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The Basic Constructive Logic for Negation-Consistency

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Abstract In this paper, consistency is understood in the standard way, i.e. as the absence of a contradiction. The basic constructive logic B_{Kc4}, which is adequate to this sense of consistency in the ternary relational semantics without a set of designated points, is defined. Then, it is shown how to define a series of logics by extending B_{Kc4} up to minimal intuitionistic logic. All logics defined in this paper are paraconsistent logics.

Keywords Constructive negation · Substructural logics · Ternary relational semantics · Paraconsistent logic

1 Introduction

This paper is a sequel to (Robles and Méndez 2007), which defined a series of logics that are included in positive intuitionistic contractionless logic extended with the constructive contraposition axioms

$$\text{(i).}\quad (A \to \neg B) \to (B \to \neg A)$$
$$\text{(ii).}\quad B \to [(A \to \neg B) \to \neg A]$$

and the EFQ ('E falso quodlibet') axioms

$$\text{(iii).}\neg A \to (A \to B)$$

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and

$$\text{(iv)}.\ A \to (\neg A \to B)$$

In (Robles and Méndez 2007), it is shown that in theories whose underlying logic is any one of that series there introduced, consistency has to be understood as the absence of the negation of a theorem and not, in general, as the absence of a contradiction. This concept of consistency is named weak consistency.

The purpose of this paper is to carry on a similar study on negation consistency as it is understood in the following definition:

Definition 1 Let a be a theory (a theory is a set of formulas closed under adjunction and provable entailment. Cf. Sect. 5). Then, a is n-consistent (negation-consistent) iff for no wff A, $A \wedge \neg A \in a$, i.e. iff a contains no contradiction (a theory is n-inconsistent—negation-inconsistent—iff it contains a contradiction).

The first aim of this paper is to introduce the logic $\mathrm{B_{Kc4}}$ ($\mathrm{B_{Kc1}}$, $\mathrm{B_{Kc2}}$ and $\mathrm{B_{Kc3}}$ are defined in (Robles and Méndez 2007) which is the basic constructive logic adequate to consistency as understood in definition 1. The logic $\mathrm{B_{Kc4}}$ is 'adequate' because it has exactly the 'syntactical power' for consistency (a syntactical notion) when it is defined as negation-consistency: it is a logic too strong for weak consistency and too weak for absolute consistency (i.e., non-triviality). The logic $\mathrm{B_{Kc4}}$ is *basic* because it is the *minimal* logic adequate to negation-consistency in the ternary relational semantics without a set of designated points (cf. Sect. 4). And it is *constructive* because it is endowed with a (weak) intuitionistic-type negation.

The logic $\mathrm{B_{Kc4}}$ is a peculiar relatively strong logic. It is not included in e.g. Lewis' modal logic S5 (A7 below is not a theorem of S5) or in relevance logic R ($\mathrm{B_{Kc4}}$ is not a relevance logic), but it does not contain such a weak logic as Routley and Meyer's basic logic B. On the other hand, though the restricted ECQ ('E contradictione quodlibet') axiom

$$\text{(v)}.\ (A \wedge \neg A) \to \neg B$$

is provable, the unrestricted ECQ axiom

$$\text{(vi)}.\ (A \wedge \neg A) \to B$$

is not derivable in $\mathrm{B_{Kc4}}$. So, it is a second aim of this paper to extend $\mathrm{B_{Kc4}}$ preserving this property, i.e. the unprovability of (vi), in order to prevent negation-inconsistency from collapsing into triviality. According to this aim, $\mathrm{B_{Kc4}}$ is extended with the contraposition axioms (i) and (ii). The resulting logic is named $\mathrm{B_{Kc5}}$. Next, $\mathrm{B_{Kc5}}$ is extended with some positive implication axioms up to minimal intuitionistic logic $\mathrm{J_m}$. Neither (vi) or (iii) and (iv) are, of course, provable in any of these logics. In this way, we get the spectrum of constructive logics in which consistency is equivalent to negation-consistency from the basic logic $\mathrm{B_{Kc4}}$ up to the 'upper bound' $\mathrm{J_m}$.

But let us take a look at these logics from another perspective. As it is known, a logic L is *paraconsistent* iff the ECQ axiom (vi) is not provable in L (see

Priest and Tanaka 2004). So, all the logics defined in this paper are paraconsistent logics. Moreover, note that the ECQ axiom (vi) is rejected not because of *ad hoc* reasons but on the basis of a precise concept of consistency. Nevertheless, the ECQ axiom (v) is provable in B_{Kc4} (and therefore, in all the logics defined in this paper), and, obviously, the axiom (v) comes close to unrestricted 'E contradictione quodlibet' (vi). Let us then qualify better the status of the logics here presented. Let us name a logic *quasi-paraconsistent* iff, though (v) is provable, (vi) is not. Then, all the logics in this paper are quasi-paraconsistent logics.

The structure of the paper is as follows. In Sect. 2, the positive logic B_{K+} introduced in (Robles and Méndez 2007) is recalled. It is the result of adding the rule K to Routley and Meyer's basic positive logic B_+. The logic B_{Kc4} and the logic B_{Kc5} are studied in Sects. 3, 4, 5 and Sects. 6, 7, respectively. In Sect. 5, it will be proved that in the present semantic context, weak consistency understood as the absence of the negation of a theorem does entail negation-consistency (in fact, negation-consistency is equivalent to weak consistency). In Sect. 6, a short discussion on the weak full reductio axioms, which, maybe unexpectedly because of the weakness of the implication connective of B_{Kc5}, are nevertheless provable in it, is included. Finally, in Sect. 8 it is shown how to extend B_{Kc4} and B_{Kc5} up to minimal intuitionistic logic J_m with some strong implication axioms. The results of (Robles and Méndez 2007) are not, on the whole, presupposed.

2 The Positive Logic B_{K+}

B_{K+} is axiomatized with

Axioms

A1. $A \to A$

A2. $(A \wedge B) \to A \quad / \quad (A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B) \quad / \quad B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

The rules of derivation are

Modus ponens (MP): $(\vdash A \ \& \ \vdash A \to B) \Rightarrow \ \vdash B$

Adjunction (Adj.): $(\vdash A \ \& \ \vdash B) \Rightarrow \ \vdash A \wedge B$

Suffixing (Suf.): $\vdash A \to B \Rightarrow \ \vdash (B \to C) \to (A \to C)$

Prefixing (Pref.): $\vdash A \to B \Rightarrow \ \vdash (C \to A) \to (C \to B)$

K: $\vdash A \Rightarrow \ \vdash B \to A$

Therefore, B_{K+} is B_+ with the addition of the K rule.

We now define the semantics for $\mathrm{B_{K+}}$. A $\mathrm{B_{K+}}$ *model* is a triple $<K, R, \vDash>$ where K is a non-empty set, and R is a ternary relation on K subject to the following definitions and postulates for all $a, b, c, d \in K$ with quantifiers ranging over K:

$$\text{d1. } a \leq b =_{df} \exists x Rxab$$
$$\text{d2. } R^2abcd =_{df} \exists x(Rabx \;\&\; Rxcd)$$
$$\text{P1. } a \leq a$$
$$\text{P2. } (a \leq b \;\&\; Rbcd) \Rightarrow Racd$$

Finally, $\vDash$ is a valuation relation from K to the sentences of the positive language satisfying the following conditions for all propositional variables p, wff A, B and $a \in K$:

(i). $(a \leq b \;\&\; a \vDash p) \Rightarrow b \vDash p$
(ii). $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$
(iii). $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$
(iv). $a \vDash A \to B$ iff for all $b, c \in K$, $(Rabc \;\&\; b \vDash A) \Rightarrow c \vDash B$

A formula A is $\mathrm{B_{K+}}$ *valid* ($\vDash_{\mathrm{B_{K+}}} A$) iff $a \vDash A$ for all $a \in K$ in all models.

In (Robles et al. 2007) or in (Robles and Méndez 2007), it is proved that $\mathrm{B_{K+}}$ is sound and complete in relation to the semantics defined above. We also note that the postulates

$$\text{P3. } Rabc \Rightarrow b \leq c$$
$$\text{P4. } (a \leq b \;\&\; b \leq c) \Rightarrow a \leq c$$

and

$$\text{P5. } R^2abcd \Rightarrow Rbcd$$

are immediate in all $\mathrm{B_{K+}}$ models.

As it is known, in the standard semantics for relevance logics (see e.g., (Routley et al. 1982), there is a set of 'designated points' in terms of which the relation $\leq$ is defined and with respect to which formulas are determined to be valid. The absence of this set in $\mathrm{B_{K+}}$ semantics (and the corresponding changes in d1 and the definition of validity) are the only but crucial differences between $\mathrm{B_+}$ models and $\mathrm{B_{K+}}$ models.

As it is shown in (Robles et al. 2007) or in (Robles and Méndez 2007), the logic $\mathrm{B_{K+}}$ is *the basic positive logic* in the ternary relational semantics when there is not a set of designated points and validity is defined in respect of all points in K. That is, $\mathrm{B_{K+}}$ is the basic positive logic in the semantics just referred to in the same sense that Routley and Meyer's $\mathrm{B_+}$ is the basic positive logic in general ternary relational semantics.

3 The Logic B_{Kc4}

We add the unary connective $\neg$ (negation) to the positive language. Then, B_{Kc4} (the logic B_{K+} with a constructive negation) can be axiomatized by adding the following axioms to B_{K+}

$$\text{A7. } \neg A \to [A \to (A \wedge \neg A)]$$
$$\text{A8. } [B \to (A \wedge \neg A)] \to \neg B$$
$$\text{A9. } (A \wedge \neg A) \to \neg(A \to A)$$

We note the following

Proposition 1 *B_{Kc4} is well axiomatized in respect of B_{K+}. That is, given B_{K+}, each negation axiom is independent of the other two axioms.*

Proof By MaGIC, the matrix generator developed by J. Slaney (see Slaney 1995). □

The following are some theorems and rules of inference of B_{Kc4} (a proof for each one of them is sketched to their right):

T1. $\vdash \neg(A \wedge \neg A)$	A8
T2. $\vdash A \to B \Rightarrow \ \vdash \neg B \to \neg A$	A7, A8
T3. $\neg A \to [A \to \neg(A \to A)]$	A7, A9
T4. $\vdash \neg B \Rightarrow \ \vdash [(A \to B) \to \neg A]$	A7, A8

$$\text{T5. } \vdash (A \to \neg A) \to \neg A$$

Proof By the theorem of B_{K+}

$$(A \to B) \to [A \to (A \wedge B)]$$

we have

$$(A \to \neg A) \to [A \to (A \wedge \neg A)]$$

Then, T5 follows by A8. □

T6. $\vdash A \Rightarrow \ \vdash \neg A \to \neg B$	K, T2
T7. $\vdash B \Rightarrow \ \vdash [(A \to \neg B) \to \neg A]$	T5, T6
T8. $\vdash [B \to \neg(A \to A)] \to \neg B$	T7
T9. $\vdash A \Rightarrow \ \vdash \neg\neg A$	A1, T7
T10. $\vdash A \Rightarrow \ \vdash \neg(B \to B) \leftrightarrow \neg A$	T6

T11. $\vdash A \Rightarrow\ \vdash [(A \to \neg B) \to \neg B]$	K, T5
T12. $\vdash (A \wedge \neg A) \to \neg B$	A9, T6
T13. $\vdash (A \wedge \neg A) \to \neg(B \to B)$	T12
T14. $\vdash A \Rightarrow\ \vdash (A \wedge \neg A) \leftrightarrow \neg A$	A1, K
T15. $\vdash \neg A \to (A \to \neg B)$	A7, T12 (or T3, T6)
T16. $\neg A \to [A \to \neg(B \to B)]$	T15
T17. $\vdash \neg A \leftrightarrow (A \to \neg A)$	T5, T15
T18. $\vdash [(A \to B) \wedge (A \to \neg B)] \to \neg A$	A8
T19. $\vdash A \to \neg B \Rightarrow\ \vdash [(A \to B) \to \neg A]$	K, T18
T20. $\vdash A \to B \Rightarrow\ \vdash [(A \to \neg B) \to \neg A]$	K, T18
T21. $\vdash (A \to B) \to \neg(A \wedge \neg B)$	T19
T22. $\vdash (A \to \neg B) \to \neg(A \wedge B)$	T20
T23. $(\neg A \vee \neg B) \to \neg(A \wedge B)$	T2
T24. $\vdash \neg(A \vee B) \to (\neg A \wedge \neg B)$	T2
T25. $\vdash (\neg A \wedge \neg B) \to \neg(A \vee B)$	A7, A8

Note that B_{Kc4} is a relatively strong logic: it has the weak contraposition axioms as rules: T2, T4, T7 (but note that $\vdash A \to \neg B \Rightarrow \vdash B \to \neg A$ is not provable (MaGIC)); double negation introduction as a rule: T9; the principle of non-contradiction: T1; specialized reductio: T5; the weak reductio axioms as rules: T19, T20, the restricted ECQ axiom T12, and the restricted EFQ axioms T15 and T6 (as a rule).

We end this section with some notes on the axiomatization of B_{Kc4} . The logic B_{Kc1} is the basic constructive logic adequate to consistency understood as the absence of the negation of a theorem (cf. Robles and Méndez 2007). It is axiomatized by adding T3 and T8 to B_{K+}. Therefore, B_{Kc1} is deductively included in B_{Kc4} The converse does not hold: A9 is not derivable in B_{Kc1} (MaGIC).

Now (I owe this point to a referee of the JoLLI), we have:

Proposition 2 *Let $B_{Kc4(ii)}$ be the result of adding A9 to B_{Kc1}. B_{Kc4} and $B_{Kc4(ii)}$ are deductively equivalent.*

Proof (a) $B_{Kc4(ii)}$ is obviously included in B_{Kc4}. (b) We prove that A7 and A8 are theorems of $B_{Kc4(ii)}$. We note the following theorems and rules of inference of B_{Kc1} (cf. Robles and Méndez 2007):

(i). $\neg A \to (A \to \neg B)$

(ii). $\vdash A \to B \Rightarrow\ \vdash \neg B \to \neg A$

(iii). $\vdash A \Rightarrow\ \neg\neg A$

(iv). $\vdash \neg B \Rightarrow\ \vdash (A \to B) \to \neg A$

and the theorem of $\mathrm{B}_{\mathrm{K}+}$

$$(\mathrm{v})\ (A \to B) \to [A \to (A \wedge B)]$$

used above in the proof of T5. Then, by (ii) and A9

$$(\mathrm{vi})\ \neg\neg(A \to A) \to \neg(A \wedge \neg A)$$

By A1, (iii) and (vi),

$$(\mathrm{vii})\ \neg(A \wedge \neg A)$$

So, A8 is derivable by (iv) and (vii). And finally, A7 is proved by (i) and (v). □

Therefore, $\mathrm{B}_{\mathrm{Kc4}}$ can be viewed as the result of adding A9 to $\mathrm{B}_{\mathrm{Kc1}}$. More exactly, we have:

Proposition 3 *Let* $B_{\mathrm{Kc4(iii)}}$ *be the result of adding the principle of non-contradiction (PNC) T1 (of* B_{Kc4}*) to* B_{Kc1}*. Then,* B_{Kc4} *and* $B_{\mathrm{Kc4(iii)}}$ *are deductively equivalent.*

Proof That $\mathrm{B}_{\mathrm{Kc4(iii)}}$ is included in $\mathrm{B}_{\mathrm{Kc4}}$ is obvious. And the converse is immediate by (i) above ($\neg A \to (A \to \neg B)$) and T1. □

Therefore, $\mathrm{B}_{\mathrm{Kc4}}$ can exactly be viewed as the result of adding the principle of non-contradiction to $\mathrm{B}_{\mathrm{Kc1}}$. Note, however, that T1 is immediate from A8 and that A9 is, given $\mathrm{B}_{\mathrm{K}+}$, independent of A7 and A8 (cf. proposition 1). We shall return to the relation between $\mathrm{B}_{\mathrm{Kc1}}$ and $\mathrm{B}_{\mathrm{Kc4}}$ at the end of Sect. 6. Meanwhile, it is of some interest, maybe, to present some other axiomatizations of $\mathrm{B}_{\mathrm{Kc4}}$ as alternatives to those discussed above. We have:

Proposition 4 *The following systems* (1) *and* (2) *are deductively equivalent to* B_{Kc4}*:*

(1.) *A7, T8 and A9*
(2.) *A8 and T3*

Proof The proof of case (1) is easy and is left to the reader. So, we prove case (2). Note that T1, T5 and T18-T22 of $\mathrm{B}_{\mathrm{Kc4}}$ are derivable by just using A8, given $\mathrm{B}_{\mathrm{K}+}$. Then, T8 follows by A1, K and T20. Therefore, system (2) includes $\mathrm{B}_{\mathrm{Kc4}}$ by T1, T3 and T8, using proposition 3. □

Finally, we remark the following facts: (a) in $\mathrm{B}_{\mathrm{Kc4(iii)}}$ of proposition 3, T1 can be substituted for any one of T12, T13, T18–T22. (b) When present, A9, T3 and T8 can be substituted for T12 (T13), T15 and T7, respectively, in all the axiomatizations discussed above. (c) A7, A8, T1, T2, T4–T11, T14, T18–T25 do not axiomatize $\mathrm{B}_{\mathrm{Kc4}}$ (MaGIC). (d) All formulations of $\mathrm{B}_{\mathrm{Kc4}}$ in propositions 2–4 are well axiomatized in the sense of proposition 1, given $\mathrm{B}_{\mathrm{K}+}$

4 Semantics for $\mathrm{B_{Kc4}}$

A $\mathrm{B_{Kc4}}$ *model* is a quadruple $\langle K, S, R, \vDash\rangle$ where S is a non-empty subset of K, and K, R and $\vDash$ are defined, similarly, as in $\mathrm{B_{K+}}$ models except for the addition of the following clause and postulates:

$$\text{(v) } a \vDash \neg A \text{ iff for all } b, c \in K, (Rabc \mathbin{\&} c \in S) \Rightarrow b \nvDash A$$

$$\text{P6. } (Rabc \mathbin{\&} c \in S) \Rightarrow a \in S$$

$$\text{P7. } a \in S \Rightarrow (\exists x \in S)Raax$$

$\vDash_{\mathrm{B_{Kc4}}} A$ (A is $\mathrm{B_{Kc4}}$ *valid*) iff $a \vDash A$ for all $a \in K$ in all models.

In order to prove soundness, we need the following two lemmas (see Robles and Méndez 2007)

Lemma 1 $(a \leq b \ \& \ a \vDash A) \Rightarrow b \vDash B$.

Proof The proof is by induction on the length of A. The conditional and negation clauses are proved with P2. □

Lemma 2 $\vDash_{\mathrm{B_{Kc4}}} A \rightarrow B$ *iff for all $a \in K$ in all models, $a \vDash A \Rightarrow a \vDash B$.*

Proof The proof is by lemma 1, P1 and d1. □

Next, we prove the theorem of semantic consistency (soundness).

Theorem 1 *If* $\vdash_{\mathrm{B_{Kc4}}} A$, *then* $\vDash_{\mathrm{B_{Kc4}}} A$.

Proof Given the semantic consistency of $\mathrm{B_{K+}}$ (see Robles and Méndez 2007), we only have to prove that A7, A8 and A9 are valid (we use lemma 2).

A7 is valid: suppose $a \vDash \neg A$, $a \nvDash A \rightarrow (A \wedge \neg A)$ for a wff A, $a \in K$ in some model. Then, $Rabc$, $b \vDash A$, $c \nvDash A \wedge \neg A$ for $b, c \in K$. Now, $c \nvDash A$ or $c \nvDash \neg A$. But by P3 (cf. Sect. 2) and $Rabc$, $b \leq c$; by $b \vDash A$ and lemma 1, $c \vDash A$. So, $c \nvDash \neg A$. Then, by clause (v), $Rcde$, $d \vDash A$ for $d \in K$ and $e \in S$. By P6, $c \in S$. On the other hand, by $a \vDash \neg A$ and clause (v), it follows that $(Raxy \mathbin{\&} y \in S) \Rightarrow x \nvDash A$ for all $x \in K$ and $y \in S$. But we have $Rabc$ and $c \in S$. So, $b \nvDash A$, which contradicts $b \vDash A$ above.

A8 is valid: suppose $a \vDash B \rightarrow (A \wedge \neg A)$, $a \nvDash \neg B$ for some wff A, B and $a \in K$ in some model. Then, $Rabc$, $b \vDash B$ for $b \in K$, $c \in S$. By clause (iv), $c \vDash A \wedge \neg A$ i.e., $c \vDash A$ and $c \vDash \neg A$, from which, by clause (v), it follows that $(Rcxy \mathbin{\&} y \in S) \Rightarrow x \nvDash A$ for all $x \in K$ and $y \in S$. But given $c \in S$, we have $Rccx$ for some $x \in S$, by P7. Then, $c \nvDash A$, contradicting $c \vDash A$ above.

A9 is valid: suppose $a \vDash A \wedge \neg A$, $a \nvDash \neg(A \rightarrow A)$ for some wff A and $a \in K$ in some model. Then, $Rabc$, $b \vDash A \rightarrow A$ for $b \in K$, $c \in S$. On the other hand, we have $a \vDash A$ and $a \vDash \neg A$ whence $(Raxy \mathbin{\&} y \in S) \Rightarrow x \nvDash A$ for all $x \in K$ and $y \in S$. But $a \in S$ ($Rabc$, $c \in S$, P6). So, $Raax$ for some $x \in S$, by P7. Then, $a \nvDash A$ contradicting $a \vDash A$ above. □

5 Completeness of $\mathrm{B_{Kc4}}$

First, we state some definitions. A *theory* is a set of formulas closed under adjunction and provable entailment (that is, a is a theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem and $A \in a$, then $B \in a$); a theory is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a theory is *regular* iff all theorems of $\mathrm{B_{Kc4}}$ belong to a; a theory is *null* iff no wff belongs to a. Finally, a is *n-consistent* iff for no wff A, $A \wedge \neg A \in a$. Next, we define the canonical model. Let K^T be the set of all theories and R^T be defined on K^T as follows: for all formulas A, B and a, $b \in K^T$, $R^T abc$ iff if $A \to B \in a$ and $A \in b$, then $B \in c$. Further, let K^C be set of all prime non-null theories, S^C the set of all prime non-null n-consistent theories and R^C the restriction of R^T to K^C. Finally, let $\vDash^C$ be defined as follows: for any wff A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the $\mathrm{B_{Kc4}}$ *canonical model* is the quadruple $\langle K^C, S^C, R^C, \vDash^C\rangle$.

We prove the completeness theorem in respect of the present semantics.

Theorem 2 *If* $\vDash_{\mathrm{B_{Kc4}}} A$*, then* $\vdash_{\mathrm{B_{Kc4}}} A$.

First, we prove some useful lemmas.

Lemma 3 *If a is a non-null theory, then a is regular.*

Proof Let $A \in a$ and B be a theorem. By the rule K, $A \to B$ is a theorem. So, $B \in a$. □

Lemma 4 *Let a, b be non-null theories. The set $x = \{B \mid \exists A[A \to B \in a \ \&\ A \in b]\}$ is a non-null theory such that $R^T abx$.*

Proof It is easy to prove that x is a theory such that $R^T abx$. We prove that x is non-null. Let $A \in b$. By lemma 3, $A \to A \in a$. So, $A \in x$ by $R^T abx$. □

Lemma 5 *Let $a \in K^T$, b a non-null element in K^T and c a prime member in K^T such that $R^T abc$. Then, $R^T axc$ for some $x \in K^C$ such that $b \subseteq x$.*

Proof It is an easy adaptation of the standard proof for relevance logics (see Robles and Méndez 2007). □

We now introduce the following definitions (see Robles and Méndez 2007):

Definition 2 A theory a is w-inconsistent$_1$ (weak inconsistent 1) iff it contains the negation of a theorem (a is w-consistent$_1$—weak consistent 1—iff it is not w-inconsistent$_1$).

Definition 3 A theory a is w-inconsistent$_2$ (weak inconsistent 2) iff for some theorem $\neg A$, $A \in a$ (a is w-consistent$_2$—weak consistent 2) iff it is not w-inconsistent$_2$.

Next, we prove some results on the relationship between n-consistency with w-consistency$_1$ and w-consistency$_2$. We have:

Proposition 5 *Let $B_{K+,\neg}$ be any negation extension of B_{K+}, and a be any $B_{K+,\neg}$ theory. Then, (a) if a is w-inconsistent$_1$, then a is n-inconsistent, and (b) if a is w-inconsistent$_2$, then a is n-inconsistent.*

Proof (a) Let $\neg A \in a$, A being a theorem. By the K rule, $\neg A \to A$ is a theorem. So, $A \in a$, and consequently, $A \wedge \neg A \in a$. (b) The proof is similar to case (a). □

Proposition 6 *Let $B_{K+,\neg}$ be any negation extension of B_{K+} in which the rule*

$$r.\ \vdash B \Rightarrow\ \vdash (A \wedge \neg A) \to \neg B$$

holds, and let a be any $B_{K+,\neg}$ theory. Then, a is n-consistent, iff a is w-consistent$_1$.

Proof By proposition 5 and the rule r. □

Proposition 7 *(a) Let $B_{K+,\neg}$ be any negation extension of B_{K+} in which the principle of non-contradiction (PNC) T1 ($\neg(A \wedge \neg A)$) holds. (b) Let $B_{K+,\neg}$ be any negation extension of B_{K+} in which there is a theorem of the form $\neg\neg A$ and T12 ($(A \wedge \neg A) \to \neg B$) holds. Now, let a be any $B_{K+,\neg}$ theory (in any of the senses (a) or (b) explained above). Then, a is n-consistent, iff a is w-consistent$_2$.*

Proof By proposition 5 and the conditions of proposition 7. □

We have the following corollary of propositions 5–7:

Proposition 8 *Let a be a B_{Kc4} theory. Then, a is n-consistent iff a is w-consistent$_1$ iff a is w-consistent$_2$.*

At the end of this section, we shall show that B_{Kc4} is indeed the basic constructive logic for n-consistency in the ternary relational semantics without a set of designated points. Therefore, in the present semantic context, n-consistency cannot be independent of w-consistency$_1$ or w-consistency$_2$. But we have proved that w-consistency$_1$ is independent of n-consistency in (Robles and Méndez 2007), and that w-consistency$_2$ is also independent of n-consistency in (Robles and Méndez, in preparation).[1]

Next, we prove the primeness lemma:

Lemma 6 (Primeness lemma) *If a is a non-null n-consistent theory, then there is a prime non-null n-consistent theory x such that $a \subseteq x$.*

Lemma 6 follows from any of the three propositions 9–11 below.

[1] In (Robles and Méndez 2007), it is incorrectly stated that w-consistency $_1$ and w-consistency$_2$ are not equivalent, the latter being entailed by the former in the context of B_{Kc1}. Now, T15 of B_{Kc4} is also a theorem of B_{Kc1} (cf. proposition 2). So, let a be a w-inconsistent$_2$ theory. Then, $A \in a$, $\neg A$ being a theorem. By T15, $A \to \neg B$ is also a theorem. Therefore, a contains every negation formula, whence it is w-inconsistent$_1$. Consequently, w-consistency$_1$ entails w-consistency$_2$, given B_{Kc1}, and thus, the two weak sense of consistency are equivalent, given B_{Kc1}.

First, we prove a proposition on the preservation of w-consistency$_1$ of theories when extended to prime theories. Let $B_{+,\neg}$ be any negation extension of (Routley and Meyer's basic positive logic) B_+ in which the rule contraposition

$$\text{con.} \ \vdash A \to B \Rightarrow \ \vdash \neg B \to \neg A$$

is provable. We note that the following De Morgan law

$$\text{dm1.} \ \vdash (\neg A \vee \neg B) \to \neg(A \wedge B)$$

is provable in $B_{+,\neg}$ (A2, A5, con). We have:

Proposition 9 *Let a be a w-consistent$_1$ $B_{+,\neg}$ theory. Then, there is some prime w-consistent$_1$ $B_{+,\neg}$ theory x such that $a \subseteq x$.*

Proof Define from a a maximal w-consistent$_1$ theory such that $a \subseteq x$. If x is not prime, then there are wff A, B such that $A \vee B \in x$, $A \notin x$, $B \notin x$. Define the set $[x, A] = \{C \mid \exists D[D \in x \ \& \vdash_{B_{+,\neg}} (A \wedge D) \to C]\}$. Define $[x, B]$ similarly. It is not difficult to prove that $[x, A]$ and $[x, B]$ are theories strictly including x. By the maximality of x, they are w-inconsistent$_1$. That is, $\neg C \in [x, A]$, $\neg D \in [x, B]$ for some theorems C, D. By definitions, we have $\vdash_{B_{+,\neg}} (A \wedge E) \to \neg C$, $\vdash_{B_{+,\neg}} (B \wedge E') \to \neg D$ for some $E, E' \in x$. By basic theorems of B_+, $\vdash_{B_{+,\neg}} [(A \vee B) \wedge (E \wedge E')] \to (\neg C \vee \neg D)$. So, $\neg C \vee \neg D \in x$ and by dm1, $\neg(C \wedge D) \in x$. But by Adj., $\vdash_{B_{+,\neg}} C \wedge D$. Therefore, if x is not prime, it would be w-inconsistent$_1$, which is impossible. □

Therefore, in any logic including B_+ plus con (or B_+ plus dm1), w-consistent$_1$ theories can be extended to prime w-consistent$_1$ theories.

Next, we prove a proposition on the preservation of w-consistency$_2$. Let $B_{+,\neg}$ be any negation extension of B_+ in which the following De Morgan law

$$\text{dm2.} \ \vdash (\neg A \wedge \neg B) \to \neg(A \vee B)$$

holds. Then, we have:

Proposition 10 *Let a be a w-consistent$_2$ $B_{+,\neg}$ theory. Then, there is some prime w-consistent$_2$ $B_{+,\neg}$ theory x such that $a \subseteq x$.*

Proof Proceed similarly, as in proposition 9 until we have to conclude that $[x, A]$ and $[x, B]$ are w-inconsistent$_2$. Then, $C \in [x, A]$, $D \in [x, B]$ for some theorems $\neg C$, $\neg D$. As in proposition 9, it follows $C \vee D \in x$, but by Adj., $\vdash_{B_{+,\neg}} \neg C \wedge \neg D$ and, consequently, $\vdash_{B_{+,\neg}} \neg(C \vee D)$ by dm2 . Therefore, x is w-inconsistent$_2$, which is impossible. □

Note that, in fact, only the De Morgan law dm2 as a rule

$$\text{rdm2.} \ \vdash (\neg A \wedge \neg B) \Rightarrow \ \vdash \neg(A \vee B)$$

is needed.

Therefore, in any logic including B_+ plus dm2 (rdm2), w-consistent$_2$ theories can be extended to prime w-consistent$_2$ theories.

Finally, we prove a proposition on the preservation of n-consistency. Let $B_{K+,\neg}$ be any negation extension of B_{K+} in which dm2 and the PNC T1 (of B_{Kc4}) hold. Then, we have:

Proposition 11 *Let a be a n-consistent $B_{K+,\neg}$ theory. Then, there is some prime n-consistent $B_{K+,\neg}$ theory x such that $a \subseteq x$.*

Proof Proceed similarly, as in the two previous propositions until one has to conclude the n-inconsistency of $[x, A]$ and $[x, B]$, i.e., $C \wedge \neg C \in [x, A]$, $D \wedge \neg D \in [x, B]$ for some wff C, D. Reasoning similarly, as above, it follows that $(C \wedge \neg C) \vee (D \wedge \neg D) \in x$. Now, by T1 and Adj., $\neg(C \wedge \neg C) \wedge \neg(D \wedge \neg D)$ is a theorem. So, $\neg[(C \wedge \neg C) \vee (D \wedge \neg D)]$ is also a theorem by dm2. As x is regular (it is non-null), $\neg[(C \wedge \neg C) \vee (D \wedge \neg D)] \in x$. Therefore, $[(C \wedge \neg C) \vee (D \wedge \neg D)] \wedge \neg[(C \wedge \neg C) \vee (D \wedge \neg D)] \in x$, contradicting the n-consistency of x. □

Note that, as in the preceding proposition, the rule rmd2 is sufficient.

Therefore, in any logic included in B_{K+} plus dm2 (rdm2) and T1, n-consistent theories can be extended to prime n-consistent theories.

Given proposition 8, lemma 6 follows by any of the propositions 9, 10 or 11.

Now, given the completeness of B_{K+} (see Robles and Méndez 2007), in order to prove that of B_{Kc4}, it is obvious that we just have to prove that P6, P7 and clause (v) are canonically valid. We now prove lemma 7 from which the canonical validity of P6 and P7 follows immediately.

Lemma 7

(1) *Let a, b be non-null elements in K^T and c be a n-consistent member in K^T such that $R^T abc$. Then, a is n-consistent as well.*
(2) *Let a be a n-consistent member in K^T. Then, there is some n-consistent member x in K^T such that $R^T aax$.*

Proof

(1) Assume the hypothesis of case 1. By reductio, suppose that for some wff A, $A \wedge \neg A \in a$. By A9, $\neg(A \to A) \in a$. Now, by A7, $\neg(A \to A) \to \{(A \to A) \to [(A \to A) \wedge \neg(A \to A)]\}$ is a theorem. So, $\{(A \to A) \to [(A \to A) \wedge \neg(A \to A)]\} \in a$. Now, $A \to A \in b$ by lemma 3. Thus, by $R^T abc$, $(A \to A) \wedge \neg(A \to A) \in c$ contradicting the n-consistency of c.[2]
(2) Suppose a is a non-null n-consistent theory. Define the non-null theory x such that $R^T aax$ (cf. lemma 4). By reductio, suppose that for some wff A, $A \wedge \neg A \in x$. Then, for some $B \in a$, $B \to (A \wedge \neg A) \in a$. But, by A8, $\neg B \in a$. Consequently, $B \wedge \neg B \in a$, contradicting the n-consistency of a. □

[2] The proof can be simplified by using proposition 8 (see Robles and Méndez 2007).

We now prove the canonical adequacy of P6 and P7. They read canonically as follows:

P6. Let $a, b \in K^C$, $c \in S^C$ and $R^C abc$. Then, $a \in S^C$.

P7. Let $a \in S^C$. Then, there is some $x \in S^C$ such that $R^C aax$.

Proof Proof of P6: Immediate from lemma 7 (1).
Proof of P7: suppose $a \in S^C$. By lemma 7 (2), there is a non-null n-consistent theory y such that $R^T aay$. By lemma 6 y is extended to a prime non-null n-consistent theory x such that $y \subseteq x$. Obviously, $R^C aax$. □

Finally, we prove that clause (v) holds canonically:

Proof *If* $a \vDash^C \neg A$*, then* $(R^C abc \ \&\ \ c \in S^C) \Rightarrow b \nvDash^C A$*:*

Suppose $\neg A \in a$, $R^C abc$, $c \in S^C$ and (by reductio) $A \in b$. By A7, $A \to (A \wedge \neg A) \in a$. Then, $(A \wedge \neg A) \in c$ contradicting the n-consistency of c.

If $a \nvDash^C \neg A$*, then there are* $b \in K^C$*,* $c \in S^C$ *such that* $R^C abc$ *and* $A \in b$*:*

Suppose $\neg A \notin a$. Define the sets $x = \{B \mid \vdash_{\mathrm{B_{Kc4}}} A \to B\}$, $y = \{B \mid \exists C[C \to B \in a \ \&\ C \in x]\}$. It is easy to show that x and y are non-null theories such that $R^T axy$ (cf. lemma 4). Moreover, $A \in x$. If y is not n-consistent, then for some wff B, $B \wedge \neg B \in y$. Then, $C \to (B \wedge \neg B) \in a$, $\vdash_{\mathrm{B_{Kc4}}} A \to C$ for some wff C. By Suf., $\vdash_{\mathrm{B_{Kc4}}} [C \to (B \wedge \neg B)] \to [A \to (B \wedge \neg B)]$. So, $A \to (B \wedge \neg B) \in a$. By A8, $\neg A \in a$ contradicting the hypothesis. Now, as y is non-null, it is extended to a prime non-null n-consistent theory c such that $y \subseteq c$. Clearly, $R^T axc$ ($R^T axy$ and $y \subseteq c$). Then, x is extended to a prime non-null theory b such that $R^C abc$ and $A \in b$ by lemma 5. □

We end this section with a brief discussion on the reasons for dubbing $\mathrm{B_{Kc4}}$ 'the basic constructive logic for n-consistency' in the ternary relational semantics without a set of designated points.

Axioms A7 and A8 are needed (and sufficient) in the proof of the canonical adequacy of clause (v). On the other hand, the 'principle of non-contradiction' (PNC) T1 and the De Morgan law T25 are sufficient to prove the primeness lemma (cf. proposition 11). Therefore, a reasonable conclusion is that A7 and A8 should suffice for axiomatizing the basic constructive logic for n-consistency (I owe this point and its development to a suggestion by a referee of the JoLLI). But surprisingly enough, T25, which is valid in the semantics by using the mere clauses (no semantic postulates being needed) is not derivable from A7 and A8 (MaGIC). Consequently, our conclusion forcibly has to be that A7, A8 and T25 (added to $\mathrm{B_{K+}}$) is the system we are searching for. And it has to be so, I think, but not in the present semantic context, as I will show in the following lines.

Given $\mathrm{B_{K+}}$ and $\mathrm{B_{K+}}$ semantics, P7 is the 'corresponding postulate' to A8. That is, the axiom is proved valid with the postulate, and this one is proved valid with the axiom. But notice that this is not the case, regarding A7 and P6: A9 is needed in the canonical proof of P6. The fact is that A7 is too weak an axiom to prove P6 canonically valid, if the positive context is $\mathrm{B_{K+}}$.

So, let us explore which postulates would validate A7. $\mathrm{B_{K+}}$ semantics being supposed (for stronger positive logics other possibilities would be open), it is easy to see that in addition to P6, the three following possibilities are obtained:

$$\text{P6(i). } (R^2abcd \ \ \& \ \ d \in S) \Rightarrow (\exists x \in S)Rabx$$
$$\text{P6(ii). } (Rabc \ \ \& \ \ Rcde \ \ \& \ \ e \in S) \Rightarrow (\exists x \in S)Racx$$
$$\text{P6(iii). } (R^2abcd \ \ \& \ \ d \in S) \Rightarrow (\exists x \in S)Racx$$

Now, given B_{K+} semantics plus P7, P6 and P6(i) are equivalent; P6(ii) is provable canonically only if A7, A8, A9 and

$$\text{(i). } \neg A \to [(A \to A) \to \neg A]$$

are present. And finally, P6(iii) is only provable if A8 and

$$\text{(ii). } \neg A \to (B \to \neg A)$$

are theorems (note that (i) and (ii) are equivalent, B_{K+} being supposed).

It is clear that P6(ii) must be rejected: we would be obliged to strengthen B_{Kc4} with (i), which is not provable in it (MaGIC). Therefore, we are left with two possibilities: P6 (P6(i)) or P6(iii). If we choose P6, A9 (in fact, the restricted ECQ axiom T12) is automatically valid; if we choose P6(iii), the restricted K axiom (ii) has to be introduced. And this axiom seems to be perhaps too much of a strengthening of A7. So, we choose P6.

As A7 and A8 (with T25) do not seem to be semantically (though syntactically they are indeed) isolated in the present semantic context, we can consider B_{Kc4} as the basic constructive logic for n-consistency in the ternary relational semantics without a set of designated points.

Anyway, what about the other possibility? As we have noted above, P7 is the corresponding postulate to A8. On the other hand, (ii) is proved valid with P6(iii), which, in its turn, is proved canonically valid with A8 and (ii). Now, let $B_{Kc4(b)}$ be axiomatized by adding A8 and (iii) to B_{K+}. What about this logic? $B_{Kc4(b)}$ is a most peculiar logic. Contrarily to what happens with B_{Kc4}, $B_{Kc4(b)}$ is undefinable (so it seems) with a propositional falsity constant. We think that it is, in fact, the basic constructive paraconsistent (in a strong sense of the concept) logic with the PNC in the ternary relational semantics without a set of designated points. It has the PNC as a theorem, and negation introduced via implication of a contradiction. But it has not the slightest flavour of the ECQ axioms and just a little scent of the EFQ axioms, namely, the rule

$$\text{(iii). } \vdash A \Rightarrow \ \vdash \neg A \to \neg B$$

Moreover, B_{Kc4} and $B_{Kc4(b)}$ are independent logics.

Could $B_{Kc4(b)}$ possibly be the basic constructive logic for n-consistency in the ternary relational semantics without a set of designated points?

We promise to discuss $B_{Kc4(b)}$ and the question just noted in another paper.

6 The Logic B_{Kc5}

The logic B_{Kc5} is the result of adding to B_{Kc4} the following axioms:

$$\text{A10. } (A \to \neg B) \to (B \to \neg A)$$
$$\text{A11. } B \to [(A \to \neg B) \to \neg A]$$

But we prove that A8 and A11 are sufficient to axiomatize B_{Kc5}.

First, we note that T1, T5, T11, T18–T22 of B_{Kc4} are derivable from B_{K+} plus A8. Next, we prove

$$\text{T26. } \vdash A \to \neg\neg A \qquad \text{A11, T11}$$
$$\text{T27. } \vdash \neg B \to [(A \to B) \to \neg A] \qquad \text{A11, T26}$$
$$\text{T28. } \vdash \neg A \to (B \to \neg A) \qquad \text{T27, K}$$

On the other hand, by the theorem of B_{K+}

$$(A \to B) \to [A \to (A \wedge B)]$$

and T28, we have

$$\text{T29. } \neg B \to [A \to (A \wedge \neg B)]$$

Now, the weak contraposition axiom

$$\text{T30. } \vdash (A \to B) \to (\neg B \to \neg A)$$

is proved as follows. By T27

$$\text{(i). } \neg(A \wedge \neg B) \to [[A \to (A \wedge \neg B)] \to \neg A]$$

Then, T30 follows by (i), T21 and T29. So, we have

$$\text{T31. } (A \to \neg B) \to (B \to \neg A) \qquad \text{T26, T30}$$

Finally, A7 is derivable from T29, and A9 by T1 and T31. Therefore, B_{Kc4} is included in B_{Kc5}.

Some other theorems of B_{Kc5} are, for example,

$$\text{T32. } \vdash A \to [(A \to \neg B) \to \neg B] \qquad \text{A11, T31}$$
$$\text{T33. } \vdash (A \wedge B) \to \neg(A \to \neg B) \qquad \text{A10, T21}$$
$$\text{T34. } \vdash (A \wedge \neg B) \to \neg(A \to B) \qquad \text{A10, T22}$$
$$\text{T35. } \vdash A \to (\neg A \to \neg B) \qquad \text{T15, T26}$$

Moreover, it is especially remarkable that the full (weak) reductio axioms are derivable:[3]

$$\text{T36.} \vdash (A \to B) \to [(A \to \neg B) \to \neg A]$$

Proof

1. $(A \wedge B) \to \neg(A \to \neg B)$	T33
2. $(A \to B) \to [A \to (A \wedge B)]$	By B_{K+}
3. $[A \to (A \wedge B)] \to [A \to \neg(A \to \neg B)]$	Pref., 1
4. $(A \to B) \to [A \to \neg(A \to \neg B)]$	(Transitivity, 2, 3)
5. $[A \to \neg(A \to \neg B)] \to [(A \to \neg B) \to \neg A]$	T31 (A10)
6. $(A \to B) \to [(A \to \neg B) \to \neg A]$	Transitivity, 4, 5

□

$$\text{T37.} \vdash (A \to \neg B) \to [(A \to B) \to \neg A]$$

Proof Similar to the proof of T36 by using T34. □

Now, on the axiomatization of B_{Kc5}, we note the following

Proposition 12

(1) *In the formulation of B_{Kc5}, A8 can be substituted by A9.*
(2) *Moreover, in the formulation of B_{Kc5}, A8 can be substituted by the PNC T1.*
(3) *Let $B_{Kc5(b)}$ the result of adding A7, A8 and A10 to B_{K+}. Then, B_{Kc5} and $B_{Kc5(b)}$ are deductively equivalent.*
(4) *T1–T3, T6, T9, T10, T12–T16, T23–T26, T28–T30, T31 (A10) and T35 are derivable from A7, A9 and A10. But A7, A9 and A10 do not axiomatize B_{Kc5}. Nevertheless, if any of T4, T5, T7, T8, T11, T17 (from right to left), T18–T22, T27, T32–T34, T36 or T37, which are not derivable from A7, A8 and A10, is added to these axioms, the resulting system is equivalent to B_{Kc5}.*
(5) *T1, T2, T4–T14, T18–T26, T30, T31 (A10), T33, T34, T36 and T37 are derivable from A8, A9 and A10. But A8, A9 and A10 do not axiomatize B_{Kc5}. However, if any of T3, T15, T16, T17 (from left to right), T27, T28, T29, T32 or T35, which are not derivable from A8, A9 and A10, is added to these axioms, the resulting system is equivalent to B_{Kc5} .*

[3] It is conjectured in (Robles et al. 2005) that the full (weak) reductio axioms T36 and T37 cannot be introduced in B_+ supplemented with the contraposition axioms A10 and T30 and the specialized law of reductio T5. The resources of the logic seem to be insufficient to prove the canonical adequacy of the corresponding semantical postulates. Moreover, in (Robles and Méndez 2004), it is conjectured that this also happens in the case of the full (strong) reductio axioms $(\neg A \to B) \to [(\neg A \to \neg B) \to A]$ and $(\neg A \to \neg B) \to [(\neg A \to B) \to A]$ in respect of Routley and Meyer's logic B. Now, interestingly enough (we think), not only can T36 and T37 be introduced in B_{Kc5}, they are also derivable.

(6) *B_{Kc5} and the equivalent formulations in* (1)*,* (2) *and* (3) *are well axiomatized in respect of B_{K+} (cf. proposition 1).*

Proof The proof of case (1) is easy and is left to the reader. Case (2) is proved as follows:

By A11

$$\text{(a). } A \to [(\neg A \to \neg A) \to \neg\neg A]$$

whence by the K rule

$$\text{(b). } A \to (B \to \neg\neg A)$$

Again, by A11 and T1,

$$\text{(c). } [B \to \neg\neg(A \wedge \neg A)] \to \neg B$$

Then,

$$\text{T12. } (A \wedge \neg A) \to \neg B$$

follows by using (b) and (c).

Next, we prove case (3). We prove that $\mathrm{B_{Kc5}}$ is included in $\mathrm{B_{Kc5(b)}}$, the converse being obvious:

By A8,

$$\text{T1. } \neg(A \wedge \neg A)$$

and by T1, K and A10,

$$\text{A9. } (A \wedge \neg A) \to \neg(A \to A)$$

So, note that $\mathrm{B_{Kc4}}$ is included in $\mathrm{B_{Kc5(b)}}$. Now, by A10,

$$\text{T26. } A \to \neg\neg A$$

and by A10 and T26,

$$\text{T30. } (A \to B) \to (\neg B \to \neg A)$$

On the other hand, by T15 and A10,

$$\text{T28. } \neg A \to (B \to \neg A)$$

whence by $\mathrm{B_{K+}}$

$$\text{T29. } \neg B \to [A \to (A \wedge \neg B)]$$

Then, T27 is proved as follows:

$$\text{(a). } \neg B \to [\neg(A \wedge \neg B) \to \neg A]$$

by T29 and T30. So, by (a) and T21,

$$\text{T27. } \neg B \to [(A \to B) \to \neg A]$$

Finally, by T26 and T27,

$$\text{A11. } B \to [(A \to \neg B) \to \neg A]$$

Regarding cases (4) and (5), the proofs of non-derivability are by MaGIC; the proofs of derivability are left to the reader: they are easy leaning on the proofs provided for B_{Kc4} and B_{Kc5} throughout the paper. Finally, case (6) is proved again with MaGIC. □

Given the axiomatization of B_{Kc4} provided in propositions 2–4 and those of B_{Kc5} in proposition 12, a number of alternative formulations of the latter logic can be provided. But, enough 'axiom chopping' having been displayed in this paper (maybe too much, in fact), this point will not be pursued here any further: it is not the aim of this paper to 'exhaustively axiomatize' (cf. Méndez 1987) B_{Kc4} (B_{Kc5}) with A7–A9 (A7–A11) and T1–T25 (T1–T37).

7 Semantics for B_{Kc5}

A B_{Kc5} *model* is a quadruple $\langle K, S, R, \vDash \rangle$ where K, S, R and $\vDash$ are defined (in a similar way) as in a B_{Kc4} model save for the addition (besides clause (v) and postulates P6 and P7) of the postulates

$$\text{P8. } (R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in S)R^2acbx$$
$$\text{P9. } (R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in S)R^2bcax$$

$\vDash_{B_{Kc5}} A$ (A is B_{Kc5} *valid*) iff $a \vDash A$ for all $a \in K$ in all models.

It is clear that in order to prove the soundness of B_{Kc5}, we just have to prove that A10 and A11 are valid and that in order to prove completeness, we have to prove that P8 and P9 hold canonically.

As remarked in (Robles and Méndez 2007), A10 and A11 are proved valid with P8 and P9, respectively. On the other hand, P8 is proved canonically valid with A7, A8 and A10, and P9 is proved canonically valid with A7, A8 and A11, in a similar way, as they were proved in (Robles and Méndez 2007).

8 Concluding Remarks: Strengthening the Logics

We show how to define some logics between B_{Kc4} and minimal intuitionistic logic from some well-known ones (cf. Robles and Méndez 2007).

Consider the following axioms and rule of inference:

A12. $(B \to C) \to [(A \to B) \to (A \to C)]$

A13. $(A \to B) \to [(B \to C) \to (A \to C)]$

A14. $[A \to (A \to B)] \to (A \to B)$

A15. $\vdash A \Rightarrow \ \vdash (A \to B) \to B$

A16. $A \to [(A \to B) \to B]$

A17. $A \to (B \to A)$

The logic TW_+ ('Contractionless Positive Ticket Entailment') is B_+ plus A12 and A13; the logic T_+ ('Positive Ticket Entailment') is TW_+ plus A14; the logic E_+ ('Positive Logic of Entailment') is T_+ plus A15; the logic R_+ ('Positive Logic of Relevance') is T_+ plus A16; the logic J_+ (Positive Intuitionistic Logic) is R_+ plus A17. Therefore, TW_{K+}, T_{K+}, E_{K+} and R_{K+} are TW_+, T_+, E_+ and R_+ plus the K rule, respectively. We note that R_{K+} and J_{K+} are deductively equivalent logics. Now, negation can be introduced in these positive logics in a similar way as it has been introduced in B_{K+}. The logics TW_{Kc4}, T_{Kc4} , E_{Kc4}, R_{Kc4} and J_{Kc4}, as well as the logics TW_{Kc5}, T_{Kc5}, E_{Kc5}, R_{Kc5} and J_{Kc5}, can also be defined in this way. It is clear that J_{Kc5} is minimal intuitionistic logic defined with a negation connective. On the relationship between these logics we note the following proposition:

Proposition 13

(1) *TW_{Kc4} and TW_{Kc5}, T_{Kc4} and T_{Kc5}, E_{Kc4} and E_{Kc5} are different logics.*

(2) *R_{Kc4} (= J_{Kc4}) is deductively equivalent to R_{Kc5} (= J_{Kc5}).*

Proof (1) By MaGIC, A10 and A11 are not derivable in E_{Kc4}.

(2) The axioms A10 and A11 can be proved in R_{Kc4} (J_{Kc4}) as follows:
By A13, A7 and A8, we have

$$\text{(a). } (A \to B) \to (\neg B \to \neg A)$$

By A12, A7 and A8,

$$\text{(b). } \neg B \to [(A \to B) \to \neg A]$$

Now, by A17 and (a)

$$\text{(c). } A \to [\neg A \to \neg(B \to B)]$$

From (c) and T8

$$\text{(d). } A \to \neg\neg A$$

Therefore, by (a) and (d),

$$\text{(e). } (A \to \neg B) \to (B \to \neg A)$$

which is A10. Similarly, by (b) and (d),

$$\text{(f). } B \to [(A \to \neg B) \to \neg A]$$

which is A11. □

As for semantics, consider the following set of postulates:

P10. $R^2abcd \Rightarrow (\exists x \in K)(Rbcx \,\&\, Raxd)$

P11. $R^2abcd \Rightarrow (\exists x \in K)(Racx \,\&\, Rbxd)$

P12. $Rabc \Rightarrow R^2abbc$

P13. $(\exists x \in K)Raxa$

P14. $Rabc \Rightarrow Rbac$

P15. $Rabc \Rightarrow a \leq c$

Given B_+, postulates P10, P11, P12, P13 and P14 are the corresponding postulates to A12, A13, A14, A15 and A16, respectively (see (Routley et al. 1982); given B_{K+}, P15 is the corresponding postulate to A17. Therefore, models for the different L_{Kc} logics are defined by simply adding the corresponding postulates to the B_{Kc} logic. Thus, T_{Kc4} models, for example, are defined by adding postulates P10, P11 and P12 to postulates P6 and P7 in B_{Kc4} models. Soundness and completeness follow from those of the B_{Kc} logics and the fact that the postulates added to the models are the corresponding postulates to the axioms added to the logic.

Now, let us briefly compare B_{Kc4} and its extensions with B_{Kc1} and its extensions.

As we have seen (cf. Sect. 3), the basic constructive logic for w-consistency$_1$ B_{Kc1} can be axiomatized by adding to B_{K+} T3 and T8 (of B_{Kc4}). Then, in (Robles and Méndez 2007), the logic B_{Kc2} is defined. It is the result of adding the contraposition axioms A10 and A11 to B_{Kc1}. Finally, in the aforementioned paper, it is shown how to extend B_{Kc2} with A12, A13, A15, A16 and A17. The contraction axiom A14 is not considered because if it is added to B_{Kc1}, w-consistency would collapse into n-consistency. We remarked (cf. proposition 3) that B_{Kc4} can be viewed as the result of adding the PNC T1 to B_{Kc1}. By case (2) of proposition 12, B_{Kc5} can similarly be understood as the result of adding T1 to B_{Kc2}. So, we have:

Proposition 14 *B_{Kc1} and B_{Kc2} are included in (but do not include) B_{Kc4} and B_{Kc5}, respectively.*

Proof T3 and T8 are theorems of B_{Kc4}, but A8, for example, is not provable in B_{Kc2} (MaGIC). □

Moreover, let S_{Kc1} (S_{Kc2}) be any extension of B_{Kc1} (B_{Kc2}) defined by adding any selection of axioms A12, A13, A15, A16 and A17; and let S_{Kc4} (S_{Kc5}) be the extension of B_{Kc4} (B_{Kc5}) defined by adding the same selection. We have:

Proposition 15 *S_{Kc1} (S_{Kc2}) is included in (but does not include) S_{Kc4} (S_{Kc5}), respectively.*

Proof (a) By proposition 14. (b) Let $\mathrm{JW}_{\mathrm{Kc2}}$ be the result of adding to $\mathrm{B}_{\mathrm{Kc2}}$ A12, A13, A15, A16 and A17. Although A7 is provable in $\mathrm{JW}_{\mathrm{Kc2}}$, A8 and A9 are not (MaGIC). □

We end by noting that the EFQ axioms (iii), (iv) and the ECQ axiom (vi) (cf. Introduction) are unprovable in all logics defined in this paper: they are not derivable in minimal intuitionistic logic $\mathrm{J_m}$. Therefore, though in $\mathrm{JW}_{\mathrm{Kc5}}$ (in fact, in $\mathrm{B}_{\mathrm{Kc4}}$) negation-inconsistency is equivalent to w-inconsistency$_1$ and w-inconsistency$_2$ (cf. proposition 8), it cannot be defined as absolute inconsistency, i.e. triviality.

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