to occur for each of the three values of δ for which results were computed in [5].

The singular perturbation technique which we use to solve this particular buckling problem can be applied to a broad class of related problems in elasticity theory. Moreover, our approach deals with certain novel aspects of bifurcation theory. In this problem, the complicated dependence of the branching behavior on the parameter δ could perhaps be anticipated; for although the spectrum of the linear eigenvalue problem associated with the linear buckling theory is discrete, the spacing of the eigenvalues is proportional to δ. Hence, one can expect a strong nonlinear coupling between the eigenfunctions when $\alpha = O(\delta)$.

This feature of "closely spaced" eigenvalues which is responsible for the limited validity of standard perturbation or iteration procedures has certain beneficial aspects. It actually allows us to obtain more information concerning the structure of the branching solution than is usually possible by means of constructive procedures. For further discussion of this point the reader may wish to refer to [8] where a simpler mathematical model is treated. The additional mathematical feature which distinguishes the present problem from that in [8] is that the ordinary differential operator defining the two-point boundary value problem for the shell problem has regular singular points at the boundary points.

The layout of this paper is as follows. In Sec. 2, we introduce the basic equations and briefly discuss the bifurcation results obtained in [2, 3]. Sec. 3 is devoted to a discussion of the proposed asymptotic integration technique. In Secs. 4 and 5 we apply this technique to the branching problem for simple eigenvalues. In Sec. 6, we provide a brief summary of our bifurcation results, including a comparison with the results of Koiter's analysis. Numerical evidence is presented which supports our conclusions.

2. Formulation. The basic equations governing the finite symmetrical deflections of thin shells of revolution under the assumption of small strain were formulated by Reissner in [6]. Following [6], the middle surface of the undeformed spherical shell of radius a and thickness h is represented in cylindrical coordinates (r, θ, z) in the parametric form $r = r_0(\xi) = a \sin \xi$, $z = z_0(\xi) = -a \cos \xi$, where ξ denotes the polar angle measured from the south pole and is in the range $0 \leq \xi \leq \pi$. Assuming axisymmetric deformations with the z-axis as axis of symmetry, Reissner formulated the problem of finite deflections in terms of a coupled pair of integro-differential equations relating to each other a basic deformation variable and a basic stress variable.

For the undeformed sphere the parameter ξ measures the angle between the radial (horizontal) direction and the ray tangent to the meridian of the middle surface in the direction of increasing ξ at any point with polar angle ξ. The corresponding angle, $\phi = \phi(\xi)$, at the displaced points (which were originally specified by ξ) on the deformed middle surface serves as the basic deformation variable. The basic stress variable $\Psi = \Psi(\xi)$ is defined by

$$\Psi = r_0 H = a \sin \xi \, H,$$

where H is the horizontal (radial) stress resultant for the deformed shell.

In this work we shall consider only a uniform (external) compressive loading. Such a loading is often interpreted in two ways: either as a pressure loading or as a centrally directed (dead) loading. (Following [3, 5] we shall assume that for the initial postbuckling problem, the governing equations are the same for the two types of loading.) If we let the parameter ρ measure the magnitude of the loading, it is easily verified that the uniformly contracted (membrane) state

$$\phi = \xi, \; \Psi = -\frac{\rho a^2}{4} \sin 2\xi$$

is a solution of Reissner's equations for all $\rho > 0$. Our interest is in studying the branching of the shell into nonspherical shapes. To facilitate the analysis we introduce new basic variables β, ψ defined by

$$\beta = \xi - \phi, \; \Psi = \frac{a^2 \rho_{cr}}{4} [-\lambda \sin 2\xi + \psi], \tag{2.1}$$

where we have set

$$\rho = \lambda \rho_{cr}, \; \rho_{cr} = 4E\delta^2 \sqrt{12(1-\nu^2)}, \; \delta = \frac{1}{\sqrt{12(1-\nu^2)}} \frac{h}{a}, \tag{2.2}$$

and where ν is Poisson's ratio and E is Young's modulus. The situation where the dimensionless load parameter $\lambda = 1$ corresponds to the classical critical pressure.

Although Reissner's equations allow for arbitrarily large deflections subject only to the condition of small strain, it turns out that it is necessary to retain only quadratic nonlinear terms involving β and ψ for our consideration of the initial branching problem. (In the case of large deflections the fully nonlinear equations should be used; see [9] for a boundary layer analysis.) The equations for the nonlinear eigenvalue problem are

$$\beta'' + \beta' \cot \xi + \left(\frac{2\lambda}{\delta} - \underline{\nu} - \cot^2 \xi\right)\beta + \frac{1}{\delta}\psi = \frac{\beta\psi}{\delta}\cot \xi,$$

$$\psi'' + \psi' \cot \xi + (\underline{\nu} - \cot^2 \xi)\psi - \frac{1}{\delta}\beta = -\frac{\beta^2}{2\delta}\cot \xi, \tag{2.3}$$

for $0 < \xi < \pi$, with boundary conditions

$$\beta = \psi = 0 \quad \text{at} \quad \xi = 0, \pi. \tag{2.4}$$

In (2.3) primes denote differentiation with respect to ξ. The system (2.3)–(2.4) is essentially the same as that considered in [3, 5].

The linear buckling theory is concerned with infinitesimal deviations from the uniformly contracted state. The corresponding linear eigenvalue problem follows upon setting the terms on right-hand side of (2.3) to zero. To simplify the analysis, we note that it can readily be shown that omission of the underlined terms in (2.3) results in a relative error of at most $O(\delta)$ as $\delta \to 0$, which is the same error implicit in the shell equations themselves. Upon setting the right-hand side of (2.3) to zero and omitting the underlined terms, we find that the resulting linear eigenvalue problem has non-zero solutions if and only if

$$\lambda = \lambda_n(\delta) \equiv \frac{\delta\mu_n}{2} + \frac{1}{2\delta\mu_n}, \qquad n = 1, 2, \ldots, \tag{2.5}$$

where

$$\mu_n \equiv n(n+1) - 1. \tag{2.6}$$

The corresponding eigenfunctions are

$$\beta(\xi) = A_n P_n^1(\cos \xi), \qquad \psi(\xi) = -\frac{A_n}{\mu_n \delta} P_n^1(\cos \xi), \qquad n = 1, 2, \ldots, \tag{2.7}$$

where $A_n \neq 0$ are arbitrary constants and $P_n^1(\cos \xi)$ is the associated Legendre function of the first kind of degree n and order 1. If n is even, the corresponding eigenfunction is symmetric with respect to $\xi = \pi/2$ and it follows from the definition of β and ξ that the associated deflection shape of the shell takes the form of either both inward or outward (depending on the sign of A_n) dimples at the poles with oscillations (ripples) in between. On the other hand, if n is odd, the corresponding eigenfunction is antisymmetric and the associated deflection shape has the form of an inward dimple at one pole and an outward bulge at the other pole with ripples in between. Furthermore, consideration of (2.5) reveals that it is possible to have multiple eigenvalues, i.e.,

$$\lambda_n = \lambda_m \quad \text{for} \quad n \neq m.$$

The lowest eigenvalue has the most physical relevance. Since we are assuming that $0 < \delta \ll 1$, the minimum value of λ_n in (2.5) will clearly correspond to a large value of μ_n, so that it proves advantageous to treat μ_n and n as continuous variables. Setting $\partial\lambda/\partial\mu_n = 0$ to find the "critical" value of λ, we thus obtain

$$\lambda_{cr} = 1$$

for

$$\mu_n = \mu_{cr} \equiv 1/\delta. \tag{2.8}$$

We interpret this result as follows. For a fixed δ, the lowest eigenvalue is greater than or equal to λ_{cr}. (Note that it is possible to choose a monotone decreasing sequence $\{\delta_m\}_{m=1}^{\infty}$, $\delta_m \to 0$, such that for each member of the sequence the corresponding lowest eigenvalue for the shell problem is $\lambda_m = 1$; namely,

$$\delta_m = \frac{1}{\mu_m} = \frac{1}{m(m+1)-1}, \qquad m = 1, 2, \dots .)$$

By using (2.5) it is possible to provide a closer relationship between λ_{cr} and the lowest eigenvalue. In fact, upon solving (2.6) for n as a function of μ_n and setting

$$n_{cr} = \frac{-1 + \sqrt{5 + 4\mu_{cr}}}{2},$$

it follows from (2.5) that

$$\lambda_n - 1 = O(\delta) \quad \text{as} \quad \delta \to 0,$$

for $n = n_{cr} + O(1)$. As a corollary we have the result that for two eigenvalues λ_n and λ_m with $n = n_{cr} + O(1)$ and $m = n_{cr} + O(1)$ their difference satisfies

$$\lambda_m - \lambda_n = O(\delta) \quad \text{as} \quad \delta \to 0.$$

This property of the spacing of the eigenvalues tending to zero as $\delta \to 0$ is common in shell stability problems and is responsible for the limited applicability of standard constructive procedures for determining the branching behavior of many important shells. The primary purpose of the remainder of this paper is to illustrate a technique for treating such singular bifurcation problems.

Having briefly discussed the classical linear buckling theory and demonstrated the feature of closely spaced eigenvalues, we turn now to the nonlinear problem. Before proceeding to analyze the system (2.3)–(2.4) a few words are in order regarding terminology. A (buckled) solution of the nonlinear eigenvalue problem represented by $\beta(\xi, \lambda)$, $\psi(\xi, \lambda)$ is said to branch from the eigenvalue λ_n if on some sufficiently small closed interval containing λ_n as an endpoint $\beta(\xi, \lambda)$, $\psi(\xi, \lambda)$ depend continuously on λ with $\beta(\xi, \lambda_n) \equiv 0$, $\psi(\xi, \lambda_n) \equiv 0$ and $\beta(\xi, \lambda) \not\equiv 0$, $\psi(\xi, \lambda) \not\equiv 0$ for $\lambda \neq \lambda_n$. If λ_n is a left (right) endpoint, we say that solutions bifurcate or branch up (down) from λ_n.

To emphasize that we are concerned with deducing the initial branching behavior, we write

$$\lambda = \lambda_n(1 + \alpha\chi), \qquad 0 < \alpha \ll 1, \qquad \chi = O(1), \tag{2.9}$$

where λ_n is an eigenvalue for the linear buckling problem defined in (2.5). At this point we introduce a change of variables suggested by (2.7) which will simplify the analysis, namely

$$y = \psi + (\beta/\mu_n\delta). \tag{2.10}$$

Upon omitting the underlined terms in (2.3), multiplying the first equation by $1/\mu_n\delta$ and adding the resulting equation to the second equation in (2.3), we obtain the system which forms the basis for what follows:

$$\beta'' + \beta' \cot \xi + (\mu_n - \cot^2 \xi)\beta + \frac{y}{\delta} = -\frac{2\alpha\chi}{\delta}\lambda_n\beta + \frac{\beta \cot \xi}{\delta}\left(y - \frac{\beta}{\mu_n\delta}\right),$$

$$y'' + y' \cot \xi + \left(\frac{1}{\mu_n\delta^2} - \cot^2 \xi\right) y = -\frac{2\alpha\chi}{\mu_n\delta^2}\lambda_n\beta - \frac{\beta^2 \cot \xi}{2\delta}\left(1 + \frac{2}{\mu_n^2\delta^2}\right) + \frac{\beta y \cot \xi}{\mu_n\delta^2} \tag{2.11}$$

for $0 < \xi < \pi$, with boundary conditions

$$\beta = y = 0 \quad \text{at} \quad \xi = 0, \pi. \tag{2.12}$$

The initial branching behavior of solutions to the nonlinear eigenvalue problem (2.11)–(2.12) can be deduced by either standard perturbation or iteration methods, and it can be shown rigorously that the resulting expansions actually are asymptotic approximations to the solutions [2]. However, these expansions involve δ as a parameter and their region of validity tends to zero as δ tends to zero. Since the reader can find a detailed account of the regular perturbation and iteration results in [2, 3], we shall be content to simply sketch those results which will be of use in the sequel.

In order to simplify the presentation we shall restrict our attention to the special case in which bifurcation occurs at the critical value

$$\lambda_n = \lambda_{cr} = 1,$$

so that $\delta = 1/\mu_n$. Suppose first that n is an even integer. Then it is not difficult to show that the initial branching behavior is given by

$$\beta \sim \alpha\beta_0 + \alpha^2\beta_1 + \alpha^3\beta_2 + \cdots, \qquad y \sim \alpha^2 y_1 + \alpha^3 y_2 + \cdots, \tag{2.13}$$

for $\alpha \to 0$. The first few terms for β are

$$\beta_0 = A_0 P_n^1(\cos \xi),$$

$$\beta_1 = A_1 + \sum_{\substack{k=1 \\ k \neq n/2}}^{n} C_k P_{2k}^1(\cos \xi), \tag{2.14}$$

where A_0 A_1, C_k are constants (which depend on n).

Now the limit $\delta \to 0$ corresponds to $n \to \infty$, since $n = (1/\sqrt{\delta}) + O(1)$ from (2.6) and (2.8). From [10] we have that

$$P_n^1(\cos \xi) = \left(\frac{2n}{\pi \sin \xi}\right)^{1/2} \cos\left[(n + \tfrac{1}{2})\xi + \frac{\pi}{4}\right] + O\left(\frac{1}{n^{1/2}}\right) \quad \text{as} \quad n \to \infty. \tag{2.15}$$

This expansion is uniformly valid on any interval $[\xi_0, \xi_\pi]$ where $0 < \xi_0 < \xi_\pi < \pi$; other expansions apply in small neighborhoods of $\xi = 0$ and $\xi = \pi$. Hopson derived (2.15) from an integral representation. It turns out that one can derive the same expansion by applying the two-variable expansion procedure (multiple scaling) discussed, for example, in [11]. From (2.15) it is clear that the two variables are ξ and $\bar{\xi} = (n + \frac{1}{2})\xi$.

After considerable algebra and the use of (2.15) one finds that

$$\beta_0 = \chi\left(\frac{32\pi}{27n\sin\xi}\right)^{1/2}\cos\left[(n+\tfrac{1}{2})\xi+\frac{\pi}{4}\right]+O\left(\frac{1}{n^{3/2}}\right), \tag{2.16}$$

sufficiently far from $\xi = 0$ and $\xi = \pi$. The dependence on δ is apparent. In order to estimate the region of validity of (2.13) it is useful to compare the size of β_1 with that of β_0. The largest contribution in the expression for β_1 in (2.14) arises from terms in the sum for $k = (n/2) + O(1)$. After a careful asymptotic evaluation one finds that

$$\beta_1 = O(\beta_0/\delta) \quad \text{as} \quad \delta \to 0, \tag{2.17}$$

uniformly on $0 \le \xi \le \pi$. Comparing with (2.13), this result strongly suggests that the region of validity of the regular perturbation expansion is

$$\alpha = o(\delta) \quad \text{as} \quad \delta \to 0. \tag{2.18}$$

Further considerations reveal that this is indeed the case.

When n is an odd integer the computations become more complicated. For this case it can be shown that the appropriate expansions are

$$\beta = \alpha^{1/2}\beta_0 + \alpha\beta_1 + \alpha^{3/2}\beta_2 + \cdots, \qquad y = \alpha y_1 + \alpha^{3/2} y_2 + \cdots, \tag{2.19}$$

with β_0 as given in (2.14). It turns out that

$$\beta_0 = O(\delta^{3/4}) \quad \text{and} \quad \beta_1 = O(\beta_0/\sqrt{\delta}) \quad \text{as} \quad \delta \to 0.$$

Thus, the expansions (2.19) have the same limited region of validity as for even values of n. Moreover, it is straightforward to show that analogous results hold for $n = n_{cr} + O(1)$.

We conclude this section with the observation that results similar to the above have been derived previously by Koiter [3] by means of a variational approach. In particular, Koiter discovered the extremely small region of validity of the standard perturbation expansions for both even and odd values of n represented by (2.18). In order to extend the branching solution beyond the region $\alpha = o(\delta)$ Koiter performed a refined asymptotic analysis based on the general theory of [4]. However, because of very serious analytical complications associated with expressing the assumed form of the solution in terms of a series of the linear eigenfunctions, he again took the solution to leading order in α to be a constant multiple of the $P_n^1(\cos\xi)$ corresponding to the lowest eigenvalue. Since he had already established that this particular functional form was inadequate for describing the structure of the solution for smaller values of α, it is questionable as to how accurate one can expect the refined analysis to be. It would seem that a more fruitful approach would be to leave the structure of the leading-order term to be determined as part of the analysis. Our method allows for this flexibility and we find, as is shown in the sequel, that this structure changes as the deviation from the branch point increases.

3. The asymptotic integration technique. The results of Sec. 2 suggest that one must seek alternative procedures to the standard perturbation and iteration methods in order to gain significant information concerning the buckling behavior of thin spherical shells. In this section we describe such a procedure which takes into account *ab initio* the fact that the branching problem depends on two small parameters, namely δ, which essentially measures the ratio of the shell thickness to the radius of the undeformed shell, and α (cf. (2.9)), which measures the deviation of the loading parameter λ from a particular eigenvalue λ_n. The crucial dependence of the branching behavior on the relationship between these two parameters is brought out by the results of Sec. 2.

Ideally one would like to have asymptotic expansions for the branching solution in the limit $\alpha \to 0$ which are uniformly valid on an interval $0 \le \delta \le \delta_0$ for some fixed δ_0. However, this does not appear to be feasible; so instead we adopt an analytical procedure which is based on the construction of asymptotic approximations to the branching solution for a fixed (order) relationship between α and δ in the limit as $\delta \to 0$. While the basic idea is quite simple, perhaps it will prove useful briefly to consider an example which will also serve to illustrate certain other features of our analysis.

Let the function u be defined by

$$u(\alpha, \delta) = \sqrt{1 + \alpha/\delta}\left(1 + \frac{\alpha^2}{\delta}\right)^{-1}, \qquad \alpha \ge 0, \qquad \delta > 0. \tag{3.1}$$

Suppose we are interested in the asymptotic behavior of u for $\alpha \to 0$ with δ small but fixed. Then, upon expanding (3.1) in a power series in α we obtain the regular perturbation expansion

$$u(\alpha, \delta) \sim 1 + \frac{\alpha}{2\delta} - \alpha^2\left(\frac{1}{8\delta^2} + \frac{1}{\delta}\right) + \cdots \quad \text{as} \quad \alpha \to 0. \tag{3.2}$$

So long as $\alpha \ll \delta$, the first few terms in (3.2) serve as a good approximation to u. However, if we desire an approximation to u for larger values of α, say $\delta \ll \alpha \ll 1$, then (3.2) clearly will be of no use. We must derive new expansions which account for the nonuniformity. This situation is similar to that discussed in Sec. 2 for the function β.

It is a straightforward matter to show how our limit procedure applies to this example. For instance, suppose we choose the order relationship $\alpha = \chi\delta^{3/2}$, $\chi = O(1)$ as $\delta \to 0$. Then substituting into (3.1) and expanding for small δ we obtain

$$u(\chi\delta^{3/2}, \delta) \sim 1 + \frac{\chi\delta^{1/2}}{2} - \frac{\chi^2\delta}{8} - \chi\delta^2 + \cdots \quad \text{as} \quad \delta \to 0. \tag{3.3}$$

If we replace χ by $\alpha/\delta^{3/2}$ and rearrange (3.3) on the basis of $\alpha \to 0$ with δ fixed, we clearly recover (3.2) (the rearrangement requires only a finite number of transpositions for each power of α). In Secs. 4 and 5 our study of the branching problem will correspond to constructing expansions for the function u for the order relationships $\alpha = O(\delta)$ and $\alpha = O(\delta^{1/2})$, respectively.

While it is obvious what one must do in this simple example, the branching problem is quite another matter. As the mathematical details are fairly complicated, we shall restrict our attention in this section to the special case of branching from the lowest eigenvalue which we take to coincide with the critical value and to be even. (This case corresponds to a study of the branching for a sequence of shells with $\delta = \delta_m = 1/\mu_m$, $m = 2, 4, \ldots$.) Moreover, we fix the order relationship between α and δ to be $\alpha = O(\delta^{3/2})$, as our intention is to reproduce the results obtained by the regular perturbation method as a test of our necessarily formal procedure.

Guided by the form of the solution to the linear buckling problem, we make the following change of variables:

$$f = \beta\sqrt{\sin \xi}, \qquad g = y\sqrt{\sin \xi}, \qquad \bar{\xi} = \xi/\varepsilon \tag{3.4}$$

with

$$\varepsilon = \frac{1}{n+\frac{1}{2}}, \qquad \delta = \frac{1}{\mu_n} = \varepsilon^2\left(1 - \frac{5\varepsilon^2}{4}\right), \qquad \alpha = \varepsilon^3, \qquad \lambda = 1 + \chi\varepsilon^3. \tag{3.5}$$

In terms of these variables the system (2.11)–(2.12) becomes

$$\begin{aligned} \ddot{f} + \left(1 - \frac{3\varepsilon^2}{4}\csc^2\xi\right)f &= \left(1 - \frac{5\varepsilon^2}{4}\right)\left[-g - 2\varepsilon^3\chi f + f(f-g)\frac{\cot\xi}{\sqrt{\sin\xi}}\right], \\ \ddot{g} + \left(1 - \frac{3\varepsilon^2}{4}\csc^2\xi\right)g &= -\left(1 - \frac{5\varepsilon^2}{4}\right)\left[2\varepsilon^2\chi f + f(\tfrac{3}{2}f - g)\frac{\cot\xi}{\sqrt{\sin\xi}}\right], \end{aligned} \tag{3.6}$$

with the boundary conditions

$$f = g = 0 \quad \text{at} \quad \bar{\xi} = 0, \qquad \pi/\varepsilon. \tag{3.7}$$

In (3.6) dots denote differentiation with respect to $\bar{\xi}$.

Our aim is to construct an asymptotic approximation to the solution of the system (3.6)–(3.7) in the limit as $\varepsilon \to 0$ (equivalently $\delta \to 0$). Since the linear operator on the left-hand side of (3.6) has regular singular points at $\xi = 0$ and $\xi = \pi$, it proves convenient to express the solution in terms of three different expansions: (1) an "inner" expansion valid near $\xi = 0$, (2) an "outer" expansion valid away from $\xi = 0$ and $\xi = \pi$ and (3) an "inner" expansion valid near $\xi = \pi$. Matching of these expansions determines the arbitrary constants which each contains.

First, we construct the first few terms in the outer expansion. The form of (3.6) suggests the appropriateness of the method of multiple scales [11, 12]. This method has been employed successfully in the treatment of other shell buckling problems [13, 14]. It turns out that we require only two variables for this problem (namely, ξ and $\bar{\xi}$), whereas the extension to the case of larger values of $|\lambda - \lambda_n|$ considered in Sec. 5 involves three variables. According to the method of multiple scales, the derivative with respect to $\bar{\xi}$ is transformed as follows:

$$\frac{d}{d\bar{\xi}} = \frac{\partial}{\partial\bar{\xi}} + \varepsilon\frac{\partial}{\partial\xi}.$$

Thus, (3.6) transforms to

$$\begin{aligned} \ddot{f} + \left(1 - \frac{3\varepsilon^2}{4}\csc^2\xi\right)f &= -2\varepsilon\dot{f}' - \varepsilon^2 f'' + \left(1 - \frac{5\varepsilon^2}{4}\right) \\ &\quad \cdot\left[-g - 2\varepsilon^3\chi f + f(f-g)\frac{\cot\xi}{\sqrt{\sin\xi}}\right], \\ \ddot{g} + \left(1 - \frac{3\varepsilon^2}{4}\csc^2\xi\right)g &= -2\varepsilon\dot{g}' - \varepsilon^2 g'' - \left(1 - \frac{5\varepsilon^2}{4}\right) \\ &\quad \cdot\left[2\varepsilon^3\chi f + f(\tfrac{3}{2}f - g)\frac{\cot\xi}{\sqrt{\sin\xi}}\right]. \end{aligned} \tag{3.8}$$

In (3.8) dots and primes denote partial differentiation with respect to $\bar{\xi}$ and ξ, respectively.

The presence of the regular singular points at $\bar{\xi} = 0$, π/ε complicates the determination of the form of the asymptotic expansions for f and g. Actually, it can be shown that the change of variables in (2.10) and (3.4) ensures that $g = o(f)$ as $\varepsilon \to 0$ provided $\lambda - \lambda_n = o(1)$. As a consequence, once we determine the order of magnitude of f, the order of magnitude of g follows from the necessity of being able to match with the inner expansions. While we could allow the order of magnitude of f to be arbitrary at the outset and then establish the correct asymptotic form as part of the analysis, it proves convenient to use the results from the regular perturbation analysis. From (2.9), (2.13), (2.16), (3.4) and (3.5) it follows that for n even with $\alpha = \chi\varepsilon^3$ we have

$$f = O(\alpha\beta_0) = O(\varepsilon^3\chi/\sqrt{n}) = O(\varepsilon^{7/2}\chi) \quad \text{as} \quad \varepsilon \to 0. \tag{3.9}$$

Starting with (3.9) we readily find that the outer expansions have the form

$$\begin{aligned} f &\sim \varepsilon^{7/2}f_0 + \varepsilon^{9/2}f_1 + \varepsilon^{11/2}\log\varepsilon\, f_2 + \varepsilon^{11/2}f_3 + \varepsilon^{13/2}\log\varepsilon\, f_4 + \varepsilon^{13/2}f_5 + \cdots, \\ g &\sim \varepsilon^{11/2}g_3 + \varepsilon^{13/2}\log\varepsilon\, g_4 + \varepsilon^{13/2}g_5 + \cdots, \end{aligned} \tag{3.10}$$

where the need for the terms involving $\log \varepsilon$ will become apparent. As noted above, the size of g is dictated by the form of the inner equations. On substituting (3.10) into (3.8) and equating the coefficients of the various terms in the asymptotic sequence appearing in (3.8) to zero, we obtain the sequence of problems:

$$\begin{array}{ll} O(\varepsilon^{7/2}): & \ddot{f}_0 + f_0 = 0 \\ O(\varepsilon^{9/2}): & \ddot{f}_1 + f_1 = -2\dot{f}'_0 \\ O(\varepsilon^{11/2}\log\varepsilon): & \ddot{f}_2 + f_2 = 0 \\ O(\varepsilon^{11/2}): & \ddot{g}_3 + g_3 = 0 \\ & \ddot{f}_3 + f_3 = -2\dot{f}'_1 - f''_0 + \tfrac{3}{4}f_0 \csc^2\xi - g_0 \\ O(\varepsilon^{13/2}\log\varepsilon): & \ddot{f}_4 + f_4 = -2\dot{f}'_2 \\ & \ddot{g}_4 + g_4 = 0 \\ O(\varepsilon^{13/2}): & \ddot{f}_5 + f_5 = -2\dot{f}'_3 - f''_1 + \tfrac{3}{4}f_1 \cos^2\xi - g_3 - 2\chi f_0 \\ \vdots & \ddot{g}_5 + g_5 = -2\dot{g}'_3 - 2\chi f_0. \end{array} \tag{3.11}$$

From the first equation in (3.11) it follows that

$$f_0(\xi, \bar{\xi}) = A_0(\xi)e^{i\bar{\xi}} + (*), \tag{3.12}$$

where A_0 is an arbitrary function and (*) denotes the complex conjugate. With this the second equation in (3.11) becomes

$$\ddot{f}_1 + f_1 = -2iA'_0 e^{i\bar{\xi}} + (*). \tag{3.13}$$

Because we require that the outer expansions be uniform except near $\bar{\xi} = 0$ and $\bar{\xi} = \pi/\varepsilon$ we cannot allow f_1 to contain "secular" terms of the form $\bar{\xi}e^{i\bar{\xi}}$ and $\bar{\xi}e^{-i\bar{\xi}}$. Thus, (3.13) implies that

$$A_0 = a_0(\text{constant}), \qquad f_1 = A_1(\xi)e^{i\bar{\xi}} + (*). \tag{3.14}$$

As the procedure should now be clear, we shall be content to simply quote the results that we need:

$$f = \varepsilon^{7/2}[a_0 e^{i\xi} + (*)] + \varepsilon^{9/2}\left[\left(a_1 + \frac{3ia_0}{8}\cot\xi + \frac{c_2\xi i}{2} - \frac{\chi a_0}{4}\xi^2\right)e^{i\xi} + (*)\right] + O(\varepsilon^{11/2}\log\varepsilon),$$
$$g = \varepsilon^{11/2}[(c_2 - i\chi a_0\xi)e^{i\xi} + (*)] + O(\varepsilon^{13/2}\log\varepsilon) + \cdots, \tag{3.15}$$

where a_0, a_1 and c_2 are arbitrary constants.

We have computed further terms in (3.15) to check the matching to higher order, but to record them here would serve no useful purpose. It is easy to verify that f_3 and g_5 in (3.10)–(3.15) involve log(sin ξ) as a factor, which indicates a potential need for the log ε terms in (3.10) for matching with the inner expansions. The reader should note that the nonlinear terms do not enter into the outer expansions until the $O(\varepsilon^7)$ terms.

Having considered the outer region in some detail, we turn next to an examination of the inner region near $\xi = 0$. It is apparent from (3.6) that the choice of $\bar{\xi} = \varepsilon\xi$ as the scaled variable in the inner region leads to a distinguished limit (cf. [11]). Expanding the variable coefficients in (3.6) gives rise to the pair of equations

$$\ddot{f} + \left[1 - \frac{3\varepsilon^2}{4}\left(\frac{1}{\varepsilon^2\bar{\xi}^2} + \frac{\varepsilon^2}{3} + O(\varepsilon^4\bar{\xi}^4)\right)\right]f$$
$$= \left(1 - \frac{5\varepsilon^2}{4}\right)\left[-g - 2\varepsilon^3\chi f + f(f-g)\frac{1}{(\varepsilon\bar{\xi})^{3/2}}\left(1 - \frac{5\varepsilon^2}{4} + O(\varepsilon^4\bar{\xi}^4)\right)\right],$$
$$\ddot{g} + \left[1 - \frac{3\varepsilon^2}{4}\left(\frac{1}{\varepsilon^2\bar{\xi}^2} + \frac{\varepsilon^2}{3} + O(\varepsilon^4\bar{\xi}^4)\right)\right]g \tag{3.16}$$
$$= -\left(1 - \frac{5\varepsilon^2}{4}\right)\left[2\varepsilon^3\chi f + f(\tfrac{3}{2}f - g)\frac{1}{(\varepsilon\bar{\xi})^{3/2}}\left(1 - \frac{\varepsilon^2\bar{\xi}^2}{4} + O(\varepsilon^4\bar{\xi}^4)\right)\right],$$

with the boundary conditions

$$f = g = 0 \quad \text{at} \quad \bar{\xi} = 0. \tag{3.17}$$

The appropriate form of the expansions for the inner region at $\xi = 0$ is

$$f(\bar{\xi}) = F(\bar{\xi}) \sim \varepsilon^{7/2}F_0(\bar{\xi}) + \varepsilon^{9/2}F_1(\bar{\xi}) + \varepsilon^{11/2}F_2(\bar{\xi}) + \cdots,$$
$$g(\bar{\xi}) = G(\bar{\xi}) \sim \varepsilon^{11/2}G_2(\bar{\xi}) + \varepsilon^{13/2}G_3(\bar{\xi}) + \cdots. \tag{3.18}$$

It is clear that we can add terms involving log ε to these expansions if it becomes necessary. Furthermore, from (3.4) it follows that β and y are larger by a factor of $O(\sqrt{n})$ in the inner region near $\xi = 0$ than in the outer region. Substituting (3.18) into (3.16)–(3.17) leads to the sequence of problems

$$O(\varepsilon^{7/2}): \quad \ddot{F}_0 = \left(1 - \frac{3}{4\bar{\xi}^2}\right)F_0 = 0 \tag{3.19}$$

$$O(\varepsilon^{9/2}): \quad \ddot{F}_1 + \left(1 - \frac{3}{4\bar{\xi}^2}\right)F_1 = 0 \tag{3.20}$$

$$O(\varepsilon^{11/2}):\quad \ddot{F}_2 + \left(1 - \frac{3}{4\bar{\xi}^2}\right)F_2 = \frac{F_0}{4} - G_2 + \frac{F_0^2}{\bar{\xi}^{3/2}} \tag{3.21}$$

$$\ddot{G}_2 + \left(1 - \frac{3}{4\bar{\xi}^2}\right)G_2 = -\frac{3}{2}\frac{F_0^2}{\bar{\xi}^{3/2}} \tag{3.22}$$

$$\vdots$$

with the associated boundary conditions

$$F_0 = F_1 = F_2 = G_2 = \cdots = 0 \quad \text{at} \quad \bar{\xi} = 0. \tag{3.23}$$

The solution of (3.19)–(3.20) subject to (3.23) is

$$F_0 = \mathscr{A}_0 \sqrt{\bar{\xi}}\, J_1(\bar{\xi}), \qquad F_1 = \mathscr{A}_1 \sqrt{\bar{\xi}}\, J_1(\bar{\xi}), \tag{3.24}$$

where $\mathscr{A}_0, \mathscr{A}_1$ are arbitrary constants and J_1 is the Bessel function of the first kind of order 1. Having determined F_0, it is convenient to express the solution of (3.22) in terms of the Green's function of the linear differential operator on the left-hand side of (3.22) for the initial-value problem on $\bar{\xi} \geq 0$. The solution of (3.22) subject to (3.23) takes the form

$$G_2 = \mathscr{D}_2 \sqrt{\bar{\xi}}\, J_1(\bar{\xi}) - \frac{3\pi}{4}\mathscr{A}_0^2 \sqrt{\bar{\xi}}\left[Y_1(\bar{\xi}) \int_0^{\bar{\xi}} J_1^3(s)\, ds - J_1(\bar{\xi}) \int_0^{\bar{\xi}} J_1^2(s) Y_1(s)\, ds \right], \tag{3.25}$$

where $\mathscr{D}_2$ is an arbitrary constant and Y_1 is the Bessel function of the second kind of order 1.

For the purpose of matching with the outer expansion we rewrite (3.25) as

$$\begin{aligned} G_2 = {} & \mathscr{D}_2 \sqrt{\bar{\xi}}\, J_1(\bar{\xi}) - \frac{3\pi}{4}\mathscr{A}_0^2 \mathscr{I}_1 \sqrt{\bar{\xi}}\, Y_1(\bar{\xi}) + \frac{3\pi}{4}\mathscr{A}_0^2 \mathscr{I}_2 \sqrt{\bar{\xi}}\, J_1(\bar{\xi}) \\ & + \frac{3\pi}{4}\sqrt{\bar{\xi}}\, \mathscr{A}_0^2 \left[Y_1(\bar{\xi}) \int_{\bar{\xi}}^{\infty} J_1^3(s)\, ds - J_1(\bar{\xi}) \int_{\bar{\xi}}^{\infty} J_1^2(s) Y_1(s)\, ds \right], \end{aligned} \tag{3.26}$$

where

$$\mathscr{I}_1 = \int_0^{\infty} J_1^3(s)\, ds, \qquad \mathscr{I}_2 = \int_0^{\infty} J_1^2(s) Y_1(s)\, ds.$$

From Whittaker and Watson [15, p. 383] we find that

$$\mathscr{I}_1 = \sqrt{3}/2\pi. \tag{3.27}$$

Finally, for completeness, we record the solution of (3.21):

$$\begin{aligned} F_2 = {} & \mathscr{A}_2 J_1(\bar{\xi}) - \tfrac{2}{3} G_2(\bar{\xi}) + \frac{\bar{\xi}^{3/2}}{2}\left[\mathscr{D}_2 + \frac{3\pi}{4}\mathscr{A}_0^2 \mathscr{I}_2 - \frac{\mathscr{A}_0}{4} \right] \dot{J}_1(\bar{\xi}) \\ & + \frac{3\pi^2 \mathscr{A}_0^2}{8} \sqrt{\bar{\xi}} \int_0^{\bar{\xi}} s[J_1(s) Y_1(\bar{\xi}) - J_1(\bar{\xi}) Y_1(s)] \\ & \cdot \left[J_1(s) \int_s^{\infty} J_1^2(t) Y_1(t)\, dt + Y_1(s) \int_0^{s} J_1^3(t)\, dt \right] ds, \end{aligned} \tag{3.28}$$

where $\mathscr{A}_2$ is an arbitrary constant.

The inner expansion at $\xi = \pi$ is intimately related to the expansion at $\xi = 0$. The appropriate scaled variable is

$$\zeta = (\pi - \xi)/\varepsilon, \tag{3.29}$$

in terms of which

$$\csc \xi = \csc \varepsilon\zeta, \qquad \cot \xi = -\cot \varepsilon\zeta.$$

Thus, the only difference in (3.6) expressed in terms of ζ instead of $\bar{\xi}$ is the sign of the nonlinear terms. It follows that the inner expansions at $\xi = \pi$ are given by

$$\begin{aligned} f = F_\pi &\sim \varepsilon^{7/2}\mathscr{A}_{0\pi}\sqrt{\zeta}\, J_1(\zeta) + \varepsilon^{9/2}\mathscr{A}_{1\pi}\sqrt{\zeta}\, J_1(\zeta) + \cdots, \\ g = G_\pi &\sim \varepsilon^{11/2}\Big\{\mathscr{D}_{2\pi}\sqrt{\zeta}\, J_1(\zeta) + \frac{3\pi}{4}\mathscr{A}^2_{0\pi}\sqrt{\zeta}\,(\mathscr{I}_1 Y_1(\zeta) - \mathscr{I}_2 J_1(\zeta)) \\ &\quad - \frac{3\pi}{4}\sqrt{\zeta}\,\mathscr{A}^2_{0\pi}[Y_1(\zeta)\int_\zeta^\infty J_1^3(s)\,ds - J_1(\zeta)\int_\zeta^\infty J_1^2(s)Y_1^2(s)\,ds]\Big\} + \cdots, \end{aligned} \tag{3.30}$$

where $\mathscr{A}_{0\pi}$, $\mathscr{A}_{1\pi}$ and $\mathscr{D}_{2\pi}$ are arbitrary constants.

It should be noted that, strictly speaking, the inner expansions are not boundary-layer expansions as they involve neither exponential decay nor a different length scale from the outer expansion. Nevertheless, we anticipate that the inner and outer expansions have overlapping regions of validity and can thus be matched in the usual manner to determine the arbitrary constants. Since the matching in this problem is relatively straightforward, we shall not introduce intermediate limits.

For the purpose of ascertaining the limiting behavior of the inner expansions, we need the following expressions [15]:

$$\begin{aligned} J_1(x) &= \frac{1}{\sqrt{\pi x}}\left[\sin x - \cos x + \frac{3}{8x}(\sin x + \cos x) + O\left(\frac{1}{x^2}\right)\right] \quad \text{as} \quad x \to +\infty, \\ Y_1(x) &= \frac{-1}{\sqrt{\pi x}}\left[\sin x + \cos x - \frac{3}{8x}(\sin x - \cos x) + O\left(\frac{1}{x^2}\right)\right] \quad \text{as} \quad x \to +\infty. \end{aligned} \tag{3.31}$$

With the aid of (3.31), we find that the inner expansions at $\xi = 0$ have the asymptotic form (for $\bar{\xi} \to \infty$)

$$\begin{aligned} F &= \varepsilon^{7/2}\frac{\mathscr{A}_0}{\sqrt{\pi}}\left[(\sin\bar{\xi} - \cos\bar{\xi}) + \frac{3}{8\bar{\xi}}(\sin\bar{\xi} + \cos\bar{\xi}) + \cdots\right] \\ &\quad + \varepsilon^{9/2}\frac{\mathscr{A}_1}{\sqrt{\pi}}[(\sin\bar{\xi} - \cos\bar{\xi}) + \cdots] + O(\varepsilon^{11/2}), \\ G &= \varepsilon^{11/2}\left\{\frac{\mathscr{D}_2}{\sqrt{\pi}}\left[(\sin\bar{\xi} - \cos\bar{\xi}) + \frac{3}{8\bar{\xi}}(\sin\bar{\xi} - \cos\bar{\xi}) + \cdots\right] + \frac{3\sqrt{\pi}}{4}\mathscr{A}_0^2\right. \\ &\quad \left.\cdot\left[\mathscr{I}_1(\sin\bar{\xi} + \cos\bar{\xi}) + \mathscr{I}_2(\sin\bar{\xi} - \cos\bar{\xi}) + O\left(\frac{1}{\bar{\xi}}\right)\right]\right\} + O(\varepsilon^{13/2}), \end{aligned} \tag{3.32}$$

where we have simply recorded enough terms to allow for matching to leading order. Similarly, the inner expansions at $\xi = \pi$ have the asymptotic form (for $\zeta \to +\infty$)

$$F_\pi = \varepsilon^{7/2}\frac{\mathscr{A}_{0\pi}}{\sqrt{\pi}}\left[(\sin\zeta - \cos\zeta) + \frac{3}{8\zeta}(\sin\zeta + \cos\zeta) + \cdots\right]$$
$$+ \varepsilon^{9/2}\frac{\mathscr{A}_{1\pi}}{\sqrt{\pi}}[(\sin\zeta - \cos\zeta) + \cdots] + O(\varepsilon^{11/2}),$$
$$G_\pi = \varepsilon^{11/2}\left\{\frac{\mathscr{D}_{2\pi}}{\sqrt{\pi}}\left[(\sin\zeta - \cos\zeta) + \frac{3}{8\zeta}(\sin\zeta - \cos\zeta) + \cdots\right]\right. \tag{3.33}$$
$$\left. - \frac{3\sqrt{\pi}}{4}\mathscr{A}_{0\pi}^2\left[\mathscr{I}_1(\sin\zeta + \cos\zeta) + \mathscr{I}_2(\sin\zeta - \cos\zeta) + O\left(\frac{1}{\zeta}\right)\right]\right\} + O(\varepsilon^{13/2}).$$

Next, we need the asymptotic behavior of the outer expansions both in the limit as $\xi \to 0$ and as $\xi \to \pi$. Setting $\xi \to \varepsilon\bar{\xi}$ in (3.15) and expanding for $\varepsilon\bar{\xi}$ small (but $\bar{\xi}$ large), we obtain the limit behavior for the outer expansions as $\xi \to 0$ as

$$f = \varepsilon^{7/2}[(a_0 + a_0^*)\cos\bar{\xi} + i(a_0 - a_0^*)\sin\bar{\xi}] + \varepsilon^{9/2}\left\{\left[(a_1 + a_1^*) + \frac{3i}{8}(a_0 - a_0^*)\right.\right.$$
$$\left.\cdot\left(\frac{1}{\varepsilon\bar{\xi}} - \frac{\varepsilon\bar{\xi}}{2} + \cdots\right) + \frac{i}{2}\varepsilon\bar{\xi}(c_2 - c_2^*) - \frac{\chi}{4}\varepsilon^2\bar{\xi}^2(a_0 + a_0^*)\right]\cos\bar{\xi}$$
$$+\left[i(a_1 - a_1^*) - \tfrac{3}{8}(a_0 + a_0^*)\left(\frac{1}{\varepsilon\bar{\xi}} - \frac{\varepsilon\bar{\xi}}{2} + \cdots\right) - \frac{\varepsilon\bar{\xi}}{2}(c_2 + c_2^*)\right. \tag{3.34}$$
$$\left.\left. - \frac{i\chi}{4}\varepsilon^2\bar{\xi}^2(a_0 - a_0^*)\right]\sin\bar{\xi}\right\} + O(\varepsilon^{11/2}\log\varepsilon), \qquad \xi \to 0,$$
$$g = \varepsilon^{11/2}\{[(c_2 + c_2^*) + i\chi\varepsilon\bar{\xi}(a_0 - a_0^*)]\cos\bar{\xi} + [i(c_2 - c_2^*) - \chi\varepsilon\bar{\xi}(a_0 + a_0^*)]\sin\bar{\xi}\}$$
$$+ O(\varepsilon^{13/2}\log\varepsilon), \qquad \xi \to 0.$$

In order to match the outer expansion as $\xi \to \pi$ to the inner expansion at $\xi = \pi$, we need the relations

$$\xi = \pi - \varepsilon\zeta, \qquad e^{i\bar{\xi}} = e^{i(-\zeta + \pi/\varepsilon)} = e^{i(n+1/2)\pi}e^{-i\zeta} = kie^{-i\zeta}, \tag{3.35}$$

where

$$k = e^{in\pi}. \tag{3.36}$$

With (3.35) we can expand (3.15) for $\varepsilon\zeta$ small (but ζ large). We obtain

$$f = \varepsilon^{7/2}k[i(a_0 - a_0^*)\cos\zeta + (a_0 + a_0^*)\sin\zeta] + \varepsilon^{9/2}k\left\{\left[i(a_1 - a_1^*) + \tfrac{3}{8}(a_0 + a_0^*)\right.\right.$$
$$\left.\cdot\left(\frac{1}{\varepsilon\zeta} - \frac{\varepsilon\zeta}{3} + \cdots\right) - \frac{(c_2 + c_2^*)}{2}(\pi - \varepsilon\zeta) - \frac{i(a_0 - a_0^*)\chi}{4}(\pi - \varepsilon\zeta)^2\right]\cos\zeta$$
$$+\left[(a_1 + a_1^*) - \frac{3i}{8}(a_0 - a_0^*)\left(\frac{1}{\varepsilon\zeta} - \frac{\varepsilon\zeta}{3} + \cdots\right) + \frac{i(c_2 - c_2^*)}{2}(\pi - \varepsilon\zeta)\right. \tag{3.37}$$
$$\left.\left. - \frac{(a_0 + a_0^*)}{4}\chi(\pi - \varepsilon\zeta)^2\right]\sin\zeta\right\} + O(\varepsilon^{11/2}\log\varepsilon), \qquad \xi \to \pi,$$
$$g = \varepsilon^{11/2}k\{[i(c_2 - c_2^*) - \chi(a_0 + a_0^*)(\pi - \varepsilon\zeta)]\cos\zeta + [(c_2 + c_2^*) + i\chi(a_0 - a_0^*)$$
$$\cdot(\pi - \varepsilon\zeta)]\sin\zeta\} + O(\varepsilon^{13/2}\log\varepsilon), \qquad \xi \to \pi.$$

We are now in a position to match the various expansions. First, we consider the matching of the inner expansion at $\xi = 0$ with the outer expansion. From (3.32) and (3.34) it follows that in order for the $\varepsilon^{7/2}\cos\bar{\xi}$ and $\varepsilon^{7/2}\sin\bar{\xi}$ terms in f to match, we must have

$$a_0 + a_0^* = -\mathscr{A}_0/\sqrt{\pi}, \qquad i(a_0 - a_0^*) = \mathscr{A}_0/\sqrt{\pi}, \tag{3.38}$$

and, in order for the $\varepsilon^{11/2}\cos\bar{\xi}$ and $\varepsilon^{11/2}\sin\bar{\xi}$ terms in g to match, we require

$$\begin{aligned} c_2 + c_2^* &= -\frac{\mathscr{D}_2}{\sqrt{\pi}} + \frac{3\sqrt{\pi}}{4}\mathscr{A}_0^2(\mathscr{I}_1 - \mathscr{I}_2), \\ i(c_2 - c_2^*) &= \frac{\mathscr{D}_2}{\sqrt{\pi}} + \frac{3\sqrt{\pi}}{4}\mathscr{A}_0^2(\mathscr{I}_1 + \mathscr{I}_2). \end{aligned} \tag{3.39}$$

Next, we record the analogous expressions for the matching of the inner expansion at $\xi = \pi$ with the outer expansion. From (3.33) and (3.37), we obtain

$$i(a_0 k - a_0^* k^*) = -\mathscr{A}_{0\pi}/\sqrt{\pi}, \qquad (ka_0 + k^* a_0^*) = \mathscr{A}_{0\pi}/\sqrt{\pi}, \tag{3.40}$$

and

$$\begin{aligned} i(c_2 k - c_2^* k^*) - \pi\chi(a_0 k + a_0^* k^*) &= -\frac{\mathscr{D}_{2\pi}}{\sqrt{\pi}} - \frac{3\sqrt{\pi}}{4}\mathscr{A}_{0\pi}^2(\mathscr{I}_1 - \mathscr{I}_2), \\ (c_2 k + c_2^* k^*) + \pi\chi i(a_0 k - a_0^* k^*) &= \frac{\mathscr{D}_{2\pi}}{\sqrt{\pi}} - \frac{3\sqrt{\pi}}{4}\mathscr{A}_{0\pi}^2(\mathscr{I}_1 + \mathscr{I}_2). \end{aligned} \tag{3.41}$$

Eqs. (3.38)–(3.41) contain sufficient information to permit us to calculate a_0, $\mathscr{A}_0$ and $\mathscr{A}_{0\pi}$. Further matching is necessary in order to determine the constants in the higher-order terms.

Now the reader should note that, while our approach to solving the nonlinear eigenvalue problem in (3.6)–(3.7) is based on the assumption that ε is small (equivalently n is large), nowhere have we made use of the fact that n is an even integer. The reason for the additional restriction on n will now be established. First, we observe that (3.38) and (3.40) imply that

$$a_0 = \mathscr{A}_0 = \mathscr{A}_{0\pi} = 0 \tag{3.42}$$

unless k is real. Thus, it follows from (3.36) that n must be an integer. In effect, the matching condition has led to the eigenvalue relation for the linear buckling problem.

Having determined that k must be real (either ± 1) it follows from (3.38)–(3.40) that

$$a_0 = -\frac{(1+i)}{2\sqrt{\pi}}\mathscr{A}_0, \qquad \mathscr{A}_{0\pi} = -k\mathscr{A}_0 \tag{3.43}$$

Using these results and eliminating $c_2 + c_2^*$ and $i(c_2 - c_2^*)$ from (3.39) and (3.41), we obtain

$$\begin{aligned} -\frac{\mathscr{D}_2}{\sqrt{\pi}} + \frac{3\sqrt{\pi}}{4}\mathscr{A}_0^2(\mathscr{I}_1 - \mathscr{I}_2) &= -\chi\sqrt{\pi}\mathscr{A}_0 + k\frac{\mathscr{D}_{2\pi}}{\sqrt{\pi}} - \frac{3\sqrt{\pi}}{4}k\mathscr{A}_0^2(\mathscr{I}_1 + \mathscr{I}_2), \\ \frac{\mathscr{D}_2}{\sqrt{\pi}} + \frac{3\sqrt{\pi}}{4}\mathscr{A}_0^2(\mathscr{I}_1 + \mathscr{I}_2) &= \chi\sqrt{\pi}\mathscr{A}_0 - k\frac{\mathscr{D}_{2\pi}}{\sqrt{\pi}} - \frac{3\sqrt{\pi}}{4}k\mathscr{A}_0^2(\mathscr{I}_1 - \mathscr{I}_2). \end{aligned} \tag{3.44}$$

Adding the two equations in (3.44) we have

$$\frac{3\sqrt{\pi}}{4}\mathscr{A}_0^2\mathscr{I}_1(1+k) = -2\chi\sqrt{\pi}\mathscr{A}_0. \tag{3.45}$$

If n is odd, then (3.34) implies that $k = -1$, and the only solution of (3.45) is $\mathscr{A}_0 = 0$. This result suggests that the order of magnitude of the leading term in the expansions in (3.10) is not valid for n odd. If, on the other hand, n is even, then $k = +1$ and it follows from (3.45) that

$$\mathscr{A}_0 = -\frac{2\chi}{3\mathscr{I}_1} = -\frac{4\chi\pi}{3\sqrt{3}} \text{ for } n \text{ even}, \tag{3.46}$$

where we have used (3.27).

In order to be able to compare the present results with those established in Sec. 2 by means of the regular perturbation method, we employ (3.4), (3.10), (3.12), (3.14), (3.43) and (3.46) to obtain

$$\begin{aligned}\beta &= \varepsilon^{7/2}\sqrt{\frac{32\pi}{27\sin\xi}}\cos\left(\bar{\xi}+\frac{\pi}{4}\right) + O(\varepsilon^{9/2}),\\ &= \alpha\chi\sqrt{\frac{32\pi}{27\,n\sin\xi}}\cos\left[(n+\tfrac{1}{2})\xi+\frac{\pi}{4}\right] + O(\alpha/n^{3/2}),\end{aligned} \tag{3.47}$$

where (3.47) holds except near $\xi = 0$ and $\xi = \pi$. The approximation in (3.47) is exactly the same as that provided by (2.13)–(2.16).

As noted above, further matching must be performed to find the other arbitrary constants. For example, upon computing f_3 and g_4 in (3.10), we are able to establish that

$$\mathscr{D}_2 = -\mathscr{D}_{2\pi} = -\frac{3\pi}{4}\mathscr{A}_0^2\mathscr{I}_2\cdot k.$$

We conclude this section with the remark that it requires approximately twice as much algebra to deduce the initial branching behavior for odd n. The appropriate expansions corresponding to (3.10) would be

$$\begin{aligned} f &\sim \varepsilon^3 f_0 + \varepsilon^{7/2} f_1 + \varepsilon^4 f_2 + \cdots,\\ g &\sim \varepsilon^{9/2} g_3 + \varepsilon^{10/2} g_4 + \cdots. \end{aligned} \tag{3.48}$$

We shall elaborate on the branching results for odd n in the next section.

4. Branching of solutions for even and odd n. Having described our asymptotic integration technique in much detail in Sec. 3, we now turn our attention to the problem of investigating the branching behavior of solutions to the nonlinear eigenvalue problem (2.11)–(2.12) in the region where the standard perturbation method breaks down; namely, $\alpha = O(\delta)$. In the next section, we shall briefly consider the nature of the solution branches for even larger deviations of the loading parameter λ from an eigenvalue λ_n. However, the order relationship $\alpha = O(\delta)$ is, in a sense, a distinguished order relationship. The solution which we construct for $\alpha = O(\delta)$ contains the standard perturbation results, whereas, for larger deviations, the corresponding leading-order term does not reduce back to the leading-order term in the standard perturbation expansion in the limit $\alpha \to 0$ with δ fixed. We shall elaborate further on this point in Sec. 5.

With regard to the model function defined in (3.1), the choice of order relationship $\alpha = O(\delta)$ for (2.11)–(2.12) corresponds to setting $\alpha = \bar{\chi}\delta$, $\bar{\chi} = O(1)$ as $\delta \to 0$, in (3.1) and expanding for small δ:

$$u(\bar{\chi}\delta, \delta) \sim \sqrt{1 + \bar{\chi}}\,[1 - \bar{\chi}^2\delta + \bar{\chi}^4\delta^2 - \cdots] \quad \text{as} \quad \delta \to 0. \tag{4.1}$$

Clearly the expansion in (4.1) contains the regular perturbation expansion in (3.2) in the sense that if we replace $\bar{\chi}$ by α/δ and expand (4.1) on the basis of $\alpha \to 0$ with δ fixed, we recover (3.2) (the rearrangement requires only a finite number of transpositions for each power of α). On the other hand, for large $\bar{\chi}$, (4.1) suggests that the leading behavior of $u(\bar{\chi}\delta, \delta)$ is given by

$$u(\bar{\chi}\delta) \approx \sqrt{\bar{\chi}}. \tag{4.2}$$

(Of course, from (3.1) we have that (4.2) holds so long as $1 \ll \bar{\chi} \ll \delta^{-1/2}$.)

Although it is possible to utilize our technique to study the branching from any eigenvalue, we shall restrict our attention to the physically relevant class of eigenvalues which are near the critical value. For a fixed value of δ, the eigenvalues λ_n are given by (2.5)–(2.6) with the lowest eigenvalue corresponding to that positive integer n which is closest to the number

$$n_{\text{cr}}(\delta) = \frac{-1 + \sqrt{5 + 4/\delta}}{2}. \tag{4.3}$$

Only if

$$\delta = \delta_m = \frac{1}{m(m+1) - 1}, \qquad m = 1, 2, \ldots$$

is n_{cr} an integer.

As the subsequent analysis will show, the behavior of the solutions branching from the lowest and neighboring eigenvalues depends in a crucial manner on how close n_{cr} actually is to the nearest positive integer. In order to incorporate this additional feature into our asymptotic integration scheme (which is based on the limit $\delta \to 0$) we shall construct approximate solutions for the monotone decreasing sequence $\{\delta_n\}_{n=1}^{\infty}$, with the δ_n so chosen that

$$n_{\text{cr}}(\delta_n) = n - \tau, \qquad n = 1, 2 \ldots,$$

with τ a fixed number. From (4.3) it follows that

$$\delta_n \doteq [(n + \tfrac{1}{2} - \tau)^2 - \tfrac{5}{4}]^{-1}, \qquad n = 1, 2, \ldots. \tag{4.4}$$

On the other hand, we observe that for a given δ, τ is prescribed uniquely to within an integer by (4.3) and that for $|\tau| < \frac{1}{2}$, the corresponding λ_n is the lowest eigenvalue.

As in Sec. 3, it proves convenient to employ

$$\varepsilon = 1/(n + \tfrac{1}{2})$$

as the basic small parameter. In terms of ε we have from (2.6) and (4.4) that

$$\frac{1}{\delta_n} = \left(\frac{1}{\varepsilon} - \tau\right)^2 - \frac{5}{4} = \frac{1}{\varepsilon^2}\,[1 - 2\varepsilon\tau + \varepsilon^2(\tau^2 - \tfrac{5}{4})], \qquad \mu_n = \frac{1}{\varepsilon^2} - \frac{5}{4}, \tag{4.5}$$

which combined with (2.5) yields

$$\lambda_n = 1 + 2\tau^2\varepsilon^2 + O(\varepsilon^3) \quad \text{as} \quad \varepsilon \to 0. \tag{4.6}$$

Our aim is to investigate the initial branching behavior for the simple eigenvalues λ_n with

$$\lambda = \lambda_n(1 + \varepsilon^2\bar{\chi}), \qquad \bar{\chi} = O(1), \qquad \text{as} \quad \varepsilon \to 0. \tag{4.7}$$

When $\tau = 0$ and $\bar{\chi} = O(\varepsilon)$ we recover the results obtained in the last section. Upon making the change of variables

$$f = \beta\sqrt{\sin\xi}, \qquad g = y\sqrt{\sin\xi}, \qquad \bar{\xi} = \xi/\varepsilon \tag{4.8}$$

and substituting (4.5)–(4.8) into (2.11)–(2.12), we obtain the system

$$\ddot{f} + \left(1 - \frac{3\varepsilon^2}{4}\csc^2\xi\right)f = [1 - 2\varepsilon\tau + O(\varepsilon^2)] \cdot \left\{-g - 2\varepsilon^2\bar{\chi}[1 + O(\varepsilon^2)]f + f[g - (1 + O(\varepsilon))f]\frac{\cot\xi}{\sqrt{\sin\xi}}\right\}, \tag{4.9}$$

$$\ddot{g} + \left(1 - 4\varepsilon\tau + 6\varepsilon^2\tau^2 - \frac{3\varepsilon^2}{4}\csc^2\xi + O(\varepsilon^3)\right)g = -[1 - 4\varepsilon\tau + O(\varepsilon^2)] \cdot \left[2\varepsilon^2\bar{\chi}(1 + O(\varepsilon^2))f + \frac{f^2\cot\xi}{2\sqrt{\sin\xi}}(3 - 2\varepsilon\tau) - \frac{fg\cot\xi}{\sqrt{\sin\xi}}\right], \tag{4.10}$$

with the boundary conditions

$$f = g = 0 \qquad \text{at} \qquad \bar{\xi} = 0, \pi/\varepsilon. \tag{4.11}$$

Following the approach developed in Sec. 3, we express the solution of (4.9)–(4.11) in terms of three expansions. The considerations of Sec. 2 suggest that we attempt outer expansions of the form (for both even and odd n)

$$f = \varepsilon^{5/2}f_0 + \varepsilon^{7/2}f_1 + \cdots, \qquad g = \varepsilon^{7/2}g_1 + \varepsilon^{9/2}g_2 + \cdots. \tag{4.12}$$

Upon substituting (4.12) into (4.9)–(4.10) and applying the two-variable procedure described in Sec. 3, we obtain

$$f_0(\xi, \bar{\xi}) = A_0(\xi)e^{i\bar{\xi}} + (*), \qquad g_1(\xi, \bar{\xi}) = C_1(\xi)e^{i\bar{\xi}} + (*). \tag{4.13}$$

In order to avoid the presence of secular terms in f_1 and g_2, we are forced to require A_0 and C_1 to satisfy

$$A_0'(\xi) = \frac{1}{2i}C_1(\xi), \qquad C_1'(\xi) = \frac{1}{2i}[-4\tau C_1 + 2\bar{\chi}A_0]. \tag{4.14}$$

Solving the equations in (4.14), we obtain

$$f = \varepsilon^{5/2}[(a_0e^{i(\omega-\tau)\xi} + b_0e^{-i(\omega+\tau)\xi})e^{i\bar{\xi}} + (*)] + O(\varepsilon^{7/2}), \tag{4.15}$$

$$g = \varepsilon^{7/2}\{2[a_0(\tau - \omega)e^{i(\omega-\tau)\xi} + b_0(\xi + \omega)e^{-i(\omega+\tau)\xi}]e^{i\bar{\xi}} + (*)\} + O(\varepsilon^{9/2}), \tag{4.16}$$

where a_0, b_0 are arbitrary complex constants and

$$\omega = \sqrt{\tau^2 + \bar{\chi}/2}. \tag{4.17}$$

To simplify the presentation, we have recorded only the leading terms in (4.15)–(4.16). It is important to note that when $\lambda - \lambda_n = O(\delta)$ the branching solution no longer has the functional form predicted by the linear buckling theory. In effect, the two-variable technique has provided a means for the reordering of the nonuniform regular perturbation expansion.

The corresponding inner expansions at $\xi = 0$ and $\xi = \pi$ are closely related to those studied in Sec. 3. It is easy to show that the inner expansions at $\xi = 0$ are given by

$$\begin{aligned} f(\xi) &= F(\bar{\xi}) = \varepsilon^{5/2}\mathscr{A}_0\sqrt{\bar{\xi}}\,J_1(\bar{\xi}) + O(\varepsilon^{7/2}), \\ g(\xi) &= G(\bar{\xi}) = \varepsilon^{7/2}G_2(\bar{\xi}) + O(\varepsilon^{9/2}) \end{aligned} \tag{4.18}$$

and that the inner expansions at $\xi = \pi$ are given by

$$\begin{aligned} f(\xi) &= F_\pi(\zeta) = \varepsilon^{5/2}\mathscr{A}_{0\pi}\sqrt{\zeta}\,J_1(\zeta) + O(\varepsilon^{7/2}), \\ g(\xi) &= G_\pi(\zeta) = \varepsilon^{7/2}G_{2\pi}(\zeta) + O(\varepsilon^{9/2}), \end{aligned} \tag{4.19}$$

where $\mathscr{A}_0$ and $\mathscr{A}_{0\pi}$ are arbitrary constants, G_2 is given by (3.26) and $G_{2\pi}$ is the function contained within the braces in (3.30).

Matching of the above equations proceeds essentially along the lines described in Sec. 3. Sparing the reader the details, we find that $\mathscr{A}_0$ and $\mathscr{A}_{0\pi}$ are related by the following pair of algebraic equations (for both real and imaginary values of ω):

$$\mathscr{A}_0 \cos \omega\pi + k\mathscr{A}_{0\pi} \cos \tau\pi = z\mathscr{A}_0^2, \tag{4.20}$$

$$\mathscr{A}_{0\pi} \cos \omega\pi + k\mathscr{A}_0 \cos \tau\pi = -z\mathscr{A}_{0\pi}^2, \tag{4.21}$$

where

$$z = \frac{3\sqrt{3}\sin \omega\pi}{16\omega}, \qquad k = \begin{cases} +1, & n \text{ even}, \\ -1, & n \text{ odd}. \end{cases} \tag{4.22}$$

As Eqs. (4.20)–(4.21) contain a wealth of information regarding the branching of solutions for complete spherical shells, we shall discuss them in some detail. First of all, it should be observed that the ansatz in (4.12) breaks down when the parameter z vanishes; namely, when

$$\omega = \sqrt{\tau^2 + \bar{\chi}/2} = p, \qquad p = 1, 2, \ldots . \tag{4.23}$$

Now the choice of the ansatz in (4.12) was based on the assumption that λ_n is a simple eigenvalue. This assumption, in turn, places a restriction on the permissible values of τ in (4.4). A careful examination of (2.11)–(2.12) suggests that we must restrict τ to lie outside of neighborhoods of $O(\varepsilon)$ about the values $\tau = m/2, m = \pm 1, \pm 2, \ldots$. These special values of τ correspond to the case of double eigenvalues in (2.5). Although it is a straightforward matter to study the branching from double eigenvalues by our method, we shall not include such a discussion for the sake of brevity.

Given the above restriction on τ, we find that (4.20)–(4.21) imply the existence of branching solutions which are described to leading order by (4.12) for sufficiently small values of $|\bar{\chi}|$. Suppose we consider an upward-branching solution (similar considerations hold for downward-branching solutions). Our results suggest that such a branching solution is actually given to leading order by (4.12) until $\bar{\chi} > 0$ reaches a neighborhood of $O(\varepsilon)$ of $\bar{\chi}_p$, where $\bar{\chi}_p$ is defined as the minimum value of $\bar{\chi} > 0$ which makes ω an integer (with τ

fixed). A study of the higher-order terms in (4.12) suggests that one should attempt new expansions in these neighborhoods with $f = O(\varepsilon^2)$. As the details of the analysis of these singular regions are quite complicated, we shall not present them here.

Having described the limitations of (4.20)–(4.21), we now proceed to solve them. First, we consider the case of *n even*. The only branching solution for this case is given by

$$\mathscr{A}_0 = -\mathscr{A}_{0\pi} = \frac{\cos \omega\pi - \cos \tau\pi}{z}. \tag{4.24}$$

It is easy to show that this solution reduces back to the branch point as $\bar{\chi} \to 0$. We find that

$$\mathscr{A}_0 \sim -4\bar{\chi}\pi/3\sqrt{3} \quad \text{as} \quad \bar{\chi} \to 0, \tag{4.25}$$

subject to the above restriction on τ. The expression in (4.25) agrees with the result in (3.46). Actually, the solution in (4.24) describes two branching solutions, one branching upward and the other branching downward. Physically, the former (latter) solution corresponds to outward (inward) deflections of the shell at both poles.

It is clear from (4.24) and the above discussion that with one exception we cannot expect (4.12) to describe the branching solution for $|\bar{\chi}| \geq 2$. However, based on practical considerations, this exceptional case is the most important. For the lowest even eigenvalue we have that $|\tau| < 1$ and that ω does not vanish for the downward-branching solution. Assuming that the lowest even eigenvalue is simple, it is straightforward to deduce from (4.24) that

$$\mathscr{A}_0 \sim 2\sqrt{\frac{-32\bar{\chi}}{27}}, \qquad \bar{\chi} \to -\infty, \qquad |\tau| < 1. \tag{4.26}$$

We conclude our discussion of n even with the observation that (4.20)–(4.21) admit the additional solution

$$\mathscr{A}_0 = \mathscr{A}_{0\pi} = \frac{\cos \omega\pi + \cos \tau\pi}{z}, \tag{4.27}$$

$$\mathscr{A}_0 = -\frac{(\cos \tau\pi + \cos \omega\pi)}{2z} \pm \frac{1}{2z}\left[(\cos \tau\pi + \cos \omega\pi)^2 - 4(\cos \tau\pi + \cos \omega\pi)\right]^{1/2}. \tag{4.28}$$

This solution does not reduce back to the branch point, and, in fact, it is complex unless $\bar{\chi}$ is sufficiently negative. We shall not discuss it further beyond pointing out that it either may be spurious or may represent a non-branching solution (such solutions were observed to exist in [5]).

Next, we consider the case of *n odd*. We note that when $k = -1$, (4.20)–(4.21) admit a solution

$$\mathscr{A}_0 = -\mathscr{A}_{0\pi} = \frac{\cos \omega\pi + \cos \tau\pi}{z}. \tag{4.29}$$

However, this solution does not reduce back to the branch point, and so remarks similar to those for (4.27)–(4.28) hold for it as well. The solution of (4.27)–(4.28) which does have the desired behavior is given by

$$\mathscr{A}_0 = \frac{\cos \omega\pi - \cos \tau\pi}{2z}\left\{1 \pm \left[1 + \frac{4\cos \tau\pi}{\cos \omega\pi - \cos \tau\pi}\right]^{1/2}\right\}, \tag{4.30}$$

$$\mathscr{A}_{0\pi} = \frac{\cos \omega\pi - \cos \tau\pi}{2z}\left\{1 \mp \left[1 + \frac{4\cos \tau\pi}{\cos \omega\pi - \cos \tau\pi}\right]^{1/2}\right\}. \tag{4.31}$$

The two possible deflection configurations of the spherical shell predicted by (4.30)–(4.31) for fixed $\bar{\chi}$ are simply reflections about the equator of one another. We shall discuss these solutions in detail in Sec. 6.

Calculation of higher-order terms in (4.12), (4.18)–(4.19) is a straightforward but tedious matter. It is of interest to record the leading terms in the outer expansion for β (and by virtue of (2.10) for ψ). For *n even* we have from (4.12) (with $\lambda = \lambda_n(1 + \varepsilon^2\bar{\chi})$, $\varepsilon = (n + \frac{1}{2})^{-1}$, $n = n_{cr} + \tau, 0 \le |\tau| < \frac{1}{2}$ or $\frac{1}{2} < |\tau| < 1$)

$$\beta \sim -\varepsilon^{5/2}\mathscr{A}_0\sqrt{\frac{2}{\pi \sin \xi}}\left[\frac{\sinh\left(\sqrt{-\left(\frac{\bar{\chi}}{2} + \tau^2\right)}(\pi - \xi)\right) + \sinh\left(\sqrt{-\left(\frac{\bar{\chi}}{2} + \tau^2\right)}\xi\right)}{\sinh\left(\sqrt{-\left(\frac{\bar{\chi}}{2} + \tau^2\right)}\pi\right)}\right] \cdot \cos\left(\frac{\xi}{\varepsilon} - \tau\xi + \frac{\pi}{4}\right) \quad \text{as} \quad \varepsilon \to 0, \tag{4.32}$$

with $\mathscr{A}_0$ defined in (4.24), and for *n odd* we have

$$\beta \sim -\varepsilon^{5/2}\sqrt{\frac{2}{\pi \sin \xi}}\left[\frac{\mathscr{A}_0 \sinh\left(\sqrt{-\left(\frac{\bar{\chi}}{2} + \tau^2\right)}(\pi - \xi)\right) + \mathscr{A}_{0\pi} \sinh\left(\sqrt{-\left(\frac{\bar{\chi}}{2} + \tau^2\right)}\xi\right)}{\sinh\left(\sqrt{-\left(\frac{\bar{\chi}}{2} + \tau^2\right)}\pi\right)}\right] \cdot \cos\left(\frac{\xi}{\varepsilon} - \tau\xi + \frac{\pi}{4}\right) \quad \text{as} \quad \varepsilon \to 0, \tag{4.33}$$

with $\mathscr{A}_0$ and $\mathscr{A}_{0\pi}$ defined in (4.30)–(4.31). The approximations in (4.32)–(4.33) are valid except in neighborhoods of $O(\varepsilon)$ of $\xi = 0$ and $\xi = \pi$.

We emphasize that upon replacing $\bar{\chi}$ by $\chi\varepsilon$ with $\chi = O(1)$ in (4.32) and using (4.25) we recover the approximation in (3.47) for n even. On the other hand, the above leading-order behavior (along with further consideration of higher-order terms in (4.12)) shows that the deflection pattern (for $\lambda - \lambda_n = O(1/n^2)$) is quite distinct from that predicted by the linear buckling theory. We defer further interpretation of these results to Sec. 6.

5. Continuation of the solution branches. Examination of the higher order terms in (4.12) reveals that the expansions for β and y become disordered when $\bar{\chi} = 0(1/\sqrt{\delta})$, i.e., when $\lambda - \lambda_n = 0(\sqrt{\delta})$. This situation is analogous to that arising in the expansion (4.1) for the simple model function $u(\alpha, \delta)$. In order to determine the asymptotic behavior of $u(\alpha, \delta)$ for larger values of α we set $\alpha = \tilde{\chi}\sqrt{\delta}$, $\tilde{\chi} = 0(1)$ as $\delta \to 0$ in (3.1) and expand for small δ:

$$u(\tilde{\chi}\sqrt{\delta}, \delta) \sim \left(\frac{\tilde{\chi}^2}{\delta}\right)^{1/4} \frac{1}{1 + \tilde{\chi}^2}\left[1 + \frac{\sqrt{\delta}}{2\tilde{\chi}} - \frac{\delta}{8\tilde{\chi}^2} + \cdots\right] \quad \text{as} \quad \delta \to 0. \tag{5.1}$$

Now, while this expansion is valid for any positive value of $\tilde{\chi}$, it does become disordered as $\tilde{\chi} \to 0$. In contrast with (4.1), we cannot recover the regular perturbation expansion (3.2) by replacing $\tilde{\chi}$ by $\alpha/\sqrt{\delta}$ and expanding (5.1) on the basis of $\alpha \to 0$ with δ fixed. In particular, the leading term in (5.1) does not reduce back to the leading term in the regular perturbation expansion. Nevertheless, the expansions in (4.1) and (5.1) do have overlapping regions of validity corresponding to $\bar{\chi} = \alpha/\delta \gg 1$ with $\tilde{\chi} = \alpha/\sqrt{\delta} \ll 1$.

For the buckling problem, the asymptotic integration technique employed in the previous sections must be modified slightly in order to continue the solution branches for $\lambda - \lambda_n = O(\sqrt{\delta})$. We shall illustrate the requisite modification by considering the special case in which

$$\lambda = \lambda_n(1 + \varepsilon\tilde{\chi}), \qquad \varepsilon = \frac{1}{n + \frac{1}{2}}, \qquad \tilde{\chi} = O(1) \tag{5.2}$$

with

$$n = n_{\text{cr}} + \tau, \qquad 0 \le |\tau| < \tfrac{1}{2} \quad \text{or} \quad \tfrac{1}{2} < |\tau| < 1, \qquad \tau \text{ fixed}, \qquad \text{as } \delta \to 0. \tag{5.3}$$

The restriction on τ in (5.2) allows us to study the downward-branching from the lowest even and odd eigenvalue while avoiding the complications associated with the branching problem for the higher eigenvalues and double eigenvalues.

The governing equations for this case are (4.9)–(4.11) with $\varepsilon^2\bar{\chi}$ replaced by $\varepsilon\tilde{\chi}$ to reflect (5.3). Now (4.26) and (4.31) suggest that $f = O(\varepsilon^2)$ for deviations of λ from λ_n in the range given in (5.3). Further study indicates that we should attempt outer expansions of the form (for both even and odd n)

$$f = \varepsilon^2 f_0 + \varepsilon^{5/2} f_1 + \varepsilon^3 f_2 + \cdots, \qquad g = \varepsilon^{5/2} g_1 + \varepsilon^3 g_2 + \cdots. \tag{5.4}$$

On substituting (5.4) into the governing equations, one finds that it is necessary to introduce a new "slow" variable, namely

$$\tilde{\xi} = \xi/\sqrt{\varepsilon}. \tag{5.5}$$

Thus, in the outer region, f and g depend on the three variables ξ, $\tilde{\xi}$ and $\bar{\xi}$. According to the method of multiple scales, the derivatives with respect to $\bar{\xi}$ in (4.9)–(4.10) are transformed as follows:

$$\frac{d}{d\bar{\xi}} = \frac{\partial}{\partial\bar{\xi}} + \sqrt{\varepsilon}\,\frac{\partial}{\partial\tilde{\xi}} + \varepsilon\,\frac{\partial}{\partial\xi}. \tag{5.6}$$

Applying the multiple scaling procedure, we obtain

$$f_0(\xi, \tilde{\xi}, \bar{\xi}) = A_0(\xi, \tilde{\xi})e^{i\bar{\xi}} + (*), \qquad g_1 = C_1(\xi, \tilde{\xi})e^{i\bar{\xi}} + (*). \tag{5.7}$$

In order to avoid the presence of secular terms of the form $\tilde{\xi}e^{\pm i\bar{\xi}}$ in f_1 and g_2, we must set

$$\partial A_0/\partial\tilde{\xi} = C_1/2i, \qquad \partial C_1/\partial\tilde{\xi} = (\tilde{\chi}/i)A_0, \tag{5.8}$$

from which it follows that

$$A_0(\xi, \tilde{\xi}) = \tilde{a}_0(\xi)\exp(\sqrt{-\tilde{\chi}/2}\,\tilde{\xi}) + \tilde{b}_0(\xi)\exp(-\sqrt{-\tilde{\chi}/2}\,\tilde{\xi}), \tag{5.9}$$

$$C_1(\xi, \tilde{\xi}) = 2i\sqrt{-\frac{\tilde{\chi}}{2}}\,[\tilde{a}_0\exp(\sqrt{-\tilde{\chi}/2}\,\tilde{\xi}) - \tilde{b}_0(\xi)\exp(-\sqrt{-\tilde{\chi}/2}\,\tilde{\xi})]. \tag{5.10}$$

In (5.9)–(5.10), the functions $\tilde{a}_0$ and $\tilde{b}_0$ are as yet undetermined. Furthermore,

$$f_1 = A_1(\xi, \tilde{\xi})e^{i\bar{\xi}} + (*), \qquad g_2 = C_2(\xi, \tilde{\xi})e^{i\bar{\xi}} + (*). \tag{5.11}$$

Continuing, we find that in order to avoid secular terms of the form $\tilde{\xi}\exp(\pm\sqrt{-\tilde{\chi}/2}\,\tilde{\xi})$ in A_1 and C_2 we must choose

$$\partial\tilde{a}_0/\partial\xi = i(\tfrac{3}{8}\tilde{\chi} - \tau)\tilde{a}_0, \qquad \partial\tilde{b}_0/\partial\xi = i(\tfrac{3}{8}\tilde{\chi} - \tau)\tilde{b}_0, \tag{5.12}$$

which implies that

$$\tilde{a}_0(\xi) = a_0 \exp\left[i\left(\frac{3\tilde{\chi}}{8} - \tau\right)\xi\right], \qquad \tilde{b}_0 = b_0 \exp[i(\tfrac{3}{8}\tilde{\chi} - \tau)\xi], \tag{5.13}$$

where a_0 and b_0 are arbitrary constants. Thus

$$\begin{aligned} f = \varepsilon^2\{[a_0 \exp(\sqrt{-\tilde{\chi}/2}\,\tilde{\xi} + i(\tfrac{3}{8}\tilde{\chi} - \tau)\xi) \\ + b_0 \exp(-\sqrt{-\tilde{\chi}/2}\,\tilde{\xi} + i(\tfrac{3}{8}\tilde{\chi} - \tau)\xi)]\exp(i\bar{\xi}) + (*)\} + O(\varepsilon^{5/2}). \end{aligned} \tag{5.14}$$

The corresponding inner expansions at $\xi = 0$ have the form

$$\begin{aligned} f(\bar{\xi}) &= F(\bar{\xi}) = \varepsilon^2\mathscr{A}_0\sqrt{\bar{\xi}}\,J_1(\bar{\xi}) + O(\varepsilon^{5/2}), \\ g(\bar{\xi}) &= G(\bar{\xi}) = \varepsilon^{5/2}G_2(\bar{\xi}) + O(\varepsilon^3), \end{aligned} \tag{5.15}$$

and the expansions at $\xi = \pi$ are given by

$$\begin{aligned} f(\bar{\xi}) &= F_\pi(\zeta) = \varepsilon^2\mathscr{A}_{0\pi}\sqrt{\zeta}\,J_1(\zeta) = O(\varepsilon^{5/2}), \\ g(\bar{\xi}) &= G_\pi(\zeta) = \varepsilon^{5/2}G_{2\pi}(\zeta) + O(\varepsilon^3), \end{aligned} \tag{5.16}$$

where $\mathscr{A}_0$ and $\mathscr{A}_{0\pi}$ are arbitrary constants, G_2 is given by (3.25) and $G_{2\pi}$ is the function contained within the braces in (3.30).

Matching of the inner and outer expansions leads to the following pair of algebraic equations relating $\mathscr{A}_0$ and $\mathscr{A}_{0\pi}$

$$\mathscr{A}_0 \cosh\left(-\sqrt{\frac{\tilde{\chi}}{2\varepsilon}}\,\pi\right) + k\mathscr{A}_{0\pi}\cos(\tfrac{3}{8}\tilde{\chi} - \tau)\pi = Z\mathscr{A}_0^2, \tag{5.17}$$

$$\mathscr{A}_{0\pi} \cosh\left(\sqrt{-\frac{\tilde{\chi}}{2\varepsilon}}\,\pi\right) + k\mathscr{A}_0\cos(\tfrac{3}{8}\tilde{\chi} - \tau)\pi = -Z\mathscr{A}_{0\pi}^2, \tag{5.18}$$

where

$$Z = \frac{3\sqrt{3}\sinh\sqrt{-\dfrac{\tilde{\chi}}{2\varepsilon}}\,\pi}{8\sqrt{-2\tilde{\chi}}}, \qquad k = \begin{cases} +1, & n \text{ even}, \\ -1, & n \text{ odd}. \end{cases} \tag{5.19}$$

We find that for *n even*

$$\mathscr{A}_0 = -\mathscr{A}_0 = 2\sqrt{-\frac{32}{27}} + O\left(\frac{1}{z}\right) \quad \text{as} \quad \varepsilon \to 0, \tag{5.20}$$

and for *n odd*, either

$$\mathscr{A}_0 = 2\sqrt{-\frac{32}{27}\tilde{\chi}} + O\left(\frac{1}{Z^2}\right), \qquad \mathscr{A}_{0\pi} = O\left(\frac{1}{Z}\right) \tag{5.21}$$

or

$$\mathscr{A}_0 = O\left(\frac{1}{Z}\right), \qquad \mathscr{A}_{0\pi} = -2\sqrt{\frac{32}{27}\tilde{\chi}} + O\left(\frac{1}{Z^2}\right) \tag{5.22}$$

as $\varepsilon \to 0$. Moreover, it is not difficult to verify that the leading-order behaviour of the outer expansions is given by ($\lambda = \lambda_n(1 + \varepsilon\tilde{\chi})$, $\varepsilon = 1/(n + \frac{1}{2})$, $n = n_{cr} + \tau$, $0 \le |\tau| < \frac{1}{2}$or $\frac{1}{2} < |\tau| < 1$)

$$\beta \sim -16\varepsilon^2\sqrt{-\frac{\tilde{\chi}}{27\pi \sin \xi}}\,[\exp(-\sqrt{-(\tilde{\chi}/2\varepsilon)}\,\xi) + \exp(\sqrt{-(\tilde{\chi}/2\varepsilon)}(\xi - \pi))] \cdot \cos\left[\frac{\xi}{\varepsilon} + (\tfrac{3}{8}\tilde{\chi} - \tau)\xi + \frac{\pi}{4}\right] \tag{5.23}$$

as $\varepsilon \to 0$ for *n even* and by either

$$\beta \sim -16\varepsilon\sqrt{\frac{-\tilde{\chi}}{27\pi \sin \xi}}\exp(-\sqrt{(-\tilde{\chi}/2\varepsilon)}\xi)\cos\left[\frac{\xi}{\varepsilon} + (\tfrac{3}{8}\tilde{\chi} - \tau)\xi + \frac{\pi}{4}\right] \quad \text{as} \quad \varepsilon \to 0, \tag{5.24}$$

or

$$\beta \sim 16\varepsilon^2\sqrt{\frac{-\tilde{\chi}}{27\pi \sin \xi}}\exp(\sqrt{(-\tilde{\chi}/2\varepsilon)}(\pi - \xi))\cos\left[\frac{\xi}{\varepsilon} + (\tfrac{3}{8}\tilde{\chi} - \tau)\xi + \frac{\pi}{4}\right] \quad \text{as} \quad \varepsilon \to 0 \tag{5.25}$$

for *n odd*. The approximations in (5.23)–(5.25) are valid except in neighborhoods of $O(\varepsilon)$ of $\xi = 0$ and $\xi = \pi$.

It is of some interest to compare the functional form of the outer approximations (4.32) and (5.23) keeping in mind that $\tilde{\chi} = \varepsilon\bar{\chi}$. For $|\tilde{\chi}| \ll 1$ the factor $\frac{3}{8}\tilde{\chi}\xi$ in the argument of the cosine in (5.23) becomes negligible, while for $|\bar{\chi}| \gg 1$ the exponential part of (4.32) approaches that of (5.23). Moreover, the limiting behavior of the coefficient $\varepsilon^{5/2}\mathscr{A}_0$ in (4.32) as $\bar{\chi} \to -\infty$ agrees with that of the coefficient in (5.23) as $\tilde{\chi} \to 0^-$. Thus, to leading order the expansions in (4.15) and (5.14) have overlapping regions of validity. Examination of higher-order terms in both expansions supports this conclusion. However, when $\tau \neq 0$, it is not difficult to show that the expansion in (5.14) becomes disordered when $\tilde{\chi} = O(\varepsilon)$. Basically the disordering arises because the approximation $\tau^2 + (\tilde{\chi}/2\varepsilon) \sim (\tilde{\chi}/2\varepsilon)$ does not hold for $\tilde{\chi} \to 0^-$ (with $\tau \neq 0$ fixed). Similar remarks hold for the *n*-odd case. The situation is very similar to that described above for the simple model function $u(\alpha, \delta)$ and it provides further evidence of the complex dependence of the branching behavior of the spherical shell on the two parameters δ and $\lambda - \lambda_n$.

We conclude this section with a few remarks on the further continuation of the solution branches. Suppose that λ satisfies the condition

$$\lambda = \lambda_n(1 + \chi_\gamma \varepsilon^\gamma), \qquad \varepsilon = \frac{1}{n + \frac{1}{2}}, \qquad \chi_\gamma = O(1), \qquad 0 < \gamma < 1. \tag{5.26}$$

Consider the inner expansions at $\xi = 0, \pi$. These expansions will have the form (say at $\xi = 0$)

$$f(\bar{\xi}) = F(\bar{\xi}) = \varepsilon^{3/2}[\varepsilon^{\gamma/2}F_0 + \varepsilon^\gamma F_1 + \cdots], \tag{5.27}$$

$$g(\bar{\xi}) = G(\bar{\xi}) = \varepsilon^{3/2}[\varepsilon^\gamma G_1 + \cdots], \tag{5.28}$$

where F_0 satisfies a linear equation. However, as $\gamma \to 0$, the nonlinear terms in the governing equations become as important as the linear terms and, as a result, (5.27)–(5.28) become of little value. When $\gamma = 0$, the problem can still be viewed as a perturbation problem with δ playing the role of the small parameter. In the inner region near $\xi = 0$ the governing equations are given (upon replacing $\varepsilon^3\chi$ by χ) by (3.16)–(3.17) for the case when $\tau = 0$. The appropriate expansions are given by

$$f(\bar{\xi}) = F(\bar{\xi}) = \varepsilon^{3/2}[F_0 + \varepsilon^2 F_1 + \cdots], \tag{5.29}$$

$$g(\bar{\xi}) = G(\bar{\xi}) = \varepsilon^{3/2}[G_0 + \varepsilon^2 G_1 + \cdots], \tag{5.30}$$

where F_0 and G_0 satisfy a coupled pair of nonlinear equations. Furthermore, F_0 and G_0 must decay exponentially as $\bar{\xi} \to \infty$ in order to match with the outer solutions. Many authors, e.g., [16, 17, 18], have considered the zeroth-order part of this nonlinear "boundary layer" problem.

6. Discussion of branching results. Our purpose in the preceeding sections was to demonstrate a method for extending the branching solutions for complete spherical shells beyond the region of validity of the standard perturbation approximations (i.e., $|\lambda - \lambda_n| \ll \delta$ or, equivalently, $|\lambda - \lambda_n| \ll 1/n^2$). As pointed out in the introduction, Koiter [3] has also developed and applied an asymptotic method to this problem. While we both recover the standard results in the limit $\lambda \to \lambda_n$, there is a considerable difference between Koiter's and our asymptotic solutions outside of the regime $|\lambda - \lambda_n| \ll \delta$. Since Koiter's and our methods of analysis are both formal it is not easy to resolve this disagreement. In this section we summarize our branching results and present numerical evidence which supports our conclusions.

First we shall discuss the branching situation for the case of the lowest *even* (simple) eigenvalue λ_n where $n = n_{cr} + \tau$ ($0 \le |\tau| < \frac{1}{2}$ or $\frac{1}{2} < |\tau| < 1$) is an even integer. It is convenient to express our results in terms of a bifurcation diagram of $\|\beta\|$ vs. λ, where we define

$$\|\beta\| \equiv \max_{0 \le \xi \le \pi} |\beta(\xi, \lambda)|. \tag{6.1}$$

The asymptotic analysis in Secs. 3–5 reveals that for sufficiently small values of $|\lambda - \lambda_n|$ the maximum of $|\beta(\xi, \lambda)|$ is taken on in the vicinity of the poles. Since the solutions which we have constructed for n even are symmetric with respect to $\xi = \pi/2$, we shall focus on the hemispherical region $0 \le \xi \le \pi/2$. Near the south pole

$$\beta(\xi, \lambda) = \frac{\mathscr{A}_0}{(n + \frac{1}{2})^2} J_1\left(\left(n + \frac{1}{2}\right)\xi\right) = O\left(\frac{\mathscr{A}_0^2}{(n + \frac{1}{2})^3}\right), \qquad \xi = O\left(\frac{1}{n + \frac{1}{2}}\right), \tag{6.2}$$

with

$$\mathscr{A}_0 = 16\omega \cdot \frac{\cos \omega\pi - \cos \tau\pi}{3\sqrt{3} \sin \omega\pi}, \qquad \omega = \left[\tau^2 + \left(n + \frac{1}{2}\right)^2 \frac{\lambda - \lambda_n}{2\lambda_n}\right]^{1/2}. \tag{6.3}$$

Away from the poles the magnitude of β is smaller than that near the poles by at least a factor of $(n + \frac{1}{2})^{-1/2}$.

As established in Sec. 4, Eqs. (6.2)–(6.3) contain the standard perturbation results. Under the restriction $|\lambda - \lambda_n| \ll 1/n^2$ (equivalently $|\lambda - \lambda_n| \ll \delta$) the leading order behavior of $\mathscr{A}_0$

is given by

$$\mathscr{A}_0 \sim \frac{4\pi}{3\sqrt{3}} \frac{\lambda_n - \lambda}{\lambda_n} (n + \tfrac{1}{2})^2, \qquad n \to \infty; \tag{6.4}$$

so that

$$\beta(\xi, \lambda) \sim \frac{4\pi}{3\sqrt{3}} \frac{\lambda_n - \lambda}{\lambda_n} J_1((n + \tfrac{1}{2})\xi), \qquad |\lambda - \lambda_n| \ll 1/n^2, \qquad \xi = O\left(\frac{1}{n + \frac{1}{2}}\right). \tag{6.5}$$

Using the fact that $\max_{x \geq 0} |J_1(x)| = 0.58187$ (occurring at $x = 1.84118$) it follows that

$$\|\beta\| \sim 1.4072 \frac{|\lambda - \lambda_n|}{\lambda_n}, \qquad |\lambda - \lambda_n| \ll 1/n^2. \tag{6.6}$$

We shall refer to (6.6) as the standard perturbation approximation to $\|\beta\|$ for n even.

As far as the structure of the bifurcation curves are concerned, the standard perturbation approximation is valid only within a neighborhood of $o(\delta)$ of the branch point where the curve is locally "linear" with an $O(1)$ slope. (Typically for thin shells δ is numerically $O(10^{-3})$.) An examination of (6.2)–(6.3) shows that near the branch point the curvature of the bifurcation curve is very large. Under the restriction $1/n^2 \ll \lambda_n - \lambda \ll 1$, we again obtain a simple expression for the leading order behavior of $\|\beta\|$ vs. λ for the downward branching solution. We find that

$$\beta(\xi, \lambda) \sim \frac{2}{(n + \frac{1}{2})} \sqrt{\frac{32}{27} \frac{\lambda_n - \lambda}{\lambda_n}}\, J_1((n + \tfrac{1}{2})\xi), \qquad \xi = O\left(\frac{1}{n + \frac{1}{2}}\right). \tag{6.7}$$

The results of Secs. 4–5 (see, in particular, (5.23)) along with further calculations indicate that β is transcendentally small away from the poles in this regime. Thus, we conclude that

$$\|\beta\| \sim \frac{1.2670}{(n + \frac{1}{2})} \sqrt{\frac{\lambda_n - \lambda}{\lambda_n}}, \qquad 1/n^2 \ll \lambda_n - \lambda \ll 1, \tag{6.8}$$

which should be compared with the standard perturbation approximation (6.6).

Eq. (6.8) has a very important consequence. One expects that the smaller $|d\|\beta\|/d\lambda|$ is (i.e., the steeper the drop of the "load-deflection" curve), the more disastrous will be the effect of axisymmetric imperfections on the buckling of thin spherical shells. On this basis we see that (6.8) describes a much greater imperfection-sensitivity than does the standard perturbation approximation. We should mention that we have carried out a detailed asymptotic analysis of the buckling of a slightly imperfect spherical shell, and after considerable calculation we derived the appropriate modification of (4.20)–(4.21). In particular we found that the standard approximation for the reduction in buckling load is only valid for extremely small initial imperfections ($O(\delta^{3/2})$ of the shell thickness). For larger (and more realistic) initial imperfections one must resort to our uniformly valid formulas which do, indeed, describe a much greater imperfection-sensitivity.

To facilitate a physical interpretation of our branching results we introduce the dimensionless, auxiliary variable $w(\xi)$ defined by

$$w(\xi) = \frac{W(\xi) - W_0}{h}, \qquad W_0 = \frac{\lambda h(1 - \nu)}{\sqrt{3(1 - \nu^2)}}, \tag{6.9}$$

where $W(\xi)$ is the radial displacement (positive inwards) of the buckled state and the quantity W_0 is the radial displacement of the uniformly contracted or unbuckled state. To the same order of accuracy as the governing equations (2.3) we obtain from [6] the following approximation to w:

$$w(\xi) = -\frac{\cos \xi}{\delta\sqrt{12(1-\nu^2)}} \int_{\pi/2}^{\xi} \left[\underline{\beta \cos \eta} + \frac{\beta^2}{2} \sin \eta + \nu\delta\psi'(\sin \eta - \beta \cos \eta) \right] d\eta$$
$$+ \frac{\sin \xi}{\sqrt{12(1-\nu^2)}} [\underline{-\psi' \sin \xi} + \lambda\nu\beta^2 \sin \xi + \nu\psi(\cos \xi + \beta \sin \xi)]. \quad (6.10)$$

Provided $|\beta|$ and $|\psi|$ are sufficiently small, the underlined terms in (6.10) yield the dominant contribution to w.

In general it is difficult to obtain a simple representation for w from (6.10). The difficulty is compounded in our case since our asymptotic solution for β (and ψ) consists of three different expansions: inner expansions valid near $\xi = 0, \pi$, and an outer expansion valid away from the poles. However, in the regime $1/n^2 \ll \lambda_n - \lambda \ll 1$ we can derive an especially simple expression for the leading order behavior of w. Upon setting $\psi \sim -\beta$, using (6.7) for β near the south pole and neglecting the exponentially small contribution of β away from the poles, we obtain from (6.10)

$$w(\xi) \sim \frac{8}{9}\sqrt{\frac{\lambda_n - \lambda}{2(1-\nu^2)\lambda_n}}\, J_0((n+\tfrac{1}{2})\xi), \qquad \xi = O\left(\frac{1}{n+\frac{1}{2}}\right), \qquad \frac{1}{n^2} \ll \lambda_n - \lambda \ll 1. \quad (6.11)$$

Moreover, w is transcendentally small away from the poles.

The conclusion we draw from (6.11) is that the deflection pattern in the post-buckled state (for n even with $1/n^2 \ll \lambda_n - \lambda \ll 1$) consists mainly of two equally deep inward dimples at the poles and little disturbance over the remainder of the shell. In particular, the dimensionless displacement at the poles is given by

$$w(0) \sim \frac{8}{9}\sqrt{\frac{\lambda_n - \lambda}{2(1-\nu^2)\lambda_n}}, \qquad \frac{1}{n^2} \ll \lambda_n - \lambda \ll 1. \quad (6.12)$$

A striking feature of the present result (6.12) is that the displacement at the poles represented as a fraction of the shell thickness is independent (to leading order) of the degree n of the buckling mode, and thus independent of the ratio h/a, in marked contrast to the corresponding result for the initial post-buckling state [3]:

$$w(0) \sim \frac{2(n+\frac{1}{2})}{9\sqrt{1-\nu^2}} \frac{\lambda_n - \lambda}{\lambda_n}, \qquad |\lambda_n - \lambda| \ll 1/n^2. \quad (6.13)$$

In order to obtain a direct check on our asymptotic results we carried out extensive numerical calculations of the branching solutions for the boundary value problem (2.3)–(2.4) for various values of δ and n. In each case the numerical solution for β and ψ was used to determine $w(\xi)$ from (6.10). We employed a recently developed code (COLSYS) which uses the method of spline-collocation at Gaussian points with a B-spline basis [19, 20]. The code uses a variable mesh and places more points in regions of rapid change in the solution.

As mentioned previously, Bauer, Reiss and Keller [5] computed branching solutions for (2.3)–(2.4) by a modified shooting technique. Unfortunately, the numerical results presented in [5] are in terms of w or integrals of w. Moreover, most of the results are for a rather thick shell ($a/h = 9.13$, corresponding to $n_{cr} \approx 5$), with few calculations recorded for thinner shells near the branch points. Still, where possible, we compared our numerical solutions with those presented in [5] and found very close agreement. The interested reader will find a detailed account of our numerical studies in [21] which, in particular, contains an interesting comparison of the solutions of the quadratic nonlinear system (2.3)–(2.4) and of Reissner's fully nonlinear equations [6] for $O(1)$ deviations $\lambda - \lambda_n$.

So as to keep the discussion brief we shall present a comparison between our asymptotic and numerical results for one case of a moderately thin shell with

$$\delta = 0.003315, \tag{6.14}$$

which corresponds to $k = 10^{-5}$ in [5] and $a/h = 91.288$ with $\nu = 0.3$. The results for this case are typical of those obtained for other values of δ. By definition

$$n_{cr} = \frac{-1 + \sqrt{5 + 4/\delta}}{2} = 16.9043, \tag{6.15}$$

so that the lowest even eigenvalue is λ_{16} and $\tau = -0.9043$. From (2.5) we obtain

$$\lambda_{16} = 1.00575. \tag{6.16}$$

The standard perturbation approximation (6.6) takes the form

$$\|\beta\| \sim 1.3992\,|\lambda - 1.00575|, \qquad |\lambda - \lambda_{16}| \ll 1/256; \tag{6.17}$$

while our extended formula is given by (using $\max_{x \geq 0} |J_1(x)| = 0.58187$)

$$\|\beta\| \sim \frac{0.58187}{(16.5)^2}\,|\mathscr{A}_0|, \qquad 0 \leq \lambda_{16} - \lambda \ll 1, \tag{6.18}$$

with $\mathscr{A}_0$ defined by (6.3).

It is clear from Fig. 1 that the region of validity of the standard perturbation approximation (6.17) is indeed restricted to a very small neighborhood of the branch point. On the other hand, we see that the extended perturbation approximation (6.18) is in reasonably close agreement with the numerical approximation—the difference being consistent with the asymptotic error bound in (6.2). In fact, the agreement is similar to that which one typically achieves by standard perturbation procedures in bifurcation problems which do not involve the complicating feature of closely spaced eigenvalues.

In order to illustrate further the accuracy of our extended perturbation solution, we present in Fig. 2 graphs of β vs. ξ for $\lambda = 0.95$ with δ given by (6.14). As we shall explain later these graphs serve to describe the solution branching from the lowest odd eigenvalue λ_{17} as well as that branching from the even eigenvalue λ_{16}. Our asymptotic solution for the interval shown in Fig. 2 consists of two parts. Near $\xi = 0$ we have the leading-order term of the *inner* expansion given by (6.7) with $n = 16$:

$$\beta(\xi, 0.95) \sim 0.031068\, J_1(16.5\xi), \tag{6.19}$$

where for practical purposes we are assuming that $1/256 \ll \lambda_{16} - 0.95 \ll 1$. Away from $\xi = 0$ the outer expansion is valid. From (5.23) it follows that the leading-order behavior of the *outer* expansion is given by

$$\beta(\xi, 9.5) \sim \frac{-0.00612}{\sqrt{\sin \xi}}\,[e^{-2.7548\xi} + e^{2.7548(\xi - \pi)}]\cos\left(17.06\xi + \frac{\pi}{4}\right). \tag{6.20}$$

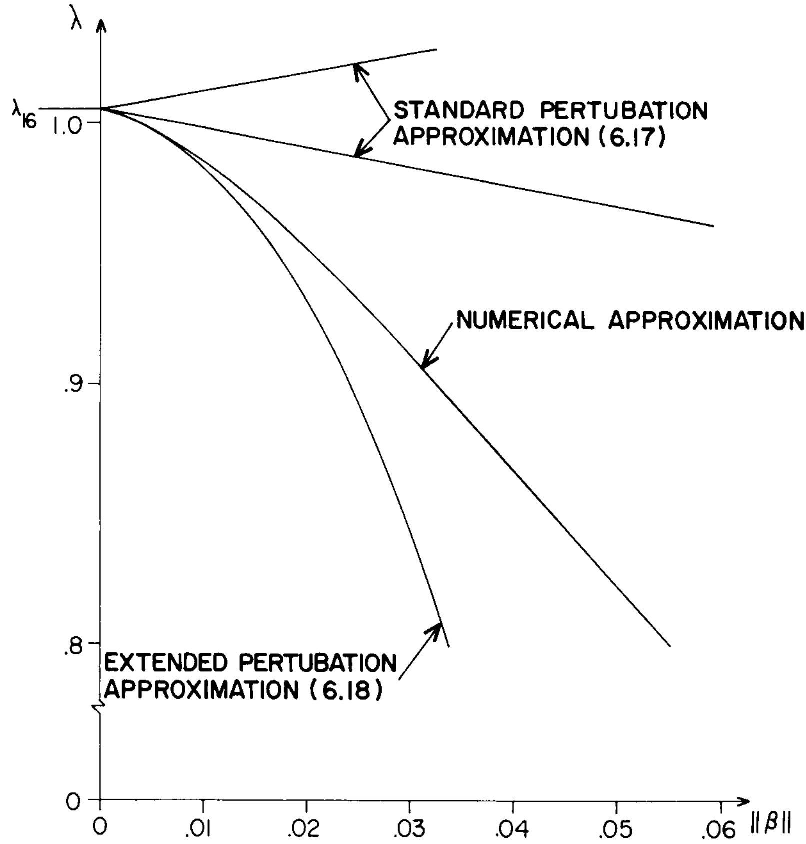

FIG. 1. Graph of λ vs. $\|\beta\|$ corresponding to the lowest even eigenvalue with $\delta = 0.003315$.

As expected, the inner expansion provides a good approximation to the branching solution near $\xi = 0$, whereas the same is true of the outer expansion away from $\xi = 0$. Observe that the ratio of the magnitude of the numerical approximation to β near $\xi = 0$ to that away from $\xi = 0$ is much greater than the corresponding ratio for the linear eigenfunction $P_{16}^1(\cos \xi)$. The presence of the exponential factor in the outer expansion (6.20) reflects this behavior as well.

Next we shall discuss the branching situation for the case of the lowest odd (simple) eigenvalue λ_n where $n = n_{cr} + \tau$ $(0 \le |\tau| < \frac{1}{2}$ or $\frac{1}{2} < |\tau| < 1)$ is an odd integer. For this case the solution of the nonlinear problem does not exhibit symmetry and so we must consider the interval $0 \le \xi \le \pi$. Near the south pole

$$\beta(\xi, \lambda) = \frac{\mathscr{A}_0}{(n + \frac{1}{2})^2} J_1((n + \tfrac{1}{2})\xi) + O\left(\frac{\mathscr{A}_0^2}{(n + \frac{1}{2})^3}\right), \qquad \xi = O\left(\frac{1}{n + \frac{1}{2}}\right), \tag{6.21}$$

with $\mathscr{A}_0$ defined in (4.30), while near the north pole

$$\beta(\xi, \lambda) = \frac{\mathscr{A}_{0\pi}}{(n + \frac{1}{2})^2} J_1((n + \tfrac{1}{2})(\pi - \xi)) + O\left(\frac{\mathscr{A}_{0\pi}^2}{(n + \frac{1}{2})^3}\right), \qquad \pi - \xi = O\left(\frac{1}{n + \frac{1}{2}}\right), \tag{6.22}$$

with $\mathscr{A}_{0\pi}$ defined in (4.31). Away from the poles the magnitude of β is smaller than that near the poles by at least a factor of $(n + \frac{1}{2})^{-1/2}$.

Under the restriction $|\lambda - \lambda_n| \ll 1/n^2$ (equivalently $|\lambda - \lambda_n| \ll \delta$) we find that the leading-order behaviors of $\mathscr{A}_0$ and $\mathscr{A}_{0\pi}$ are given by

$$\mathscr{A}_0 \sim \mathscr{A}_{0\pi} \sim \pm \frac{8\pi(n+\frac{1}{2})}{3\sqrt{3}} \sqrt{\frac{(\lambda_n - \lambda)\tau \cot \tau\pi}{\pi\lambda_n}}, \qquad n \to \infty. \tag{6.23}$$

We should point out that Koiter [3] essentially derived (6.23) for the special case $\tau = 0$. Without loss of generality we shall consider only the solution corresponding to the plus sign in (6.23) in what follows. For this solution the associated deflection shape has the form of an inward dimple at the south pole and an outward bulge at the north pole. It follows from (6.1) and (6.21)–(6.23) that

$$\|\beta\| \sim \frac{1.5878}{(n+\frac{1}{2})} \sqrt{\frac{(\lambda_n - \lambda)\tau \cot \tau\pi}{\lambda_n}}, \qquad |\lambda - \lambda_n| \ll 1/n^2. \tag{6.24}$$

We shall refer to (6.24) as the standard perturbation approximation to $\|\beta\|$ for n odd. (Actually (6.24) holds for all odd eigenvalues provided $|\tau| \neq m/2$, $m = 1, 2, \ldots$.)

The standard perturbation approximation serves to describe the structure of the bifurcation curve only within a neighborhood of $o(\delta)$ of the branch point where the curve is locally "parabolic," with a very large curvature (due to the factor $(n + \frac{1}{2})^{-1}$.) There is

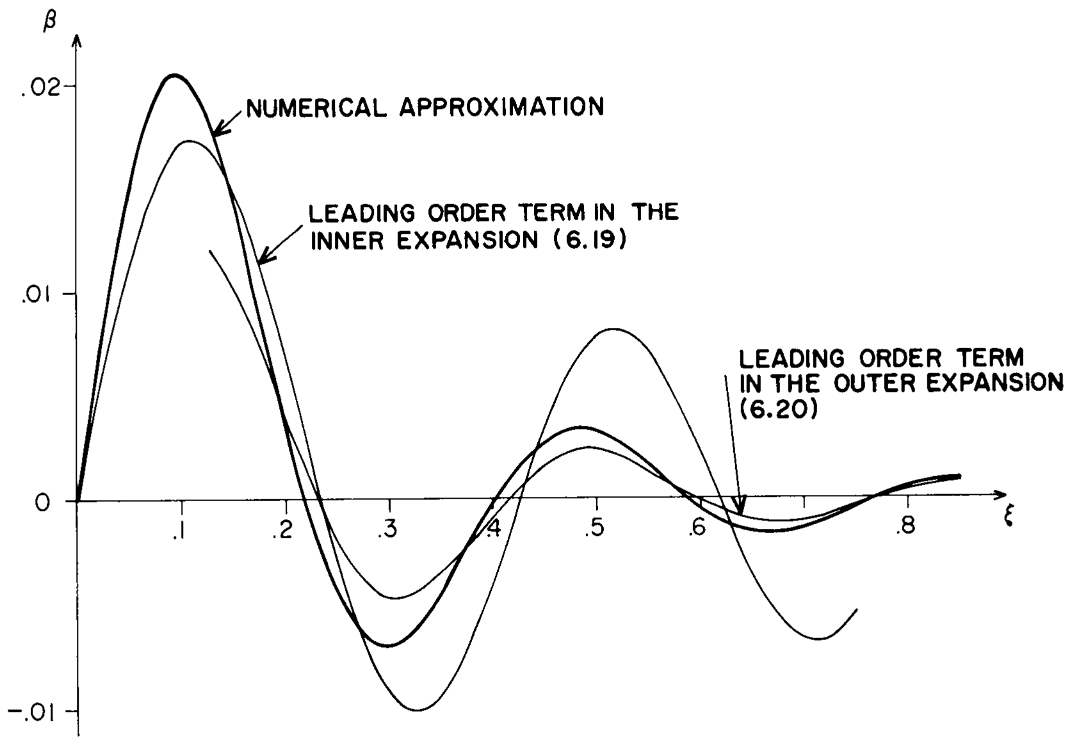

FIG. 2. Graph of β vs. ξ for $\lambda = 0.95$ corresponding to the lowest even and odd eigenvalues (λ_{16} and λ_{17}, respectively) with $\delta = 0.003315$.

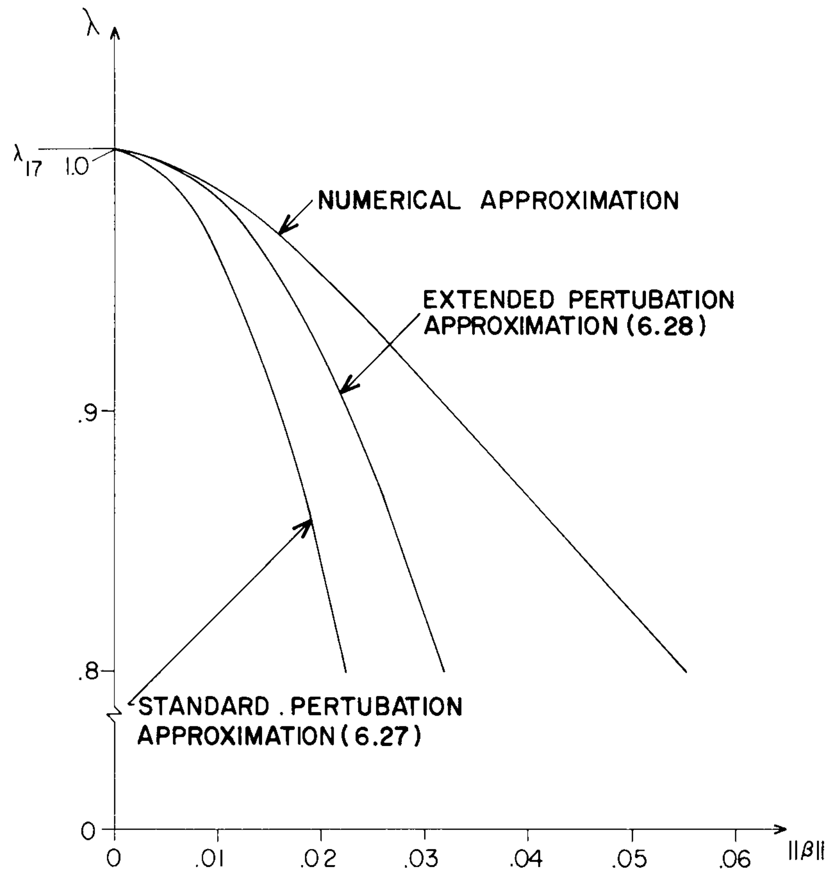

FIG. 3. Graph of λ vs. $\|\beta\|$ corresponding to the lowest odd eigenvalue with $\delta = 0.003315$.

another important difference between the n even and odd cases as recorded in (6.6) and (6.24), respectively. For sufficiently small deviations $|\lambda - \lambda_n|$ the branching solutions for n even have a weak dependence on τ, whereas the branching behavior for n odd depends crucially on the deviation of λ_n from the critical value. Clearly the value of τ affects the amplitude of the branching solution; but, even more importantly, it determines the direction of the branching. For example, if $|\tau| < \frac{1}{2}$, corresponding to the lowest eigenvalue being odd, then it follows from (6.24) that the branching is downward (see Fig. 3). On the other hand, for $\frac{1}{2} < |\tau| < 1$, the branching is upward. Of course, whenever the lowest odd eigenvalue corresponds to $\frac{1}{2} < |\tau| < 1$, the lowest eigenvalue is even, so that the branching from the lowest eigenvalue is always downward.

As far as the authors are aware, the result in (6.24) which predicts this phenomenon of alternating upward and downward branching from the odd eigenvalues is new. We should mention that formula (6.24) correctly predicts the direction of the branching both in our numerical studies and in those of Bauer, Reiss and Keller [5]. It is interesting that calculation of the energy expression in [5] indicates that in many instances the upward-branching solutions are stable equilibrium solutions.

Under the restriction $|\tau| < \frac{1}{2}$ and $1/n^2 \ll \lambda_n - \lambda \ll 1$ we are able to compute a simple asymptotic approximation to $\mathscr{A}_0$ and $\mathscr{A}_{0\pi}$. From (4.30) and (4.31) we obtain

$$\mathscr{A}_0 \sim 2(n+\tfrac{1}{2})\sqrt{\frac{32}{27}\frac{\lambda_n-\lambda}{\lambda}}, \quad \mathscr{A}_{0\pi} = O\left(\exp\left(-\pi(n+\tfrac{1}{2})\sqrt{\frac{\lambda_n-\lambda}{\lambda_n}}\right)\right), \quad \frac{1}{n^2} \ll \lambda_n - \lambda \ll 1, \tag{6.25}$$

which agrees, as it should, with (5.21)–(5.22). Comparing (6.21) and (6.25) with (6.7), we conclude that in the regime $1/n^2 \ll \lambda_n - \lambda \ll 1$ the downward-branching solutions for the lowest even and odd eigenvalues have the same leading-order behavior near the south pole. Also, in both cases β is transcendentally small away from the poles (see (5.23)–(5.24)). However, whereas β is symmetrical with respect to $\xi = \pi/2$ for n even, for n odd we have that β is transcendentally small near the north pole. It is clear that (6.8) also serves to describe the bifurcation curve in the case of n odd with $|\tau| < \frac{1}{2}$.

The above result has a very interesting physical interpretation. It is not difficult to see that when the (loading) deviation $\lambda_n - \lambda$ is large compared to h/a then the radial displacement near the south pole is given by (6.11). Furthermore, w is transcendentally small away from the south pole. The ripples between the poles and the outward bulge at the north pole have disappeared. Thus the deflection pattern in the post-buckled state (for n odd, with $|\tau| < \frac{1}{2}$ and $1/n^2 \ll \lambda_n - \lambda \ll 1$) consists mainly of a *single* inward dimple and little disturbance over the remainder of the shell. Such a deflection pattern is often observed in experiments. Moreover, our numerical results [21] as well as those of Bauer, Reiss and Keller [5] (see, in particular, their Fig. 7a) confirm this behavior. In fact, it is shown in [9, 21] that this deflection pattern is maintained until $\lambda = O(\sqrt{\delta})$ at which point the dimple takes on the form of a large inverted spherical cap joined to the remainder of the essentially undeformed spherical shell by a boundary layer region.

For completeness we shall briefly compare our asymptotic and numerical results for the value of δ given in (6.14). The lowest odd eigenvalue is

$$\lambda_{17} = 1.00006, \tag{6.26}$$

with $\tau = 0.0957$. The standard perturbation approximation in (6.24) takes the form

$$\|\beta\| \sim 0.05041\sqrt{1.00006 - \lambda}, \qquad 0 \le \lambda_{17} - \lambda \ll 1/289; \tag{6.27}$$

while our extended formula is given by

$$\|\beta\| \sim \frac{0.58187}{(17.5)^2}|\mathscr{A}_0|, \qquad 0 \le \lambda_{17} - \lambda \ll 1, \tag{6.28}$$

with $\mathscr{A}_0$ defined by (4.30).

From Fig. 3 we observe that the extended approximation is in reasonably close agreement with the numerical approximation in the region specified in (6.28). For thinner shells (corresponding to smaller values of δ and larger values of n) the agreement is even better. It is also clear from Figs. 1 and 3 that outside of the immediate vicinity of the branch points the numerical approximations to the bifurcation curves for the lowest even and odd eigenvalues are almost identical. In fact, this behavior is maintained until λ is approximately 0.1. Moreover, the spatial structures of the branching solutions for the lowest even and odd eigenvalues are very similar near the south pole. For example, the numerical approximation of the solution branching from λ_{17} with $\lambda = 0.95$ agrees to within 0.1% with that portion of the corresponding curve for λ_{16} shown in Fig. 2 (see also Table 1). As pointed out above, our extended perturbation approximations exhibit this behavior as well.

We conclude our discussion with a few remarks regarding Koiter's refined asymptotic analysis as presented in [3]. While we are in basic agreement with Koiter regarding the qualitative nature of the post-buckling behavior for even eigenvalues, there is a considerable difference between our results for odd eigenvalues. He finds that (for $\tau = 0, n = n_{cr}$) the radial displacement at the poles satisfies

$$w(0) \sim w(\pi) \sim \frac{\pi(n + \frac{1}{2})(1 - \lambda)}{9\sqrt{1 - \nu^2}}, \qquad \frac{1}{n^2} \ll 1 - \lambda \ll 1, \tag{6.29}$$

for *both* even and odd eigenvalues. In other words, he contends that the post-buckled state associated with the lowest eigenvalue always consists of two equally deep inward dimples at the poles in the regime designated in (6.29). Although it is very difficult to resolve this disagreement, we suspect that the expansion in (8.6) in [3] is nonuniform in δ, just as the standard perturbation expansion itself is. (Koiter actually alludes to this possibility on page 100 of [3].)

In Table 1 we present numerical evidence supporting our claim that the post-buckling state for the lowest odd eigenvalue (with $|\tau| < \frac{1}{2}$) involves a single inward dimple. The value of δ is that given in (6.14). The fact that our branching results both agree with numerical calculations and accurately predict the deflection patterns observed in experiments provides a convincing argument for their validity. Thus, one can with confidence apply our asymptotic integration technique to more complicated problems (involving several parameters or non-axisymmetrical buckling shapes).

TABLE 1. Radial deflection at the poles for solutions bifurcating from the lowest two eigenvalues $\lambda_{17} = 1.00006$ and $\lambda_{16} = 1.00575$ with $\delta = 0.003315$.

λ	$w(0)$ for λ_{16}	$w(0)$ for λ_{17}	$w(\pi)$ for λ_{17}
1.001890	$0.2592 \cdot 10^{-1}$	—	—
1.000015	$0.4105 \cdot 10^{-1}$	$0.3431 \cdot 10^{-2}$	$-0.2848 \cdot 10^{-2}$
0.999015	$0.4671 \cdot 10^{-1}$	$0.2049 \cdot 10^{-1}$	$-0.8136 \cdot 10^{-2}$
0.99	$0.8249 \cdot 10^{-1}$	$0.7907 \cdot 10^{-1}$	$-0.2505 \cdot 10^{-2}$
0.975	0.1322	0.1312	$-0.4028 \cdot 10^{-3}$
0.95	0.2017	0.2012	$-0.1351 \cdot 10^{-4}$
0.9	0.3254	0.3246	$-0.5853 \cdot 10^{-5}$
0.7	0.8367	0.8351	$-0.5168 \cdot 10^{-6}$
0.5	1.587	1.584	$0.8867 \cdot 10^{-8}$
0.3	3.385	3.379	$0.1758 \cdot 10^{-7}$

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