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# THE BALANCED-PROJECTIVE DIMENSION OF ABELIAN $p$-GROUPS

BY

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ABSTRACT. The balanced-projective dimension of every abelian $p$-group is determined by means of a structural property that generalizes the third axiom of countability. As a corollary to our general structure theorem, we show for $\lambda = \omega_n$ that every $p^\lambda$-high subgroup of a $p$-group $G$ has balanced-projective dimension exactly $n$ whenever $G$ has cardinality $\aleph_n$ but $p^\lambda G \neq 0$. Our characterization of balanced-projective dimension also leads to new classes of groups where different infinite dimensions are distinguished.

**0. Introduction.** We consider throughout $p$-primary abelian groups, or equivalently, torsion modules over the integers localized at $p$. With possible generalizations in mind, we shall refer to them as "modules".

Recall that a submodule $N$ of $M$ is said to be *isotype* if $p^\alpha M \cap N = p^\alpha N$ for all ordinals $\alpha$, and *nice* if $\langle p^\alpha M, N\rangle/N = p^\alpha(M/N)$ for all $\alpha$. If $N$ is both isotype and nice, it is said to be *balanced* in $M$, and the corresponding exact sequence $0 \to N \to M \to M/N \to 0$ is called *balanced-exact*. This notion gives rise to a relative homological algebra. The category of balanced-exact sequences has enough projectives, these are the totally projective $p$-groups. The *balanced-projective dimension* (b.p.d.) of a module $M$ can be defined in the usual fashion: There is an exact sequence

$$\cdots \to T_n \xrightarrow{\delta_n} T_{n-1} \to \cdots \to T_1 \xrightarrow{\delta_1} T_0 \xrightarrow{\delta_0} M \to 0,$$

where $T_n$ is totally projective for each $n$ and the image of $\delta_n$ is balanced in $T_{n-1}$. We set b.p.d. $M = n$ if $n$ is the smallest index $\geqslant 0$ with $\operatorname{Im}\delta_n$ totally projective, and define b.p.d. $M = \infty$ if no such $n$ exists. An obvious version of Schanuel's lemma guarantees that $n$ is well defined. Evidently, b.p.d. $M = 0$ if and only if $M$ is totally projective.

We are able to characterize, for each $n$, the modules that have balanced-projective dimension $n$. Our characterization (Theorem 4.5) generalizes the well-known characterization of totally projective $p$-groups as modules satisfying the third axiom of countability **[H1]**. Two other versions of the third axiom concept **[H2]** also carry over to the more general case (Theorem 3.2).

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As an application, we show that there are modules having arbitrary balanced-projective dimension.

Our characterization leads to some new classes of modules (abelian $p$-groups). The classes are labeled by ordinals with larger ordinals corresponding to classes of more complicated modules. The class labeled with a finite ordinal $n$ consists of those modules whose balanced-projective dimension is $n$. We show that there are non-empty classes for arbitrarily large ordinals. This means roughly that not only are there modules with infinite balanced-projective dimension but indeed with arbitrarily large *infinite* balanced-projective dimension (when the balanced-projective dimension is viewed as the characterization given herein). It may come as a surprise that such modules can in fact be found among the balanced submodules of totally projectives.

**1. Families of submodules.**

DEFINITION 1.1. Let $\kappa$ denote an infinite cardinal. By an $H(\kappa)$-family in the module $M$ is meant a collection $\mathscr{C}$ of submodules of $M$ such that

$H1$. $0 \in \mathscr{C}$;

$H2$. $\mathscr{C}$ is closed under module union, i.e. $\sum N_i \in \mathscr{C}$ if $N_i \in \mathscr{C}$ for each $i$;

$H3$. if $C \in \mathscr{C}$ and if $A$ is any submodule of $M$ of cardinality $\leqslant \kappa$, then there is a $B \in \mathscr{C}$ that contains both $C$ and $A$ such that $B/C$ has cardinality at most $\kappa$.

It is readily checked that $H3$ can be replaced by the apparently weaker, but—in the presence of $H1$ and $H2$—equivalent condition:

$H3'$. if $A$ is a submodule of $M$ whose cardinality does not exceed $\kappa$, then there exists a submodule $B \in \mathscr{C}$ containing $A$ whose cardinality is likewise at most $\kappa$.

A $G(\kappa)$-*family* in $M$ is defined analogously with $H2$ replaced by the following condition.

$G2$. $\mathscr{C}$ is closed under unions of chains.

Here, however, $H3'$ is no longer applicable.

Finally, by an $F(\kappa)$-*family* in $M$ is meant a well-ordered ascending chain of submodules that is continuous, begins at 0, ends at $M$, and the quotient of any two adjacent members has cardinality at most $\kappa$.

Evidently, every $H(\kappa)$-family is a $G(\kappa)$-family, and every $G(\kappa)$-family contains $F(\kappa)$-families. We wish to establish a few easy lemmas on these families which will be needed later on.

LEMMA 1.2. *The intersection of any two $H(\kappa)$-families ($G(\kappa)$-families) in $M$ is again one.*

PROOF. If $\mathscr{C}$ and $\mathscr{D}$ are the two families in $M$, then $\mathscr{C} \cap \mathscr{D}$ obviously satisfies $H1$–$H2$ ($H1$–$G2$ resp.). To verify $H3$ for $\mathscr{C} \cap \mathscr{D}$, suppose that $C \in \mathscr{C} \cap \mathscr{D}$ and suppose $A$ is a submodule of $M$ of cardinality $\leqslant \kappa$. There is a chain $\langle C, A \rangle \subseteq B_1 \subseteq C_1 \subseteq B_2 \subseteq C_2 \subseteq \cdots$ of submodules of cardinalities $\leqslant \kappa$ over $C$ with $B_n \in \mathscr{C}$ and $C_n \in \mathscr{D}$. Now $\bigcup B_n = \bigcup C_n \in \mathscr{C} \cap \mathscr{D}$ is as desired.

LEMMA 1.3. *Let $\beta$: $B \twoheadrightarrow C$ be an epimorphism.*

(a) *If $\mathscr{B}$ is an $H(\kappa)$-family in $B$, then there is an $H(\kappa)$-family $\mathscr{C}$ in $C$ such that $\beta(\mathscr{B}) = \mathscr{C}$.*

(b) *If $\mathscr{C}$ is an $H(\kappa)$ (or $G(\kappa)$)-family of submodules in $C$, then there is an $H(\kappa)$ (or $G(\kappa)$)-family $\mathscr{B}$ in $B$ such that $\beta(\mathscr{B}) = \mathscr{C}$.*

PROOF. (a) Given $\mathscr{B}$, define $\mathscr{C} = \{\beta B' : B' \in \mathscr{B}\}$. Then $H1$ and $H3'$ are obvious for $\mathscr{C}$, while $H2$ follows at once from $\Sigma(\beta B_i') = \beta(\Sigma B_i')$.

(b) Given $\mathscr{C}$, define $\mathscr{B} = \{B' \subseteq B : \beta B' \in \mathscr{C}\}$. It is straightforward to see that $\mathscr{B}$ is as desired.

It is now easy to derive the following

LEMMA 1.4. *Let $\beta: B \twoheadrightarrow C$ be an epimorphism and $\mathscr{B}$, $\mathscr{C}$ be $H(\kappa)$-families in $B$ and $C$, respectively. Then there exist subfamilies $\mathscr{B}' \subseteq \mathscr{B}$ and $\mathscr{C}' \subseteq \mathscr{C}$ which are themselves $H(\kappa)$-families and satisfy $\beta\mathscr{B}' = \mathscr{C}'$.*

PROOF. According to the proof of Lemma 1.3, there are $H(\kappa)$-families $\mathscr{B}_1$, $\mathscr{C}_1$ in $B$ and $C$ respectively such that $\beta\mathscr{B}_1 = \mathscr{C}$ and $\beta\mathscr{B} = \mathscr{C}_1$. By Lemma 1.2, $\mathscr{B}' = \mathscr{B} \cap \mathscr{B}_1$ and $\mathscr{C}' = \mathscr{C} \cap \mathscr{C}_1$ are likewise $H(\kappa)$-families. They evidently satisfy $\beta\mathscr{B}' = \mathscr{C}'$.

LEMMA 1.5. *Let $A$ be a submodule of $B$. If $\mathscr{A}$ is a $G(\kappa)$-family in $A$, then there is a $G(\kappa)$-family $\mathscr{B}$ in $B$ satisfying $\mathscr{A} = \{A \cap B' : B' \in \mathscr{B}\}$ (which we write briefly as $\mathscr{A} = A \cap \mathscr{B}$).*

PROOF. Set $\mathscr{B} = \{B' \subseteq B : A \cap B' \in \mathscr{A}\}$. Obviously, conditions $H1$ and $G2$ hold for $\mathscr{B}$, so we only need to show that $\mathscr{B}$ satisfies condition $H3$ in order to conclude that $\mathscr{B}$ is a $G(\kappa)$-family. Suppose that $H \in \mathscr{B}$. Consequently, $H \cap A \in \mathscr{A}$. Let $K$ be a submodule of $B$ of cardinality not exceeding $\kappa$. Since $\langle H, K\rangle \cap A$ has cardinality at most $\kappa$ over $H \cap A$ and since $H \cap A \in \mathscr{A}$, there exists $A_0 \in \mathscr{A}$ containing $\langle H, K\rangle \cap A$ that still has cardinality at most $\kappa$ over $H \cap A$. Observe that $\langle H, A_0, K\rangle \cap A = A_0 \in \mathscr{A}$ and that $\langle H, A_0, K\rangle$ has cardinality at most $\kappa$ over $H$. Therefore, $\langle H, A_0, K\rangle \in \mathscr{B}$ and $\mathscr{B}$ satisfies $H3$. Since $\mathscr{A} \subseteq \mathscr{B}$, it is immediate that $\mathscr{A} = \{A \cap B' : B' \in \mathscr{B}\}$.

We can now establish the following

LEMMA 1.6. *Let $A$ be a submodule of $B$ and let $\mathscr{A}$ and $\mathscr{B}$, respectively, be $G(\kappa)$-families in $A$ and $B$. There exist subfamilies $\mathscr{A}' \subseteq \mathscr{A}$ and $\mathscr{B}' \subseteq \mathscr{B}$ such that $\mathscr{A}'$ is an $F(\kappa)$-family in $A$ while $\mathscr{B}'$ remains a $G(\kappa)$-family in $B$ with the property that $\mathscr{A}' \subseteq A \cap \mathscr{B}' \subseteq \mathscr{A}$.*

PROOF. First, an application of Lemma 1.5 yields a $G(\kappa)$-family $\mathscr{B}_0'$ in $B$ that satisfies $\mathscr{A} = A \cap \mathscr{B}_0'$. According to Lemma 1.2, $\mathscr{B}' = \mathscr{B}_0' \cap \mathscr{B}$ survives as a $G(\kappa)$-family in $B$. Clearly,

$$A \cap \mathscr{B}' = \{A \cap N : N \in \mathscr{B}'\} \subseteq A \cap \mathscr{B}_0' = \mathscr{A}.$$

To find a suitable $F(\kappa)$-family $\mathscr{A}'$ in $A$ and finish the proof, we simply construct an $F(\kappa)$-subfamily $\mathscr{B}''$ of $\mathscr{B}'$ and set $\mathscr{A}' = \{A \cap N : N \in \mathscr{B}''\} = A \cap \mathscr{B}''$.

The next lemma is of considerable interest and will prove to be useful in a later section. The main part of its proof is a generalization of the proof of Lemma 1.5.

LEMMA 1.7. *Let*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots, \qquad \alpha < \kappa,$$

*be a well-ordered continuous ascending chain of submodules of $M$ whose union is $M$. If $\mathscr{C}_\alpha$ is a $G(\kappa)$-family in $M_\alpha$ for each $\alpha$, then*

$$\mathscr{C} = \{N \subseteq M: N \cap M_\alpha \in \mathscr{C}_\alpha \text{ for each } \alpha\}$$

*is a $G(\kappa)$-family in $M$.*

PROOF. Since conditions $H1$ and $G2$ are obvious for $\mathscr{C}$, we only need to verify $H3$. If $X \subseteq M$, denote $X \cap M_\alpha$ simply by $X_\alpha$. Now suppose that $N \in \mathscr{C}$ and that $K \subseteq M$ has cardinality at most $\kappa$. By hypothesis, $N_\alpha \in \mathscr{C}_\alpha$ and the cardinality of $K_\alpha$ does not exceed $\kappa$. Moreover, $|\langle N, K\rangle_\alpha / N_\alpha| \leqslant \kappa$. Therefore, there exists $L(\alpha) \in \mathscr{C}_\alpha$ containing $\langle N, K\rangle_\alpha$ with $|L(\alpha)/N_\alpha| \leqslant \kappa$. Set $L = \langle L(\alpha)\rangle$ for $\alpha < \kappa$, and observe that $\langle N, K\rangle \subseteq L$ since $\langle N, K\rangle_\alpha \subseteq L(\alpha)$. Moreover, $|L/N| \leqslant \kappa$ since $|L(\alpha)/N_\alpha| \leqslant \kappa$. Trivially, $L(\alpha) \subseteq L_\alpha$, but we desire equality (in order to conclude that $L \in \mathscr{C}$). If $L^0 = \langle N, K\rangle$ and $L^1 = L$, we have already observed that $L^0_\alpha \subseteq L^1(\alpha) \subseteq L^1_\alpha$, and the middle term belongs to $\mathscr{C}_\alpha$. Letting $L^{n+1}$ replace $L^n$, we can inductively construct an ascending sequence of submodules $L^n$ of $M$ so that $|L^n/N| \leqslant \kappa$ and so that $L^n_\alpha \subseteq L^{n+1}(\alpha) \subseteq L^{n+1}_\alpha$, where $L^{n+1}(\alpha) \in \mathscr{C}_\alpha$. Finally, define $L^\omega = \bigcup L^n$ and notice that $L^\omega_\alpha = \bigcup L^n(\alpha)$ belongs to $\mathscr{C}_\alpha$. Thus $L^\omega$ is a member of $\mathscr{C}$. Since $L^\omega \supseteq L^0 \supseteq \langle N, K\rangle$ and since $|L^\omega/N| \leqslant \kappa$, $\mathscr{C}$ satisfies $H3$ and the lemma is proved.

**2. Separability and compatibility.** As indicated in the introduction, we shall need to generalize the third axiom of countability so that it applies to an arbitrary infinite cardinal $\kappa$. In order to do this, we first need the notion of separability used in [**H3** and **H4**], which generalizes the concept of a nice submodule. The height of an element $x$ in $M$ is denoted by $|x|$, and we write $|x|_M$ if it is not clear from the context what the containing module $M$ is. Likewise, if $N$ is a submodule or even a subset of $M$, we let

$$|x + N| = \sup\{|x + y|: y \in N\}$$

whenever $x \in M$.

DEFINITION 2.1. Let $\kappa$ be an arbitrary cardinal. A submodule $N$ of $M$ is *$\kappa$-separable* in $M$ if, for each $x \in M$,

$$|x + N| = |x + S|$$

for some subset $S$ of $M$ whose cardinality does not exceed $\kappa$.

It is convenient to consider also the coset valuation on a quotient module defined by

$$\|x + N\| = \sup\{|x + y| + 1: y \in N\}.$$

The following proposition is immediate.

PROPOSITION 2.2. *A submodule $N$ of $M$ is $\kappa$-separable if and only if* $\operatorname{cof}(\|x + N\|) \leqslant \kappa$; *in this connection, the cofinality of an isolated ordinal is understood to be zero.*

Using the preceding notion of separability and the notion of an $H(\kappa)$-family in Definition 1.1, we obtain in a natural fashion a generalization of the third axiom of countability (called Axiom 3 groups in [**G**]).

DEFINITION 2.3. Let $\kappa = \aleph_\alpha$ be an infinite cardinal. A module $M$ satisfies *Axiom* $3{:}\kappa$ (and $M$ is said to be an Axiom $3{:}\kappa$ module) if $M$ has an $H(\kappa)$-family of $\kappa$-separable submodules. If $\kappa = \aleph_{-1}$ is finite, $M$ satisfies Axiom $3{:}\kappa$ provided that $M$ has an $H(\aleph_0)$-family of $\kappa$-separable submodules; this is precisely the third axiom of countability with respect to nice submodules.

Some basic facts about separability and Axiom $3{:}\kappa$ modules are established next.

PROPOSITION 2.4. *Suppose that $N$ is a nice* ($= \aleph_{-1}$*-separable*) *submodule of $M$. If $N \subseteq N' \subseteq M$, then $N'$ is $\kappa$-separable in $M$ if and only if $N'/N$ is $\kappa$-separable in $M/N$.*

PROOF. The proposition is well known for the case $\kappa = \aleph_{-1}$ is finite. The general case will follow from Proposition 2.2 if we can demonstrate that $\|x + N'\|_M = \|x + N + N'/N\|_{M/N}$ for each $x \in M$. However, for $x$ fixed, $y \in N'$ and $z \in N$, we evidently have

$$\begin{aligned}\|x + N + N'/N\|_{M/N} &= \sup\{|x + y + N|_{M/N} + 1\}\\ &= \sup\{|x + y + z|_M + 1\} = \|x + N'\|_M.\end{aligned}$$

Another situation that commonly arises where a submodule is $\kappa$-separable is described in the following proposition.

PROPOSITION 2.5. *Suppose that $N$ is $\kappa$-separable in $M$. If $N \subseteq N' \subseteq M$ and $N'/N$ has cardinality not exceeding $\kappa$, then $N'$ is also $\kappa$-separable.*

PROOF. Assume that $N'$ is not $\kappa$-separable. By Proposition 2.2, $\operatorname{cof}(\|x + N'\|)$ exceeds $\kappa$ for some $x \in M$. Since there are at most $\kappa$ cosets of $N$ in $N'$, it follows that $\|x + y + N\| = \|x + N'\|$ for some $y \in N'$. This, however, is impossible since $N$ is $\kappa$-separable and $\operatorname{cof}(\|x + y + N\|) \leqslant \kappa$.

We say that a module $N$ is *absolutely* $\kappa$-separable if $N$ is $\kappa$-separable in any module in which it appears as an isotype submodule. The next result is fundamental to our determination of those modules that have balanced-projective dimension $n$.

THEOREM 2.6. *If $N$ is a module that satisfies Axiom* $3{:}\kappa$, *then $N$ is absolutely $\kappa^+$-separable, where $\kappa^+$ as usual denotes the smallest cardinal greater than $\kappa$.*

PROOF. If $\kappa$ is finite, $N$ is totally projective and the result is contained in [**H4**]. Suppose that $\kappa$ is infinite and that $N$ is an Axiom $3{:}\kappa$ module that is an isotype submodule of $M$. Assume that $\operatorname{cof}(\|x + N\|) > \kappa^+$ for some $x \in M$. Let $\mathscr{C}$ be an $H(\kappa)$-family of $\kappa$-separable submodules of $N$. Observe that if $\alpha < \|x + N\|$, there must exist $y \in N$ such that $|x + y| > \alpha$. On the other hand, $\|x + A\| < \|x + N\|$ if $A \subseteq N$ and if the cardinality of $A$ does not exceed $\kappa^+$. Due to the properties of an $H(\kappa)$-family, there must exist consequently a continuous ascending chain indexed by $\kappa^+$ of submodules $A_\alpha \in \mathscr{C}$ so that

$$\|x + A_0\| < \|x + A_1\| < \cdots < \|x + A_\alpha\| < \cdots < \|x + N\|.$$

For convenience, let $\beta = \kappa^+$ and set $A = \bigcup_{\alpha<\beta} A_\alpha$. Again, $\|x + A\| < \|x + N\|$ since $\mathrm{cof}(\|x + A\|) = \kappa^+$ and $\mathrm{cof}(\|x + N\|) > \kappa^+$. Since $\|x + A\| < \|x + N\|$, there exists $y \in N$ such that $|x - y| > \|x + A\| > \|x + A_\alpha\|$ for each $\alpha < \kappa^+$. Thus $\|y + A\| = \|x + A\|$, and $\mathrm{cof}(\|y + A\|) = \mathrm{cof}(\|x + A\|) = \kappa^+$. This, however, immediately yields a contradiction since $\|y + A\| = \|y + A\|_M = \|y + A\|_N$ and since $A \in \mathscr{C}$ is $\kappa$-separable in $N$. We conclude therefore that $\mathrm{cof}(\|x + N\|) \leqslant \kappa^+$ and that $N$ is $\kappa^+$-separable in $M$.

As in **[H3]** we say that two submodules $A$ and $B$ of $M$ are *compatible* and write $A\|B$ if for each pair $(a, b) \in A \times B$, there exists $c \in A \cap B$ such that $|a + c|_M \geqslant |a + b|_M$. By an argument similar to the proof of Lemma 1 in **[H3]**, we can easily establish the following

PROPOSITION 2.7. *Let $\kappa$ be an infinite cardinal and let $N$ be $\kappa$-separable in $M$. If $A$ is any submodule of $M$, there exists a submodule $B \supseteq A$ such that $B\|N$ and the cardinality of $B$ does not exceed $\kappa|A|$.*

Here we have denoted the cardinality of $A$ by $|A|$, but there should be no danger of confusion with the height valuation.

The following result will prove very useful.

PROPOSITION 2.8. *Let $N$ be a nice submodule of $M$ and let $N'$ and $B$ be arbitrary submodules of $M$. If the conditions*

(2.8.1) $$N \subseteq N',$$

(2.8.2) $$B\|N,$$

(2.8.3) $$(N'/N)\|(\langle B, N\rangle/N)$$

*are satisfied, then $B\|N'$.*

PROOF. Suppose that $|b + y| = \lambda$, where $b \in B$ and $y \in N'$. According to condition (2.8.3), there exists $x \in M$ such that $|b + x + N|_{M/N} \geqslant \lambda$ with $x + N \in N'/N \cap \langle B, N\rangle/N$. Obviously, we can choose the representative $x$ of the coset $x + N$ so that $x \in B \cap N'$. Since $N$ is nice in $M$ and since $|b + x + N|_{M/N} \geqslant \lambda$, there exists $z \in N$ such that $|b + x + z|_M \geqslant \lambda$. Since $B\|N$, condition (2.8.2) implies that $|b + x + c| \geqslant \lambda$ for some $c \in B \cap N$. Observe that $w = x + c$ is contained in $B \cap N'$ since $x$ and $c$ are. Therefore $|b + w| \geqslant \lambda$ means that $B\|N'$, and the proposition is proved.

**3. Equivalent axioms.** In §1, we defined $H(\kappa)$-, $G(\kappa)$- and $F(\kappa)$-families of submodules. From these different kinds of families stem three different but closely related axioms. For simplicity and agreement with **[H2]** these axioms are denoted as Axioms H, G, and F. If $\kappa = \aleph_{-1}$ is finite, it is understood that Axiom H, Axiom G, and Axiom F take on their usual meaning **[H2]**. The axioms are generalized to include $\kappa = \aleph_\alpha$ $(\alpha \geqslant 0)$ as follows (with $\kappa$ fixed).

*Axiom* H = Axiom 3:$\kappa$. $M$ has an $H(\kappa)$-family of $\kappa$-separable submodules.

*Axiom* G. $M$ has a $G(\kappa)$-family of $\kappa$-separable submodules.

*Axiom* F. $M$ has an $F(\kappa)$-family of $\kappa$-separable submodules.

Although we will not state it as a formal axiom, there is another property of $M$

that is closely related to the above properties, the existence of a composition series. A continuous (there are no jumps at limit ordinals) ascending chain of $\kappa$-separable submodules

(3.1) $$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots, \qquad \alpha < \tau,$$

is called a *composition series* (of $\kappa$-separable submodules) if the chain begins with 0, ends with $M$ (in the sense that $M = \bigcup_{\alpha<\tau} N_\alpha$), and $N_{\alpha+1}/N_\alpha$ is cyclic for each $\alpha$. It should be observed that an $F(\kappa^+)$-family of $\kappa$-separable submodules can be refined to a composition series by an application of Proposition 2.5.

In order to characterize those modules having a given balanced-projective dimension, the following result is essential. It generalizes the main theorem of [**H2**].

THEOREM 3.2. *Axioms* F, G *and* H *are all equivalent.*

PROOF. It is immediate that Axiom H implies Axiom G and that Axiom G implies Axiom F. Moreover, we have observed that Axiom F yields a composition series. Thus it suffices to show that the existence of a composition series (of $\kappa$-separable submodules) implies that $M$ satisfies Axiom 3:$\kappa$, and we may assume that $\kappa$ is infinite since the result is well known if $\kappa$ is finite.

Suppose that

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots, \qquad \alpha < \tau,$$

is a composition series of $\kappa$-separable submodules $N_\alpha$ of $M$. For each $\alpha$ such that $\alpha + 1 < \tau$, let $N_{\alpha+1} = \langle N_\alpha, x_\alpha \rangle$. We know that $px_\alpha \in N_\alpha$ and that $N_\beta = \langle x_\alpha\colon \alpha < \beta \rangle$. Moreover, if $\mathrm{cof}(\|x_\alpha + N_\alpha\|) = 0$, we can choose $x_\alpha$ so that

$$|x_\alpha| = |x_\alpha + N_\alpha| = \sup\{|x_\alpha + n_\alpha|\colon n_\alpha \in N_\alpha\};$$

assume throughout that $x_\alpha$ has been so chosen. Once the $x_\alpha$'s have been selected (and we assume now they have been chosen once for all) observe that each element $x$ in $M$ can be represented uniquely as

(*) $$x = c_0 x_{\alpha(0)} + c_1 x_{\alpha(1)} + \cdots + c_n x_{\alpha(n)},$$

where the $c_i$'s are integers such that $1 \leqslant c_i < p$ and where $\alpha(0) < \alpha(1) < \cdots < \alpha(n)$; let us agree that the vacuous sum represents zero. The representation (*) is called the *standard* representation of $x$. For agreement with [**H2**] let $T = \tau$, the index for the composition series of $M$. Define a subset $S$ of $T$ to be a *closed* subset if it enjoys the following properties:

(a) If $\alpha \in S$, then the standard representation of $px_\alpha$ involves no elements of $T$ outside of $S$. In other words, $\alpha(i) \in S$ for each $i$ in the standard representation of $px_\alpha$.

(b) If $\alpha \in S$ and we define $N_\alpha(S) = \langle x_\gamma\colon \gamma \in S \text{ and } \gamma < \alpha \rangle$, then $\|x_\alpha + N_\alpha\| = \|x_\alpha + N_\alpha(S)\|$.

A few observations are made. First, $N_\alpha(S) \subseteq N_\alpha$ since $N_\alpha = \langle x_\gamma\colon \gamma < \alpha \rangle$. Therefore, equality in (b) is tantamount to the inequality $\leqslant$. It is now easy to see that the union of any number of closed subsets of $T$ is again closed. Another essential feature of the closed sets of $T$ is described in the following assertion.

*Claim.* If $R$ is any subset of $T$ of cardinality not exceeding $\kappa$, there exists a closed subset $S$ of $T$ containing $R$ of cardinality not exceeding $\kappa$.

Our argument supporting this claim is brief. Since the standard representation of an element $x$ requires only a finite number of ordinals $\alpha(i)$, obviously the closure of an arbitrary subset in regard to property (a) alone does not transcend the cardinality of the set if the set is infinite. Therefore, since (a) is an inductive property, we can concentrate solely on property (b). However, since $\operatorname{cof}(\|x_\alpha + N_\alpha\|) \leqslant \kappa$ for each $\alpha \in T$, it is clear that we can choose $S_1 \subseteq T$ of cardinality not exceeding $\kappa$ with $S_1 \supseteq R$ so that

$$\operatorname{cof}\left(\left\|x_\alpha + N_\alpha(S_1)\right\|\right) \geqslant \operatorname{cof}\left(\left\|x_\alpha + N_\alpha\right\|\right)$$

for a given $\alpha \in R$. In fact, we can select $S_1$ so that it is true for all $\alpha \in R$. Letting $R = S_0$ and inductively repeating the process for $S_{i+1}$ instead of $S_i$ for $i < \omega$, we obtain the desired set $S = \bigcup S_i$; namely, $S$ has property (b) and the cardinality of $S$ does not exceed $\kappa$.

Define the collection $\mathscr{C}$ of submodules of $M$ by $N \in \mathscr{C}$ if and only if, for some closed subset $S$ of $T$,

$$N = \langle x_\alpha \colon \alpha \in S \rangle.$$

We denote this special submodule of $M$ by $M(S)$. Naturally, the empty set is considered to be a closed set, so $0 = M(\varnothing)$ belongs to $\mathscr{C}$. Moreover, $M(S) = \langle M(S_i) \rangle$ if $S_i$ is closed for each $i$ and $S = \bigcup S_i$. We have demonstrated also that if $A$ is any submodule of $M$ of cardinality not exceeding $\kappa$, then $A \subseteq M(S)$ for some closed subset $S$ of cardinality not exceeding $\kappa$. To prove that $M$ satisfies Axiom H = Axiom $3{:}\kappa$, it suffices now to prove that $M(S)$ is, in fact, $\kappa$-separable in $M$ whenever $S$ is a closed subset of $T$.

Let $N = M(S) = \langle x_\alpha \colon \alpha \in S \rangle$, where $S$ is a closed subset of $T$, and assume that $N$ is not $\kappa$-separable in $M$. Let $\operatorname{cof}(\|y + N\|) > \kappa$ for some $y \in M$. Among all such choices of $y$, choose one that produces a minimal $\alpha(n)$ in its standard representation

$$y = c_0 x_{\alpha(0)} + c_1 x_{\alpha(1)} + \cdots + c_n x_{\alpha(n)}.$$

Obviously, $\alpha(n) \notin S$, for if $\alpha(n) \in S$ then the last term in the standard representation of $y$ can be deleted (without destroying the required property of $y$). Likewise, if $\|y + N\| = \mu$, then $|x_{\alpha(n)}| < \mu$ for the same reason. Indeed, we can say more. Let $\lambda = \|x_{\alpha(n)} + N_{\alpha(n)}\|$. Since $\operatorname{cof}(\mu) > \kappa$ and $N_{\alpha(n)}$ is $\kappa$-separable in $M$ (being one of the terms in the composition series), we know that $\lambda \neq \mu$. If $\lambda > \mu$, then $\|x_{\alpha(n)} + N_{\alpha(n)}\| \geqslant \mu + 1$ implies that $|x_{\alpha(n)} + z| \geqslant \mu$ for some $z \in N_{\alpha(n)}$. Consequently, if $y' = y - (c_n x_{\alpha(n)} + c_n z)$, then

$$\|y' + N\| = \|y + N\| = \mu.$$

But $y' \in N_{\alpha(n)}$, and therefore its standard representation involves only $\alpha$'s less than $\alpha(n)$. However, this is contrary to the choice of $y$ and $\alpha(n)$, so we conclude that $\lambda < \mu$. As we have noted, $\alpha(n) \notin S$. Since $\alpha(n) \notin S$ and since $S$ is closed, no element $x$ belonging to $N = M(S)$ uses the generator $x_{\alpha(n)}$ in its standard representation. Thus, in particular

$$\left\|y + N_{\alpha(n)}\right\| = \left\|x_{\alpha(n)} + N_{\alpha(n)}\right\| = \lambda \geqslant \left\|y + \left(N \cap N_{\alpha(n)+1}\right)\right\|$$

since $N \cap N_{\alpha(n)+1} \subseteq N_{\alpha(n)}$. Because $\lambda < \mu$, certainly there must exist $w \in N$ such that $|y + w| \geqslant \lambda$, but $w$ cannot reside in $N_{\alpha(n)+1}$. Among all possible choices for $w$ choose one that produces a minimal $\beta(m)$, where the standard representation of $w$ is

$$w = d_0 x_{\beta(0)} + d_1 x_{\beta(1)} + \cdots + d_m x_{\beta(m)}$$

(with $1 \leqslant d_i < p$ and $\beta(0) < \beta(1) < \cdots < \beta(m)$). Note that we have made sure that $\beta(m) > \alpha(n)$. Since $|y + w| \geqslant \lambda$, we conclude that $\|x_{\beta(m)} + N_{\beta(m)}\| \geqslant \lambda + 1$ and consequently $\|x_{\beta(m)} + N_{\beta(m)}(S)\| \geqslant \lambda + 1$. This means that $|x_{\beta(m)} + v| \geqslant \lambda$ for some $v \in N_{\beta(m)}(S)$. Since $w' = w - (d_m x_{\beta(m)} + d_m v) \in N_{\beta(m)}(S)$ and since $|y + w'| \geqslant \lambda$, we have a contradiction on the choice of $w$ and $\beta(m)$. Therefore, $N$ must be $\kappa$-separable in $M$, and the theorem is proved.

**4. A determination of the balanced-projective dimension.** The main theorem of this section establishes a necessary and sufficient structural condition on a module $M$ in order that b.p.d. $M = n$. First, we need some preliminary results.

LEMMA 4.1. *Let $B \rightarrowtail T \twoheadrightarrow M$ be balanced exact. Suppose that $\mathscr{C}_T$ and $\mathscr{C}_M$ are $H(\kappa)$-families of $T$ and $M$, respectively, where $\kappa$ is an infinite cardinal. If $\mathscr{C}_T$ maps onto $\mathscr{C}_M$ in the sense that $\mathscr{C}_M = \{\langle N, B\rangle/B\colon N \in \mathscr{C}_T\}$ and if the members of $\mathscr{C}_T$ are nice and those of $\mathscr{C}_M$ are $\kappa$-separable, there exists a $G(\kappa)$-subfamily $\mathscr{C}'_T$ of $\mathscr{C}_T$ such that $N \| B$ for each $N \in \mathscr{C}'_T$.*

PROOF. Define $\mathscr{C}'_T = \{N \in \mathscr{C}_T\colon N\|B\}$. Obviously $\mathscr{C}'_T$ satisfies conditions $H1$ and $G2$. The nontrivial part is to show that $\mathscr{C}'_T$ satisfies condition $H3$. Let $N \in \mathscr{C}'_T$ and suppose that $N \subseteq A \subseteq T$ with $|A/N| \leqslant \kappa$. By hypothesis, $\langle N, B\rangle/B \in \mathscr{C}_M$. Thus $\langle N, B\rangle/B$ is $\kappa$-separable in $M$. According to Proposition 2.4, $\langle N, B\rangle/N$ is $\kappa$-separable in $T/N$ because both $B$ and $N$ are nice in $T$. By virtue of Proposition 2.7, there is an extension $N'$ of $A$ such that $N'/N \| \langle B, N\rangle/N$ and $N'/N$ has cardinality not exceeding $\kappa$. In view of Proposition 2.8, we conclude that $N'\|B$. Since $N \in \mathscr{C}_T$, we can manage to choose $N' \in \mathscr{C}_T$, for we can capture $N'$ with a member $A'$ of $\mathscr{C}_T$ so that $A'/N$ still has cardinality at most $\kappa$. Then we repeat the process (with $A'$ replacing $A$) obtaining a sequence

$$N \subseteq A \subseteq N' \subseteq A' \subseteq N'' \subseteq A'' \subseteq \cdots,$$

where the $N$'s are compatible with $B$ and the $A$'s, with the exception of the first, belong to $\mathscr{C}_T$. Since both properties are inductive, the union of the sequence has the desired property for $N'$. Since we can choose $N' \in \mathscr{C}_T$ and since $N'\|B$, we observe that $N' \in \mathscr{C}'_T$ and the lemma is proved.

THEOREM 4.2. *Let $B \rightarrowtail T \twoheadrightarrow M$ be balanced exact, where $T$ is totally projective. If $M$ satisfies Axiom $3{:}\kappa^+$, then $B$ must satisfy Axiom $3{:}\kappa$.*

PROOF. If $\kappa = \aleph_{-1}$ is finite, we interpret $\kappa^+$ to be $\aleph_0$. In this special case the theorem is proved in **[H4]**, so assume that $\kappa$ is infinite. Let $\mathscr{C}_T$ and $\mathscr{C}_M$ denote $H(\kappa^+)$-families of nice submodules of $T$ and $\kappa^+$-separable submodule of $M$, respectively. By Lemma 1.4, without loss of generality we may assume that $\mathscr{C}_M = \{\langle N, B\rangle/B\colon N \in \mathscr{C}_T\}$. Furthermore, by Lemma 4.1, there is a $G(\kappa^+)$-subfamily $\mathscr{C}'_T$

of $\mathscr{C}_T$ such that $N\|B$ for each $N \in \mathscr{C}'_T$. Let $\mathscr{D}_T$ be an $F(\kappa^+)$-subfamily of $\mathscr{C}'_T$. Observe that $\mathscr{C}_B = \{B \cap N\colon N \in \mathscr{D}_T\}$ is an $F(\kappa^+)$-family of submodules of $B$. It is easy to show that the members of $\mathscr{C}_B$ are nice in $B$. Indeed if $x \in B$ and $|x + B \cap N|_B = \mu$, then $|x + N|_T \geq \mu$ so $|x + y|_T \geq \mu$ for some $y \in N$ since $N \in \mathscr{C}_T$ is nice in $T$. But $|x + y|_T \geq \mu$ implies that $|x + z|_T \geq \mu$ for $z \in B \cap N$, and $|x + z|_B \geq \mu$ since $B$ is isotype.

Since $\mathscr{C}_B$ is an $F(\kappa^+)$-family of nice submodules, $\mathscr{C}_B$ can be refined (by Proposition 2.5) to an $F(\kappa)$-family of $\kappa$-separable submodules. Theorem 3.2 implies that $B$ is an Axiom 3: $\kappa$ module.

The converse of Theorem 4.2 is also valid. Before we prove the converse we need one more preliminary result.

LEMMA 4.3. *Let $B \rightarrowtail T \twoheadrightarrow M$ be balanced exact. Suppose that $\mathscr{C}_B$ and $\mathscr{C}_T$ are $G(\kappa)$-families of $B$ and $T$, respectively, where $\kappa$ is an infinite cardinal. If $B \cap \mathscr{C}_T = \{B \cap N\colon N \in \mathscr{C}_T\} \subseteq \mathscr{C}_B$ and the members of $\mathscr{C}_T$ are nice and those of $\mathscr{C}_B$ are $\kappa$-separable, there exists a $G(\kappa^+)$-subfamily $\mathscr{C}'_T$ of $\mathscr{C}_T$ such that $N\|B$ for each $N \in \mathscr{C}'_T$.*

PROOF. Define $\mathscr{C}'_T = \{N \in \mathscr{C}_T\colon N\|B\}$. Clearly $\mathscr{C}'_T$ satisfies conditions $H1$ and $G2$. To show that $\mathscr{C}'_T$ satisfies condition $H3$ for $\kappa^+$, assume that $N \in \mathscr{C}'_T$ and that $N \subseteq A \subseteq T$ with $A/N$ having cardinality not exceeding $\kappa^+$. Since $N \in \mathscr{C}'_T \subseteq \mathscr{C}_T$, we know that $N \cap B \in \mathscr{C}_B$. Therefore, $B/(N \cap B)$ has a $G(\kappa)$-family of $\kappa$-separable submodules; namely, such a family is

$$\mathscr{C}_{B/N \cap B} = \{C/N \cap B\colon C \in \mathscr{C}_B \text{ and } C \supseteq N \cap B\}.$$

Theorem 2.6 implies that $B/B \cap N$ is absolutely $\kappa^+$-separable, and so is $\langle B, N\rangle/N \cong B/B \cap N$.

We next observe that $\langle B, N\rangle/N$ is isotype in $T/N$. Suppose that $b + N \in p^\lambda(T/N)$. Since $N$ is nice in $T$, we know that $p^\lambda(T/N) = \langle p^\lambda T, N\rangle/N$ and $|b + x|_T \geq \lambda$ for some $x \in N$. Recall that $N\|B$, so $|b + c|_B = |b + c|_N \geq \lambda$ for some $c \in B \cap N$. It quickly follows that $b + N \in p^\lambda(\langle B, N\rangle/N)$ since $b + N = b + c + N$ and since $b + c \in p^\lambda B$. We have shown that $\langle B, N\rangle/N$ is isotype in $T/N$ and therefore $\kappa^+$-separable. Now invoke Proposition 2.7 to obtain an extension $N'$ of $A$ (with $N \subseteq A \subseteq N' \subseteq T$) such that $N'/N\|\langle B, N\rangle/N$ in $T/N$, where $N'/N$ has cardinality not exceeding $\kappa^+$. Proposition 2.8 yields $N'\|B$. Again, by an argument similar to that used in the proof of Lemma 4.1, we can choose $N' \in \mathscr{C}_T$ without sacrificing $N'\|B$. Thus we can choose $N' \in \mathscr{C}'_T$. Hence $\mathscr{C}'_T$ is a $G(\kappa^+)$-subfamily of $\mathscr{C}_T$ with $N\|B$ for each $N \in \mathscr{C}'_T$.

THEOREM 4.4. *Let $B \rightarrowtail T \twoheadrightarrow M$ be balanced exact, where $T$ is totally projective. If $B$ satisfies Axiom 3: $\kappa$, then $M$ must satisfy Axiom 3: $\kappa^+$.*

PROOF. As in Theorem 4.2 we may assume that $\kappa$ is infinite, for the result follows from **[H4]** when $\kappa$ is finite. Let $\mathscr{C}_B$ and $\mathscr{C}_T$ denote $G(\kappa)$-families of $\kappa$-separable submodules of $B$ and nice submodules of $T$, respectively. By Lemma 1.6, there is an $F(\kappa)$-subfamily $\mathscr{C}'_B$ of $\mathscr{C}_B$ and a $G(\kappa)$-subfamily $\mathscr{C}'_T$ of $\mathscr{C}_T$ such that $\mathscr{C}'_B \subseteq B \cap \mathscr{C}'_T \subseteq \mathscr{C}_B$. According to Lemma 4.3, there exists a $G(\kappa^+)$-subfamily $\mathscr{C}''_T$ of $\mathscr{C}'_T$ such that $N\|B$ for each $N \in \mathscr{C}''_T$. Let $\mathscr{D}_T$ be an $F(\kappa^+)$-subfamily of $\mathscr{C}''_T$.

Define $\mathscr{C}_M = \{\langle N, B\rangle/B\colon N \in \mathscr{D}_T\}$. Clearly, $\mathscr{C}_M$ is an $F(\kappa^+)$-family in $M$, but we need to show that its members are $\kappa^+$-separable. However, $\langle N, B\rangle/N \cong B/B \cap N$ is an Axiom 3:$\kappa$ module as in the proof of Lemma 4.3, so $\langle N, B\rangle/N$ is absolutely $\kappa^+$-separable in $T/N$ and is isotype (by the same argument of Lemma 4.3). Thus $\langle N,B\rangle/N$ is $\kappa^+$-separable in $T/N$. We switch back to $\langle B, N\rangle/B$. Since $N$ and $B$ are both nice, Proposition 2.4 implies that $\langle N, B\rangle/B$ is $\kappa^+$-separable in $M = T/B$ since $\langle N, B\rangle/N$ is $\kappa^+$-separable in $T/N$. This completes the proof of the theorem.

We are now ready for the main result.

THEOREM 4.5. *For each $n \geqslant 0$, it is true that* b.p.d. $M \leqslant n$ *if and only if $M$ satisfies Axiom* 3: $\aleph_{n-1}$.

PROOF. Since $M$ satisfies Axiom 3: $\aleph_{-1}$ if and only if $M$ is totally projective and since this is precisely when b.p.d. $M = 0$, the theorem holds for $n = 0$. The proof proceeds by induction on $n \geqslant 1$. Let $B \rightarrowtail T \twoheadrightarrow M$ be a balanced-exact sequence where $T$ is totally projective.

First, assume that b.p.d. $M \leqslant n$. Then b.p.d. $B \leqslant n - 1$, and the induction hypothesis implies that $B$ satisfies Axiom 3: $\aleph_{n-2}$. Theorem 4.4 asserts that $M$ satisfies Axiom 3: $\aleph_{n-1}$. Conversely, suppose that $M$ satisfies Axiom 3: $\aleph_{n-1}$. By Theorem 4.2, $B$ satisfies Axiom 3: $\aleph_{n-2}$. The induction hypothesis yields b.p.d. $B \leqslant n - 1$, so b.p.d. $M \leqslant n$.

**5. Balanced-projective dimensions of unions of chains.** It is frequently useful to have an estimate for the balanced-projective dimension of a union of chains if the balanced-projective dimensions of the members of the chains are known.

An obvious modification of Auslander's lemma [A] yields the following result:

LEMMA 5.1. *Let*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \subseteq M_\lambda = M \qquad (\alpha < \lambda) \tag{1}$$

*be a well-ordered continuous ascending chain of submodules of $M$. If each $M_\alpha$ $(\alpha < \lambda)$ is balanced in $M$ and if* b.p.d. $M_{\alpha+1}/M_\alpha \leqslant n$ *for all $\alpha < \lambda$, then* b.p.d. $M \leqslant n$.

In an alternative version of this lemma, b.p.d. $M_{\alpha+1}/M_\alpha \leqslant n$ is replaced by b.p.d. $M_\alpha \leqslant n - 1$ for $\alpha < \lambda$. Recall that Hill **[H5]** has shown that a stronger statement holds whenever the $M_\alpha$'s are totally projective and $\lambda = \omega_0$. Even if the $M_\alpha$'s are just isotype in $M$, then $M$ has to be totally projective as well.

We intend to prove two theorems in the same vein. The first shows that in Lemma 5.1, balancedness can be weakened. We follow an argument used by Simmons **[S]**.

THEOREM 5.2. *Let $\lambda$ be a limit ordinal and* (1) *a well-ordered continuous ascending chain of submodules of $M$ such that each $M_\alpha$ $(\alpha < \lambda)$ is isotype in $M$. If* b.p.d. $M_\alpha \leqslant n - 1$ *for all $\alpha < \lambda$, then* b.p.d. $M \leqslant n$.

PROOF. Let $0 \to B \to T \to M \to 0$ be the canonical balanced-projective resolution of $M$. Observe that if we take the canonical balanced-projective resolution $0 \to B_\alpha \to T_\alpha \to M_\alpha \to 0$ of an isotype submodule $M_\alpha$ of $M$, then $T_\alpha$ will be a summand of the totally projective module $T$. Thus $B_\alpha$ can be viewed as a submodule of $B$. This gives rise to a direct system of balanced-exact sequences

$$\begin{array}{ccccccccc} 0 & \to & B_\alpha & \to & T_\alpha & \to & M_\alpha & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B_\beta & \to & T_\beta & \to & M_\beta & \to & 0 \end{array} \qquad (\alpha < \beta < \lambda)$$

whose limit is the given balanced-projective resolution for $M$ since evidently both $\bigcup T_\alpha = T$ and $\bigcup M_\alpha = M$.

Since $T_\alpha$ is a summand of $T_\beta$, we conclude that $B_\alpha$ is balanced in $B_\beta$. Thus $0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_\alpha \subseteq \cdots \subseteq B_\lambda = B$ is a well-ordered continuous ascending chain of balanced submodules. By the alternative version of Auslander's lemma (or by induction on $N$), it follows that

$$\text{b.p.d. } B \leqslant 1 + \sup_{\alpha<\lambda} \text{b.p.d. } B_\alpha.$$

If b.p.d. $M_\alpha = 0$ for all $\alpha < \lambda$, then for each $\alpha$, $B_\alpha$ is a summand of $T_\alpha$, so of $T_\beta$ and hence of $B_\beta$ $(\alpha < \beta < \lambda)$. By Lemma 5.1 (or almost any other consideration), b.p.d. $B = 0$ and we obtain b.p.d. $M \leqslant 1$. If $n \geqslant 2$ and b.p.d. $M_\alpha \leqslant n - 1$ for $\alpha < \lambda$, then b.p.d. $B_\alpha \leqslant n - 2$, and therefore b.p.d. $B \leqslant n - 1$. This implies b.p.d. $M \leqslant n$, as claimed.

The next result improves on the upper bound given in Theorem 5.2 for the balanced-projective dimension in case the chains are relatively short.

THEOREM 5.3. *Let $\lambda$ be a limit ordinal with* $\operatorname{cof}(\lambda) \leqslant \omega_{n-1}$ $(1 \leqslant n < \omega)$. *If* (1) *is a well-ordered continuous ascending chain of submodules of $M$ such that $M_\alpha$ is isotype in $M$ and* b.p.d. $M_\alpha \leqslant n$ *for each* $\alpha < \lambda$, *then* b.p.d. $M \leqslant n$.

PROOF. Set $\kappa = \aleph_{n-1}$. By taking a cofinal subset of the $M_\alpha$'s we may assume without loss of generality that $\lambda \leqslant \omega_{n-1}$. By Theorem 4.5, $M_\alpha$ has an $H(\kappa)$-family $\mathscr{C}_\alpha$ of $\kappa$-separable submodules. Lemma 1.7 ensures that

$$\mathscr{C} = \{N \subseteq M\colon N \cap M_\alpha \in \mathscr{C}_\alpha \text{ for } \alpha < \lambda\}$$

is a $G(\kappa)$-family in $M$. It remains to show that the members of $\mathscr{C}$ are $\kappa$-separable in $M$. By way of contradiction, assume that $N \in \mathscr{C}$ and that $\operatorname{cof}(\|x + N\|) \geqslant \omega_n$ for some $x \in M$. For $\gamma < \operatorname{cof}(\|x + N\|)$, choose $x_\gamma \in N$ so that $\sup_\gamma\{|x + x_\gamma|\} = \|x + N\|$. Since there are at least $\omega_n$ of the elements $x_\gamma$ and not more than $\omega_{n-1}$ of the submodules $M_\alpha$, clearly most of the $x_\gamma$'s must be contained in a single $M_\alpha$. More precisely, there exists $\alpha < \omega_{n-1}$ such that $M_\alpha$ contains a cofinal subset of the $x_\gamma$'s. Since the $M_\alpha$'s ascend, obviously we can choose $\alpha$ so that $M_\alpha$ also contains the fixed element $x$. However, this is impossible since $N \cap M_\alpha$ is $\kappa$-separable in $M_\alpha$ by virtue of $N \cap M_\alpha \in \mathscr{C}_\alpha$, and the theorem is proved.

**6. Applications.** Our results suggest the idea of introducing classes of abelian $p$-groups which generalize the notion of balanced-projective dimension. The classes are labeled by ordinal numbers.

For a module $M$, consider the cardinals $\kappa$ such that $M$ has an $H(\kappa)$-family of $\kappa$-separable submodules. Such $\kappa$ does exist, e.g. $\kappa = |M|$ is such. If $\aleph_\alpha = \kappa$ has this property and $\alpha$ is minimal, $\alpha \geqslant -1$, define the *class of separability*, or simply, the *class of $M$* as cl $M = \alpha + 1$.

Our Theorem 4.5 asserts that $\operatorname{cl} M = \text{b.p.d. } M$ whenever the latter is finite. Note that $\operatorname{cl} M \geqslant 0$, and $\operatorname{cl} M = 0$ precisely when $M$ is totally projective.

The following upper bound for $\operatorname{cl} M$ is relevant.

THEOREM 6.1. *If $\alpha \geqslant 0$ and a module $M$ has cardinality $\aleph_\alpha$ or is of length $< \omega_\alpha$, then $\operatorname{cl} M \leqslant \alpha + 1$. Moreover, $\operatorname{cl} M \leqslant \alpha$ if $\alpha$ is not a limit ordinal.*

PROOF. A module $M$ of cardinality not exceeding $\aleph_\alpha$ is the union of a suitable well-ordered continuous ascending chain of submodules, all of which have cardinality less than $\aleph_\alpha$. By Proposition 2.5, each of the submodules is $\aleph_\alpha$-separable. Theorem 3.2 implies that $\operatorname{cl} M \leqslant \alpha + 1$. Moreover, $\operatorname{cl} M \leqslant \alpha$ if $\alpha$ is isolated.

In a module of length $< \omega_\alpha$, each submodule is evidently $\aleph_\beta$-separable for some $\beta < \alpha$. Hence the claim is immediate.

Observe that, for $\alpha = 0$, Theorem 6.1 includes the well-known facts that both countable and bounded $p$-groups are totally projective. The next result is of utmost importance. It establishes the existence of nonempty classes of separability for arbitrarily large ordinals. In particular, it implies the existence of $p$-groups of balanced-projective dimension $n$ for every integer $n \geqslant 0$ as well as for $n = \infty$.

THEOREM 6.2. *Let $\alpha$ be an isolated ordinal and $G$ a $p$-group of cardinality $\aleph_\alpha$ such that $p^{\omega_\alpha} G \neq 0$; there always exists such a $G$ even among the totally projectives. The class of any $p^{\omega_\alpha}$-high subgroup $H$ of $G$ is precisely $\alpha$.*

PROOF. Since any subgroup $H$ of $G$ has cardinality not exceeding $\aleph_\alpha$, Theorem 6.1 implies $\operatorname{cl} H \leqslant \alpha$. On the other hand, $H$ being a $p^{\omega_\alpha}$-high subgroup is isotype in $G$. It stays isotype under the canonical map $G \twoheadrightarrow G/p^{\omega_\alpha}G$. Moreover, $G = \langle H, p^\gamma G\rangle$ for each $\gamma < \omega_\alpha$. We conclude that $|g + H| = \omega_\alpha$ for every $g \in G/p^{\omega_\alpha}G$ which is not contained in $H$—and there is one since $p^{\omega_\alpha}G \neq 0$. Hence $H$ cannot be $\aleph_{\alpha-1}$-separable in $G/p^{\omega_\alpha}G$. Thus, by Theorem 2.6, $H$ cannot contain any $H(\kappa)$-family of $\kappa$-separable submodules if $\kappa < \aleph_{\alpha-1}$. It follows that $\operatorname{cl} H$ is not less than $\alpha$, and the theorem is proved.

COROLLARY 6.3. *If $n$ is a nonnegative integer and $G$ has cardinality $\aleph_n$ but $p^{\omega_n}G \neq 0$, then any $p^{\omega_n}$-high subgroup of $G$ has balanced-projective dimension exactly $n$.*

It should be pointed out that even balanced subgroups associated with totally projective $p$-groups belong to classes of separability of arbitrarily large ordinals. In fact, if $B$ is balanced in a totally projective $p$-group $T$, then $\operatorname{cl} B = (\operatorname{cl} T/B) - 1$ whenever $(\operatorname{cl} T/B) - 1$ is not a limit ordinal.

The results herein are also applicable to the category $\mathscr{V}_p$ of valuated $p$-groups. According to **[RW]** the projective dimension of the abelian $p$-groups $G$ computed in $\mathscr{V}_p$ is precisely one more than its balanced-projective dimension (in the category of nonvaluated groups). Thus, for example, Theorem 6.2 yields the following corollary.

COROLLARY 6.4. *There exist groups in $\mathscr{V}_p$ having arbitrary projective dimension.*

REMARK. The preceding corollary can also be derived from the results of Hill and White **[HW]**.

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