# The Bakry-Emery Ricci tensor and its applications to some compactness theorems 

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#### Abstract

Let $(M, g)$ be a complete and connected Riemannian manifold of dimension $n$. By using the Bakry-Emery Ricci curvature tensor on $M$, we prove two theorems which correspond to the Myers compactness theorem.


Keywords Laplacian of distance function • Index form • Diameter estimate

## 1 Introduction

The purpose of this article is to generalize the well-known Myers compactness theorem [5] by using the Bakry-Emery Ricci curvature (see [1,4] and [6]) on a complete and connected Riemannian manifold ( $M, g$ ) of dimension $n$.

In [6] (p. 380, Theorem 1.4), G. Wei and W. Wylie assumed that the Bakry-Emery Ricci curvature has a positive lower bound, i.e.,

$$
\begin{equation*}
\operatorname{Ric}+\operatorname{Hess}(\phi) \geq(n-1) H>0 \tag{1}
\end{equation*}
$$

and also assumed that $|\phi| \leq k$, where $\phi \in \mathcal{C}^{\infty}(M)$ is a smooth function. Under these assumptions, they proved that $M$ is compact and diameter has the upper bound

$$
\begin{equation*}
\operatorname{diam}(M) \leq \frac{\pi}{\sqrt{H}}+\frac{4 k}{(n-1) \sqrt{H}} . \tag{2}
\end{equation*}
$$

In the following, we consider the same assumptions given by G. Wei and W. Wylie, but, for the diameter of $M$, we obtain a different upper bound which can be compared with (2):

Theorem 1 Let $(M, g)$ be a complete and connected Riemannian manifold of dimension $n$. If $(M, g)$ admits a smooth function $\phi \in \mathcal{C}^{\infty}(M)$ satisfying the inequalities

$$
\begin{equation*}
\operatorname{Ric}+\operatorname{Hess}(\phi) \geq(n-1) H>0 \tag{3}
\end{equation*}
$$

[^0]and $|\phi| \leq k$, then $M$ is compact and the diameter satisfies
\[

$$
\begin{equation*}
\operatorname{diam}(M) \leq \frac{\pi}{\sqrt{H}} \sqrt{1+\frac{2 \sqrt{2} k}{n-1}} . \tag{4}
\end{equation*}
$$

\]

Comparing (2) and (4), we see that, when the positive constant $k$ satisfies

$$
\begin{equation*}
k>\frac{(n-1) \pi}{8}(\sqrt{2} \pi-4), \tag{5}
\end{equation*}
$$

the upper bound (4) is sharper than the upper bound (2).
Instead of the assumption $|\phi| \leq k$ given in Theorem 1, we can assume that $g(\nabla \phi, \nabla \phi) \in$ $\mathcal{C}^{\infty}(M)$ has an upper bound: In [2], Fernández-López and García-Río proved that, if $(M, g)$ admits a vector field $V$ satisfying the inequality Ric $+\mathcal{L}_{V} g \geq c>0$ where $\mathcal{L}_{V}$ denotes the Lie derivative, and $\sqrt{g(V, V)}$ has an upper bound, then $M$ is compact. However, no an upper bound to the diameter of $M$ is given in [2]. In [3], such a bound was obtained for the diameter of $M$. Namely, if we have the inequalities Ric $+\mathcal{L}_{V} g \geq(n-1) H>0$ and $\sqrt{g(V, V)} \leq \gamma$, then $M$ is compact and has

$$
\begin{equation*}
\operatorname{diam}(M) \leq \frac{\pi}{(n-1) H}\left(\sqrt{2} \gamma+\sqrt{2 \gamma^{2}+(n-1)^{2} H}\right) \tag{6}
\end{equation*}
$$

(see [3]). When the vector field $V$ is taken to be $V=\frac{1}{2} \nabla \phi$, the above inequalities yield Ric $+\operatorname{Hess}(\phi) \geq(n-1) H>0$ and $g(\nabla \phi, \nabla \phi) \leq 4 \gamma^{2}$. The inequality (6) still holds and, under the $K=4 \gamma^{2}$, it can be also written as

$$
\begin{equation*}
\operatorname{diam}(M) \leq \frac{\pi}{(n-1) H}\left(\sqrt{\frac{K}{2}}+\sqrt{\frac{K}{2}+(n-1)^{2} H}\right) \tag{7}
\end{equation*}
$$

In the following Theorem $2, g(\nabla \phi, \nabla \phi) \in \mathcal{C}^{\infty}(M)$ has again an upper bound, but now it depends on both the positive constant $K$ and a distance function $r=d(., p)$ with respect to a fixed point $p \in M$. We obtain an upper bound for the diameter of $M$. It can be compared with the above bound (7):

Theorem 2 Let $(M, g)$ be a complete and connected Riemannian manifold of dimension $n$, and let $r$ be the distance function with respect to a fixed point $p \in M$, i.e., $r(x)=d(x, p)$. Suppose that $(M, g)$ admits a smooth function $\phi \in \mathcal{C}^{\infty}(M)$ such that

$$
\begin{equation*}
(g(\nabla \phi, \nabla \phi))(x) \leq \frac{K}{r^{2}(x)} \tag{8}
\end{equation*}
$$

for all $x \in M-\{p\}$, where $K$ is a positive constant. If $(M, g)$ has the inequality

$$
\begin{equation*}
\operatorname{Ric}+\operatorname{Hess}(\phi) \geq(n-1) H>0, \tag{9}
\end{equation*}
$$

then $M$ is compact and the diameter from $p$ satisfies

$$
\begin{equation*}
\operatorname{diam}_{p}(M) \leq \sqrt{4 \sqrt{K}+n-1} \frac{\pi}{\sqrt{(n-1) H}} \tag{10}
\end{equation*}
$$

In the Theorem 2, the diameter bound is given with respect to the point $p \in M$. In other words, the bound is for " $\operatorname{diam}_{p}(M)$ " not for "diam $(M)$ ". But, by using the triangle inequality, we get

$$
\begin{align*}
\operatorname{diam}(M) & =d\left(p^{\prime}, q^{\prime}\right) \leq d\left(p^{\prime}, p\right)+d\left(q^{\prime}, p\right) \\
& \leq \sqrt{4 \sqrt{K}+n-1} \frac{2 \pi}{\sqrt{(n-1) H}} \tag{11}
\end{align*}
$$

where the distance between the points $p^{\prime}$ and $q^{\prime}\left(p^{\prime}, q^{\prime} \in M\right)$ gives the diameter of $M$. For the case $p \neq p^{\prime}$ and $p \neq q^{\prime}$, comparing (7) and (11), we see that, when the positive constant $\sqrt{K}$ satisfies

$$
\begin{equation*}
\sqrt{K} \geq 16(n-1) H\left(1+\sqrt{1+\frac{3}{128 H}}\right) \tag{12}
\end{equation*}
$$

the upper bound (11) is sharper than the upper bound (7). If $p \in M$ directly gives the diameter of $M$, i.e., $p=p^{\prime}$ (or $p=q^{\prime}$ ), then we can compare the bounds (7) and (10). In this special case where $\operatorname{diam}_{p}(M)=\operatorname{diam}(M)$, we see that, when the positive constant $\sqrt{K}$ satisfies

$$
\begin{equation*}
\sqrt{K} \geq 8(n-1) H \tag{13}
\end{equation*}
$$

the upper bound (10) is sharper than the upper bound (7).
In order to prove the Theorem 1, we use the index form I of a minimizing unit speed geodesic segment. To prove the Theorem 2, we establish a comparison estimate for a modified Laplacian operator.

## 2 Proofs of the theorems

The gradient, Hessian and Laplacian of any smooth function $f \in \mathcal{C}^{\infty}(M)$ are defined by $g(\nabla f, V)=V(f),(\operatorname{Hess}(f))(V, W)=g\left(\nabla_{V} \nabla f, W\right)$ and $\Delta f=\operatorname{tr}(\nabla \nabla f)$ for all vector field $V, W$, respectively. For a distance function $r(x)=d(x, p)$ where $p \in M$ is a fixed point, it is well-known that $r$ is only smooth on $M-\left(C_{p} \cup\{p\}\right)$ where $C_{p}$ denotes the cut locus of the point $p \in M$. In addition to this fact, we have $\nabla r=\partial_{r}$ in the adapted coordinates with respect to the $r$, and also have $g(\nabla r, \nabla r)=1$ where $r$ is smooth.

Proof of Theorem 1 Let $p, q \in M$ and let $\sigma$ be a minimizing unit speed geodesic segment from $p$ to $q$ of length $\ell$. Considering a parallel orthonormal frame $\left\{E_{1}=\dot{\sigma}, E_{2}, \ldots, E_{n}\right\}$ along $\sigma$ and a smooth function $f \in \mathcal{C}^{\infty}([0, \ell])$ such that $f(0)=f(\ell)=0$, we have

$$
\begin{equation*}
\mathrm{I}\left(f E_{i}, f E_{i}\right)=\int_{0}^{\ell}\left(g\left(\dot{f} E_{i}, \dot{f} E_{i}\right)-g\left(R\left(f E_{i}, \dot{\sigma}\right) \dot{\sigma}, f E_{i}\right)\right) d t \tag{14}
\end{equation*}
$$

where I denotes the index form of $\sigma$. From (14), we obtain

$$
\begin{equation*}
\sum_{i=2}^{n} \mathrm{I}\left(f E_{i}, f E_{i}\right)=\int_{0}^{\ell}\left((n-1) \dot{f}^{2}-f^{2} \operatorname{Ric}(\dot{\sigma}, \dot{\sigma})\right) d t \tag{15}
\end{equation*}
$$

by $g(R(\dot{\sigma}, \dot{\sigma}) \dot{\sigma}, \dot{\sigma})=0$. Using the assumption (3) given in Theorem 1 in the integral expression (15), we get

$$
\begin{align*}
\sum_{i=2}^{n} \mathrm{I}\left(f E_{i}, f E_{i}\right) & \leq \int_{0}^{\ell}\left((n-1)\left(\dot{f}^{2}-H f^{2}\right)+f^{2}(\operatorname{Hess}(\phi))(\dot{\sigma}, \dot{\sigma})\right) d t \\
& =\int_{0}^{\ell}\left((n-1)\left(\dot{f}^{2}-H f^{2}\right)+f^{2} g\left(\nabla_{\dot{\sigma}} \nabla \phi, \dot{\sigma}\right)\right) d t \\
& =\int_{0}^{\ell}\left((n-1)\left(\dot{f}^{2}-H f^{2}\right)+f^{2} \dot{\sigma}(g(\nabla \phi, \dot{\sigma}))\right) d t \tag{16}
\end{align*}
$$

where we have used the parallelism of the metric tensor $g$ and $\nabla_{\dot{\sigma}} \dot{\sigma}=0$. In the expression (16), the term $f^{2} \dot{\sigma}(g(V, \dot{\sigma}))$ equals to

$$
\begin{equation*}
f^{2} \dot{\sigma}(g(\nabla \phi, \dot{\sigma}))=f^{2} \frac{d}{d t}(g(\nabla \phi, \dot{\sigma})(\sigma(t))) . \tag{17}
\end{equation*}
$$

When $g(\nabla \phi, \dot{\sigma})(\sigma(t))$ is denoted by $g(\nabla \phi, \dot{\sigma})$ for short, the expression (17) can be written as

$$
\begin{equation*}
f^{2} \dot{\sigma}(g(\nabla \phi, \dot{\sigma}))=-2 f \dot{f} g(\nabla \phi, \dot{\sigma})+\frac{d}{d t}\left(f^{2} g(\nabla \phi, \dot{\sigma})\right) . \tag{18}
\end{equation*}
$$

Here we also have $g(\nabla \phi, \dot{\sigma})=\dot{\sigma}(\phi)=\frac{d}{d t} \phi(\sigma(t))\left(=\frac{d \phi}{d t}\right.$ for short). Thus, the equation (18) yields

$$
\begin{align*}
f^{2} \dot{\sigma}(g(\nabla \phi, \dot{\sigma})) & =-2 f \dot{f} \frac{d \phi}{d t}+\frac{d}{d t}\left(f^{2} g(\nabla \phi, \dot{\sigma})\right) \\
& =2 \phi \frac{d}{d t}(f \dot{f})-2 \frac{d}{d t}(\phi f \dot{f})+\frac{d}{d t}\left(f^{2} g(\nabla \phi, \dot{\sigma})\right) \tag{19}
\end{align*}
$$

Integrating both sides of (19), we obtain

$$
\begin{align*}
\int_{0}^{\ell} f^{2} \dot{\sigma}(g(\nabla \phi, \dot{\sigma})) d t & =\int_{0}^{\ell} 2 \phi \frac{d}{d t}(f \dot{f}) d t-\left.2(\phi f \dot{f})\right|_{0} ^{\ell}+\left.\left(f^{2} g(\nabla \phi, \dot{\sigma})\right)\right|_{0} ^{\ell} \\
& =2 \int_{0}^{\ell} \phi \frac{d}{d t}(f \dot{f}) d t \tag{20}
\end{align*}
$$

because of $f(0)=f(\ell)=0$. Now if we take $P=\phi$ and $Q=\frac{d}{d t}(f \dot{f})$, then we have, from the Cauchy-Schwarz inequality,

$$
\begin{align*}
\int_{0}^{\ell} P Q d t \leq\left(\int_{0}^{\ell} P^{2} d t\right)^{1 / 2}\left(\int_{0}^{\ell} Q^{2} d t\right)^{1 / 2}  \tag{21}\\
\int_{0}^{\ell} \phi \frac{d}{d t}(f \dot{f}) d t \leq\left(\int_{0}^{\ell} \phi^{2} d t\right)^{1 / 2}\left(\int_{0}^{\ell}\left(\frac{d}{d t}(f \dot{f})\right)^{2} d t\right)^{1 / 2} . \tag{22}
\end{align*}
$$

Using the assumption $|\phi| \leq k$ given in Theorem 1 in (22), we obtain

$$
\begin{equation*}
\int_{0}^{\ell} \phi \frac{d}{d t}(f \dot{f}) d t \leq k \sqrt{\ell}\left(\int_{0}^{\ell}\left(\frac{d}{d t}(f \dot{f})\right)^{2} d t\right)^{1 / 2} \tag{23}
\end{equation*}
$$

Thus, by using (23), the equation (20) yields

$$
\begin{equation*}
\int_{0}^{\ell} f^{2} \dot{\sigma}(g(\nabla \phi, \dot{\sigma})) d t \leq 2 k \sqrt{\ell}\left(\int_{0}^{\ell}\left(\frac{d}{d t}(f \dot{f})\right)^{2} d t\right)^{1 / 2} \tag{24}
\end{equation*}
$$

By virtue of (24), the inequality (16) becomes

$$
\begin{align*}
\sum_{i=2}^{n} \mathrm{I}\left(f E_{i}, f E_{i}\right) \leq & \int_{0}^{\ell}\left((n-1)\left(\dot{f}^{2}-H f^{2}\right)\right) d t \\
& +2 k \sqrt{\ell}\left(\int_{0}^{\ell}\left(\frac{d}{d t}(f \dot{f})\right)^{2} d t\right)^{1 / 2} . \tag{25}
\end{align*}
$$

In (25), if the function $f$ is taken to be $f(t)=\sin \left(\frac{\pi}{\ell} t\right)$, then we get

$$
\begin{align*}
\sum_{i=2}^{n} \mathrm{I}\left(f E_{i}, f E_{i}\right) \leq & (n-1) \int_{0}^{\ell}\left(\frac{\pi^{2}}{\ell^{2}} \cos ^{2}\left(\frac{\pi}{\ell} t\right)-H \sin ^{2}\left(\frac{\pi}{\ell} t\right)\right) d t \\
& +\frac{2 k \pi^{2}}{\ell \sqrt{\ell}}\left(\int_{0}^{\ell} \cos ^{2}\left(\frac{2 \pi}{\ell} t\right) d t\right)^{1 / 2} \tag{26}
\end{align*}
$$

and consequently

$$
\begin{equation*}
\sum_{i=2}^{n} \mathrm{I}\left(f E_{i}, f E_{i}\right) \leq-\frac{1}{2 \ell}\left((n-1) H \ell^{2}-2 \sqrt{2} k \pi^{2}-(n-1) \pi^{2}\right) \tag{27}
\end{equation*}
$$

Here, if $(n-1) H \ell^{2}-2 \sqrt{2} k \pi^{2}-(n-1) \pi^{2}>0$, then one has

$$
\begin{equation*}
\sum_{i=2}^{n} \mathrm{I}\left(f E_{i}, f E_{i}\right)=\mathrm{I}\left(f E_{2}, f E_{2}\right)+\mathrm{I}\left(f E_{3}, f E_{3}\right) \ldots \mathrm{I}\left(f E_{n}, f E_{n}\right)<0 \tag{28}
\end{equation*}
$$

which implies $\mathrm{I}\left(f E_{m}, f E_{m}\right)<0$, for some $2 \leq m \leq n$,. Namely, the index form I is not positive semi-definite. However, this result contradicts with $\sigma$ being minimizing geodesic. Hence, we must take

$$
\begin{equation*}
(n-1) H \ell^{2}-2 \sqrt{2} k \pi^{2}-(n-1) \pi^{2} \leq 0 . \tag{29}
\end{equation*}
$$

This inequality gives

$$
\begin{equation*}
\ell \leq \frac{\pi}{\sqrt{H}} \sqrt{1+\frac{2 \sqrt{2} k}{n-1}} \tag{30}
\end{equation*}
$$

Thus, we have proved Theorem 1.

Proof of Theorem 2 To prove Theorem 2, we consider a modified Laplace operator $\widetilde{\Delta}$ defined by

$$
\begin{equation*}
\widetilde{\Delta} f=\Delta f-g(\nabla \phi, \nabla f)+F(f), \tag{31}
\end{equation*}
$$

where $\phi \in \mathcal{C}^{\infty}(M)$ is given in Theorem 2 and $F$ is a real valued smooth function defined on a subset of real line, and $F(f)$ denotes $F \circ f$. In the equation (31), when $f$ is taken to be the distance function $r$ given in Theorem 2, we obtain, on $M-\left(C_{p} \cup\{p\}\right)$,

$$
\begin{align*}
g(\nabla r, \nabla \widetilde{\Delta} r) & =g(\nabla r, \nabla \Delta r-\nabla g(\nabla \phi, \nabla r)+\nabla F(r)) \\
& =g(\nabla r, \nabla \Delta r)-g(\nabla r, \nabla g(\nabla \phi, \nabla r))+F^{\prime}(r) \\
& =g(\nabla r, \nabla \Delta r)-(\operatorname{Hess}(\phi))(\nabla r, \nabla r)+F^{\prime}(r) \tag{32}
\end{align*}
$$

where $F^{\prime}(r)=\frac{d}{d r} F(r)$. On the other hand, we have the well-known inequality

$$
\begin{equation*}
0 \geq \operatorname{Ric}(\nabla r, \nabla r)+\frac{1}{n-1}(\Delta r)^{2}+g(\nabla r, \nabla \Delta r) \tag{33}
\end{equation*}
$$

on $M-\left(C_{p} \cup\{p\}\right)$. From (32) and (33), we find

$$
\begin{align*}
0 \geq & \operatorname{Ric}(\nabla r, \nabla r)+(\operatorname{Hess}(\phi))(\nabla r, \nabla r)-F^{\prime}(r) \\
& +\frac{1}{n-1}(\Delta r)^{2}+g(\nabla r, \nabla \widetilde{\Delta} r) . \tag{34}
\end{align*}
$$

It is obvious that we have

$$
\begin{equation*}
\Delta r=\widetilde{\Delta} r+g(\nabla \phi, \nabla r)-F(r) \tag{35}
\end{equation*}
$$

by (31). Inserting (35) into (34), we obtain

$$
\begin{align*}
0 \geq & \operatorname{Ric}(\nabla r, \nabla r)+(\operatorname{Hess}(\phi))(\nabla r, \nabla r)-F^{\prime}(r) \\
& +\frac{1}{n-1}(\widetilde{\Delta} r+g(\nabla \phi, \nabla r)-F(r))^{2}+g(\nabla r, \nabla \widetilde{\Delta} r) . \tag{36}
\end{align*}
$$

By virtue of the inequality $(a \mp b)^{2} \geq \frac{1}{\gamma+1} a^{2}-\frac{1}{\gamma} b^{2}$ for all real numbers $a, b$ and positive real number $\gamma>0$, we obtain

$$
\begin{align*}
(\tilde{\Delta} r+g(\nabla \phi, \nabla r)-F(r))^{2} \geq & \frac{1}{\gamma+1}(\tilde{\Delta} r+g(\nabla \phi, \nabla r))^{2} \\
& -\frac{1}{\gamma}(F(r))^{2} . \tag{37}
\end{align*}
$$

Using the same inequality, for the term " $(\widetilde{\Delta} r+g(\nabla \phi, \nabla r))^{2} "$ in the above inequality, we get

$$
\begin{align*}
(\widetilde{\Delta} r+g(\nabla \phi, \nabla r)-F(r))^{2} \geq & \frac{1}{(\gamma+1) \eta+\gamma+1}(\widetilde{\Delta} r)^{2}-\frac{1}{\gamma}(F(r))^{2} \\
& -\frac{1}{(\gamma+1) \eta}(g(\nabla \phi, \nabla r))^{2} \tag{38}
\end{align*}
$$

for all $\gamma, \eta>0$. Inserting (38) into (36) and denoting $\alpha=(n-1) \gamma>0, \beta=(n-1)(\gamma+1) \eta>0$ we obtain, on $M-\left(C_{p} \cup\{p\}\right)$,

$$
\begin{align*}
0 \geq & \operatorname{Ric}(\nabla r, \nabla r)+(\operatorname{Hess}(\phi))(\nabla r, \nabla r)+\frac{1}{\alpha+\beta+n-1}(\widetilde{\Delta} r)^{2} \\
& +g(\nabla r, \nabla \widetilde{\Delta} r)-F^{\prime}(r)-\frac{1}{\alpha}(F(r))^{2}-\frac{1}{\beta}(g(\nabla \phi, \nabla r))^{2} . \tag{39}
\end{align*}
$$

From the Cauchy-Schwarz inequality, we have

$$
\begin{equation*}
(g(\nabla \phi, \nabla r))^{2} \leq g(\nabla \phi, \nabla \phi) g(\nabla r, \nabla r)=g(\nabla \phi, \nabla \phi), \tag{40}
\end{equation*}
$$

which implies

$$
\begin{equation*}
-\frac{1}{\beta}(g(\nabla \phi, \nabla r))^{2} \geq-\frac{1}{\beta} g(\nabla \phi, \nabla \phi) . \tag{41}
\end{equation*}
$$

Using (41) in (39), one has

$$
\begin{align*}
0 \geq & \operatorname{Ric}(\nabla r, \nabla r)+(\operatorname{Hess}(\phi))(\nabla r, \nabla r)+\frac{1}{\alpha+\beta+n-1}(\widetilde{\Delta} r)^{2} \\
& +g(\nabla r, \nabla \widetilde{\Delta} r)-F^{\prime}(r)-\frac{1}{\alpha}(F(r))^{2}-\frac{1}{\beta} g(\nabla \phi, \nabla \phi) . \tag{42}
\end{align*}
$$

From the assumption (8) given in Theorem 2, we obtain, on $M-\left(C_{p} \cup\{p\}\right)$,

$$
\begin{align*}
0 \geq & \operatorname{Ric}(\nabla r, \nabla r)+(\operatorname{Hess}(\phi))(\nabla r, \nabla r)+\frac{1}{\alpha+\beta+n-1}(\widetilde{\Delta} r)^{2} \\
& +g(\nabla r, \nabla \widetilde{\Delta} r)-F^{\prime}(r)-\frac{1}{\alpha}(F(r))^{2}-\frac{K}{\beta r^{2}} . \tag{43}
\end{align*}
$$

In the above expression, if we take $\beta=\frac{4 K}{\alpha}$ and $F(r)=\frac{\alpha}{2 r}$, then the inequality (43) yields

$$
\begin{align*}
0 \geq & \operatorname{Ric}(\nabla r, \nabla r)+(\operatorname{Hess}(\phi))(\nabla r, \nabla r) \\
& +\frac{\alpha}{\alpha^{2}+(n-1) \alpha+4 K}(\widetilde{\Delta} r)^{2}+g(\nabla r, \nabla \widetilde{\Delta} r) . \tag{44}
\end{align*}
$$

Applying the assumption (9) given in Theorem 2 to (44), it follows that

$$
\begin{equation*}
0 \geq \partial_{r}(\tilde{\Delta} r)+\frac{\alpha}{\alpha^{2}+(n-1) \alpha+4 K}(\tilde{\Delta} r)^{2}+(n-1) H . \tag{45}
\end{equation*}
$$

Because of $\widetilde{\Delta} r=\Delta r-g(\nabla \phi, \nabla r)+\frac{\alpha}{2 r}$, we have

$$
\begin{align*}
\lim _{r \rightarrow 0^{+}} r \widetilde{\Delta} r & =\lim _{r \rightarrow 0^{+}}\left(r \Delta r-r g(\nabla \phi, \nabla r)+\frac{\alpha}{2}\right)  \tag{46}\\
& =n-1+\frac{\alpha}{2} \leq \frac{\alpha^{2}+(n-1) \alpha+4 K}{\alpha} . \tag{47}
\end{align*}
$$

Thus, with the aid of the well-known Sturm-Liouville comparison argument, we obtain

$$
\begin{equation*}
\widetilde{\Delta} r \leq \sqrt{(n-1) H\left(n-1+\alpha+\frac{4 K}{\alpha}\right)} \cot \left(\frac{\sqrt{\alpha(n-1) H}}{\sqrt{\alpha^{2}+(n-1) \alpha+4 K}} r\right) \tag{48}
\end{equation*}
$$

on $M-\left(C_{p} \cup\{p\}\right)$. To conclude the proof of Theorem 2, we can use the same arguments given in [7]: Let $q \in M$ and let $\sigma$ be a minimizing unit speed geodesic segment from $p$ to $q$ where the point $p \in M$ is given in Theorem 2. Assume that

$$
\begin{equation*}
d(p, q)>\frac{\sqrt{\alpha^{2}+(n-1) \alpha+4 K}}{\sqrt{\alpha(n-1) H}} \pi . \tag{49}
\end{equation*}
$$

Then, since $\sigma$ is a minimizing unit speed geodesic segment from $p \in M$ to $q \in M$, we have the fact

$$
\begin{equation*}
\sigma\left(\frac{\sqrt{\alpha^{2}+(n-1) \alpha+4 K}}{\sqrt{\alpha(n-1) H}} \pi\right) \in M-\left(C_{p} \cup\{p\}\right) . \tag{50}
\end{equation*}
$$

Thus the distance function $r$ is smooth at this point. Namely, at this point, left hand side of (48) is a constant. But it is obvious that, when

$$
\begin{equation*}
r \rightarrow\left(\frac{\sqrt{\alpha^{2}+(n-1) \alpha+4 K}}{\sqrt{\alpha(n-1) H}} \pi\right)^{-} \tag{51}
\end{equation*}
$$

right hand side of (48) goes to $-\infty$. This is a contradiction. Hence must be

$$
\begin{equation*}
d(p, q) \leq \frac{\sqrt{\alpha^{2}+(n-1) \alpha+4 K}}{\sqrt{\alpha(n-1) H}} \pi . \tag{52}
\end{equation*}
$$

Here $\alpha=2 \sqrt{K}$ gives the minimum value of right hand side of (52). Inserting $\alpha=2 \sqrt{K}$ into (52), we find

$$
\begin{equation*}
d(p, q) \leq \sqrt{4 \sqrt{K}+n-1} \frac{\pi}{\sqrt{(n-1) H}} \tag{53}
\end{equation*}
$$

Thus, we have proved Theorem 2.
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