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THE BASIC LAPLACIAN OF A RIEMANNIAN FOLIATION

By Efton Park and Ken Richardson

Abstract. We study the basic Laplacian on Riemannian foliations by writing the basic Laplacian in terms of the orthogonal projection from square-integrable forms to basic square-integrable forms. Using a geometric interpretation of this projection, we relate the ordinary Laplacian to the basic Laplacian. Among other results, we show the existence of the basic heat kernel and establish estimates for the eigenvalues of the basic Laplacian.

Introduction. Let M be a compact oriented manifold and let $\mathcal{F}$ be a transversally oriented foliation on M. A foliation $\mathcal{F}$ is a Riemannian foliation if there exists a Riemannian metric on M with the property that the distance from one leaf of $\mathcal{F}$ to another is locally constant; such a metric is called a bundle-like metric for $\mathcal{F}$. Associated to $\mathcal{F}$ are the space of basic forms:

$$\Omega_B^*(M) = \Omega_B^*(M, \mathcal{F}) = \{\omega \in \Omega^*(M): \ i(X)\omega = 0, i(X)d\omega = 0 \text{ for all } X \in \Gamma(T\mathcal{F})\},$$

where $i(X)$ is the interior product with the vector field X and $\Gamma(T\mathcal{F})$ denotes the sections of the distribution $T\mathcal{F}$ associated to $\mathcal{F}$. The exterior derivative d maps basic forms to basic forms; let d_B denote d restricted to $\Omega_B^*(M)$. The basic Laplacian is the operator $\Delta_B = d_B\delta_B + \delta_B d_B$ on basic forms, where δ_B is the adjoint of d_B on $\Omega_B^*(M)$. The analytic and geometric properties of this operator have been studied by several researchers. In [5], the basic Laplacian was studied as an operator on basic functions (i.e., functions that are constant on leaves of $\mathcal{F}$), and the author proved the existence of the heat kernel in this case. In [13], the existence of the heat kernel on basic forms was proved for the case where the mean curvature form of the foliation is basic. There are also "basic" Hodge theorems, for example [6] and [10]. However, the proof of the Hodge theorem in [6] does not yield various estimates that are important in applications, while the theorem proved in [10] has the same restriction as the results in [13], namely that the authors require the mean curvature form to be basic.

In this paper, we study the basic Laplacian on forms, without any restriction on the mean curvature. We prove the existence and uniqueness of the heat kernel for Δ_B on forms for any Riemannian foliation, and we write down an explicit formula for the heat kernel. We also present a proof of the Hodge theorem for

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basic forms that is similar to the proof in [10] and does not require basic mean curvature.

We have used several techniques in obtaining our results. The main new idea in our work is in taking a Hilbert space approach to the study of the basic Laplacian. It follows easily from the theory of operators on Hilbert space that the adjoint δ_B of d_B can be expressed as $\delta_B = P\delta$, where δ is the adjoint of d and P is the orthogonal projection of the Hilbert space $L^2(\Omega^*(M))$ of all L^2-forms on M to the Hilbert subspace $L^2_B(\Omega^*(M))$ spanned by the smooth basic forms on M. We show that this projection, which we call the basic projection, can be computed in terms of geometric data, and this is a crucial fact in the proofs of our results. We also employ two techniques that have been used by other researchers in this area. First, we study the Riemannian foliation $\mathcal{F}$ by lifting it to a transversely parallelizable foliation $\widehat{\mathcal{F}}$ on the oriented orthonormal transverse frame bundle $\widehat{M}$ of $(M, \mathcal{F})$. Second, we use a device that is also used in [10]: we show that Δ_B is the restriction to basic forms of a (non-selfadjoint) elliptic operator on all forms. This is an important technical point, because the space of basic forms is not the set of all sections of any vector bundle, and therefore the usual theory of elliptic operators does not apply directly to Δ_B.

This paper is organized as follows. In Section 1, we first define the basic projection for the transversely parallelizable foliation $(\widehat{M}, \widehat{\mathcal{F}})$, and give a geometric formula for it. We then use the basic projection on $(\widehat{M}, \widehat{\mathcal{F}})$ to define the basic projection P for the original foliation $(M, \mathcal{F})$. We prove that P maps smooth forms to smooth forms and maps basic forms to themselves. We also record several formulas that express how P acts on specific kinds of forms.

In Section 2, we examine relationships among the basic projection P, the exterior derivative d and its adjoint δ, and the basic Hodge star operator. We give an explicit formula for the commutator $[P, \delta]$, and by taking adjoints also get a formula for $[P, d]$. From these formulas we show that on basic forms, the basic Laplacian can be written as $\Delta_B = \Delta + \epsilon d + d\epsilon$, where Δ is the usual Laplacian on M and ϵ is an order zero map from the space of basic forms to its perpendicular space. The expression on the right-hand side makes sense for all forms and therefore gives us a strongly elliptic operator $\widetilde{\Delta}$ on $\Omega^*(M)$ that equals Δ_B when restricted to basic forms.

In Section 3, we give a new proof of the Hodge decomposition theorem for basic forms. We prove the existence and uniqueness of the heat kernel $K^k_B(t, x, y)$ of the basic Laplacian on k-forms, and we prove that $K^k_B(t, x, y) = P_x P_y K^k_{\widetilde{\Delta}^*}(t, x, y)$, where P_x and P_y are the basic projections in the x and y variables, respectively, and where $K^k_{\widetilde{\Delta}^*}(t, x, y)$ is the heat kernel of $\widetilde{\Delta}^*$. From this we get a basic Green's operator G_B that is related to the Green's operator $\widetilde{G}$ of $\widetilde{\Delta}^*$ by the formula $G_B = P\widetilde{G}$. We also obtain an estimate for the growth of the eigenvalues of the basic Laplacian. Suppose the space of basic k-forms is infinite-dimensional. Let $\lambda^{B,k}_0 \leq \lambda^{B,k}_1 \leq \lambda^{B,k}_2 \leq \cdots$ be the eigenvalues of Δ_B, and let $\lambda^{\overline{\Delta},k}_0 \leq \lambda^{\overline{\Delta},k}_1 \leq$

$\lambda_2^{\overline{\Delta},k} \leq \cdots$ be the eigenvalues of $\overline{\Delta} = \Delta - \epsilon^* \epsilon$. We prove that $\lambda_j^{B,k} \geq \lambda_j^{\overline{\Delta},k}$ for all j, which implies that there exists a positive constant C such that $\lambda_j^{B,k} \geq Cj^{2/n}$ for large j.

In Section 4, we consider special cases of our results. We first look at the basic Laplacian on functions, and show the formulas for the various Laplacians we construct in Section 2 simplify in this case. Next, let $\{\lambda_j^{\Delta}\}$ be the nondecreasing sequence of eigenvalues of Δ on smooth functions, and let $\{\lambda_j^B\}$ be the nondecreasing sequence of eigenvalues of Δ_B on smooth basic functions. Then when the space of smooth basic functions is infinite-dimensional, we prove that $\lambda_j^B \geq \lambda_j^{\Delta}$ for all j, and therefore $\operatorname{tr}_{L^2_B}(e^{-t\Delta_B}) \leq \operatorname{tr}_{L^2}(e^{-t\Delta})$ for $t > 0$. We also apply our results to Riemannian foliations where the mean curvature form is basic; this is the situation studied in [10] and [13]. Under this hypothesis, we obtain the following results. We show that the basic Laplacian and the ordinary Laplacian (as well as the other Laplacians we define in Section 2) coincide as operators on smooth basic functions. We then prove that the spectrum of Δ_B as an operator on basic functions is contained in the spectrum of Δ as an operator on functions (also proved in [10]) and that the heat kernel $K_B(t,x,y)$ of Δ_B on basic functions equals $P_x P_y K_{\Delta}(t,x,y)$, where $K_{\Delta}(t,x,y)$ is the heat kernel of Δ on functions. Let $\{\lambda_j^{\Delta,k}\}$ be the nondecreasing sequence of eigenvalues of Δ on smooth k-forms $\mathbf{\Omega}^k$, and let $\{\lambda_j^{B,k}\}$ be the nondecreasing sequence of eigenvalues of Δ_B on smooth basic k-forms $\mathbf{\Omega}_B^k$. For $k = 1$, we prove that when the space of smooth basic 1-forms is infinite-dimensional, $\lambda_j^{B,1} \geq \lambda_j^{\Delta,1}$ for all j, and hence $\operatorname{tr}_{L^2(\Omega_B^1)}(e^{-t\Delta_B}) \leq \operatorname{tr}_{L^2(\Omega^1)}(e^{-t\Delta})$ for $t > 0$. If $k > 1$ and the space of smooth basic k-forms is infinite-dimensional, we show that $\lambda_j^{B,k} \geq \lambda_j^{\Delta,k} - \max_{x \in M}\{(\phi_0, \phi_0)_x\}$, where ϕ_0 is a form defined in Section 2 that comes from Rummler's formula, and where $(\ ,\ )_x$ denotes the pointwise inner product of forms at a point x in M. Finally, we show that when the foliation has codimension 1 and the space of smooth basic 1-forms is infinite-dimensional, $\lambda_j^{B,k} \geq \lambda_j^{\Delta,k} - \max_{x \in M}\{(P\kappa - \kappa, P\kappa - \kappa)_x\}$; this last inequality does not require that the mean curvature be basic.

This work can be viewed as an extension of the spectral analysis of Riemannian submersions; see, for example, [12], [2], [15], and [8]. We should also mention some important results that are not directly related to our work. In [1], the author showed that the basic component of the mean curvature form (in our terminology, the projection of the mean curvature form) is closed and defines a class in basic cohomology that is independent of the choice of bundle-like metric. Furthermore, any representative of this cohomology class can be realized as the projection of the mean curvature form of some bundle-like metric. In [3], the author proves that for any Riemannian foliation on a compact manifold, there exists a bundle-like metric for which the mean curvature form is basic. These results can be used to study topological aspects of Riemannian foliations by capitalizing on the wealth of literature concerning foliations with basic mean curvature. In our work, we are interested in fixing a bundle-like metric and studying the relationship between the Riemannian geometry of the manifold and the transverse geometry of the foliation. Other researchers have been interested in these geometric aspects as well ([14]).

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1. The basic projection. Throughout this paper, M will be a closed, oriented Riemannian manifold, and $\mathcal{F}$ will be a transversally oriented Riemannian foliation on M. Let $\Omega^*(M)$ be the space of smooth forms on M, and let $\Omega^*_B(M)$ be the space of smooth basic forms. The exterior derivative d restricts to a map d_B from $\Omega^*_B(M)$ to itself, and we wish to write down the formal adjoint of d_B.

Suppose $\Omega^*(M)$ is endowed with the usual L^2-inner product $\langle\ ,\ \rangle$ induced from the metric on M, and let $L^2(\Omega^*(M))$ be the completion of $\Omega^*(M)$. Let $L^2(\Omega^*_B(M))$ be the closed subspace of $L^2(\Omega^*(M))$ spanned by $\Omega^*_B(M)$, let P_B be the orthogonal projection on $L^2(\Omega^*(M))$ with range $L^2(\Omega^*_B(M))$, and let δ be the formal adjoint of d. Then for any two smooth basic forms α and β,

$$\langle \alpha, P_B\delta\beta\rangle = \langle P_B\alpha, \delta\beta\rangle = \langle \alpha, \delta\beta\rangle = \langle d\alpha, \beta\rangle = \langle d_B\alpha, \beta\rangle,$$

so $P_B\delta$ is the Hilbert space adjoint δ_B of d_B (strictly speaking, when working with unbounded operators, one must explicitly specify their domains. However, we are ultimately only interested in smooth forms, and the domains of each of these operators include the appropriate smooth forms).

While the Hilbert space approach provides a simple formula for δ_B, this formula is not very useful from a geometric point of view. For example, it is not at all obvious that δ_B maps smooth basic forms to smooth basic forms. For this reason, it is desirable to have another way to compute δ_B. We accomplish this by demonstrating that the projection P_B can be computed in terms of geometric data.

Let $T\mathcal{F}$ be the p-dimensional subdistribution of TM corresponding to $\mathcal{F}$, and let $N\mathcal{F}$ be the orthogonal distribution to $T\mathcal{F}$. Let $\widehat{M}$ be the oriented orthonormal transverse frame bundle of $(M, \mathcal{F})$, and let π be the natural projection $\pi\colon \widehat{M} \to M$. The manifold $\widehat{M}$ is a principal $SO(q)$-bundle over M, where q is the codimension of $\mathcal{F}$. Associated to $\mathcal{F}$ is the *lifted foliation* $\widehat{\mathcal{F}}$ on $\widehat{M}$. The lifted foliation is transversely parallelizable, and the closures of its leaves are the fibers of a fiber bundle $\pi_B\colon \widehat{M} \to W$. (See [11] for details.) We let $\overline{\mathcal{F}}$ denote the foliation of $\widehat{M}$ by leaf closures of $\widehat{\mathcal{F}}$.

Endow $\widehat{M}$ with the metric $g_M + g_{SO(q)}$, where g_M is the (lifted) metric from M on vectors in the manifold directions, and $g_{SO(q)}$ is a normalized left-invariant metric on the fibers.

Let $C^\infty_B(\widehat{M})$ denote the space of smooth basic functions on $\widehat{M}$. We define a map A from $C^\infty(\widehat{M})$ to functions on $\widehat{M}$ by averaging along the leaves of $\overline{\mathcal{F}}$. More precisely, let f be a smooth function on $\widehat{M}$, and choose $\hat{x}$ in $\widehat{M}$. Let $\overline{L}_{\hat{x}}$ be the leaf of $\overline{\mathcal{F}}$ containing $\hat{x}$, and let ω be any form on $\widehat{M}$ which restricts to the volume form on the tangent spaces of leaves of $\overline{\mathcal{F}}$ near $\overline{L}_{\hat{x}}$. Define $Af(\hat{x})$ by

$$Af(\hat{x}) = \frac{\displaystyle\int_{\overline{L}_{\hat{x}}} f(u)\omega(u)}{\displaystyle\int_{\overline{L}_{\hat{x}}} \omega(u)}.$$

Since the leaf $\overline{L}_{\hat{x}}$ is compact, the integrals above converge. The function Af is necessarily basic, because it is constant along the leaves. It is also clear that A is the identity on basic functions, since basic functions are constant on the leaf closures. Furthermore, suppose that f_1 is a smooth basic function on $\widehat{M}$ and f_2 is any smooth function on $\widehat{M}$. Then $A(f_1 f_2) = f_1 A(f_2)$. One of the most crucial properties of A is:

PROPOSITION 1.1. *A maps smooth functions to smooth basic functions.*

Proof. The preceding remarks show that A maps smooth functions to basic functions, so we need only verify that A maps smooth functions to smooth functions.

We begin by describing the coordinates that we will use to evaluate the integrals. The leaves of $\overline{\mathcal{F}}$ are closed submanifolds of $\widehat{M}$, and all of the leaves are isometric to one another. Given a leaf $\overline{L}$ of $\overline{\mathcal{F}}$, there exists a tubular neighborhood of $\overline{L}$ that is a union of leaves of $\overline{\mathcal{F}}$ and is diffeomorphic to $\overline{L} \times U$, where U is an ϵ-ball transverse to $\overline{L}$ (see [11]).

Fix a point in $\widehat{M}$. Let $\overline{L}$ be the leaf of $\overline{\mathcal{F}}$ containing that point, and choose a coordinate atlas for $\overline{L}$. Let z be any point in a tubular neighborhood T of $\overline{L}$, and write z in coordinates (x, y). Here y is the point of intersection of U and the leaf of $\overline{\mathcal{F}}$ that contains z, and x represents the local coordinates of the unique point in the leaf $\overline{L}$ that is closest to z. Since there is no leaf holonomy, the y-coordinates are globally well-defined on the tubular neighborhood. Let $\mathcal{U}$ be a finite simple open cover of T, and let Φ be a smooth, finite partition of unity subordinate to $\mathcal{U}$. Let $\overline{p}$ be the dimension of $\overline{\mathcal{F}}$. Then, for any $f \in C^\infty(\widehat{M})$ and $(x, y) \in T$,

$$Af(x,y) = Af(y) = \frac{\displaystyle\sum_{U\in\mathcal{U}} \sum_{\phi\in\Phi} \int_{\mathbb{R}^{\overline{p}}_x} \phi(x,y)\, f_U(x,y)\, \omega_U(x,y)}{\displaystyle\sum_{U\in\mathcal{U}} \sum_{\phi\in\Phi} \int_{\mathbb{R}^{\overline{p}}_x} \phi(x,y)\, \omega_U(x,y)}.$$

Here the U subscripts denote composition with the inverse of the coordinate chart on U. Observe that all of the functions in the integrands are smooth, bounded functions, all of whose derivatives are also smooth and bounded. Furthermore, the support of each integrand is compact. By the Lebesgue dominated convergence theorem, the expression above is continuous in y, and we can differentiate under the integral with respect to any of the y coordinates. Similarly, all of the y-coordinate derivatives of Af are continuous in y. Thus A maps smooth functions to smooth basic functions, as desired. □

Let π^* denote the pullback map induced by $\pi\colon \widehat{M} \to M$. This extends to a bounded linear operator from $L^2(\Omega^*(M))$ to $L^2(\Omega^*(\widehat{M}))$ that we will also denote by π^*; note that the asterisk does *not* denote adjoint here. Let $\eta\colon L^2(\Omega^*(\widehat{M})) \to L^2(\Omega^*(M))$ be the adjoint of π^*.

LEMMA 1.2. *The map π^* is an isometry from $L^2(\Omega^*(M))$ to $L^2(\Omega^*(\widehat{M}))$, and thus $\eta\pi^*$ is the identity on $L^2(\Omega^*(M))$.*

Proof. We have chosen the metric on $\widehat{M}$ so that the induced measure on $\widehat{M}$ is a product of the measures on M and $SO(q)$. Thus the volume form $\widehat{dV}$ on $\widehat{M}$ is the product of the volume form $d\mu$ on $SO(q)$ and the volume form dV on M. Let $(\ ,\ )$ denote the pointwise inner product of forms defined in the usual way (see, for example [3]) and fix a measureable section $s\colon M \to \widehat{M}$. Then for α in $L^2(\Omega^*(M))$,

$$\begin{aligned}
\langle \pi^*\alpha, \pi^*\alpha\rangle &= \int_{\widehat{M}} (\pi^*\alpha, \pi^*\alpha)\,\widehat{dV} \\
&= \int_M \int_{SO(q)} \Big(\pi^*\alpha\,(s\,(x)\,u)\,, \pi^*\alpha\,(s\,(x)\,u) \Big)\; d\mu(u)\,dV(x) \\
&= \int_M \int_{SO(q)} \Big(\alpha\,(\pi\,(s\,(x)\,u))\,, \alpha\,(\pi\,(s\,(x)\,u)) \Big)\; d\mu(u)\,dV(x) \\
&= \int_M (\alpha\,(x)\,, \alpha\,(x)) \int_{SO(q)} d\mu(u)\,dV(x) \\
&= \int_M (\alpha\,(x)\,, \alpha\,(x))\; dV(x) = \langle \alpha, \alpha\rangle.
\end{aligned}$$

Therefore π^* is an isometry from $L^2(\Omega^*(M))$ to $L^2(\Omega^*(\widehat{M}))$, and the desired property of its adjoint η follows. □

Remark. If f is a smooth function on $\widehat{M}$, for each $x \in M$, $\eta f(x)$ is the average of f on $\pi^{-1}(x)$. Similarly, there is a geometric interpretation of η on forms. Because we do not need these facts in this paper, we will not dwell on them here.

LEMMA 1.3. *Let f be a smooth function on $\widehat{M}$, and let $\widehat{dV}$ be the volume form on $\widehat{M}$. Then*

$$\int_{\widehat{M}} f\,\widehat{dV} = \int_{\widehat{M}} Af\,\widehat{dV}.$$

Proof. Let $\mathcal{T} = \{T_1, \ldots, T_k\}$ be a finite open cover of $\widehat{M}$ by tubular neighborhoods of leaves of $\overline{\mathcal{F}}$. Let $\Phi_B = \{\phi_1, \ldots, \phi_k\}$ be a basic partition of unity subordinate to $\mathcal{T}$. (To construct such a partition of unity, simply apply A to the functions in any partition of unity subordinate to $\mathcal{T}$.) Let $\mathcal{O}_i$ be a finite open cover of T_i by simple open sets, and let $\Psi_i = \{\psi_U | U \in O_i\}$ be a partition of unity subordinate to $\mathcal{O}_i$. If $\widehat{dV}$ is the volume form on $\widehat{M}$, it can be written locally as $\widehat{dV} = \pm\omega \wedge \nu$, where ν is a transverse volume form which does not depend on the leaf coordinates and ω is a form which restricts to the volume form of the leaves of $\overline{\mathcal{F}}$ (see [17]). Let $\overline{p}$ and $\overline{q}$ be the dimension and codimension of the foliation $\overline{\mathcal{F}}$, respectively. Then

$$\begin{aligned}
\int_{\widehat{M}} f\,\widehat{dV} &= \sum_{T_i \in \mathcal{T}} \sum_{U \in \mathcal{O}_i} \int_{\mathbb{R}^{\overline{q}}} \phi_i(y) \int_{\mathbb{R}^{\overline{p}}} \psi_U(x,y)\, f_U(x,y)\, |\omega_U(x,y)|\, |\nu_i(y)| \\
&= \sum_{T_i \in \mathcal{O}_i} \int_{\mathbb{R}^{\overline{q}}} \phi_i(y) Af(y) \left(\sum_{U \in \mathcal{O}_i} \int_{\mathbb{R}^{\overline{p}}} \psi_U(x,y)\, |\omega_U(x,y)| \right) |\nu_i(y)| \\
&= \sum_{T_i \in \mathcal{T}} \sum_{U \in \mathcal{O}_i} \int_{\mathbb{R}^{\overline{q}}} \phi_i(y) \int_{\mathbb{R}^{\overline{p}}} \psi_U(x,y)\, Af_U(y)\, |\omega_U(x,y)|\, |\nu_i(y)| \\
&= \int_{\widehat{M}} Af\, \widehat{dV}.
\end{aligned}$$

In the above calculation, the U subscripts denote composition with the inverse of the coordinate chart on U, the i subscripts denote the composition with the inverse of the transverse ball coordinate chart on the tubular neighborhood T_i, and we use the absolute value signs to avoid orientation issues. □

Let $L^2(M)$ and $L^2_B(M)$ be the completions of $C^\infty(M)$ and $C^\infty_B(M)$, respectively. We define $L^2(\widehat{M})$ and $L^2_B(\widehat{M})$ in a similar fashion.

LEMMA 1.4. *The map A extends to a projection on $L^2(\widehat{M})$.*

Proof. It is clear from the definition of A that $A^2 f = Af$ for $f \in C^\infty(\widehat{M})$. Next, for f_1 and f_2 in $C^\infty(\widehat{M})$,

$$\langle Af_1, f_2 \rangle = \int_{\widehat{M}} (Af_1) f_2\, \widehat{dV} = \int_{\widehat{M}} A\left((Af_1) f_2\right) \widehat{dV} = \int_{\widehat{M}} (Af_1)(Af_2)\, \widehat{dV},$$

where the second equality is a consequence of Lemma 1.3 and the third equality follows from the comments preceding Proposition 1.1. Therefore $\langle Af_1, f_2 \rangle = \langle Af_1, Af_2 \rangle$, and by symmetry, $\langle Af_1, Af_2 \rangle = \langle f_1, Af_2 \rangle$. Hence A is formally self-adjoint. Then for any smooth f,

$$||Af||^2 = \langle Af, Af \rangle = \langle A^2 f, f \rangle = \langle Af, f \rangle \leq ||Af|| \cdot ||f||,$$

whence $||Af|| \leq ||f||$. Thus A is a bounded operator, and therefore extends continuously to a projection on $L^2(\widehat{M})$. □

Definition. The *basic projection* on functions is the linear map $P\colon C^\infty(M) \to C^\infty_B(M)$ defined by $P = \eta A \pi^*$.

It is not immediately apparent that $\mathrm{Im}(P) \subset C^\infty_B(M)$, but the following proposition will address this. We now investigate the relationship between P and the Hilbert space projection P_B.

PROPOSITION 1.5. *We have the following:*

(1) *For any $f \in C(M)$ and $\hat{x} \in \widehat{M}$, $A\pi^* f(\hat{x})$ = the average of f over the leaf closure containing $\pi(\hat{x})$.*

(2) *For any $f \in L^2(M)$, $A\pi^* f \in Im(\pi^*)$.*

(3) *P maps smooth functions to smooth basic functions.*

(4) *For any $f \in C(M)$ and $x \in M$, $Pf(x)$ = the average of f over the leaf closure containing x.*

(5) *For any $f \in C(M)$, $\int_M f\, dV = \int_M Pf\, dV$.*

Proof. Choose $x \in M$, and let K be the closure of the leaf of $\mathcal{F}$ that contains x. Choose any $\hat{x} \in \pi^{-1}(x)$, and let $\overline{K}$ be the leaf of $\overline{\mathcal{F}}$ containing $\hat{x}$. As shown in [11], $\overline{K}$ is a principal subbundle of $\pi^{-1}(K)$ over K whose structure group is a compact Lie subgroup H of $SO(q)$.

Let $\{\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_s, E_1, \ldots, E_k, E_{k+1}, \ldots, E_q\}$ be a basis of $N\widehat{\mathcal{F}}$ at $\hat{x}$, chosen as follows. First, let $\{\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_s\}$ be an orthonormal basis of the tangent space to the fiber of $\widehat{M}$ with the property that the vectors $\lambda_{r+1}, \ldots, \lambda_s$ span the tangent space of the component of $\overline{K} \cap \pi^{-1}(x)$ which contains $\hat{x}$. This latter set of vectors can be viewed as a basis of the Lie algebra $\mathfrak{h}_0$ of H_0, where H_0 denotes the connected component of the identity in H. We remark that if H is a finite group, H_0 is trivial, and $r = s$. Next, we require that $\{E_1, \ldots, E_q\}$ be an orthogonal set of horizontal vectors such that $\{\pi_* E_1, \ldots, \pi_* E_k\}$, respectively $\{\pi_* E_{k+1}, \ldots, \pi_* E_q\}$, restricts to an orthonormal basis of NK, respectively $TK \cap N\mathcal{F}$, at $\pi(\hat{x}) = x$.

We can extend all of these vectors in the basis to be foliate vector fields on $\widehat{M}$ which are global sections of $N\widehat{F}$, by choosing the fundamental fields associated to the vectors tangent to the fibers and the basic fields associated to the $\{E_i\}$. (See [11] for details.) On $\pi^{-1}(K)$, the basic fields are twisted by the $SO(q)$ action on the fibers and parallel along the leaves. For simplicity, we will use the same notation for the extended vector fields as the vectors at $\hat{x}$. Observe that the properties stated in the previous paragraph remain true for the vector fields along $\overline{K}$. Specifically, $\{\lambda_1,\ldots,\lambda_r,\lambda_{r+1},\ldots,\lambda_s\}$ forms an orthonormal basis of the tangent space to each fiber, $\lambda_{r+1},\ldots,\lambda_s$ span the tangent space of each component of $\overline{K}\cap\pi^{-1}(y)$ for all $y\in K$, and $\{\pi_*(E_1(\hat{x})),\ldots,\pi_*(E_k(\hat{x}))\}$, respectively $\{\pi_*(E_{k+1}(\hat{x})),\ldots,\pi_*(E_q(\hat{x}))\}$, restricts to an orthonormal basis of NK, respectively $TK\cap N\mathcal{F}$, at each point $\pi(\hat{x})$ of K. The above facts follow from the properties of bundle-like metrics and the induced adapted connection on $\widehat{M}$. (Again, see [11].)

Fix a global measureable section F of $\overline{K}$ over K which is smooth off a set of measure zero in K. Then we can describe a point y of $\overline{K}$ with (not necessarily smooth) coordinates $(h,u)\in H\times K$, where $u=\pi(y)$ and $F_u h=y$. The fibers are transverse to the span of the $\{E_i\}$, and the distribution spanned by $\lambda_{r+1},\ldots,\lambda_s$ is clearly involutive. Recall that E_i^* is the 1-form defined by $E_i^*(V)=\langle E_i,V\rangle$ for all $V\in T_z\widehat{M}$, $z\in\widehat{M}$. The form $C|\lambda_{r+1}^*\wedge\cdots\wedge\lambda_s^*\wedge E_{k+1}^*\wedge\cdots\wedge E_q^*\wedge\pi^*\chi_{\mathcal{F}}|$ restricts to the volume form of $\overline{K}$, where $\chi_{\mathcal{F}}$ is the characteristic form of the foliation $\mathcal{F}$. The characteristic form restricts to the volume form of the leaves of $\mathcal{F}$ and is locally given by $V_1^*\wedge\cdots\wedge V_p^*$, where $\{V_1,\ldots,V_p\}$ is a local orthonormal basis for $T\mathcal{F}$. (See [17].) The positive number C depends only on the norms of the E_i and the angle between $\overline{K}\cap\pi^{-1}(x)$ and the span of $\{E_{k+1},\ldots,E_q\}$. Because the E_i are parallel along the leaves and $SO(q)$ acts on $\widehat{M}$ by isometries, C is a constant on $\overline{K}$. Note that $\lambda_{r+1}^*\wedge\ldots\wedge\lambda_s^*$ restricts to the volume form of each component of $\pi^{-1}(u)\cap\overline{K}$ for $u\in K$. The volume of this set is independent of u, because the transverse metric on $\widehat{M}$ takes pairs of foliate vector fields to basic functions, which are constant on $\overline{K}$. Let $R_h\colon\ \pi^{-1}K\to\pi^{-1}K$ denote right multiplication by $h\in H$. For each $h\in H$ and $\hat{x}\in\overline{K}$, $\{(R_h)_*E_{k+1}(\hat{x}),\ldots,(R_h)_*E_q(\hat{x})\}$ is an oriented orthonormal frame for the distribution spanned by $\{E_{k+1},\ldots,E_q\}$. Let $\nu_K=(\pi_*E_{k+1}(\hat{x}))^*\wedge\ldots\left(\pi_*E_q(\hat{x})\right)^*$. By construction, this form on K is well-defined independent of $\hat{x}\in\pi^{-1}(x)$ and is the transversal volume form of the restriction of the foliation $\mathcal{F}$ to K. Thus the pullback of this form is a function of u alone. From the comments above, we see that $C|\lambda_{r+1}^*\wedge\ldots\wedge\lambda_s^*\wedge\pi^*\nu_K\wedge\pi^*\chi_{\mathcal{F}}|$ restricts to the volume form of $\overline{K}$, for some constant C.

Let h_1 be the identity element of H, and choose elements $h_2,\ldots h_m$ of H so that H is the disjoint union of the left cosets $h_1H_0,h_2H_0,\ldots,h_mH_0$. Then for any continuous function f on M,

$$
\begin{aligned}
A\pi^*f(\hat{x}) &= \frac{\displaystyle\int_{\overline{K}}\pi^*f}{\displaystyle\int_{\overline{K}}1}\\
&= \frac{\displaystyle\int_K\sum_{i=1}^m\left(\int_{H_0}\pi^*f(h_ih,u)\,C|\lambda_{r+1}^*\wedge\cdots\wedge\lambda_s^*(h_ih,u)|\right)|\pi^*\nu_K(u)\wedge\pi^*\chi_{\mathcal{F}}(u)|}{\displaystyle\int_K\sum_{i=1}^m\left(\int_{H_0}C|\lambda_{r+1}^*\wedge\cdots\wedge\lambda_s^*(h_ih,u)|\right)|\pi^*\nu_K(u)\wedge\pi^*\chi_{\mathcal{F}}(u)|}
\end{aligned}
$$

$$= \frac{\displaystyle\int_K \sum_{i=1}^{m} \left(\int_{H_0} f(u) \left|\lambda_{r+1}^* \wedge \cdots \wedge \lambda_s^*(h_i h, u)\right| \right) \left|\pi^* \nu_K(u) \wedge \pi^* \chi_{\mathcal{F}}(u)\right|}{\displaystyle\int_K \sum_{i=1}^{m} \left(\int_{H_0} \left|\lambda_{r+1}^* \wedge \cdots \wedge \lambda_s^*(h_i h, u)\right| \right) \left|\pi^* \nu_K(u) \wedge \pi^* \chi_{\mathcal{F}}(u)\right|}$$

$$= \frac{\displaystyle\int_K \sum_{i=1}^{m} \mathrm{Vol}(h_i H_0) f(u) \left|\pi^* \nu_K(u) \wedge \pi^* \chi_{\mathcal{F}}(u)\right|}{\displaystyle\int_K \sum_{i=1}^{m} \mathrm{Vol}(h_i H_0) \pi^* \nu_K(u) \wedge \pi^* \chi_{\mathcal{F}}(u)|}$$

$$= \frac{\displaystyle\int_K m\mathrm{Vol}(H_0) f(u) \left|\nu_K(u) \wedge \chi_{\mathcal{F}}(u)\right|}{\displaystyle\int_K m\mathrm{Vol}(H_0) \left|\nu_K(u) \wedge \chi_{\mathcal{F}}(u)\right|}$$

$$= \frac{\displaystyle\int_K f(u) \left|\nu_K(u) \wedge \chi_{\mathcal{F}}(u)\right|}{\displaystyle\int_K \left|\nu_K(u) \wedge \chi_{\mathcal{F}}(u)\right|} = \quad \text{the average of } f \text{ over } K.$$

We have used the fact that left multiplication by a group element is an isometry on H. We remark that the integral over H_0 does not appear in the above calculation if H_0 is trivial. Thus the proof of (1) is complete. (1) implies that (2) is true for continuous functions, since (1) implies that $A\pi^* f(\hat{x})$ depends only on $\pi(\hat{x})$. Since π^* is an isometry, its range is closed, and therefore (2) is true for all f in $L^2(M)$. Statement (4) immediately follows from (1) and (2). To prove (3), take $f \in C^\infty(M)$. Proposition 1.1 implies that $A\pi^* f \in C^\infty(M)$, which by (2) can be written in the form $\pi^* g$ for some smooth function g on M. Lemma 1.2 then yields $Pf = \eta A\pi^* f = g \in C^\infty(M)$. Statement (4) implies that Pf is constant on each leaf closure and hence basic, and therefore (3) is proved. To prove (5), let f be a continuous function on M. Then

$$\begin{aligned}
\int_M f\, dV &= \int_{\widehat{M}} \pi^* f\, d\widehat{V} = \int_{\widehat{M}} A\pi^* f\, d\widehat{V} \\
&= \int_{\widehat{M}} \pi^*(\eta A\pi^* f)\, d\widehat{V} = \int_M \eta A\pi^* f\, dV = \int_M Pf\, dV,
\end{aligned}$$

where first and fourth equality follow because $d\widehat{V}$ is the product of the volume forms on M and $SO(q)$, and because the volume of $SO(q)$ is 1. The second equality is a consequence of Lemma 1.3, while the third equality follows from (2) and Lemma 1.2. Finally, the last equality above just involves the definition of P. □

PROPOSITION 1.6. *Let f be a smooth function on M. Then $Pf = P_B f$.*

Proof. First, $P = \eta A\pi^*$ extends to a bounded linear operator P on $L^2(M)$, since it is the composition of maps with this property. Second, P is an idempotent; i.e.,

$P^2 = P$. To show this, it suffices to show that P is the identity on basic functions. Given a function $f \in C^\infty_B(M)$, $\pi^* f$ is basic on $\widehat{M}$, because the leaf closures of $\widehat{M}$ cover the leaf closures of M. Then $A\pi^* f = \pi^* f$, because A is the identity on basic functions upstairs. Thus $Pf = \eta A\pi^* f = \eta\pi^* f = f$. Third, P is self-adjoint; this follows immediately from the proof of Lemma 1.4 and the formula for the adjoint of a composition of operators. Hence P is a Hilbert space projection on $L^2(M)$, and is therefore completely determined by its range. But it is evident that the range of P (restricted to smooth functions) is the space of smooth basic functions on M, and therefore the range of the extension of P is $L^2_B(M)$. This is precisely the range of the operator P_B restricted to $L^2(M)$, so (the extension of) P and P_B agree on $L^2(M)$, from whence the desired result follows. $\square$

In light of Proposition 1.6, we will henceforth denote both the operators P and P_B by just P.

We now extend the domain of the projection P to include forms. First, we extend the map A to a map on forms. We will use the notation of the proof of Proposition 1.5. Because $\widehat{\mathcal{F}}$ is transversely parallelizable, the basic forms are precisely the forms which take sets of foliate vector fields to basic functions. It follows that the space of smooth basic forms on $\widehat{M}$ is a free module over $C^\infty_B(M)$, generated by the one-forms $\omega_1 = \lambda_1^*, \ldots, \omega_s = \lambda_s^*, \omega_{s+1} = E_1^*, \ldots, \omega_{s+l} = E_l^*$ and the various wedge products of these forms. Thus we can write any smooth basic form on $\widehat{M}$ uniquely as $\sum_I f_I\omega_I$, where $I = \{i_1, \ldots, i_k\}$ $(k < s+l)$ is a multi-index, f_I is basic, and $\omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$. Observe that the basis $\{\omega_I\}$ is orthonormal.

We now define the map $\tilde{A}$: $\Omega^*(\widehat{M}) \to \Omega^*_B(\widehat{M})$. Given any smooth form $\alpha \in \Omega^*(\widehat{M})$, define

$$\tilde{A}\alpha = \sum_I A\left((\alpha, \omega_I)\right)\omega_I,$$

where the A inside the summation is the functional studied earlier in this section, and (,) is the pointwise inner product of forms. This definition is independent of the choice of orthonormal basis $\{\omega_I\}$. To see this, let $\{\gamma_I\}$ be any other orthonormal basis. Then $\gamma_I = \sum M_{IJ}\omega_J$, for some orthogonal matrix M_{IJ} of basic functions, and

$$\begin{aligned}
\sum_I A\left((\alpha, \gamma_I)\right)\gamma_I &= \sum_{I,J,K} A\left((\alpha, M_{IJ}\omega_J)\right) M_{IK}\omega_K \\
&= \sum_{I,J,K} A\left(M_{IJ}\left(\alpha, \omega_J\right)\right) M_{IK}\omega_K \\
&= \sum_{I,J,K} A\left((\alpha, \omega_J)\right) M_{IJ}M_{IK}\omega_K = \sum_J A\left((\alpha, \omega_J)\right)\omega_J.
\end{aligned}$$

The calculation above used the remarks preceding Proposition 1.1.

LEMMA 1.7. *We have the following:*

(1) *$\tilde{A}$ maps smooth forms to smooth basic forms.*

(2) *$\tilde{A}$ is the identity on smooth basic forms.*

(3) *For all smooth forms α and smooth basic forms β, $\tilde{A}(\beta \wedge \alpha) = \beta \wedge \tilde{A}(\alpha)$.*

(4) *For all smooth forms α and smooth basic forms β, $\left(\tilde{A}\alpha, \beta\right) = A\left((\alpha, \beta)\right)$.*

(5) *$\tilde{A}$ is formally self-adjoint.*

(6) *$\tilde{A}$ extends to a projection on $L^2\left(\Omega^*(\widehat{M})\right)$.*

(7) *For any form $\gamma \in \Omega^*(M)$, $\tilde{A}\pi^*\gamma \in \mathrm{Im}\ \pi^*$.*

Proof. (1) through (4) follow from the definition of $\tilde{A}$ and the corresponding facts for the map A on functions. Observe that for any $\alpha, \phi \in \Omega^*(\widehat{M})$,

$$\langle \tilde{A}\alpha, \phi\rangle = \int_{\widehat{M}} (\tilde{A}\alpha, \phi)\, \widehat{dV} = \int_{\widehat{M}} A\,(\tilde{A}\alpha, \phi)\, \widehat{dV} = \int_{\widehat{M}} (\tilde{A}\alpha, \tilde{A}\phi)\, \widehat{dV}.$$

The second equality follows from Lemma 1.3, and the third equality follows from (4) above. Items (5) and (6) follow from the above observation and an argument analogous to the proof of Lemma 1.4. Now, in the definition of $\tilde{A}$, the forms ω_I could be replaced by ${R_g}^*\omega_I$ for a fixed $g \in SO(q)$, because right multiplication is an isometry of $\widehat{M}$ which maps leaves to leaves. Also, note that a form α on $\widehat{M}$ is the pullback of a form on M if and only if, in a local trivialization, the form does not depend on the fiber coordinates. Such forms are characterized by the following two properties: (1) For any vector λ tangent to a fiber in $\widehat{M}$, $i(\lambda)\alpha = 0$. (2) For any $g \in SO(q)$, ${R_g}^*\alpha = \alpha$. Suppose that α is such a form, and consider the form $\tilde{A}\alpha$. By the definition of $\tilde{A}$, the first property is satisfied by the form $\tilde{A}\alpha$. Fix $g \in SO(q)$. Then

$$\begin{aligned} {R_g}^*\tilde{A}\alpha &= \sum_I {R_g}^* A\left((\alpha, \omega_I)\right) {R_g}^*\omega_I \\ &= \sum_I A\left(\left({R_g}^*\alpha, {R_g}^*\omega_I\right)\right) {R_g}^*\omega_I \\ &= \sum_I A\left(\left(\alpha, {R_g}^*\omega_I\right)\right) {R_g}^*\omega_I = \tilde{A}\alpha. \end{aligned}$$

Therefore, the second property is satisfied by the form $\tilde{A}\alpha$, so $\tilde{A}\alpha \in \mathrm{Im}\,\pi^*$; this proves (7). $\square$

Definition. The *basic projection* on forms is the linear map $P\colon\ \Omega^*(M) \to \Omega^*_B(M)$ defined by $P = \eta\tilde{A}\pi^*$.

Note that we have used the same symbol that we use for the basic projection on functions. Again, it is not immediately clear that P maps smooth forms to smooth basic forms. We have:

LEMMA 1.8. *P maps smooth forms to smooth basic forms, and for every $\alpha \in \Omega^*(M)$, $P\alpha = P_B\alpha$.*

Proof. Lemma 1.7 shows that $\tilde{A}\pi^*\alpha = \pi^*\gamma$ for some $\gamma \in \Omega^*(M)$ such that $\pi^*\gamma$ is basic. The definition of a basic form in terms of interior products implies that γ is basic. By Lemma 1.2, $P\alpha = \gamma \in \Omega_B^*(M)$. The proof of Proposition 1.6 can easily be modified to show that $P\alpha = P_B\alpha$. $\square$

LEMMA 1.9. *If $\beta \in \Omega^*(M)$ is basic, then $\pi^*\beta$ is basic.*

Proof. For a given point in M, pick a simple open neighborhood U over which $\widehat{M}$ is trivial, and choose coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ on U, where the x coordinates are in the leaf directions and the y coordinates are in transverse directions. On U, β is a linear combination of forms $f\, dy_{j_1} \wedge dy_{j_2} \ldots \wedge dy_{j_k}$ where the function f is independent of the x-coordinates. Then on $\pi^{-1}(U) \cong U \times SO(q)$, $\pi^*\beta$ is a linear combination of forms with the same local expression, and therefore $\pi^*\beta$ is basic. $\square$

PROPOSITION 1.10. *P is the unique linear map from $\Omega^*(M)$ to $\Omega_B^*(M)$ such that $(P\alpha, \beta) = P(\alpha, \beta)$ for every smooth form α and basic form β.*

Proof. We first show that P satisfies the statement. If $\beta \in \Omega_B^*(M)$, $\pi^*\beta \in \Omega_B^*(\widehat{M})$ by Lemma 1.9. Then

$$\begin{aligned} \pi^*(P\alpha, \beta) &= \pi^*(\eta\tilde{A}\pi^*\alpha, \beta) \\ &= (\tilde{A}\pi^*\alpha, \pi^*\beta) \\ &= A(\pi^*\alpha, \pi^*\beta) \\ &= A\pi^*(\alpha, \beta), \end{aligned}$$

where the second equality follows from Lemma 1.7. If we apply η to both sides of the equation above, we have $(P\alpha, \beta) = P(\alpha, \beta)$. Suppose another map Q has this property. Then $(Q\alpha - P\alpha, \beta) = 0$ for any smooth form α and every basic form β. Since P and Q must map to basic forms, we conclude that $P = Q$. $\square$

PROPOSITION 1.11. *If $\alpha \in \Omega^*(M)$ and $\beta \in \Omega_B^*(M)$, then $P(\beta \wedge \alpha) = \beta \wedge P(\alpha)$.*

Proof. We have the following string of equalities:

$$\begin{aligned} \pi^*(P(\beta \wedge \alpha)) &= \tilde{A}\pi^*(\beta \wedge \alpha) && \text{by Lemma 1.7} \\ &= \tilde{A}(\pi^*\beta \wedge \pi^*\alpha) \\ &= \pi^*\beta \wedge \tilde{A}\pi^*\alpha && \text{by Lemmas 1.7, 1.9} \\ &= \pi^*\beta \wedge \pi^*\eta\tilde{A}\pi^*\alpha && \text{by Lemma 1.7} \\ &= \pi^*(\beta \wedge P\alpha). \end{aligned}$$

To complete the proof, apply η to both sides. $\square$

2. The basic Laplacian. In Section 1, we showed that the adjoint δ_B of the restriction d_B of the exterior derivative to $\Omega_B^*(M)$ is given by the formula $\delta_B = P\delta$, where δ is the formal adjoint of d on $L^2(\Omega^*(M))$. In this section, we find δ_B in terms of geometrically computable quantities.

We use the notation of Section 1. Let n be the dimension of M, and let p be the rank of $T\mathcal{F}$. Recall that $\chi_{\mathcal{F}}$ is the characteristic form of the foliation, which is defined in [17] and introduced in the proof of Proposition 1.5. Let $\bar{*}\colon\ \Omega^k(M) \to \Omega^{n-p-k}(M)$ be the pointwise operator defined by

$$\bar{*}\gamma = (-1)^{p(n-p-k)} * (\gamma \wedge \chi_{\mathcal{F}}) \text{ for all } \gamma \in \Omega^k(M),$$

where $*$ is the Hodge star operator. The operator $\bar{*}$ maps basic forms to basic forms, and it has the property that $*\beta = \bar{*}\beta \wedge \chi_{\mathcal{F}}$ for a basic form β (See [17] for details). It is also elementary to show that for any k-form α and any basic k-form β, $\beta \wedge \bar{*}\alpha = (\beta, \alpha)\nu$, where ν is the transversal volume form and $(\ ,\ )$ is the pointwise inner product of forms. Next, let W denote the vector space of smooth forms ω with the property that at each point of M and for all vectors v tangent to the leaves of $\mathcal{F}$, $i(v)\omega = 0$, where $i(v)$ denotes the interior product with the vector v. If α is a k-form in W and β is a basic k-form, then we have in addition that $\alpha \wedge \bar{*}\beta = \beta \wedge \bar{*}\alpha$. Also, for any $\alpha \in W$, $\bar{*}^2\alpha = (-1)^{k(n-p-k)}\alpha$.

LEMMA 2.1. *P commutes with $\bar{*}$.*

Proof. Let γ be any smooth k-form. Using the pointwise metric $(,)$, we can decompose γ as $\gamma = \gamma^W + \gamma^\perp$, where $\gamma^W \in W$ and $\gamma^\perp$ is orthogonal to W. Then we have that $\bar{*}\gamma^\perp = 0$ and $P\gamma^\perp = 0$, using the definitions. It clearly suffices to show that P commutes with $\bar{*}$ on smooth basic k-forms for each k. If there are no basic k-forms, then $P\bar{*}\gamma = \bar{*}P\gamma = 0$. Otherwise, for any basic k-form β,

$$\begin{aligned}
(\beta, \bar{*}P\gamma)\nu &= \left(\beta, \bar{*}P\gamma^W\right)\nu \\
&= \beta \wedge \bar{*}^2 P\gamma^W \\
&= (-1)^{k(n-p-k)}\beta \wedge P\gamma^W \\
&= P\left((-1)^{k(n-p-k)}\,\beta \wedge \gamma^W\right) && \text{by Proposition 1.11} \\
&= P\left(\beta, \bar{*}\gamma^W\right)\nu \\
&= \left(\beta, P\bar{*}\gamma^W\right)\nu && \text{by Proposition 1.10} \\
&= (\beta, P\bar{*}\gamma)\nu.
\end{aligned}$$

$\square$

The mean curvature form κ of the foliation is defined at each point x of M to be the dual of the mean curvature vector $\sum_{i=1}^{p}\left(\nabla_{E_i}E_i\right)^\perp$ of the leaf containing x. Here $\{E_1, \ldots, E_p\}$ is a local orthonormal basis of $T\mathcal{F}$, ∇ is the Levi-Civita connection on M, and $\perp$ denotes projection onto $N\mathcal{F}$. Rummler's formula states that $d\chi_{\mathcal{F}} = -\kappa \wedge \chi_{\mathcal{F}} + \varphi_0$, where φ_0 is a $(p+1)$-form with the property that $i(v_1)\dots i(v_p)\varphi_0 = 0$ for any set $\{v_j\}$ of p vectors in $T\mathcal{F}$. This implies that if α is a basic $(n-p-1)$-form, $\alpha \wedge \varphi_0 = 0$. (See [16] and [17].)

Proposition 2.2. *If β is a basic k-form,*

$$\delta_B \beta = (-1)^{(n-p)(k+1)+1}\, \bar{*}(d - (P\kappa)\wedge)\, \bar{*}\beta.$$

Remark. When κ is basic, this formula reduces to Theorem 12.10 in [17].

Proof. Let γ be any basic $(k-1)$-form, and let β be any basic k-form. Then

$$\begin{aligned}
\langle d\gamma, \beta\rangle &= \int_M d\gamma \wedge *\beta = \int_M d\gamma \wedge \bar{*}\beta \wedge \chi_{\mathcal{F}} \\
&= (-1)^k \int_M \gamma \wedge d\,(\bar{*}\beta \wedge \chi_{\mathcal{F}}) \\
&= (-1)^k \int_M \gamma \wedge \Big(d\bar{*}\beta \wedge \chi_{\mathcal{F}} + (-1)^{n-p-k}\, \bar{*}\beta \wedge d\chi_{\mathcal{F}} \Big) \\
&= (-1)^k \int_M \gamma \wedge \Big(d\bar{*}\beta \wedge \chi_{\mathcal{F}} + (-1)^{n-p-k}\, \bar{*}\beta \wedge (-\kappa \wedge \chi_{\mathcal{F}} + \varphi_0) \Big) \\
&= (-1)^k \int_M \gamma \wedge \Big(d\bar{*}\beta \wedge \chi_{\mathcal{F}} + (-1)^{n-p-k+1}\, \bar{*}\beta \wedge \kappa \wedge \chi_{\mathcal{F}} \Big) \\
&\quad + (-1)^{n-p-k} \gamma \wedge \bar{*}\beta \wedge \varphi_0 \\
&= (-1)^k \int_M \gamma \wedge (d\bar{*}\beta \wedge \chi_{\mathcal{F}} - \kappa \wedge \bar{*}\beta \wedge \chi_{\mathcal{F}}) + (-1)^{n-p-k} \gamma \wedge \bar{*}\beta \wedge \varphi_0.
\end{aligned}$$

Since $\gamma \wedge \bar{*}\beta$ is a basic $(n-p-1)$-form, its wedge product with φ_0 is 0. Thus,

$$\begin{aligned}
\langle d\gamma, \beta\rangle &= (-1)^k \int_M \gamma \wedge \big((d\bar{*}\beta - \kappa \wedge \bar{*}\beta) \wedge \chi_{\mathcal{F}} \big) \\
&= (-1)^{k+(k-1)(n-k+1)} \int_M \gamma \wedge *^2 \big((d\bar{*}\beta - \kappa \wedge \bar{*}\beta) \wedge \chi_{\mathcal{F}} \big) \\
&= (-1)^{k+(k-1)(n-k+1)+p(n-p-(n-p-k+1))} \int_M \gamma \wedge * \big(\bar{*}(d\bar{*}\beta) - \bar{*}(\kappa \wedge \bar{*}\beta) \big) \\
&= (-1)^{(n-p)(k+1)+1} \int_M \gamma \wedge * \big(\bar{*}(d\bar{*}\beta) - \bar{*}(\kappa \wedge \bar{*}\beta) \big) \\
&= (-1)^{(n-p)(k+1)+1} \int_M \big(\gamma, \bar{*}(d\bar{*}\beta) - \bar{*}(\kappa \wedge \bar{*}\beta) \big) dV.
\end{aligned}$$

Now, using item (5) of Proposition 1.5, we can apply P to the integrand of the last integral.

$$\langle d\gamma, \beta\rangle = (-1)^{(n-p)(k+1)+1} \int_M P\big(\gamma, \bar{*}(d\bar{*}\beta) - \bar{*}(\kappa \wedge \bar{*}\beta) \big) dV$$

$$= (-1)^{(n-p)(k+1)+1} \int_M \left(\gamma, P\bar{*}(d\bar{*}\beta) - P\bar{*}(\kappa \wedge \bar{*}\beta)\right) dV$$
$$= (-1)^{(n-p)(k+1)+1} \int_M \left(\gamma, \bar{*}(d\bar{*}\beta) - \bar{*}\big((P\kappa) \wedge \bar{*}\beta\big)\right) dV.$$

Here we have used Propositions 1.10 and 1.11 and Lemma 2.1. Since the above is true for all $\gamma \in \Omega_B^*(M)$, the proposition is proved. □

COROLLARY 2.3. *If β is a basic k-form, then*

$$\delta_B \beta = \left((-1)^{(n-p)(k+1)+1}\, \bar{*} d \bar{*} + (P\kappa) \lrcorner \right) \beta,$$

where the symbol $\lrcorner$ denotes the interior product.

Proof. Let β be a basic k-form and α be a basic $(k-1)$-form. By Proposition 2.2, it suffices to show that $(-1)^{(n-p)(k+1)} \bar{*}\big((P\kappa) \wedge (\bar{*}\beta)\big) = (P\kappa)\lrcorner\beta$.

$$\begin{aligned}
(\alpha, \bar{*}((P\kappa) \wedge \bar{*}\beta))\, dV &= \alpha \wedge * (\bar{*}((P\kappa) \wedge \bar{*}\beta)) \\
&= \alpha \wedge \bar{*}(\bar{*}((P\kappa) \wedge \bar{*}\beta)) \wedge \chi_{\mathcal{F}} \\
&= (-1)^{(n-p-k+1)(k-1)} \alpha \wedge (P\kappa) \wedge \bar{*}\beta \wedge \chi_{\mathcal{F}} \\
&= (-1)^{(n-p-k+1)(k-1)} \alpha \wedge (P\kappa) \wedge *\beta \\
&= (-1)^{(n-p-k+1)(k-1)+k-1} (P\kappa) \wedge \alpha \wedge *\beta \\
&= (-1)^{(n-p-k+1)(k-1)+k-1} ((P\kappa) \wedge \alpha, \beta)\,.
\end{aligned}$$

Because the interior product is by definition the pointwise adjoint of the wedge product and because $(n-p-k+1)(k-1)+k-1 \equiv (n-p)(k+1) \bmod 2$, the result follows. □

PROPOSITION 2.4. *The following operator equation holds:*

$$P\delta = \delta P + (P\kappa - \kappa)\lrcorner \circ P + (-1)^p (\varphi_0 \lrcorner)(\chi_{\mathcal{F}} \wedge) \circ P.$$

Proof. Let α be a $(k-1)$-form and let β be a k-form. Then $\langle \alpha, \delta P\beta \rangle = \langle d\alpha, P\beta \rangle$, and since $P\beta$ is basic, we can use the first six lines of the calculation in the proof of Proposition 2.2 to get

$$\begin{aligned}
\langle \alpha, \delta P\beta \rangle &= (-1)^k \int_M \alpha \wedge (d\bar{*}P\beta \wedge \chi_{\mathcal{F}} - \kappa \wedge \bar{*}P\beta \wedge \chi_{\mathcal{F}}) \\
&\quad + (-1)^{n-p-k} \alpha \wedge \bar{*}P\beta \wedge \varphi_0 \\
&= (-1)^k \int_M \alpha \wedge \left(d\bar{*}P\beta \wedge \chi_{\mathcal{F}} - (P\kappa) \wedge \bar{*}P\beta \wedge \chi_{\mathcal{F}} \right) \\
&\quad + \alpha \wedge (P\kappa - \kappa) \wedge \bar{*}P\beta \wedge \chi_{\mathcal{F}} + (-1)^{(n-p-k)p} \alpha \wedge \varphi_0 \wedge \bar{*}P\beta.
\end{aligned}$$

The first term in the integral can be simplified using manipulations similar to those in the proof of Proposition 2.2 to give

$$\begin{aligned}\langle\alpha,\delta P\beta\rangle &= \int_M \alpha\wedge *\delta_B P\beta \\ &\quad + (-1)^k\alpha\wedge(P\kappa-\kappa)\wedge *P\beta + (-1)^k\alpha\wedge\varphi_0\wedge *(P\beta\wedge\chi_{\mathcal F}) \\ &= \int_M \alpha\wedge *\delta_B P\beta + (-1)^k\alpha\wedge(P\kappa-\kappa)\wedge *P\beta \\ &\quad + (-1)^{k(p+1)}\alpha\wedge\varphi_0\wedge *(\chi_{\mathcal F}\wedge P\beta) \\ &= \langle\alpha,\delta_B P\beta\rangle + (-1)^{(k-1)(n-k+1)+k}\int_M \alpha\wedge *^2\left((P\kappa-\kappa)\wedge *P\beta\right) \\ &\quad + (-1)^{(k-1)(n-k+1)+k(p+1)}\int_M \alpha\wedge *^2\left(\varphi_0\wedge *(\chi_{\mathcal F}\wedge P\beta)\right).\end{aligned}$$

For an r-form γ and an s-form ω, $\gamma\lrcorner\omega = (-1)^{(s-r)(n-s)} * (\gamma\wedge *\omega)$ by a straightforward calculation. Using this formula, the above expression simplifies to

$$\langle\alpha,\delta P\beta\rangle = \langle\alpha,\delta_B P\beta\rangle - \langle\alpha,(P\kappa-\kappa)\lrcorner P\beta\rangle - (-1)^p\langle\alpha,\varphi_0\lrcorner(\chi_{\mathcal F}\wedge P\beta)\rangle.$$

Thus, we have

$$\delta P = \delta_B P - (P\kappa-\kappa)\lrcorner\circ P - (-1)^p(\varphi_0\lrcorner)(\chi_{\mathcal F}\wedge)\circ P.$$

Noting that $\delta_B = P\delta$ and solving for $P\delta P$, we obtain the desired result, except that the left-hand side of the equation is $P\delta P$. Since $\Omega_B^*(M)$ is an invariant subspace for d, $PdP = dP$, and thus $P\delta P = P\delta$. □

COROLLARY 2.5. $dP = Pd + P\circ\left((P\kappa-\kappa)\wedge\right) + (-1)^p P\circ(\chi_{\mathcal F}\lrcorner)(\varphi_0\wedge)$.

Proof. Take adjoints in Proposition 2.4. □

We write the formula in Proposition 2.4 as $P\delta = \delta P + \epsilon P$, where $\epsilon = (P\kappa - \kappa)\lrcorner + (-1)^p(\varphi_0\lrcorner)(\chi_{\mathcal F}\wedge)$. Note that ϵ is a 0^{th} order operator that maps $\Omega_B^*(M)$ into $\Omega_B^*(M)^\perp$; to see this, apply P to both sides of the operator equation for $P\delta P$, and observe that ϵ is a pointwise operator. Hence $P\epsilon P = P\epsilon^* P = 0$.

Now we can find a formula for the basic Laplacian in terms of d and δ. The basic Laplacian is the map Δ_B: $\Omega_B^*(M)\to\Omega_B^*(M)$ defined by $\Delta_B = \delta_B d + d\delta_B$. Using Proposition 2.4 and the fact that $\Omega_B^*(M)$ is an invariant subspace for d, we have, for all $\beta\in\Omega_B^*(M)$,

$$\begin{aligned}\Delta_B\beta &= (P\delta d + dP\delta)\beta = (P\delta Pd + dP\delta P)\beta \\ &= ((\delta P+\epsilon P)\,d + d\,(\delta P+\epsilon P))\,\beta \\ &= ((\delta+\epsilon)\,d + d\,(\delta+\epsilon))\,\beta\end{aligned}$$

$$\begin{aligned} &= (\delta d + d\delta + \epsilon d + d\epsilon)\,\beta \\ &= (\Delta + \epsilon d + d\epsilon)\,\beta. \end{aligned}$$

In summary:

THEOREM 2.6. *The basic Laplacian Δ_B, as a map on $\Omega_B^*(M)$, satisfies the equation*

$$\Delta_B = \Delta + \epsilon d + d\epsilon,$$

where ϵ is a 0^{th} order operator mapping $\Omega_B^*(M)$ to $\Omega_B^*(M)^\perp$ defined by

$$\epsilon = (P\kappa - \kappa)\lrcorner + (-1)^p(\varphi_0\lrcorner)(\chi_{\mathcal{F}}\wedge).$$

THEOREM 2.7.

$$\Delta_B P = \widetilde{\Delta} P = P\widetilde{\Delta}^* = P\overline{\Delta}P,$$

where $\widetilde{\Delta} = \Delta + \epsilon d + d\epsilon$, $\widetilde{\Delta}^* = \Delta + \delta\epsilon^* + \epsilon^*\delta$ is the adjoint of $\widetilde{\Delta}$, and $\overline{\Delta} = \Delta - \epsilon^*\epsilon$. (The operator ϵ is defined in the previous theorem.)

Proof. The first equality is Theorem 2.6. Since $P\Delta_B P = \Delta_B P = \widetilde{\Delta}P$, this operator is self-adjoint. By taking adjoints, the second equality follows. Next, we observe that

$$\begin{aligned} \widetilde{\Delta}P &= P\widetilde{\Delta}P = P(\Delta + \epsilon d + d\epsilon)P \\ &= P\Delta P + P\epsilon dP + Pd\epsilon P \\ &= P\Delta P + P\epsilon PdP + (dP - P\epsilon^*)\,\epsilon P \quad \text{by Corollary 2.5} \\ &= P\Delta P - P\epsilon^*\epsilon P = P\overline{\Delta}P. \end{aligned}$$

The last line follows from the fact that $P\epsilon P = 0$. □

Observe that $\widetilde{\Delta}$ and $\widetilde{\Delta}^*$ are strongly elliptic operators on $L^2(\Omega^*(M))$, and that $\overline{\Delta}$ is (essentially) self-adjoint on $L^2(\Omega^*(M))$.

3. The spectrum of Δ_B and the basic heat kernel. In this section, we use the formulas from Section 2 to prove results concerning the spectrum of Δ_B. Let Ω_B^k be the space of smooth basic k-forms on M, and let $L^2\left(\Omega_B^k\right)$ be its Hilbert space completion.

PROPOSITION 3.1. *There exists a complete orthonormal basis of $L^2\left(\Omega_B^k\right)$ consisting of smooth eigenforms of Δ_B, and the eigenspaces of Δ_B are finite dimensional.*

Proof. Note that Δ_B is a positive, symmetric operator on $\Omega_B^k \subset L^2\left(\Omega_B^k\right)$. The Friedrichs' extension L of $\Delta_B + I$ is a self-adjoint operator, which by definition is the adjoint of $\Delta_B + I$ restricted to the domain $\widetilde{D} = D \cap H_1$, where D is the domain of $(\Delta_B + I)^*$, and H_1 is the Sobolev space $H_1\left(\Omega_B^k\right)$ ([19]). L has a lower bound of 1 and is thus injective, so L^{-1} exists as a bounded map from $L^2\left(\Omega_B^k\right)$ into $\widetilde{D}$. By Rellich's lemma, the inclusion of H_1 into $L^2\left(\Omega_B^k\right)$ is compact, so the operator $L^{-1}\colon L^2\left(\Omega_B^k\right) \to L^2\left(\Omega_B^k\right)$ is a compact self-adjoint operator. Thus there exists a complete orthonormal basis of $L^2\left(\Omega_B^k\right)$ consisting of eigenforms of L^{-1}, and these eigenspaces are finite dimensional. If α is an eigenform with eigenvalue λ, then for each $\beta \in \Omega_B^k$,

$$\begin{aligned} \left\langle \lambda^{-1}\alpha, \beta \right\rangle &= \langle L\alpha, \beta\rangle = \langle \alpha, L^*\beta\rangle = \left\langle \alpha, (\Delta_B + I)\,|^*_{\widetilde{D}}\beta \right\rangle \\ &= \left\langle \alpha, \left(\widetilde{\Delta} + I\right)\Big|^*_{\widetilde{D}}\,\beta \right\rangle \qquad \text{by Theorem 2.6} \\ &= \left\langle \left(\widetilde{\Delta} + I\right)\alpha, \beta \right\rangle, \end{aligned}$$

where $\widetilde{\Delta} = \Delta + d\epsilon + \epsilon d$, as in the last section. The operator $\widetilde{\Delta}$ is second-order, so it maps $H_1\left(\Omega_B^k\right)$ to $H_{-1}\left(\Omega_B^k\right)$. Since the L^2-inner product is a nondegenerate pairing on $H_{-1}\left(\Omega_B^k\right) \times H_1\left(\Omega_B^k\right)$ and since Ω_B^k is dense in $H_1\left(\Omega_B^k\right)$, the preceding computation shows that

$$\left(\widetilde{\Delta} - \lambda^{-1} + 1\right)\alpha = 0.$$

Elliptic regularity implies that α is smooth, and the equality above implies that α is an eigenform of $\widetilde{\Delta}$ (and hence Δ_B) with eigenvalue $\lambda^{-1} - 1$. Thus the λ-eigenspaces of L are contained in the $\left(\lambda^{-1} - 1\right)$-eigenspaces of Δ_B. It is easily shown that eigenforms for Δ_B must likewise be eigenforms for L, so these eigenspaces are equal. ◻

We now give a new proof of the de Rham-Hodge decomposition of basic forms. In [10], the authors showed that if an r^{th} order smooth differential operator $L\colon \Omega^*(M) \to \Omega^*(M)$ which maps $\Omega_B^*(M)$ to itself has the property that $L^* - \widetilde{\epsilon}$ leaves $\Omega_B^*(M)$ invariant for some $(r-1)^{\text{th}}$ order operator $\widetilde{\epsilon}$ mapping $\Omega_B^*(M)$ into $\Omega_B^*(M)^{\perp}$, then

$$\Omega_B^*(M) = \Delta_{LB}\Omega_B^*(M) \oplus \ker \Delta_{LB},$$

where Δ_{LB} is the operator $LL^* + L^*L - \widetilde{\epsilon}L - L\widetilde{\epsilon}$ restricted to $\Omega_B^*(M)$. In addition, this decomposition is the intersection of the decomposition

$$\Omega^*(M) = \left(LL^* + L^*L - \widetilde{\epsilon}L - L\widetilde{\epsilon}\right)^* \Omega^*(M) \oplus \ker\left(LL^* + L^*L - \widetilde{\epsilon}L - L\widetilde{\epsilon}\right)$$

with $\Omega_B^*(M)$. If we let $L = d$ and $\widetilde{\epsilon} = -\epsilon$ and use the fact that $\operatorname{im}\Delta_B = \operatorname{im} d_B \oplus \operatorname{im}\delta_B$, we obtain the following:

THEOREM 3.2. *Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a closed, oriented manifold. Then $\ker\Delta_B$ is finite dimensional, and we have the following orthogonal decomposition:*

$$\Omega_B^k = \operatorname{im} d_B \oplus \operatorname{im}\delta_B \oplus \ker\Delta_B.$$

This is the intersection of the decomposition $\Omega^k = \widetilde{\Delta}^*\left(\Omega^k\right) \oplus \ker\widetilde{\Delta}|_{\Omega^k}$ with Ω_B^k.

Let $E_\lambda(L)$ denote the λ-eigenspace of the operator L.

PROPOSITION 3.3. *$E_\lambda(\Delta_B) = E_\lambda\left(\widetilde{\Delta}\right) \cap \Omega_B^*(M)$. The spectrum of Δ_B is contained in the real spectrum of $\widetilde{\Delta}$. The basic projection P maps $E_\lambda\left(\widetilde{\Delta}^*\right)$ into $E_\lambda(\Delta_B)$.*

Proof. Suppose that α is an eigenform of Δ_B with eigenvalue λ. Then $\lambda\alpha = \Delta_B\alpha = \widetilde{\Delta}\alpha$ by Theorem 2.6. Next, suppose that β is an eigenform of $\widetilde{\Delta}^*$ with eigenvalue μ. Then $\mu P\beta = P\widetilde{\Delta}^*\beta = \Delta_B P\beta$ by Theorem 2.7, so that either $P\beta = 0$ or $P\beta$ is an eigenform of Δ_B with eigenvalue μ. The conclusion follows from these two observations. $\square$

We have no reason to expect that P maps $E_\lambda\left(\widetilde{\Delta}^*\right)$ onto $E_\lambda(\Delta_B)$. Also, observe that Proposition 3.3 implies that if λ is not in the spectrum of Δ_B, then P maps $E_\lambda\left(\widetilde{\Delta}^*\right)$ to 0.

Let $0 \le \lambda_0^{B,k} \le \lambda_1^{B,k} \le \lambda_2^{B,k} \le \ldots$ be the eigenvalues of Δ_B on Ω_B^k, and let $\lambda_0^{\overline{\Delta},k} \le \lambda_1^{\overline{\Delta},k} \le \lambda_2^{\overline{\Delta},k} \le \ldots$ be the eigenvalues of $\overline{\Delta} = \Delta - \epsilon^*\epsilon$ on Ω^k.

PROPOSITION 3.4. *If Ω_B^k is infinite-dimensional, then $\lambda_j^{B,k} \ge \lambda_j^{\overline{\Delta},k}$ for all $j \ge 0$. Thus for large j there is a positive constant C such that $\lambda_j^{B,k} \ge Cj^{2/n}$, where $n = \dim M$.*

Proof. The operator $\overline{\Delta}$ is self-adjoint and elliptic and thus has smooth eigenforms. Observe that we can define $\lambda_j^{\overline{\Delta},k}$ using the Rayleigh quotient:

$$\begin{aligned}
\lambda_j^{\overline{\Delta},k} &= \sup_{\substack{V\subset\Omega^k\\ \operatorname{codim} V\le j}} \inf_{\alpha\in V} \frac{\left\langle\overline{\Delta}\alpha,\alpha\right\rangle}{\langle\alpha,\alpha\rangle} \le \sup_{\substack{V\subset\Omega^k\\ \operatorname{codim} V\le j}} \inf_{\substack{\alpha\in V\\ \alpha=P\alpha}} \frac{\left\langle\overline{\Delta}\alpha,\alpha\right\rangle}{\langle\alpha,\alpha\rangle}\\
&= \sup_{\substack{V\subset\Omega^k\\ \operatorname{codim} V\le j}} \inf_{\substack{\alpha\in V\\ \alpha=P\alpha}} \frac{\left\langle\overline{\Delta}P\alpha,P\alpha\right\rangle}{\langle\alpha,\alpha\rangle} = \sup_{\substack{V\subset\Omega^k\\ \operatorname{codim} V\le j}} \inf_{\substack{\alpha\in V\\ \alpha=P\alpha}} \frac{\left\langle P\overline{\Delta}P\alpha,\alpha\right\rangle}{\langle\alpha,\alpha\rangle}\\
&= \sup_{\substack{V\subset\Omega^k\\ \operatorname{codim} V\le j}} \inf_{\substack{\alpha\in V\\ \alpha=P\alpha}} \frac{\langle\Delta_B P\alpha,\alpha\rangle}{\langle\alpha,\alpha\rangle} = \sup_{\substack{V\subset\Omega_B^k\\ \operatorname{codim} V\le j}} \inf_{\alpha\in V} \frac{\langle\Delta_B\alpha,\alpha\rangle}{\langle\alpha,\alpha\rangle} = \lambda_j^{B,k}.
\end{aligned}$$

Because $\overline{\Delta}$ is a selfadjoint second-order elliptic operator on $\Omega^*(M)$, for large j there is a $C > 0$ such that $\lambda_j^{\overline{\Delta},k} \geq Cj^{2/n}$. (See, for example, [7].) □

We now define the basic heat kernel $K_B^k(t,x,y)$ on k-forms (that is, the fundamental solution to the basic heat equation). The basic heat kernel $K_B^k(t,x,y)$ is a double form of bidegree (k,k) on $M \times M$ for each $t > 0$ that is basic on each factor (i.e. $P_x K_B^k(t,x,y) = P_y K_B^k(t,x,y) = K_B^k(t,x,y) \in \Omega_B^k(M) \oplus \Omega_B^k(M)$) and is a solution to the system

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta_{B,x}\right) K_B^k(t,x,y) &= 0 \qquad \text{for } t > 0 \\ \lim_{t\to 0^+} \int_{M_y} K_B^k(t,x,y) \wedge *\beta(y) &= \beta(x) \quad \text{for all basic } k\text{-forms } \beta. \end{aligned}$$

If such a basic heat kernel exists, it can be used to solve the following initial value problem. Given $\beta_0 \in \Omega_B^k(M)$, consider the system

$$\left(\frac{\partial}{\partial t} + \Delta_B\right) \beta(t,x) = 0, \qquad \lim_{t\to 0^+} \beta(t,x) = \beta_0(x).$$

The solution is given by $\beta(t,x) = \int_{M_y} K_B^k(t,x,y) \wedge *\beta_0(y)$. We show that $K_B^k(t,x,y)$ exists and is unique and give two different expressions for it.

Let $\widetilde{\Delta}^* = \Delta + \delta\epsilon^* + \epsilon^*\delta$ as in Section 2. There is a unique fundamental solution $K^k_{\widetilde{\Delta}^*}(t,x,y)$ for the heat operator $\frac{\partial}{\partial t} + \widetilde{\Delta}^*$ acting on $\Omega^k(M)$ ([9]). Let $P_x\colon L^2(\Omega^*(M)) \otimes L^2(\Omega^*(M)) \to L^2\left(\Omega_B^*(M)\right) \otimes L^2(\Omega^*(M))$ be the basic projection on the first factor, and define $P_y\colon L^2(\Omega^*(M)) \otimes L^2(\Omega^*(M)) \to L^2(\Omega^*(M)) \otimes L^2\left(\Omega_B^*(M)\right)$ similarly.

THEOREM 3.5. *The basic heat kernel $K_B^k(t,x,y)$ exists and is a smooth double form on $M \times M$ that depends smoothly on t. It is unique and satisfies*

$$K_B^k(t,x,y) = P_x P_y K^k_{\widetilde{\Delta}^*}(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \alpha_j(x) \otimes \alpha_j(y),$$

where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues of Δ_B corresponding to the orthonormal basis of eigenforms $\{\alpha_1(x), \alpha_2(x), \ldots\}$. (If $\Omega_B^k(M)$ is finite dimensional, the sum is finite.)

Proof. The form $K^k_{\widetilde{\Delta}^*}(t,x,y)$ is the unique solution to the system

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \widetilde{\Delta}_x^*\right) K^k_{\widetilde{\Delta}^*}(t,x,y) &= 0 \qquad \text{for } t > 0 \\ \lim_{t\to 0^+} \int_{M_y} K^k_{\widetilde{\Delta}^*}(t,x,y) \wedge *\alpha(y) &= \alpha(x) \quad \text{for all } k\text{-forms } \alpha. \end{aligned}$$

Applying the operator P_xP_y to both sides of the first equation above, we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t}P_xP_y + P_x\widetilde{\Delta}_x^*P_y\right) K^k_{\widetilde{\Delta}^*}(t,x,y) &= 0 \quad \text{for } t>0, \text{ and}\\ \left(\frac{\partial}{\partial t}P_xP_y + \Delta_{B,x}P_xP_y\right) K^k_{\widetilde{\Delta}^*}(t,x,y) &= 0 \quad \text{by Theorem 2.7.}\end{aligned}$$

This implies that $P_xP_yK^k_{\widetilde{\Delta}^*}(t,x,y)$ solves the basic heat equation for $t>0$. From the initial condition satisfied by $K^k_{\widetilde{\Delta}^*}(t,x,y)$, we get

$$\lim_{t\to 0^+}\int_{M_y} K^k_{\widetilde{\Delta}^*}(t,x,y)\wedge *_y\beta(y) = \lim_{t\to 0^+}\int_{M_y}\left(K^k_{\widetilde{\Delta}^*}(t,x,y),\beta(y)\right)_y dV_y = \beta(x)$$

for all smooth basic forms β; the y subscript denotes the pointwise inner product on the second term in the tensor product. By Proposition 1.5, we can apply P_y to the integrand without changing the integral, so

$$\begin{aligned}\beta(x) &= \lim_{t\to 0^+}\int_{M_y} P_y\left(K^k_{\widetilde{\Delta}^*}(t,x,y),\beta(y)\right)_y dV_y \quad \text{or}\\ \beta(x) &= \lim_{t\to 0^+}\int_{M_y}\left(P_yK^k_{\widetilde{\Delta}^*}(t,x,y),\beta(y)\right)_y dV_y \quad \text{by Proposition 1.10.}\end{aligned}$$

Applying P_x to both sides of the above equation, we obtain

$$\begin{aligned}\beta(x) &= \lim_{t\to 0^+}\int_{M_y} P_x\left(P_yK^k_{\widetilde{\Delta}^*}(t,x,y),\beta(y)\right)_y dV_y\\ &= \lim_{t\to 0^+}\int_{M_y}\left(P_xP_yK^k_{\widetilde{\Delta}^*}(t,x,y),\beta(y)\right)_y dV_y\\ &= \lim_{t\to 0^+}\int_{M_y} P_xP_yK^k_{\widetilde{\Delta}^*}(t,x,y)\wedge *\beta(y).\end{aligned}$$

In the above calculation, we used the fact that β is basic. Fubini's Theorem allows us to move P_x inside the integral, since we are integrating over a compact set and the functions involved are all bounded. Thus $P_xP_yK^k_{\widetilde{\Delta}^*}(t,x,y)$ satisfies the definition of the basic heat kernel. Since P_x and P_y map smooth forms to smooth forms, the smoothness of $P_xP_yK^k_{\widetilde{\Delta}^*}(t,x,y)$ follows from the smoothness of $K^k_{\widetilde{\Delta}^*}(t,x,y)$.

We now rewrite the heat kernel. Let $K(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\alpha_j(x)\otimes\alpha_j(y)$, where λ_j and α_j are defined as in the statement of the theorem. Given a nonnegative integer m, choose a positive integer s so that $2s > m+\frac{n}{2}$. The Sobolev embedding theorem implies that there is a constant C so that

$$\|\alpha_j\|_{C^m} \le C\left(\|\widetilde{\Delta}\alpha_j\|_2 + \|\alpha_j\|_2\right) = C\left(\lambda_j^s + 1\right).$$

(Recall that by Theorem 2.6, $\widetilde{\Delta}\alpha_j = \Delta_B\alpha_j$.) Hence for some positive constants C_2 and C_3,

$$\begin{aligned}\|K(t,x,y)\|_{C^m} &\le \sum_{j=1}^{\infty} e^{-\lambda_j t} C^2 \left(\lambda_j^s + 1\right)^2 \le \sum_{j=1}^{\infty} e^{-\lambda_j t/2} C^2 C_2 t^s \\ &\le C^2 C_2 t^{-s} \sum_{j=1}^{\infty} e^{-C_3 j^{2/n} t/2} < \infty\end{aligned}$$

for $t > 0$. Thus $K(t,x,y)$ is well-defined and smooth for $t > 0$. Next, it is easy to check that $K(t,x,y)$ satisfies the definition of the basic heat kernel, using the fact that $\{\alpha_j\}$ forms a complete orthonormal basis of $L^2\left(\Omega_B^k\right)$.

To complete the proof, it suffices to show that the basic heat kernel is unique. Let $K_B(t,x,y)$ be any basic heat kernel. We may write $K_B(t,x,y)$ in the form $\sum_{i,j} C_{ij}(t)\alpha_i(x) \otimes \alpha_j(y)$ for some functions $C_{ij}(t)$, again using the fact that $\{\alpha_j\}$ forms a complete orthonormal basis of $L^2\left(\Omega_B^k\right)$. Next, consider the initial value problem

$$\left(\frac{\partial}{\partial t} + \Delta_B\right)\beta(t,x) = 0, \quad \lim_{t\to 0^+}\beta(t,x) = \alpha_j(x).$$

Using the existence of $K_B(t,x,y)$, we can write down the solution:

$$\beta(t,x) = \int_{M_y} K_B^k(t,x,y) \wedge *\alpha_j(y) = \sum_i C_{ij}(t)\alpha_i(x).$$

Using Theorem 2.6, we note that $\beta(t,x)$ is also a solution to the initial value problem

$$\left(\frac{\partial}{\partial t} + \widetilde{\Delta}\right)\beta(t,x) = 0, \quad \lim_{t\to 0^+}\beta(t,x) = \alpha_j(x).$$

Since this problem has a unique solution in $L^2(\mathbb{R}) \otimes L^2(\Omega^k)$ ([9]), we conclude that the coefficients $C_{ij}(t)$ are uniquely determined. Thus, the basic heat kernel is unique. □

Define the basic heat operator $e^{-t\Delta_B^k}$ on basic k-forms by

$$e^{-t\Delta_B^k}\beta = \int_{M_y} K_B^k(t,x,y) \wedge *\beta(y).$$

We now state some results that follow from the existence and uniqueness of the basic heat kernel. The proofs of these results are identical to the proofs in [10], where they were shown for the case in which the mean curvature form is basic.

COROLLARY 3.6. *Let β be a basic k-form. As $t \to \infty$, $e^{-t\Delta_B^k}\beta$ converges uniformly to a form $H_B\beta$ that is Δ_B-harmonic.*

COROLLARY 3.7. *The operator G_B: $\Omega_B^k(M) \to \Omega_B^k(M)$ given by*

$$G_B\,\alpha = \int_0^\infty \left(e^{-t\Delta_B^k}\alpha - H_B\,\alpha\right) dt \text{ for } \alpha \in \Omega_B^k(M)$$

is well defined and satisfies

$$\Delta_B G_B = I - H_B, \quad G_B|_{\ker \Delta_B} = 0.$$

The equations above uniquely define G_B on $\Omega_B^k(M)$.

We call G_B the basic Green's operator. Using the basic projection, we derive an alternate expression for the basic Green's operator. Define a Green's operator $\widetilde{G}$: $\Omega^k(M) \to \Omega^k(M)$ for $\widetilde{\Delta}^*$ in the following way. Given $\alpha \in \Omega^k(M)$, write $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in \ker\widetilde{\Delta}$ and $\alpha_2 \in \left(\ker\widetilde{\Delta}\right)^\perp$; note that these forms are smooth by elliptic regularity. By [18], Theorem 4.11, p. 140, there is a unique form β orthogonal to $\ker\widetilde{\Delta}^*$ such that $\widetilde{\Delta}^*\beta = \alpha_2$. Define $\widetilde{G}\alpha = \beta$.

PROPOSITION 3.8. *$G_B = P\widetilde{G}$.*

Proof. By definition, for all $\alpha \in \Omega_B^k(M)$,

$$\widetilde{\Delta}^*\widetilde{G}\alpha = \alpha - H_B\alpha.$$

(The orthogonal projection of α onto the $\ker\widetilde{\Delta}$ is $H_B\alpha$, because $\Omega_B^k(M) \cap \ker\widetilde{\Delta} = \ker\Delta_B$ by Proposition 3.3.) Then

$$\alpha - H_B\alpha = P(\alpha - H_B\alpha) = P\widetilde{\Delta}^*\widetilde{G}\alpha = \Delta_B P\widetilde{G}\alpha,$$

where the first equality is true because $\alpha - H_B\alpha$ is basic and the last equality follows from Theorem 2.7. By definition, $P\widetilde{G}$ is the basic Green's operator. □

4. Applications and special cases. In this section, we prove some results about the basic Laplacian on functions and the basic Laplacian for foliations with basic mean curvature form.

First consider Δ_B on functions. The operator $\epsilon = (P\kappa - \kappa)\lrcorner + (-1)^p(\varphi_0\lrcorner)(\chi_{\mathcal{F}}\wedge)$ is zero on functions, and the second term is zero on basic 1-forms. Using Theorem 2.7, we immediately obtain the following:

PROPOSITION 4.1. *The operators $\Delta_B, \widetilde{\Delta}, \widetilde{\Delta}^*$, and $\overline{\Delta}$ have the following expressions as operators on $C_B^\infty(M)$:*

$$\Delta_B = \widetilde{\Delta} = \Delta + \epsilon d = \Delta + ((P\kappa - \kappa)\lrcorner) \circ d$$

$$\widetilde{\Delta}^* = \Delta + \delta\epsilon^* = \Delta + \delta \circ ((P\kappa - \kappa)\wedge)$$
$$\overline{\Delta} = \Delta.$$

Let $\{\lambda_j^B\}$, respectively $\{\lambda_j^\Delta\}$, be the (nondecreasing) sequence of eigenvalues of Δ_B on $C_B^\infty(M)$, respectively Δ on $C^\infty(M)$. Proposition 3.4 immediately yields

PROPOSITION 4.2. *If $C_B^\infty(M)$ is infinite-dimensional, then $\lambda_j^B \geq \lambda_j^\Delta$ for all j. Thus, for all $t > 0$, $\mathrm{tr}_{L_B^2}\left(e^{-t\Delta_B}\right) \leq \mathrm{tr}_{L^2}\left(e^{-t\Delta}\right)$.*

We now apply our results to Riemannian foliations for which the mean curvature form κ is basic. Then $P\kappa = \kappa$, so $\epsilon = (-1)^p(\varphi_0 \lrcorner)(\chi_{\mathcal{F}}\wedge)$, which is zero on all functions and on basic 1-forms. Proposition 4.1 implies

PROPOSITION 4.3. *If κ is basic, $\Delta_B = \widetilde{\Delta} = \widetilde{\Delta}^* = \overline{\Delta} = \Delta$ as operators on $C_B^\infty(M)$.*

By this Proposition and Theorem 3.5, we have

COROLLARY 4.4. *Suppose κ is basic, and consider Δ_B and Δ as operators on $L_B^2(M)$ and $L^2(M)$, respectively, with heat kernels $K_B(t,x,y)$ and $K_\Delta(t,x,y)$, respectively. Then*

$$\mathrm{spec}(\Delta_B) \subset \mathrm{spec}(\Delta)$$
$$K_B(t,x,y) = P_x P_y K_\Delta(t,x,y).$$

Let $\{\lambda_j^{B,1}\}$, respectively $\{\lambda_j^{\Delta,1}\}$, be the nondecreasing sequence of eigenvalues of Δ_B on $\Omega_B^1(M)$, respectively Δ on $\Omega^1(M)$. Because $\epsilon = 0$ on basic 1-forms, we have

PROPOSITION 4.5. *If κ is basic, $\overline{\Delta} = \Delta$ on basic 1-forms, and $\lambda_j^{B,1} \geq \lambda_j^{\Delta,1}$ if $\Omega_B^1(M)$ is infinite-dimensional. In addition, for all $t > 0$, $\mathrm{tr}_{L^2\left(\Omega_B^1\right)}\left(e^{-t\Delta_B}\right) \leq \mathrm{tr}_{L^2(\Omega^1)}\left(e^{-t\Delta}\right)$.*

Consider the operator $\epsilon^*\epsilon$ when the mean curvature is a basic form. Then $\epsilon^*\epsilon = (\chi_{\mathcal{F}}\lrcorner)(\varphi_0\wedge)(\varphi_0\lrcorner)(\chi_{\mathcal{F}}\wedge)$. A calculation shows that for any $\gamma \in \Omega^l$ and $\omega \in \Omega^m$, $(\gamma\lrcorner\gamma \wedge \omega, \omega) \leq (\gamma,\gamma)(\omega,\omega)$ and $(\gamma \wedge \gamma\lrcorner\omega, \omega) \leq (\gamma,\gamma)(\omega,\omega)$. Thus, for any basic k-form β,

$$\langle \epsilon^*\epsilon\beta, \beta\rangle \leq \langle (\chi_{\mathcal{F}}, \chi_{\mathcal{F}})(\varphi_0, \varphi_0)\beta, \beta\rangle = \langle (\varphi_0, \varphi_0)\beta, \beta\rangle \leq \max_{x\in M}\{(\varphi_0, \varphi_0)_x\}\langle \beta, \beta\rangle;$$

we have used the fact that the pointwise inner product $(\chi_{\mathcal{F}}, \chi_{\mathcal{F}})$ is identically 1. Therefore, $\overline{\Delta} - \max_{x\in M}\{(\varphi_0, \varphi_0)_x\} \leq \Delta_B$ as operators on $\Omega_B^*(M)$. Let $\{\lambda_j^{B,k}\}$, respectively $\{\lambda_j^{\Delta,k}\}$, be the (nondecreasing) sequence of eigenvalues of Δ_B on $\Omega_B^k(M)$, respectively Δ on $\Omega^k(M)$. Using Proposition 3.4, we obtain the following:

PROPOSITION 4.6. *Suppose κ is a basic form. If $\Omega_B^k(M)$ is infinite-dimensional,*

$$\lambda_j^{B,k} \geq \lambda_j^{\Delta,k} - \max_{x \in M}\{(\varphi_0, \varphi_0)_x\}.$$

Observe that for codimension 1 foliations the form φ_0 is zero, even when the mean curvature form is not basic. Thus ϵ simplifies to $\epsilon = (P\kappa - \kappa)\lrcorner$. By an analysis similar to that used for Proposition 4.6, we obtain:

PROPOSITION 4.7. *For any codimension 1 Riemannian foliation such that $\Omega_B^1(M)$ is infinite-dimensional,*

$$\lambda_j^{B,1} \geq \lambda_j^{\Delta,1} - \max_{x \in M}\{(P\kappa - \kappa, P\kappa - \kappa)_x\}.$$

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