The Basic Theory of Partial α-Recursive Operators (*) (**).

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Summary. – *In this paper, we investigate the theory of partial α-recursive operators and functionals, α an admissible ordinal, which are defined in terms of α-enumeration reducibility. The theory bifurcates into the study of weak operators and functionals, and of operators and functionals proper. The status of the representative theorems of the classical theory (when $\alpha = \omega$) is examined relative to both kinds of operators and functionals. Especial attention is given to the difficulties, when such exist, encountered in generalizing a classical result, whether simple or profound, to level α. In the course of the investigation we are led to consider briefly topics such as the structure theory of completely recursively enumerable classes of α-recursively enumerable sets. This is natural since this theory bears on the properties of effective operations at level α. The paper provides the framework for the further investigation of this and allied topics.*

1. – Introduction.

The subject of partial recursive operators offers an inviting prospect for generalization to α-recursion theory. There is on hand a fairly well developed classical theory with a stock of representative theorems, including among others the First Recursion Theorem, the Myhill-Shepherdson Theorem, the Kreisel-Lacombe-Shoenfield Theorem, the Fundamental Operator Theorem, various theorems about limit functionals, and, toward the boundaries of the theory, the impressive theorem of Friedberg on the existence of a Banach-Mazur functional that coincides with no recursive functional on the class of recursive functions. Also very visible are the well-known ties of the theory of operators to the very extensive theory of relative recursion. Indeed, whole tracts of the latter topic may be cast in the guise of results about partial recursive operators; viz., a function f is partial recursive in the total function g if and only if there is a partial recursive operator F such that $F(g) = f$. Now the theory of relative recursion at level α, α an arbitrary admissible ordinal, has been vigorously developed, particularly α-degree theory. On seeking to generalize the theory of operators to level α, one should therefore anticipate contacts

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with the results, definitional problems, and peculiar difficulties of relative α-recursion theory. And all this refers only to those topics whose possibility for investigation is clear a priori. There is the further possibility that as the theory is developed at level $\alpha > \omega$, surprises may be encountered that open up whole new vistas. For example, to lift the Myhill-Shepherdson or Kreisel-Lacombe-Shoenfield Theorem (hereafter often referred to as the « MS theorem » and « KLS theorem » respectively) one is forced to consider effective operations at level α. The definition of an α-effective operation could not be more straightforward: one simply repeats the definition at $\alpha = \omega$. Thus, in attempting to generalize the MS theorem or KLS theorem at level $\alpha > \omega$, one's attention is naturally not concentrated on a given effective operation, but rather on the partial α-recursive (or α-recursive) operator (if the effective operation is function-valued) or functional (if the effective operation is ordinal-valued) alleged to be an extension of the given effective operation, and on whether, how, and to what extent the proof of the existence of such an operator or functional differs from the ω-proof. We will be led to consider two types of operators and functionals, a weak version and a version in a proper sense. In the case of operators, the ease and even the validity of the generalization of the ω-theorem may depend strongly on which version is being considered; that is, on whether the generalization pertains to weak (partial) α-recursive operators or to (partial) α-recursive operators in the proper sense.

Since virtually nothing has been published about the general theory of partial α-recursive operators, questions abound. First of all, which results remain valid at level for all admissible α? Among these which are such that their verification at level α, given the ω-instance of the result, is routine, and which, though valid for all or many α, require innovative methods to be proved? Once the proper definition and formulation at level α have been chosen, which properties, true for $\alpha = \omega$, simply fail to hold for some admissible α? Can the ordinals α for which the ω-result fails be suitably characterized? What notable differences in structure are implied by the divergencies from the ω-case?

In this paper, we make a mere beginning towards the resolution of these questions. In fact we develop what we have called the « basic theory of partial α-recursive operators », concentrating on the central body of theory and representative theorems of the subject. Our investigation uncovers and defines various difficulties in the theory of effective operations encountered at level α. The detailed study and solution of some of these difficulties must be postponed to a subsequent work. For comprehensiveness we also survey several of our results about operators or functionals that have appeared elsewhere.

2. – Background.

Our approach to the theory is modelled on that taken by Rogers for $\alpha = \omega$ in his well-known book [13]. This permits a development extrinsic in its exposition to any particular formalism such as generalized Turing machines or Kripke's equation calculus (these formalisms, of course, retain their importance) and yet attaining an acceptable standard of precision. Moreover, this path parallels that most often taken in the allied subject of generalized relative recursion, at least as presented in ordinal recursion theory, e.g. [14, 15]. Accordingly, our definition of a partial α-recursive operator will be based upon the concept of α-enumeration reducibility. This leads to some proliferation in our terminology, which should not be surprising. It is a fact that the theory of relative recursion can be formulated in terms of operators; for example, a total function f is recursive in a total function g if and only if there is a recursive operator F such that $F(g) = f$. Now, it is an oft-repeated story regarding the search for a suitable analogue of Turing reducibility in L_α that the more straightforward and initially favored formulation of « α-recursive in » was found to be seriously defective: it is not transitive, as was first shown by Driscoll for the metarecursive case, $\alpha = \omega_1^{CK}$ [5]. The crucial distinction in the subsequent version that has met with general acceptance is the fact that the basic units of discourse used in relative « computations » are not simply ordinals $\beta < \alpha$, but arbitrary members of L_α, that is, α-finite sets. The first version, now dubbed « weakly α-recursive in » has not suffered total eclipse, but has survived as a useful technical tool, particularly in negative results: in a demonstration that a set A is not α-recursive in a set B one often finds established the stronger result that A is not weakly α-recursive in B, as the weaker reducibility notion is less cumbersome to employ. The reader may anticipate, then, that this biformity will recur in our theory of effective operators in L_α; many of our notions of enumeration reducibility, enumeration operator, partial recursive operator or functional etc. will have two versions, one labelled « weak » and a stronger version more adequately representative of the corresponding notion in classical recursion theory ($\alpha = \omega$). This is an opportune point to introduce a distinction in terminology. For us, operators are function-valued mappings defined on a class of functions; functionals are ordinal-valued mappings defined on a class of functions.)

Once one has stated a suitable formulation at level α of a classical result, the fact that the formulation concerns the weak version of operators or functionals, need not reveal much about its relative ease of demonstration. Various combinations are possible. A weak formulation (i.e. a formulation involving the weak notions of operator or functional) may be trivially true given the ω-case, and the proper formulation may be equally easy to demonstrate, or quite difficult, perhaps, so far as we know, demonstrable only under certain added conditions. Or, the weak formulation may itself be difficult to establish, and once established the proper formulation may or may not be easily derivable from the weak. For various admissible α, the weak formulation may be true and easily demonstrable, while the proper formulation is false. However, one more or less general rule is this: once a formulation about weak functionals has been established, the passage to a proof of the corresponding formulation about weak operators is often trivial (as is the case when $\alpha = \omega$), but for $\alpha > \omega$ the passage from a result about proper functionals to the corresponding result about proper operators may well be non-trivial, in contrast to the usual situation when $\alpha = \omega$[1].

A word about style. Since we have anticipated a readership for this paper that may be unacquainted with α-recursion theory, we have tarried a bit longer over some proofs than would otherwise have been our practice, with a view toward pointing out, for example, that an utterly banal situation at level ω may become an obstacle at level $\alpha > \omega$; to indicate that a seemingly obvious, hastily made formulation of a known result simply may not work at level α; and to note why various modifications have to be made in a demonstration to guarantee success at level α. For the same reason, we have included in the next section many basic definitions of α-recursion theory, though these are easily available in the literature, e.g. [14, 15].

3. – Basic definitions and notational conventions.

A-recursion theory can be approached by two paths: via Gödel's hierarchy of constructible sets or by means of Kripke's equation calculus. We shall define the notions of admissible ordinal and partial α-recursive function in terms of the former approach, but first a brief word about some notational conventions. We shall write the bounded quantifiers $(x)_{<\varepsilon}$, $(Ey)_{<\delta}$ as $(x < \varepsilon)$ $(Ey < \sigma)$, and similarly with their variants, e.g. $(x)_{\leq\varepsilon}$ as $(x \leq \varepsilon)$ etc. If f is any mapping (function, operator, functional), « dom (f) » designates the domain of f, and « ran (f) » the range of f. « $A \subseteq B$ » means A is a subset of B, and « $A \subset B$ » means A is a proper subset of B. Similarly for functions f and g. If A is a set and f a mapping $f[A]$ is the set which is the image of A under f. We write « iff » for « if and only if. » If f is a function with domain D and β is an ordinal, $f{\upharpoonright}\beta$ is the restriction of f to $D \cap \beta$. If A is a set, $A \subseteq \alpha$, $\bar{A}$ is the complement of A in α, $\bar{A} = \alpha - A$. We write « $f(x)\downarrow$ » if f is a function, operator, etc. and $x \in \mathrm{dom}\,(f)$; « $f(x)\uparrow$ » means $x \notin \mathrm{dom}\,(f)$.

In the statement of some of our theorems about operators or functionals, the word « weak » occurs enclosed within parentheses. By this practice we mean to refer to two theorems, the statement of one is to be read including the enclosed word « weak », the statement of the second is to be read omitting the enclosed « weak ».

Presupposing familiarity (or acquaintance) with Gödel's L_α hierarchy, we define the $\Sigma_n, \Pi_n, \Delta_n$ hierarchy of formulas of set theory. A formula Φ with parameters in L_α is Σ_0 over $L_\alpha(\Sigma_0/L_\alpha)$ and Π_0 over $L_\alpha(\Pi_0/L_\alpha)$ if Φ contains no unbounded quantifiers. For $n \geq 1$, a formula Φ is $\Sigma_n(\Pi_n)$ over L_α (notation: Σ_n/L_α and Π_n/L_α, respectively) if Φ consists of a single existential (universal) quantifier prefixed to a formula that is $\Pi_{n-1}(\Sigma_{n-1})$ over L_α. Φ is Δ_n over $L_\alpha(\Delta_n/L_\alpha)$ iff it is both Σ_n over L_α and Π_n

(1) The KLS theorem relating effective operations and α-recursive operators, as opposed to the same theorem relating effective operations and weak α-recursive operators, is a good example of this situation.

over L_α. A relation (or predicate) P is $\Sigma_n(\Pi_n, \Delta_n)$ if iff it is definablei n L_α by a formula that is $\Sigma_n/L_\alpha(\Pi_n/L_\alpha. \Delta_n/L_\alpha)$. A function is $\Sigma_n(\Pi_n, \Delta_n)$ if its graph is $\Sigma_n(\Pi_n, \Delta_n)$. The ordinal α is *Σ_1-admissible* (briefly, *admissible*) if the axiom scheme of replacement of ZF for formulas that are Σ_1/L_α is satisfied in L_α. Throughout the remainder of this paper it is assumed that α is an arbitrary admissible ordinal.

A partial function $f\colon \alpha \to \alpha$ is *partial α-recursive* if it has a Σ_1/L_α definition; f is *α-recursive* if it is partial *α-recursive* and total on α. A set $S \subseteq \alpha$ is *α-recursively enumerable* (α-re) if it is the domain of a partial α-recursive function. (This is equivalent to saying that S has a Σ_1-definition over L_α.) S is α-recursive if S and $\bar{S}$, the complement of S in α, are α-re. (Thus, S is Δ_1/L_α.) Let us note that every α-re set is range of a $1-1$ partial α-recursive function whose domain is an ordinal $\gamma \leq \alpha$.

A subset K of α is *α-finite* if K is α-recursive and bounded in α. (As is well known this is equivalent to $K \in L_\alpha$.) The definition of admissible ordinal may be rephrased as a basic principle of α-recursion theory: if f is partial α-recursive and K is an α-finite subset of the domain of f, then $f[K]$ is α-finite.

We make use of the binary α-recursive function k of SACKS [14] such that

(i) if $k(\beta, \eta) = 0$, then $\beta < \eta$;

(ii) if K is an α-finite set, then there is an unique $\eta < \alpha$ such that $K = \{\beta | k(\beta, \eta) = 0\}$, and η is called the *canonical index of* K: $K = K_\eta$. We also find it convenient to use Σ_0, rudimentary (in the sense of JENSEN [8]) pairing functions τ, π_1, π_2 such that for all $\beta < \alpha$, $\tau(\pi_1(\beta), \pi_2(\beta)) = \beta$, and for all $\beta, \gamma < \alpha$, $\pi_1\tau(\beta, \gamma) = \beta$ and $\pi_2\tau(\beta, \gamma) = \gamma$, which ar e uniform for all admissible α. We often write $\langle x, y\rangle$ instead of $\tau(x, y)$, though we also use $\langle x, y\rangle$ as the ordered pair of x and y. An *α-finite function* is one whose graph is an α-finite set; an *α-finite sequence* is an α-finite function f such that dom $(f) = \gamma$ for some $\gamma < \alpha$.

The *Σ_1-projectum* α^* of α is the least β such that there is a $1-1$ α-recursive function $\Lambda\colon \alpha \to \beta$. The *$\Sigma_2$-cofinality* λ of α is the least ν such that there is a Σ_2-function with domain ν and range unbounded in α. The primary facts about α^* and λ, which are obvious from the definitions, are these:

(1) if $\beta < \alpha^*$, then every α-re subset of β is α-finite;

(2) if $\nu < \lambda$, then every Σ_2 function with domain ν is bounded in α. Recall too that the Σ_2-cofinality of α equals the Σ_2-cofinality of α^* [15]. Also, if f is a partial function from α into α, then the following are equivalent: (i) f is Σ_2; (ii) f is weakly α-recursive in O'; (iii) there is a binary α-recursive function g such that for all β $f(\beta) \simeq \lim_\sigma g(\sigma, \beta)$, where « $\simeq$ » denotes strong equality [15, p. 171].

As in [14, 15], we employ the function k to give a uniform enumeration of all the α-re sets. There is an α-recursive function r: $\alpha \times \alpha^*$ such that

(1) $K_{r(\sigma,\varepsilon)} \subseteq K_{r(\sigma',\varepsilon)} \subseteq \sigma'$ whenever $\sigma < \sigma' < \alpha$;

(2) $\bigcup \{K_{r(\sigma,\varepsilon)} | \sigma < \alpha\}$ ranges over the α-re sets as ε ranges over α^*.

We set $W^\sigma_\varepsilon = K_{r(\sigma,\varepsilon)}$ and $W_\varepsilon = \bigcup\{W^\sigma_\varepsilon | \sigma < \alpha\}$, and say that ε is an *index* of $W = W_\varepsilon$.

A partial function f is *weakly α-recursive in* a set B ($f \leq_{w\alpha} B$) if there is an $\varepsilon < \alpha^*$ such that for all γ and δ

$$f(\gamma) = \delta \leftrightarrow (E\xi)(E\eta)[\langle\gamma, \delta, \xi, \eta\rangle \in W_\varepsilon \,\&\, K_\xi \subseteq B \,\&\, K_\eta \subseteq \bar{B}].$$

If f and g are functions, then we say $f \leq_{w\alpha} g$ iff f is weakly α-recursive in the graph of g. A set A is weakly α-recursive in a set B if $C_A \leq_{w\alpha} B$, where C_A is the characteristic function of A. For $\alpha =$ the first nonrecursive ordinal, $\alpha = \omega_1^{CK}$, the metarecursive case, it was shown by Driscoll that the relation «$\leq_{w\alpha}$» is not transitive, and in fact not transitive on the ω_1-re sets [5]. Driscoll's argument extends to many admissible α. Shore characterized those admissible α for which «$\leq_{w\alpha}$» is not transitive on the α-re sets, namely those α for which there is more than one nonhyperregular α-re degree [17].

A set A *is α-recursive in a set* B ($A \leq_\alpha B$) if there is an $\varepsilon < \alpha^*$ such that for all γ and δ

$$(K_\gamma \subseteq A \,\&\, K_\delta \subseteq \bar{A}) \leftrightarrow (E\xi)(E\eta)[\langle\gamma, \delta, \xi, \eta\rangle \in W_\varepsilon \,\&\, K_\xi \subseteq B \,\&\, K_\eta \subseteq \bar{B}].$$

Similarly for functions f and g, etc. A set $A \subseteq \alpha$ is *α-regular* (briefly, *regular*) if $A \cap \beta$ is α-finite for all $\beta < \alpha$. A is *α-hyperregular* (briefly, *hyperregular*) if $f[\beta]$ is bounded whenever $\beta < \alpha$, $f: \beta \to \alpha$ and $f \leq_{w\alpha} A$.

We come now to the definitions that are especially germane to the subject of this paper. A set A is *weakly α-enumeration reducible to* a set B ($A \leq_{w\alpha e} B$) iff

$$(E\varepsilon < \alpha^*)(x)[x \in A \leftrightarrow (E\eta)[\langle x, \eta\rangle \in W_\varepsilon \,\&\, K_\eta \subseteq B].$$

The mapping thus defined by any $\varepsilon < \alpha^*$ from 2^α into 2^α is said to be a *weak α-enumeration operator Φ^w_ε with index ε*. A set A is *α-enumeration reducible to* a set B ($A \leq_{\alpha e} B$) iff

$$(E\varepsilon < \alpha^*)(\delta)[K_\delta \subseteq A \leftrightarrow (E\eta)[\langle\delta, \eta\rangle \in W_\varepsilon \,\&\, K_\eta \subseteq B].$$

Thus, $A \leq_{\alpha e} B$ if there is an enumeration procedure such that (1) a listing of an α-finite subset of the oracle-set B results in a listing of an α-finite subset of A, and (2) any α-finite subset of A can be obtained via the procedure by listing some α-finite subset of B.

Let us define for each $\varepsilon < \alpha^*$ a mapping Φ_ε from 2^α into 2^α by

$$\Phi_\varepsilon(A) = \bigcup\{K_\delta | (E\eta)[\langle\delta, \eta\rangle \in W_\varepsilon \,\&\, K_\eta \subseteq A]\}.$$

Recall that a set A is single-valued (sv) if

$$(x)(y)\big[\big(x \in A \ \&\ y \in A \ \&\ \pi_1(x) = \pi_1(y)\big) \to \pi_2 x = \pi_2 y\big]\,.$$

We define $V_\varepsilon = \{A | A \text{ is sv and } \Phi_\varepsilon(A) \text{ is sv}\}$.

We say the mapping Φ_ε is an *α-enumeration operator with index ε* if for all $A \in V_\varepsilon$, $\Phi_\varepsilon(A) \leq_{\alpha e} A$.

For completeness we include a definition of operators and functionals based on enumeration operators; as already stated our exposition mainly follows that of ROGERS [13], where $\alpha = \omega$.

Let $\mathfrak{F}$, $\mathfrak{F}\mathfrak{R}$, $\mathfrak{T}$, and $\mathfrak{R}$ be the classes of unary partial functions (from α into α), partial α-recursive functions, total functions, and α-recursive functions, respectively, let Π be the class of all single valued sets $A \in 2^\alpha$. The α-recursive pairing function τ provides a $1-1$ map of $\mathfrak{F}$ onto Π. A *functional operator* F is mapping from a subset of $\mathfrak{F}$ into $\mathfrak{F}$. Every functional operator F determines a mapping Γ from a subset of Π into Π and vice-versa, by means of the equations $\Gamma = \tau F \tau^{-1}$ and $F = \tau^{-1}\Gamma\tau$.

Consider a mapping Φ from 2^α into 2^α. We define Φ_Π as follows: (i) dom $(\Phi_\Pi) = \Phi^{-1}[\Pi] \cap \Pi$; and (ii) for each $A \in \operatorname{dom}(\Phi_\Pi)$, $\Phi_\Pi(A) = \Phi(A)$. We say Φ *defines* the functional operator F, where $F = \tau^{-1}\Phi_\Pi\tau$.

F is a *weak partial α-recursive operator* if (i) F is a functional operator, and (ii)$_w$ for some ε, Φ^w_ε defines F; F is a *partial α-recursive operator* if (i) holds and in place of (ii)$_w$ we have (ii): for some ε, Φ_ε defines F where Φ_ε is an α-enumeration operator. F is a (weak) *α-recursive operator* if (i) F is a (weak) partial α-recursive operator, and (ii) dom $(F) = \mathfrak{F}$. F is a *(weak) general α-recursive operator* if (i) F is a (weak) partial α-recursive operator, (ii) $\mathfrak{T} \subseteq \operatorname{dom}(F)$, and (iii) $F[\mathfrak{T}] \subseteq \mathfrak{T}$.

A *functional on* $\mathfrak{F}$ is a single valued subset of $\mathfrak{F} \times (\alpha + 1)$. If F is a functional, $\operatorname{dom}_w(F) =$ the weak domain of $F = F^{-1}(\alpha + 1)$; dom $(F) =$ the strong domain of $F = F^{-1}(\alpha)$. If $f \in \operatorname{dom}(F)$, we put $F(f) = \beta$, where $\langle f, \beta\rangle \in F$.

If Φ^w is a weak α-enumeration operator, then Φ^w determines a functional F on $\mathfrak{F}$ as follows:

$$\operatorname{dom}_w(F) = \{f | \Phi^w(\tau[f]) \text{ has at most one member}\}$$

and

$$\operatorname{dom}(F) = \{f | \Phi^w(\tau[f]) \text{ has exactly one member}\}\,.$$

If $f \in \operatorname{dom}(F)$, then $F(f) =$ the unique member of $\Phi^w(\tau[f])$. Similarly, every α-enumeration operator Φ determines a functional F on $\mathfrak{F}$.

If F is determined by a (weak) α-enumeration operator, then F is a *(weak) partial α-recursive functional*. F is a *(weak) α-recursive functional* if F is a (weak) partial α-recursive functional and $\operatorname{dom}_w(F) = \mathfrak{F}$ (or dom $(F) = \mathfrak{F}$). F is a *(weak) general α-recursive functional* if F is a (weak) partial α-recursive functional and $\mathfrak{T} \subseteq \operatorname{dom}(F)$.

4. – Elementary propositions and simple theorems.

The propositions listed in this section are «elementary» in that they are in character close to the definitions, and easy consequences of the same. As such they are as important as the definitions, though their truth is obvious, given the definitions. We call «simple theorems» those assertions which have easy proofs, but contain elements a bit more distant from the definitions. They may relate operators or functionals to some concept that is already well established in the literature. Their proofs are simple in one of two respects. The demonstration may be just plain simple, with little or no reference to prior results, or, given the ω-case of the theorem, it may amount to a virtual copy of the ω-proof. We give the proof in one or two instances to illustrate the use of the definitions.

PROPOSITION 1. – *Let Φ^w_ε be a weak α-enumeration operator and $\Phi_{\varepsilon'}$, an α-enumeration operator. Then*

(i) $A \subseteq B \to \big((\Phi^w_\varepsilon(A) \subseteq \Phi^w_\varepsilon(B)\big) \,\&\, \big(\Phi_{\varepsilon'}(A) \subseteq \Phi_{\varepsilon'}(B)\big)$;

(ii) $x \in \Phi^w_\varepsilon(A) \to (E\eta)[K_\eta \subseteq A \,\&\, x \in \Phi^w(K_\eta)]$;

(iii) $K_\delta \subseteq \Phi_{\varepsilon'}(A) \to (E\eta)[K_\eta \subseteq A \,\&\, K_\delta \subseteq \Phi_{\varepsilon'}(K_\eta)]$, if $A \in V_{\varepsilon'}$.

PROPOSITION 2. – *If F is a (weak) partial α-recursive operator and $\mathcal{F} \subseteq \operatorname{dom}(F)$, then F is a (weak) α-recursive operator.*

PROOF. – The proof is as given in ROGERS [13] for $\alpha = \omega$.

COROLLARY 1. – Every (weak) general α-recursive operator is a (weak) *α-recursive operator.*

PROPOSITION 3. – *Let Φ_ε and $\Phi_{\varepsilon'}$ be α-enumeration operators. Consider the composite map $\Phi = \Phi_\varepsilon \circ \Phi_{\varepsilon'}$. Then* (i) $\Phi(A) = \Phi_{c(\varepsilon,\varepsilon')}(A)$ *for all $A \in V_{\varepsilon'}$, where $c(\varepsilon, \varepsilon')$ is any α-recursive function such that*

$$\langle \delta, \eta\rangle \in W_{c(\varepsilon,\varepsilon')} \leftrightarrow (E\gamma)[\langle \delta, \gamma\rangle \in W_\varepsilon \,\&\, \langle \gamma, \eta\rangle \in W_{\varepsilon'}];$$

and (ii) *$\Phi_{c(\varepsilon,\varepsilon')}$ is an α-enumeration operator if $V_{c(\varepsilon,\varepsilon')} \subseteq V_{\varepsilon'}$.*

PROOF. – To justify that $\Phi_{c(\varepsilon,\varepsilon')}(A) = \Phi_\varepsilon(\Phi_{\varepsilon'}(A))$ for all $A \in V_{\varepsilon'}$, one first observes that

$$\begin{aligned}
&\bigcup \{K_\delta | (E\eta)[\langle \delta, \eta\rangle \in W_{c(\varepsilon,\varepsilon')} \,\&\, K_\eta \subseteq A] = \\
&\bigcup \{K_\delta | (E\eta)(E\gamma)[\langle \delta, \gamma\rangle \in W_\varepsilon \,\&\, \langle \gamma, \eta\rangle \in W_{\varepsilon'} \,\&\, K_\eta \subseteq A] \subseteq \\
&\bigcup \{K_\delta | (E\gamma)[\langle \delta, \gamma\rangle \in W_\varepsilon \,\&\, K_\gamma \subseteq \Phi_{\varepsilon'}(A)]\}
\end{aligned}$$

for all sets $A \in 2^\alpha$, so that $\Phi_{c(\varepsilon,\varepsilon')}(A) \subseteq \Phi_\varepsilon(\Phi_{\varepsilon'}(A))$ for all $A \in 2^\alpha$. Then using the hypothesis that $\Phi_{\varepsilon'}$ is an α-enumeration operator, one shows that

$$\bigcup \{K_\delta | (E\gamma)[\langle\delta, \gamma\rangle \in W_\varepsilon \,\&\, K_\gamma \subseteq \Phi_{\varepsilon'}(A)]\} \subseteq$$
$$\bigcup \{K_\delta | (E\eta)[\langle\delta, \eta\rangle \in W_{c(\varepsilon,\varepsilon')} \,\&\, K_\eta \subseteq A]\} \quad \text{for all } A \in V_{\varepsilon'},$$

so that $\Phi_\varepsilon(\Phi_{\varepsilon'}(A)) \subseteq \Phi_{c(\varepsilon,\varepsilon')}(A)$ for all $A \in V_{\varepsilon'}$. Hence $(\Phi_\varepsilon \circ \Phi_{\varepsilon'})(A) = \Phi_{c(\varepsilon,\varepsilon')}(A)$ if $A \in V_{\varepsilon'}$.

Now, assume $V_{c(\varepsilon,\varepsilon')} \subseteq V_{\varepsilon'}$, and suppose that $A \in V_{c(\varepsilon,\varepsilon')}$, so that $\Phi_{c(\varepsilon,\varepsilon')}(A) = \Phi_\varepsilon(\Phi_{\varepsilon'}(A))$. By our assumption, $\Phi_{\varepsilon'}(A) \in V_\varepsilon$. Hence, by the above

$$K_\delta \subseteq \Phi_{c(\varepsilon,\varepsilon')}(A) \leftrightarrow (E\gamma)[\langle\delta, \gamma\rangle \in W_\varepsilon \,\&\, K_\gamma \subseteq \Phi_{\varepsilon'}(A)]$$

and

$$K_\gamma \subseteq \Phi_{\varepsilon'}(A) \leftrightarrow (E\eta)[\langle\gamma, \eta\rangle \in W_{\varepsilon'} \,\&\, K_\eta \subseteq A],$$

using that Φ_ε and $\Phi_{\varepsilon'}$ are α-enumeration operators. So,

$$K_\delta \subseteq \Phi_{c(\varepsilon,\varepsilon')}(A) \leftrightarrow (E\gamma)(E\eta)[\langle\delta, \gamma\rangle \in W_\varepsilon \,\&\, \langle\gamma, \eta\rangle \in W_{\varepsilon'} \,\&\, K_\eta \subseteq A] \leftrightarrow$$
$$\leftrightarrow (E\eta)[\langle\delta, \eta\rangle \in W_{c(\varepsilon,\varepsilon')} \,\&\, K_\eta \subseteq A].$$

Therefore, $\Phi_{c(\varepsilon,\varepsilon')}$ is an α-enumeration operator if $V_{c(\varepsilon,\varepsilon')} \subseteq V_{\varepsilon'}$. Q.E.D.

We remark that the hypoteesis $V_{c(\varepsilon,\varepsilon')} \subseteq V_{\varepsilon'}$ of Proposition 3 is satisfied whenever $\Phi_\varepsilon^{-1}[\Pi_\alpha] \subseteq V_\varepsilon$ (and hence $\Phi_\varepsilon^{-1}[\Pi_\alpha] = V_\varepsilon$).

Let us recall that if $A \in 2^\alpha$, C_A is the characteristic function of A. As in ROGERS [13] when $\alpha = \omega$, if f and g are functions we shall frequently write $f \leq_{\alpha e} g$ and $f \leq_{w\alpha e} g$ in place of $\tau[f] \leq_{\alpha e} \tau[g]$ and $\tau[f] \leq_{w\alpha e} \tau[g]$, respectively.

THEOREM 1. – *If A and B are sets, $A \leq_{w\alpha} B$ iff $C_A \leq_{w\alpha e} C_B$.*

PROOF. – The proof is as for $\alpha = \omega$.

Proceding as when $\alpha = \omega$, one proves

COROLLARY 2. – *If f and g are total functions, then*

$$f \leq_{w\alpha} g \leftrightarrow f \leq_{w\alpha e} g$$

Driscoll showed that for $\alpha = \omega_1^{CK}$, there are α-re sets D_1, B, and D_2 such that $D_1 \leq_{w\alpha} B$, $B \leq_{w\alpha} D_2$, but $D_1 \nleq_{w\alpha} D_2$. Driscoll's demonstration, though only enunciated and carried out for the metarecursive case, generalizes to many admissible $\alpha \geq \omega_1^{CK}$. Finally, by virtue of Shore's characterization of those admissible α such that $\leq_{w\alpha}$ is not transitive on the α-re sets [17] and Proposition 3 we deduce, COROLLARY 3. – *The relation $\leq_{w\alpha e}$ is not transitive on the class of (graphs of) characteristic functions of α-re sets exactly for those α for which there is more than one non-hyperregular α-re degree.*

We restate this last fact in terms of enumeration operators.

PROPOSITION 4. – *Let α be as referred to in Corollary 3. Then there are weak α-enumeration operators $\Phi^w_{\varepsilon_1}$ and $\Phi^w_{\varepsilon_2}$ such that composite map $\Phi^w_{\varepsilon_1}\circ\Phi^w_{\varepsilon_2}$ is not a weak α-enumeration operator: there is in fact no index ε such that for all characteristic functions C_A of α-re sets A, $\Phi^w_\varepsilon(C_A) = \Phi^w_{\varepsilon_1}(\Phi^w_{\varepsilon_2}(C_A))$.*

PROOF. – Take α-re sets C, B, A such that $B \leq_{w\alpha} A$ and $C \leq_{w\alpha} B$, but $C \nleq_{w\alpha} A$. By definition of weak α-enumeration operator, there are weak α-enumeration operators $\Phi^w_{\varepsilon_1}$ and $\Phi^w_{\varepsilon_2}$ such that $C_B = \Phi^w_{\varepsilon_1}(C_A)$ and $C_C = \Phi^w_{\varepsilon_2}(C_B)$. Thus, $C_C = (\Phi^w_{\varepsilon_1}\circ\Phi^w_{\varepsilon_2})(C_A)$. But there can be no Φ^w_ε such that $C_C = \Phi^w_\varepsilon(C_A)$, since if there were, then $C_C \leq_{w\alpha e} C_A$, and hence $C \leq_{w\alpha} A$. Q.E.D.

THEOREM 2. – *For all α, $A \leq_\alpha B \leftrightarrow C_A \leq_{\alpha e} C_B$.*

PROOF. – There are α-recursive functions h, h_1, and h_2 such that for all ξ and η, $K_{h(\xi,\eta)} = (K_\xi \times \{1\}) \cup (K_\eta \times \{0\})$, $K_{h_1(\xi)} = \{\beta | \langle\beta, 1\rangle \in K_\xi$, $K_{h_2(\eta)} = \{\beta | \langle\beta, 0\rangle \in K_\eta\}$. If $A \leq_\alpha B$ via ε, then from W_ε such that

$$K_\gamma \subseteq A \;\&\; K_\delta \subseteq \bar{A} \leftrightarrow (E\xi)(E\eta)[\langle\gamma, \delta, \xi, \eta\rangle \in W_\varepsilon \;\&\; K_\xi \subseteq B \;\&\; K_\eta \subseteq \bar{B}]$$

one defines

$$W_{g(\varepsilon)}\{\langle\mu, \nu\rangle | (E\gamma, \delta, \xi, \eta)[\mu = h(\gamma, \delta) \;\&\; \nu = h(\xi, \eta) \;\&\; \langle\gamma, \delta, \xi, \eta\rangle \in W_\varepsilon]\}\,.$$

Then $C_A \leq_{\alpha e} C_B$ via $g(\varepsilon)$, g a suitable α-recursive function. If $C_A \leq_{\alpha e} C_B$ via ε so that $K_\mu \subseteq C_A \leftrightarrow (E\nu)[\langle\mu, \nu\rangle \in W_\varepsilon \;\&\; K_\nu \subseteq C_B]$, one defines $W_{f(\varepsilon)} = \{\langle\gamma, \delta, \xi, \eta\rangle | \langle h(\gamma, \delta), h(\xi, \eta)\rangle \in W_\varepsilon\}$ where f is an α-recursive function. Then $A \leq_\alpha B$ via $f(\varepsilon)$. Q.E.D.

Since for arbitrary α Corollary 2 follows easily from Theorem 1 much as when $\alpha = \omega$, and Theorem 2 holds, one might at first naturally expect that for all α the analogous corollary about α-enumeration reducibility holds as well: if f and g are total functions, then $f \leq^{\urcorner}_\alpha g \leftrightarrow f \leq_{\alpha e} g$. But we conjecture that for some, or various, α, this is false.

5. – The first recursion theorem.

There is no difficulty to speak of in lifting this theorem to an arbitrary level α. We therefore give, beyond the statement of the theorem and several corollaries, but a brief indication of the proof.

THEOREM 3. – *Let $\Phi_\varepsilon(\Phi^w_\varepsilon)$ be a (weak) α-enumeration operator. Then there is a set A such that*

(i) $\Phi_\varepsilon(A) = A$ $(\Phi^w_\varepsilon(A) = A)$;

(ii) $(B)[\Phi_\varepsilon(B) = B \to A \subseteq B]$

(*Likewise, with Φ^w_ε in place of Φ*)

(iii) A is α-re.

PROOF. – We define the sequence of sets: $A_0 = \phi$, $A_\gamma = \Phi_\varepsilon(A_\beta)$ if $\nu = \beta + 1$, $A_\gamma = \bigcup_{\beta<\gamma} \Phi(A_\beta)$ if γ is a limit ordinal, and put $A = \bigcup_\gamma A_\gamma$. It follows easily that there is a binary α-recursive function g such that for all γ, $x \in A_\gamma \leftrightarrow x \in W_{g(\varepsilon,\gamma)}$. Hence A is α-re, and if $\beta < \gamma$, then $\Phi_\varepsilon(A_\beta) \subseteq \Phi_\varepsilon(A_\gamma)$. From this (i) and (ii) follow.

OBSERVATION 1. – The proof is the same for α-enumeration operators and weak α-enumeration operators. Thus, the theorem holds for mappings Φ_ε defined by $\Phi_\varepsilon(\beta) = \bigcup \{K_\delta | (E\eta)[\langle\delta, \eta\rangle \in W_\varepsilon \,\&\, K_\eta \subseteq B]$, even if Φ_ε is not an α-enumeration operator.

As in the ω-case, an index of A can be obtained uniformly from any index ε of Φ, as is clear from the above. So, we have the

COROLLARY 4. – *There is an α-recursive function s such that for all ε*, (i) $\Phi_\varepsilon(W_{s(\varepsilon)}) = W_{s(\varepsilon)}$, and (ii) $(B)[\Phi_\varepsilon(B) = B \to W_{s(\varepsilon)} \subseteq B]$.

For (weak) α-recursive operators we have

THEOREM 4. – (*The First Recursion Theorem.*) *There is an α-recursive function t such that for all ε, if $\Phi_\varepsilon(\Phi^w_\varepsilon)$ defines a (weak) α-recursive operator, then*

(i) $F(\varphi_{t(\varepsilon)}) = \varphi_{t(\varepsilon)}$,

and

(ii) $(\psi)[F(\psi) = \psi \to \varphi_{t(\varepsilon)} \subseteq \psi]$.

6. – Extension theorems and effective operations.

In classical recursion theory ($\alpha = \omega$), there are two principal extension theorems relating partial recursive operators and functionals to effective operations. These are the Myhill-Shepherdson (MS) theorem and the Kreisel-Lacombe-Shoenfield (KLS) theorem. The former says that every function-valued (ordinal-valued) mapping $\mathcal{F}$ on $\mathcal{FR}$ is a function-valued (ordinal-valued) effective operation on $\mathcal{FR}$ iff $\mathcal{F}$ is the restriction to $\mathcal{FR}$ of a recursive operator (functional) on $\mathcal{F}$. The latter asserts: Let $\mathcal{A}$ be a class of recursive functions with a recursively dense base and let $\mathcal{F}$ be a mapping on $\mathcal{A}$ into $\mathcal{R}$ (into the set N of natural members); $\mathcal{F}$ is an effective operation from $\mathcal{A}$ to $\mathcal{R}$ (from $\mathcal{A}$ to N) iff $\mathcal{F}$ is the restriction to $\mathcal{A}$ of a recursive operator (functional).

The definitions of effective operations are stated in terms of indices of partial recursive functions, and thus generalize immediately to level α. But there is just a bit more to consider here than first meets the eye. The theory of effective operations may be regarded as a part of the theory of completely α-recursively enumerable classes of α-re sets (or of partial α-recursive functions). A first inspection of the situation seems to indicate the presence of anomalies in the theory of these classes. The proofs of certain basic, elementary lemmas pertaining to the structure of these classes break down when naively lifted to level α. But, as kindly pointed out to us by L. HARRINGTON, a very simple modification of the proofs rectifies matters. We briefly consider the situation.

Let $\mathcal{Q}$ be a class of α-re sets, or (exclusively) of partial α-recursive functions. To use an old notation [2], let $\theta\mathcal{Q} = \{\varepsilon | W_\varepsilon \in \mathcal{Q}\}$ $(= \{\varepsilon | \varphi_\varepsilon \in \mathcal{Q}\})$. $\mathcal{Q}$ is *α-recursively enumerable* if there is an α-re set S such that $\mathcal{Q} = \{W_\varepsilon | \varepsilon \in S\}$; $\mathcal{Q}$ is *completely α-recursively enumerable* (cα-re, or, just cre, with α understood) if $\theta\mathcal{Q}$ is α-re; $\mathcal{Q}$ is *completely α-recursive* (or *α-decidable*, or *decidable*) if $\theta\mathcal{Q}$ is α-recursive. A *weak array* is an re class $\mathcal{Q}$ of α-finite sets (functions). A *strong array* is a class $\mathcal{Q}$ of α-finite sets (functions) such that the set of canonical indices of members of $\mathcal{Q}$ is α-re. The well-known theorem of Rice for $\alpha = \omega$ asserts that the only decidable classes are the empty class $\emptyset$ and $\mathcal{E}_\alpha$, the class of all re sets [12]. There are proofs of Rice's theorem that make use only of (1) the existence of a partial recursive function universal for the binary partial recursive functions, (2) a simple instance of the iteration theorem, (3) the existence of a nonrecursive, re set [2]. Thus, Rice's theorem holds for all α.

THEOREM 5. – (*Rice's Theorem.*) *For all α, the only decidable classes are ϕ and $\mathcal{E}_\alpha$.*

Rice's original proof of this theorem proceeded very differently from that mentioned above, and was important in initiating study of the structure of cre classe [12]. Subsequently, MYHILL and SHEPHERDSON [11], in proving that a cre class $\mathcal{C}$ of partial recursive functions consists exactly of all partial recursive extensions of the (finite) functions belonging to some strong array used the following lemma: if $f \in \mathcal{C}$ and $\mathcal{C}$ is cre, then there is a finite subfunction g of f such that $g \in \mathcal{C}$. Let us recall the proof. Suppose $f \in \mathcal{C}$ but no finite subfunction g of f is in $\mathcal{C}$. As $\mathcal{C}$ is cre, $\varphi_n \in \mathcal{C} \leftrightarrow n \in S$, where S $(= \theta\mathcal{C})$ is re. Let p be a recursive function enumerating an re nonrecursive set B. Define $f_m(x) = f(x)$ if $(y \leq x)[m \neq p(y)]$, and $f_m(x)\uparrow$ if $(Ey \leq x)[m = p(y)]$. Hence, $m \notin B \leftrightarrow f_m = f \leftrightarrow f_m \in \mathcal{C}$. There is a recursive function e such that $f_m = \varphi_{e(m)}$. So, $m \notin B \leftrightarrow e(m) \in S$. This implies that $\bar{B}$ is re, and hence B is recursive, contrary to the hypothesis. Let us examine the equivalence, $m \notin B \leftrightarrow f_m = f$ for arbitrary $\alpha > \omega$. Surely, $m \notin B \rightarrow f_m = f$, as before. Suppose now that $m \in B$, and x_0 is the least ordinal x such that $p(x) = m$. Then dom $(f_m) =$ $=$ dom $(f) \cap x_0$, and *if* dom (f) *is regular*, then dom (f_m) is α-finite, and it follows that f_m is an α-finite function and hence, $f_m \neq f$ and $f_m \notin \mathcal{C}$ by hypothesis. But what may we conclude if dom (f) is not regular? Only that dom (f_m) is bounded, and hence we may not infer that f_m is α-finite.

But there is no real obstacle here. Let ε_1 and ε_2 be indices of the graph of the above f and of B, respectively. We redefine the function f_m as follows

$$f_m(x) = y \leftrightarrow (E\sigma)[\langle x, y\rangle \in W^\sigma_{\varepsilon_1} \,\&\, m \notin W^\sigma_{\varepsilon_2}]\,.$$

Clearly, if $m \notin B$, $f_m = f$ as before. Suppose $m \in B$ and let σ_0 be the least σ such that $m \in W^\sigma_{\varepsilon_2}$. Then

$$f_m(x) = y \leftrightarrow (E\sigma < \sigma_0)[\langle x, y\rangle \in W^\sigma_{\varepsilon_1}]\,.$$

Thus, if $m \in B$, f_m is α-finite. Now, if $f \in \mathcal{C}$ and no finite subfunction g of f is in $\mathcal{C}$, we have $m \notin B \leftrightarrow f_m = f \leftrightarrow f_m \in \mathcal{C}$ and $m \notin B \leftrightarrow e(m) \in S$, where e is α-recursive. Thus, the lemma holds at level α. Similarly, the following lemma generalizes, moreover with virtually no modification of the proof given in [11]: If $f \in \mathcal{Q}$, where $\mathcal{Q}$ is cre, then every partial α-recursive extension g of f also belongs to $\mathcal{Q}$. From these facts we deduce

THEOREM 6. – *A class $\mathcal{Q}$ is cre iff $\mathcal{Q}$ consists precisely of all α-re supersets of the α-finite sets constituting some strong array.*

6.1. *Definitions and properties of effective operations.*

An ordinal-valued mapping $\mathcal{F}$ on a class $\mathcal{Q}$ of partial α-recursive functions ($\mathcal{F}\colon \mathcal{Q} \to \alpha$) is said to be *an (ordinal-valued) effective operation (briefly, an o.e.o.)* on $\mathcal{Q}$ if there is a partial α-recursive function ψ such that $(x)[\varphi_x \in \mathcal{Q} \leftrightarrow \psi(x)\!\downarrow \,\&\, \mathcal{F}(\varphi_x) = \psi(x)]$. Thus, $\mathcal{Q}$ is cre.

Suppose instead that $\mathcal{F}$ is function-valued, $\mathcal{F}\colon \mathcal{Q} \to \mathcal{P}\mathcal{R}$. $\mathcal{F}$ is a *function-valued effective operation on* $\mathcal{Q}$ (briefly, an f.e.o.) if there is a partial α-recursive function f such that $(x)[\varphi_x \in \mathcal{Q} \leftrightarrow f(x)\!\downarrow \,\&\, \mathcal{F}(\varphi_x) = \varphi_{f(x)}]$. Such an f is said to be *extensional for* $\mathcal{Q}$.

Again, the definition implies that $\mathcal{Q}$ is cre. It follows that if $\varphi_x \in \mathcal{Q}$ and $\varphi_x \subseteq \varphi_{x'}$, then $\varphi_{x'} \in \mathcal{Q}$, as the reader can readily check. Another familiar assertion is: Fin $(\mathcal{Q}, \alpha) \leftrightarrow \mathcal{Q}$ is a class of partial α-recursive functions and for each $\varphi_x \in \mathcal{Q}$, there is an α-finite $\varphi_{x'} \subseteq \varphi_x$ in $\mathcal{Q}$.

Closely related to Fin $(\mathcal{Q}, \alpha)$ are the following statements about effective operations $\mathcal{F}$.

$\mathrm{Fin}_1(\mathcal{F}, \alpha) \leftrightarrow \mathcal{F}$ is an o.e.o. and for all $\varphi_x \in \mathcal{Q}$, $\mathcal{Q} = \mathrm{dom}(\mathcal{F})$, there is an α-finite $\varphi_{x'} \subseteq \varphi_x$ in $\mathcal{Q}$ such that $\mathcal{F}(\varphi_{x'}) = \mathcal{F}(\varphi_x)$;

$\mathrm{Fin}_2(\mathcal{F}, \alpha) \leftrightarrow \mathcal{F}$ is an f.e.o. and for all $\langle y, z\rangle \in \mathcal{F}(\varphi_x)$, there is an α-finite $\varphi_{x'} \subseteq \varphi_x$ such that $\langle y, z\rangle \in \mathcal{F}(\varphi_{x'})$.

Now, if $\mathcal{F}$ is an effective operation with domain $\mathcal{Q}$, then Fin $(\mathcal{Q}, \alpha) \to \mathrm{Fin}_1(\mathcal{F}, \alpha)$

if $\mathcal{F}$ is an o.e.o., and Fin $(\mathcal{Q}, \alpha) \rightarrow \mathrm{Fin}_2(\mathcal{F}, \alpha)$ if $\mathcal{F}$ is a f.e.o. Also, we note that from $\mathrm{Fin}_2(\mathcal{F}, \alpha)$ there follows, $\mathrm{Fin}_3(\mathcal{F}, \alpha) \leftrightarrow \mathcal{F}$ is an f.e.o., and for all $\varphi_x \in \mathrm{dom}(\mathcal{F})$ if φ_u is an α-finite subfunction of $\mathcal{F}(\varphi_x)$, there is an α-finite subfunction $\varphi_{x'}$ of $\varphi_{x'}$, $\varphi_{x'} \in \mathrm{dom}(\mathcal{F})$, such that $\varphi_u \subseteq \mathcal{F}(\varphi_{x'})$.

So, if $\mathcal{Q}$ is cre and $\mathcal{F}$ is an effective operation here are four assertions, true for arbitrary $\alpha \geq \omega$.

THEOREM 7. – (*The MS theorem for ordinal-valued effective operations.*)

Let $\mathcal{Q}$ *be a cre class of partial* α*-recursive functions and* $\mathcal{F}$ *a map from* $\mathcal{Q}$ *into* α. *Then* $\mathcal{F}$ *is an o.e.o. having property* $\mathrm{Fin}_1(\mathcal{F}, \alpha)$ *iff* $\mathcal{F}$ *is the restriction to* $\mathcal{Q}$ *of a (weak)* α*-recursive functional on* $\mathfrak{T}$.

PROOF. – If $\mathcal{F}$ is the restriction to $\mathcal{Q}$ of a recursive functional, then that $\mathcal{F}$ is an o.e.o. satisfying $\mathrm{Fin}_1(\mathcal{F}, \alpha)$ is clear. So, let $\mathcal{F}$ be an o.e.o. on $\mathcal{Q}$ satisfying $\mathrm{Fin}_1(\mathcal{F}, \alpha)$: $\mathcal{F}(\varphi_x) = \psi(x)$, ψ partial α-recursive. Imitating [13], it is easy to show that there are an α-recursive set B that contains exactly one index for every α-finite function, and an α-recursive function g such that for all $y \in B$, $\varphi_y = \tau^{-1}[K_{g(y)}]$. We define an re set W with index ε by

$$\langle \delta, \eta \rangle \in W_\varepsilon \leftrightarrow (Ey)[y \in B \;\&\; g(y) = \eta \;\&\; (Ex)[K_\delta = \{x\} \;\&\; x = \psi(y)]] \vee$$
$$\vee [K_\delta = \emptyset \;\&\; K_\eta = \emptyset]\,.$$

Consider the mapping:

$$\Phi_\varepsilon(A) = \bigcup \{K_\delta | (E\eta)[\langle \delta, \eta \rangle \in W_\varepsilon \;\&\; K_\eta \subseteq A]\}\,.$$

Then it follows that for all *sv* sets A, (1) $\Phi_\varepsilon(A)$ is *sv*, and (2) $\Phi_\varepsilon(A) \leq_{\alpha e} A$ via ε. (With respect to (1), to insure consistency one must verify such things as: if A is *sv*, $\varphi_{y_1} \subseteq A$, $\varphi_{y_2} \subseteq A$, $y_1, y_2 \in B$ and $\psi(y_1)\downarrow$, $\psi(y_2)\downarrow$, then $\psi(y_1) = \psi(y_2)$; etc.) Thus, Φ_ε defines on α-recursive functional F_ε. And, if $\varphi_x \in \mathcal{Q}$, then $\mathcal{F}(\varphi_x) = F_\varepsilon(\varphi_x)$. (To verify this, one must show that for $\varphi_x \in \mathcal{Q}$, there is an α-finite $\varphi_y \subseteq \varphi_x$ such that $\mathcal{F}(\varphi_y) = \mathcal{F}(\varphi_x)$. Here one uses the condition $\mathrm{Fin}_1(\mathcal{F}, \alpha)$, which follows from Fin $(\mathcal{Q}, \alpha)$.)
Q.E.D.

OBSERVATION 2. – With regard to weak α-recursive functionals, the proof is even more simple.

THEOREM 8. – (*The MS theorem for function-valued effective operations.*)

Let $\mathcal{Q}$ *be a cre class of partial* α*-recursive functions. Then* (1) *any partial* α*-recursive function* f *that is extensional for* $\mathcal{Q}$ *determines an f.e.o.* $\mathcal{F}$ *satisfying* $\mathrm{Fin}_2(\mathcal{F}, \alpha)$ *that is the restriction to* $\mathcal{Q}$ *of a weak* α*-recursive operator* F, *and* (2) *any (weak)* α*-recursive operator* F *determines an* α*-recursive* f *that is extensional for* $\mathcal{Q}$.

PROOF. – Let f be a partial α-recursive function that is extensional for $\mathcal{Q}$. Then since Fin $(\mathcal{Q}, \alpha)$, f determines an f.e.o. $\mathcal{F}$ on $\mathcal{Q}$ satisfying $\mathrm{Fin}_2(\mathcal{F}, \alpha)$: $\mathcal{F}(\varphi_x) = \varphi_{f(x)}$.

Let d be an α-recursive function such that $B = \operatorname{ran}(d)$. Thus for all α-finite functions φ with graph K_η, $K_\eta = \tau[\varphi_{d(\eta)}]$. We define an α-re set W_ε by

$$\langle\langle y, z\rangle, \eta\rangle \in W_\varepsilon \leftrightarrow K_\eta \text{ is sv } \& \langle y, z\rangle \in \varphi_{fd(\eta)} \& d(\eta) \in \theta\mathcal{Q}\,.$$

It follows as in the ω-case that for all *sv* sets A, $\Phi^w_\varepsilon(A)$ is sv, where Φ^w_ε is the weak α-enumeration operator defined by W_ε. Let F^w_ε be the weak α-recursive operator defined by Φ^w_ε. Then it follows, again as in the ω-case, that for all $\varphi_x \in \mathcal{Q}$, $\mathcal{F}(\varphi_x) = F^w_\varepsilon(\varphi_x)$. It is in demonstrating that $\varphi_{f(x)}(y) = z \to F^w_\varepsilon(\varphi_x)(y) = z$ that we use $\operatorname{Fin}(\mathcal{Q}, \alpha)$.

On the other hand, suppose F^w is a weak α-recursive operator. Then by a corollary to the First Recursion Theorem (actually an observation on the proof of that theorem), there is an α-recursive function g suchthat $F^w_\varepsilon(\varphi_x) = \varphi_{g(x,\varepsilon)}$ (cf. [13], p. 195). We take f by $f(x) = g(x, \varepsilon)$ for all x. Of course, all this holds if F_ε is an α-recursive operator. Q.E.D.

COROLLARY 5. – *Let $\mathcal{F}$ be a f.e.o. on a cre class $\mathcal{Q}$ of partial α-recrusive functions. (Thus, $\mathcal{F}$ satisfies* $\operatorname{Fin}_2(\mathcal{F}, \alpha)$.) *Then $\mathcal{F}$ is the restriction to $\mathcal{Q}$ of a f.e.o. $\mathcal{F}'$ on $\mathcal{P}\mathcal{R}$.*

PROOF. – Let f be the extensional partial α-recursive function that determines $\mathcal{F}$. Then there is a weak α-recursive operator F^w_ε such that for all $\varphi_x \in \mathcal{Q}$, $F^w_\varepsilon(\varphi_x) = \mathcal{F}(\varphi_x)$. F^w_ε in turn determines an extensional α-recursive f' such that $F^w_\varepsilon(\varphi_x) = \varphi_{f'(x)}$ for all φ_x. Thus, $\mathcal{F}(\varphi_x) = \mathcal{F}'(\varphi_x)$ for all φ_x in $\mathcal{Q}$, where by definition $\mathcal{F}'(\varphi_x) = \varphi_{f'(x)}$.

OBSERVATION 3. – We are not too pleased with our version of the MS theorem for f.e.o.'s. It concerns only weak partial α-recursive operators, seemingly quite a limitation.

6.2. *Effective operations on classes of α-recursive functions.*

Let $\mathcal{Q}$ be a class of α-recursive functions. In this section we consider effective operations on $\mathcal{Q}$. Accordingly, we modify our earlier definition, and no longer require that $\mathcal{Q}$ be cre.

A mapping $\mathcal{F}$ from $\mathcal{Q}$ to α is an *ordinal-valued effective operation on* $\mathcal{Q}$ (o.e.o.) if there is a partial α-recursive function ψ such that $\varphi_x \in \mathcal{Q} \to [\psi(x)\downarrow \& \mathcal{F}(\varphi_x) = \psi(x)]$. A mapping $\mathcal{F}$ from $\mathcal{Q}$ into $\mathcal{P}\mathcal{R}$ is a function-valued effective operation on $\mathcal{Q}$ (f.e.o.) if there is a partial α-recursive function f such that $\varphi_x \in \mathcal{Q} \to [f(x)\downarrow \& \mathcal{F}(\varphi_x) = \varphi_{f(x)}]$; f is said, as previously, to be *extensional* for $\mathcal{Q}$. If $\mathcal{Q} = \mathcal{R}$, $\mathcal{F}$ is said to be an *effective operation total on* $\mathcal{R}$.

A principal theorem in this context is that of KREISEL, LACOMBE, and SHOENFIELD [9], which answered one case of a question posed by MYHILL and SHEPHERDSON [11]. In our terminology, the latter authors had asked: is every effective operation $\mathcal{F}$ that is total on $\mathcal{R}$ the restriction of a recursive operator? KREISEL, LACOMBE, and SHOENFIELD were able to supply an affirmative answer, allowing in fact domains $\mathfrak{Q}$ of $\mathcal{F}$ other than $\mathfrak{R}$ that share a certain topological property with $\mathfrak{R}$, provided the range of $\mathcal{F}$ is a subclass of $\mathfrak{R}$. We shall obtain in this section a generalization of a somewhat weakened Kreisel, Lacombe, Shoenfield (KLS) Theorem to level α. It is somewhat amusing that we have not, in contrast, obtained a comparable version of the MS Theorem relative to α-recursive operators, though at $\alpha = \omega$ the proof of the KLS Theorem is certainly more technical and complicated than that of the MS Theorem. This is a reflection of the fact that the graphs of α-recursive functions, as well, of course, as the domains of these functions, are trivially regular sets.

In our paper [3], we needed a version at level α of the KLS Theorem in order to prove the Operator Gap Theorem. We remarked that the demonstration of the needed version, given the ω-case, was relatively trivial. In terms of our present development the employed instance of the KLS Theorem was that for function-valued effective operations and weak α-recursive operators. Indeed, the version of the KLS Theorem for ordinal valued effective operations and weak α-recursive functionals just lifts without resistance to level α. And, the KLS Theorem for f.e.o.'s and weak α-recursive operators is easily obtained from that for o.e.o.'s and weak α-recursive functionals. The proof of the theorem for f.e.o.'s and α-recursive operators is not trivial, and this point illustrates a general situation.

As we have already remarked, if for $\alpha = \omega$ one has proved a theorem about partial recursive functionals, the demonstration of the analogous theorem for partial recursive operators is often a simple matter. In particular, this is typically the case if the domain of the partial recursive functional is a class of effectively computable functions, such as the partial recursive or recursive functions, when the derivation of the operator-version of the theorem in question from the proved functional-version is an almost mechanical chore, and thus sometimes omitted in the literature, as in [9]. For arbitrary admissible $\alpha > \omega$, this is no longer quite the case. The KLS Theorem serves as a convenient illustration of this situation, an instance in which the derivation of the theorem even for *limited* α-recursive operators from the case for α-recursive functionals illustrates the obstacles, but falls within manageable bounds. Thus, we first indicate the proof of the theorem for α-recursive functionals, which is straightforward from the proof for $\alpha = \omega$, and then derive the theorem for limited α-recursive operators in more detail.

A class $\mathfrak{Q}$ of α-recursive functions has an *α-recursively dense base B* if B is an α-re set such that

(1) $\nu \in B \to \varphi_\nu \in \mathfrak{Q}$, and

(2) for all y,

$$\varphi_\varepsilon \in \mathfrak{Q} \to (E\nu)[\nu \in B \mathbin{\&} (x)[x \leq y \to \varphi_\varepsilon(x) = \varphi_\nu(x)]] .$$

A class $\mathfrak{Q}$ of α-recursive functions is *α-recursive (α-re)* if there is an α-recursive (α-re) set Q such that $\varphi_\varepsilon \in \mathfrak{Q} \leftrightarrow \varepsilon \in Q$, whenever φ_ε is α-recursive.

THEOREM 9. – (*The* KLS *Theorem for ordinal-valued effective operations.*)

If a class $\mathfrak{Q}$ *of* α*-recursive functions has an* α*-recursively dense base* B *and* $\mathcal{F}$ *is an ordinal-valued effective operation defined by the partial* α*-recursive function* ψ*, such that* dom $(\mathcal{F}) = \mathfrak{Q}$*, then there is an* α*-recursive functional* F *such that* $F(\varphi_\varepsilon) = \mathcal{F}(\varphi_\varepsilon)$ *for all* $\varphi_\varepsilon \in \mathfrak{Q}$.

PROOF. – Let $\mathfrak{Q}$, B, and ψ be as in the statement of the theorem. As the proof is largely a close imitation of that of KREISEL, LACOMBE, and SHOENFIELD in [9], and presents no difficulties peculiar to admissible $\alpha > \omega$, we present only so much of it as is required to define the pertinent α-enumeration operator. Let $T(z, x, y)$ be an α-recursive predicate that is universal for the α-re sets. Also, we may assume that for all z and x, if there is a y such that $T(z, x, y)$, then y is unique (e.g., $T(z, x, y) \leftrightarrow x \in W_z^y \,\&\, (y' < y)[x \notin W_z^{y'}]$).

We define the diagonal set $D = \{x | (Ey) T(x, x, y)\}$ and observe that if S is any α-re set and ε is any index of S ($S = W_\varepsilon$), then $\varepsilon \in S \cap D$ or $\varepsilon \notin S \cup D$. Let b be a $1-1$ partial α-recursive function with domain $\sigma \leq \alpha$ that enumerates B. Following [9], we define below a partial α-recursive function $t(\varepsilon, \delta, x)$ such that for each pair δ, ε the partial α-recursive function $t_{\varepsilon,\delta}(x) = t(\varepsilon, \delta, x)$ has the following property: if $\varphi_\delta \in \mathfrak{Q}$, then $t_{\varepsilon,\delta} = \varphi_\delta$ if $\varepsilon \notin D$, and $\mathcal{F}(t_\varepsilon) \neq \mathcal{F}(\varphi_\delta)$ if $\varepsilon \in D$.

We put $y(\varepsilon, x) = \min_{y \leqslant x} T(\varepsilon, \varepsilon, y)$, and define

$$\beta(\delta, \varepsilon, x) = \min_\beta \big[(z < y(\varepsilon, x))[\varphi_\delta(z) = \varphi_{b(\beta)}(z)] \,\&\, \psi(b(\beta)) \neq \psi(\delta)\big]$$

if

$$(Ey \leq x)\big[T(\varepsilon, \varepsilon, y) \,\&\, (z < y)\varphi_\delta(z)\!\downarrow \,\&\, (E\beta < \sigma)(z < y)[\varphi_{b(\beta)}(z) = \\ = \varphi_\delta(z)] \,\&\, \psi(\delta)\!\downarrow \,\&\, \psi(b(\beta)) \neq \psi(\delta)\big]$$

$\beta(\delta, \varepsilon, x)\!\uparrow$ otherwise.

By making use of the fact that for all $\beta < \sigma$, $\varphi_{b(\beta)}$ is α-recursive and $\psi(b(\beta))$ is defined, it is not difficult to see that $\beta(\delta, \varepsilon, x)$ is partial α-recursive, as can be rigorously demonstrated. Also, let us notice that if $\beta(\delta, \varepsilon, x)\!\downarrow$ then $\beta(\delta, \varepsilon, z) = \beta(\delta, \varepsilon, x)$ for all $z \geq y(\varepsilon, x)$; for if $(Ey \leq x) T(\varepsilon, \varepsilon, y)$, then $y(\varepsilon, x) = y(\varepsilon, x')$ for all $x' \geq x$.

Now we define $t_{\varepsilon,\delta}(x) = t(\varepsilon, \delta, x)$:

$$t_{\varepsilon,\delta}(x) = \varphi_\delta(x) \qquad \text{if } \sim (Ey \leq x) T(\varepsilon, \varepsilon, y)$$

$$t_{\varepsilon,\delta}(x) \stackrel{\cdot}{=} \varphi_{b(\beta(\delta,\varepsilon,x))}(x) \qquad \text{if } \beta(\delta, \varepsilon, x)\!\downarrow .$$

By definition of the function $\beta(\delta, \varepsilon, x)$, if $T(\varepsilon, \varepsilon, y)$ then if $\beta(\delta, \varepsilon, x)$ is defined (and hence necessarily $\beta(\delta, \varepsilon, x) = \beta(\delta, \varepsilon, z)$ for all $z \geq y(\varepsilon, x)$), $t_{\varepsilon,\delta}(z) = \varphi_{b(\beta(\delta,\varepsilon,x))}(z)$ for all z.

There is a binary α-recursive function g such that for all ε, δ $\varphi_{g(\varepsilon,\delta)} = t_{\varepsilon,\delta}$. For each δ, let $C_\delta = \{\varepsilon | \psi(\delta)\!\downarrow \,\&\, \psi(g(\varepsilon, \delta))\!\downarrow \,\&\, \psi(\delta) = \psi(g(\varepsilon, \delta))\}$; so C_δ is α-re. Let h be an α-recursive function such that for all δ, $C_\delta = W_{h(\delta)}$. Then as in [9] it follows that

$$(*) \qquad \varphi_\delta \in \mathfrak{Q} \to h(\delta) \in D ,$$

$$(**) \qquad [\psi(\delta)\downarrow \;\&\; T(h(\delta), h(\delta), y) \;\&\; \nu \in B \;\&\; (x < y)(\varphi_\delta(x) = \varphi_\nu(x))] \to \psi(\delta) = \psi(\nu) .$$

If $\nu \in B$, then $\varphi_\nu \in \mathfrak{Q}$, and hence $(Ey)T(h(\nu), h(\nu), y)$. We put

$$\Gamma = \{\gamma | \mathrm{Seq}\,(\gamma) \;\&\; (E\nu)(Ey)[\nu \in B \;\&\; T(h(\nu), h(\nu), y) \;\&\; \mathrm{dom}\,(K_\gamma) = y \;\&\; \\ \&\; (x < y)[K_\gamma(x) = \varphi_\nu(x)]]\} .$$

Next, we define the partial α-recursive function ψ':

$$\psi'(\delta) = \psi(\delta) \quad \text{if} \quad \delta \in \mathrm{dom}\,(\psi) \;\&\; (E\nu)(Ey)[\nu \in B \;\&\; T(h(\delta), h(\delta), y) \;\&\; \\ \&\; (x < y)[\varphi_\delta(x) = \varphi_\varepsilon(x)]] \;\&\; \delta \notin \mathrm{ran}\,(d)$$

$\psi'(d(\gamma)) = \psi(\nu)$ if $\gamma \in \Gamma$ and $\nu \in B$ and $T(h(\nu), h(\nu), y)$ for suitable ν, y. More precisely,

$$\psi'(d(\gamma)) = \psi(b\pi_1(\beta_0)) \ \text{ if } \ \gamma \in \Gamma,$$

where d is as in Theorem 8,

$$\beta_0 = \min_\beta \left[T\big(h(b\pi_1(\beta)), h(b\pi_1(\beta)), \pi_2(\beta)\big) \;\&\; \mathrm{dom}\,(K_\gamma) = \pi_2(\beta) \;\&\; \\ \&\; (x < \pi_2(\beta))[K_\gamma(x) = \varphi_{b\pi_1(\beta)}(x)]\right] .$$

Let us notice that by $(**)$, if $\psi'(\delta)\downarrow$, $\psi'(d(\gamma))\downarrow$ and $\varphi_\delta \subseteq \varphi_{d(\gamma)}$ or $\varphi_{d(\gamma)} \subseteq \varphi_\delta$, then $\psi'(\delta) = \psi'(d(\gamma))$. Also, if $\varphi_\delta \in \mathfrak{Q}$, then $\psi(\delta) = \psi'(\delta)$. Note that we do not claim that $\psi'(\delta_1) = \psi'(\delta_2)$ for all $\delta_1, \delta_2 \in \mathrm{dom}\,(\psi)$ such that $\varphi_{\delta_1} = \varphi_{\delta_2}$. Finally, we define the α-re set W

$$\langle \varrho, \gamma \rangle \in W \leftrightarrow \big(\gamma \in \Gamma \;\&\; K_\varrho = \{\psi'(d(\gamma))\}\big) \vee (K_\varrho = K_\gamma = \emptyset) .$$

An index ε of W can be obtained via an α-recursive function of indices of ψ, d, h, and B. It is easy to see that ε defines an α-enumeration operator Φ_ε, and that Φ_ε defines an α-recursive functional F_ε. It follows, much as in [9], that for all $\varphi_\delta \in \mathfrak{Q}$, $F_\varepsilon(\varphi_\delta) = \mathcal{F}(\varphi_\delta)$. Q.E.D.

OBSERVATION 4. – Just as is done in [9] for $\alpha = \omega$, one can prove that every α-re class of α-recursive functions has an α-recursively dense base. Hence, we have the

COROLLARY 6. – *If $\mathcal{Q}$ is a α-re class of α-recursive functions and $\mathcal{F}$ is an ordinal-valued effective operation with domain $\mathcal{Q}$, then there is an α-recursive functional F such that for all $\varphi_\varepsilon \in \mathcal{Q}$, $F(\varphi_\varepsilon) = \mathcal{F}(\varphi_\varepsilon)$.*

OBSERVATION 5. – An example of Myhill shows that for $\alpha = \omega$, Theorem 9 need not hold if $\mathcal{Q}$ does not have a recursively dense base. Let us define $\mathcal{F}(\varphi) = 0$ if φ is the zero function, and $\mathcal{F}(\varphi) = 1$ if (i) φ is α-recursive and (ii) if $\varphi_\varepsilon = \varphi \to (Ex)[x \leq \varepsilon \,\&\, \varphi_\varepsilon(x) \neq 0]$. Otherwise, $\mathcal{F}(\varphi)$ is undefined. The domain $\mathcal{Q}$ of $\mathcal{F}$ is a class of α-recursive functions.

We set

$$\psi(\varepsilon) = \min_u \Big[\Big((x \leq \varepsilon)\big(\varphi_\varepsilon(x)\downarrow \,\&\, \varphi_\varepsilon(x) = 0 \,\&\, u = 0\big) \vee \Big((Ex \leq \varepsilon)\big(\varphi_\varepsilon(x)\downarrow \,\&\, \varphi_\varepsilon(x)\big) \neq 0\Big) \,\& \\ \& \, u = 1\Big)\Big]\,.$$

ψ is partial α-recursive and for all $\varphi_\varepsilon \in \mathcal{Q}$, $\psi(\varepsilon)\downarrow$ and $\mathcal{F}(\varphi_\varepsilon) = \psi(\varepsilon)$. Thus, $\mathcal{F}$ is an effective operation on $\mathcal{Q}$. If there were an α-recursive functional F that agrees with $\mathcal{F}$ on $\mathcal{Q}$, then as the zero function $\varphi \in \mathcal{Q}$, there would be a μ_0 such that if $f(x) = 0$ for all $x < \mu_0$, then $F(f) = 0$. Now, for each μ, let us define the set S_μ by $S_\mu = \{\varepsilon | \varepsilon \leq \mu \,\&\, \varphi_\varepsilon(\mu)\downarrow\}$. Clearly, S_μ is α-re. Consider the function f_μ such that $f_\mu(x) = 0$ if $x \neq \mu$ and

$$f_\mu(\mu) = \min_\beta \big[(\varepsilon)\big(\varepsilon \in S_\mu \to \beta \neq \varphi_\varepsilon(\mu)\big) \,\&\, \beta \neq 0\big]\,.$$

The point is that if f_{μ_0} is α-recursive, and $f_{\mu_0} = \varphi_{\varepsilon_0}$, then $\mu_0 < \varepsilon_0$, and as $f_{\mu_0}(\mu_0) \neq 0$, $F(f_{\mu_0}) = 1$. But, also $F(f_{\varepsilon_0}) = 0$ since $f_{\mu_0}(x) = 0$ for all $x < \mu_0$, and we have a contradiction. Thus, no recursive functional F exists that coincides with $\mathcal{F}$ on $\mathcal{Q}$.

Now, if $\alpha = \omega$ or $\alpha^* = \alpha$, then S_{μ_0} is α-finite and f_{μ_0} is in fact α-recursive, so that Myhill's counterexample applies. But if $\alpha > \omega$ and $\alpha^* < \alpha$, then S_{μ_0} may not be α-finite, and f_{μ_0} not α-recursive. Indeed, if it should be the case that $\mu_0 \geq \alpha^*$ and $\alpha^* < \alpha$, then

$$S_{\mu_0} = \{\varepsilon \leq \mu_0 | \varphi_\varepsilon(p_0)\downarrow\} = \{\varepsilon | \varphi_\varepsilon(\mu_0)\downarrow\}\,,$$

and hence S_μ is not α-recursive, as is easily shown. In this case, Myhill's example fails to apply.

We show how to render Myhill's example applicable when $\alpha^* < \alpha$. Let Λ be a $1-1$ α-recursive function that projects α into α^*. We may assume that $\Lambda(0) = 0$. Observe that for all α, $\alpha^{**} = \alpha^*$. Recall also that the pairing functions τ, π_1, and π_2 are uniform for all admissible α. We may assume that $\tau(0,0) = 0$, $\pi_1(1) \neq 0$, $\pi_2(1) \neq 0$. Now, assume that $\alpha^* < \alpha$. We define $\mathcal{F}(\varphi) = 0$ if φ is the zero function; $\mathcal{F}(\varphi) = 1$ if φ is α-recursive, and $\varphi = \varphi_\varepsilon \to (Ex < \alpha^*)[\varphi_\varepsilon(x) \neq 0]$. The partial

α-recursive function ψ defined by

$$\psi(\varepsilon) = \begin{cases} 1 & \text{if } (x < \alpha^*)[\varphi_\varepsilon(x) = 0] \\ 0 & \text{if } (Ex < \alpha^*)[\varphi_\varepsilon(x) \neq 0] \\ \uparrow & \text{otherwise} \end{cases}$$

determines $\mathcal{F}$ as an effective operation on the domain $\mathfrak{Q}$ of $\mathcal{F}$.

If $A \subseteq \alpha$, then we define the set $\Lambda\{A\}$ in α^* as follows:

$$z \in \Lambda\{A\} \leftrightarrow (Ex, y)[\tau(x, y) \in A \;\&\; \pi_1 z = \Lambda(x) \;\&\; \pi_2 z = \Lambda(y)] \,.$$

Also, if $A \subseteq \alpha^*$, we define $\Lambda^{-1}\{A\} \subseteq \alpha$ by

$$z \in \Lambda^{-1}\{A\} \leftrightarrow (Eu)[u \in A \;\&\; \pi_1 u \in \operatorname{ran}(\Lambda) \;\&\; \pi_2 u \in \operatorname{ran}(\Lambda) \;\&$$
$$\& \; z = \tau(\Lambda^{-1}(\pi_1 u), \Lambda^{-1}(\pi_2 u))] \,.$$

Similarly, if f is a function in α, we define the function $\Lambda\{f\}$ by:

$$\langle u, v\rangle \in \Lambda\{f\} \leftrightarrow (Ex, y)[\langle x, y\rangle \in f \;\&\; u = \Lambda(x) \;\&\; v = \Lambda(y)];$$

and $\Lambda^{-1}\{f\}$ analogously for f a function in α^*. Let $\mathfrak{Q}'$ be the class of all α^*-recursive functions f such that for some $g \in \mathfrak{Q}$, $\Lambda\{g\}$ is a subfunction of f and for all $x \notin \operatorname{ran}(\Lambda)$, $f(x) \neq 0 \to g(x) \neq 0$. Thus, the zero function in α^* is a member of $\mathfrak{Q}'$. We define $\mathcal{F}^*$ by $\mathcal{F}^*(\varphi) = \tau(\Lambda(\pi_1 0), \Lambda(\pi_2 0))$ if $\varphi \in \mathfrak{Q}'$ and $\varphi = \varphi_\varepsilon \to (x \leq \varepsilon)[\varphi_\varepsilon(x) = 0]$; define $\mathcal{F}^*(\varphi) = \tau(\Lambda(\pi_1 1), \Lambda(\pi_2 1)$ if $\varphi \in \mathfrak{Q}'$ and $\varphi = \varphi_\varepsilon \to (Ex \leq \varepsilon)[\varphi_\varepsilon(x) \neq 0]$. (Here ε is an index of φ_ε in α^*.) This defines the domain $\mathfrak{Q}^* \subseteq \mathfrak{Q}'$ of $\mathcal{F}^*$ as well, and $\mathcal{F}^*$ is an effective operation on $\mathfrak{Q}^*$.

Now suppose there were an α-recursive functional F that agrees with $\mathcal{F}$ on $\mathfrak{Q}$. Let F be defined by the α-enumeration operator Φ_ε. Thus, we may assume that for all $\langle \delta, \eta\rangle \in W_\varepsilon$, K_δ is a singleton or $K_\delta = \emptyset$. We define W_{ε^*} in α^* by

$$\langle \delta^*, \eta^*\rangle \in W_{\varepsilon^*} \leftrightarrow (E\delta, \eta)[\langle \delta, \eta\rangle \in W_\varepsilon \;\&\; K_{\delta^*} = \Lambda\{K_\delta\} \;\&\; K_{\eta^*} = \Lambda\{K_\eta\}]$$

(ε^* an index in α^*). Then Φ_{ε^*} is an α^*-enumeration operator (though this is not quite obvious) defining an α^*-recursive functional F^* such that $F^*(\varphi) = \mathcal{F}^*(\varphi)$ for all φ in $\mathfrak{Q}^*$.

For each $\nu < \alpha^*$ we define the α-recursive functions g_ν: $g_\nu(x) = 0$ if $x \neq \nu$, $g_\nu(\nu) \neq 0$. For each $\mu < \alpha^*$, we define the α^*-finite set $S_\mu = \{\varepsilon \leq \mu | \varphi_\varepsilon(\mu)\downarrow\}$, ε here an index in α^*, also we set $f_{\mu,\beta}(x) = 0$ if $x \neq \mu$, $f_{\mu,\beta}(\mu) = \beta \neq 0$ for each $\mu < \alpha^*$, where $\beta \neq \varphi_\varepsilon(\mu)$ for all $\varepsilon \in S_\mu$. For such μ and β, $f_{\mu,\beta}$ is α^*-recursive. Since the zero function in α^*

is a member of $\mathfrak{Q}^*$, we have that there is a μ_0 such that $F^*(f) = 0$ for all f sich that $f(x) = 0$ for all $x < \mu_0$. We may assume that $\mu_0 = \Lambda(\nu_0)$ for some $\nu_0 < \alpha^*$ and as the set $\{\varphi_\varepsilon(\mu_0) | \varepsilon \in S_{\mu_0}\}$ is α^*-finite, we may take $\beta = f_{\mu_0,\beta}(\mu_0)$ so that $\beta \in \operatorname{ran}(\Lambda)$. Consider the function g_{ν_0} such that $g_{\nu_0}(x) = 0$ if $x \neq \nu_0$, $g_{\nu_0}(\nu_0) = \Lambda^{-1}(\beta)$. Then $g_{\nu_0} \in \mathfrak{Q}$ and $f_{\mu_0,\beta} \in \mathfrak{Q}'$. Moreover, if $f_{\mu_0,\beta} = \varphi_\varepsilon$, ε an index in α^*, then $\mu_0 < \varepsilon$. Hence, $f_{\mu_0,\beta} \in \mathfrak{Q}^*$, and $\mathcal{F}^*(f_{\mu_0,\beta}) = \tau(\Lambda(\pi_1 1), \Lambda(\pi_2 1) \neq 0$. But $F^*(f_{\mu_0,\beta}) = 0$. Consequently the α-recursive functional F, assumed to agree with $\mathcal{F}$ on $\mathfrak{Q}$, does not exist.

OBSERVATION 6. – Given the proof of Theorem 9, the proof of the theorem and its corollary for weak α-recursive functionals is trivial.

Unfortunately, at this writing we have not been able to obtain the wholly general form of the KLS theorem for function-valued effective operations and α-recursive operators. (From Theorem 9 the KLS theorem for f.e.o.'s and weak α-recursive operators follows easily). We do obtain a version of some generality, however. Towards proving this version, we first modify the set W_ε and the associated α-enumeration operator Φ_ε defined above in the case of ordinal valued effective operations. For each $v < \alpha$, we define an analogous $W_{\varepsilon(v)}$ and $\Phi_{\varepsilon(v)}$.

Let us notice that for each v, the function $\psi(\delta) = \varphi_{f(\delta)}(v)$ is partial α-recursive and defines an ordinal-valued effective operation $\mathcal{F}_v$ on $\mathfrak{Q}$. Analogous to the partial α-recursive functions $\beta(\delta, \varepsilon, x)$, $t_{\varepsilon,\delta}(x)$, and $g(\varepsilon, \delta)$ defined above, we have the partial α-recursive functions $\beta_v(\varepsilon, \delta, x)$, $t_{v,\varepsilon,\delta}(x)$, and $g(\varepsilon, \delta, v)$. In place of the set C_δ, we have analogously defined set $C_{\delta,v}$. Since all these procedures are uniform in v, there is a binary α-recursive function $h(\delta, v)$ such that $C_{\delta,v} = W_{h(\delta,v)}$. The analogues of properties (*) and (**) hold; for example, for all v, $\varphi_\delta \in \mathfrak{Q} \rightarrow h(\delta, v) \in D$; and for all v, we have

$$(**)_v \qquad [\varphi_{f(\delta)}(v)\downarrow \;\&\; T(h(\delta, v), h(\delta, v), y) \;\&\; \nu \in B \;\&\; (x < y)(\varphi_\delta(x) = \\ = \varphi_\nu(x))] \rightarrow \varphi_{f(\delta)}(v) = \varphi_{f(\nu)}(v)\,.$$

Our definition of the set $W_{\varepsilon(v)}$, which contains redundant information as far as the KLS Theorem for ordinal-valued effective operations is concerned, is tailored to proving our present version of the theorem for function-valued operations. The definition of $W_{\varepsilon(v)}$ involves the use of a function f_v which is obtained from the f given with an f.e.o. much as ψ' was obtained from ψ earlier.

Thus, we let n be an α-recursive function such that for all v and all $\nu \in B$, $n(\nu)$ is an index of the α-finite function $\varphi_{f(\nu)}$ restricted to $v + 1$: $\varphi_{n(\nu)} = \varphi_{f(\nu)} \upharpoonright v + 1$. We next define for each v the partial α-recursive function f_v.

$$f_v(\delta) = f(\delta) \quad \text{if} \quad \delta \in \operatorname{dom}(\varphi_{f(\delta)}(v)) \;\&\; (E\nu(Ey)[\nu \in B \;\&\; T(h(\delta, v), h(\delta, v), y) \;\& \\ \& \; (x < y)[\varphi_\delta(x) = \varphi_\nu(x)]] \;\&\; \delta \notin \operatorname{ran}(d)$$

$f_v(d(\gamma)) = n(\nu)$ for suitable $\nu \in B$, if $\gamma \in \Gamma_v$, where

$$\Gamma_v = \Big\{\gamma | \text{Seq}(\gamma) \ \& \ (E\nu)\big[\nu \in B \ \& \ (u \leq v)(Ey)[T(h(\nu, u), h(\nu, u), y) \ \& \ \& \ (x < \bar{y})[\varphi_\nu(x) = K_\gamma(x)] \ \& \ \text{dom}\,(K_\gamma) \supseteq \bar{y}]\big]\Big\},$$

where $\bar{y} = \sup \{y | T(h(\nu, u), h(\nu, u), y),\ u \leq v,\ \nu \in B\ (\nu \text{ fixed})\}$.
We put $\Gamma = \bigcup \{\Gamma_v\}$. For such v, Γ_v, and hence Γ, is clearly α-re. Since the relation $(u \leq v)(Ey)T(h(\nu, u), h(\nu, u), y)$ is Σ_1, the «suitable ν» in the definition of $f_v(d(\gamma))$ can be precisely determined by a routine use of pairing functions and the enumerating function b of B. If $f_v(\delta)\downarrow$, $f_v(d(\gamma))\downarrow$ and $\varphi_\delta \subseteq \varphi_{d(\gamma)}$ or $\varphi_{d(\gamma)} \subseteq \varphi_{\delta'}$ then, as with ψ' earlier, by use of $(**)$ v, we have that $\varphi_{f_v(\delta)}(v) = \varphi_{f_v d(\gamma)}(v)$. Let us notice that if $u \leq v$, then $\Gamma_v \subseteq \Gamma_u$. Suppose for fixed δ, we have $(u \leq v)[\delta \in \text{dom}\,(\varphi_{f(\delta)}(u))]$ and $(E\nu)\big[\nu \in B \ \& \ (u \leq v)(Ey)[T(h(\nu, u), h(\nu, u), y)] \ \& \ (x < y)[\varphi_\delta(x) = \varphi_\nu(x)]\big]$; then we further note that

$$\gamma \in \Gamma_v \ \& \ (\varphi_\delta \subseteq \varphi_{d(\gamma)} \vee \varphi_{d(\gamma)} \subseteq \varphi_\delta) \rightarrow (u \leq v)[\varphi_{f_v(\delta)}(u) = \varphi_{f_v d(\gamma)}(u)].$$

This follows by use of $(**)$ u, for each $u \leq v$. Subsequently, since all needed consistency properties are preserved, we write simply «f», suppressing the subscript here and in the sequels, to make the notation a bit simpler. We define for each v, the α-re set $W_{\varepsilon(v)}$, where ε is a suitable α-recursive function

$$\langle \varrho, \gamma \rangle \in W_{\varepsilon)v)} \leftrightarrow \gamma \in \Gamma_v \ \& \ (u \leq v)[\varphi_{fd(\gamma)}(u)\downarrow \ \& \ K_\varrho = \{\varphi_{fd(\gamma)}(v)\}] \vee [K_\varrho = K_\gamma = \emptyset].$$

One shows as before that for each v, there is an α-recursive functional F_v such that for all $\varphi_\delta \in \mathfrak{Q}$, $\mathcal{F}_v(\varphi_\delta) = F_v(\varphi_\delta)$.

Consider an α-re set W_ε and a mapping Φ_ε from 2^α into 2^α as defined earlier; that is

$$\Phi_\varepsilon(A) = \bigcup \{K_\delta | (E\eta)[\langle \delta, \eta \rangle \in W_\varepsilon \ \& \ K_\eta \subseteq A\}.$$

Consider the property:

$(***)$
(1) For all sv sets A, $\Phi_\varepsilon(A)$ is sv: $V_\varepsilon = \Pi$.

(2) For all sv sets A such that dom $(\Phi_\varepsilon(A))$ has a largest member or dom $(\Phi_\varepsilon(A))$ is unbounded, for all θ, $K_\theta \subseteq \Phi_\varepsilon(A) \leftrightarrow (E\gamma)[\langle \theta, \gamma \rangle \in W_\varepsilon \ \& \ \& \ K_\gamma \subseteq A]$,

Let F_ε be the functional operator defined by a mapping having the property $(***)$. For convenience (ad only for convenience), we say that F_ε is a limited *α-recursive operator*.

THEOREM 10. – (*A partial* KLS *theorem for function-valued effective operators.*) *Let the class* $\mathcal{Q}$ *of* α*-recursive functions have an* α*-recursively dense base* B, *and let* f *be a partial* α*-recursive function that defines the effective operation* $\mathcal{F}$ *from* $\mathcal{Q}$ *into* $\mathcal{R}$. *Then there is a limited* α*-recursive operator* F *that agrees with* $\mathcal{F}$ *on* $\mathcal{Q}$.

PROOF. – We first define the set $W_{\varepsilon'}$ by $\langle\theta, \gamma\rangle \in W_{\varepsilon'} \leftrightarrow (K_\theta$ is sv & dom (K_θ) contains a biggest element & $(v)[v \in \text{dom}\,(K_\theta) \to (E\varrho)[\langle\varrho, \gamma\rangle \in W_{\varepsilon(v)} \,\&\, K_\theta \subseteq \varphi_{fd(\gamma)}]]) \vee$ $\vee[K_\theta = K_\gamma = \emptyset]$. Next, we define W_ε by

$$\langle\theta, \gamma\rangle \in W_\varepsilon \leftrightarrow (v)[v \in \text{dom}\,(K_\theta) \to (E_{\theta'})(E_{\gamma'})[v \in \text{dom}\,(K_{\theta'}) \,\&$$
$$\& \, K_{\theta'} \subseteq K_\theta \,\&\, K_{\gamma'} \subseteq K_\gamma \,\&\, \langle\theta', \gamma'\rangle \in W_{\varepsilon'}] \,\&\, K_\theta \quad \text{is } sv \,\&\, \text{Seq}\,(\gamma)]\,.$$

W_ε is clearly α-re, and an index ε of W_ε can be obtained via an α-recursive function of indices of f, d, and B. Also, $W_{\varepsilon'} \subseteq W_\varepsilon$. Consider the mapping Φ_ε associated with W_ε.

LEMMA 1. – *If* A *is* sv, *then* $\Phi_\varepsilon(A)$ *is sv.*

PROOF. – Suppose $\langle v, z\rangle \in \Phi_\varepsilon(A)$ and $\langle v, z'\rangle \in \Phi_\varepsilon(A)$. Then for some $\theta_1, \gamma_1, \theta_2, \gamma_2$, $\langle v, z\rangle \in K_{\theta_1} \,\&\, K_{\gamma_1} \subseteq A \,\&\, \langle\theta_1, \gamma_1\rangle \in W_\varepsilon$ and $\langle v, z'\rangle \in K_{\theta_2} \,\&\, K_{\gamma_2} \subseteq A \,\&\, \langle\theta_2, \gamma_2\rangle \in W_\varepsilon$. Hence, for some $\theta_1', \gamma_1', \theta_2', \gamma_2'$, $\langle v, z\rangle \in K_{\theta_1'} \,\&\, K_{\gamma_1'} \subseteq A \,\&\, \langle\theta_1', \gamma_1'\rangle \in W_{\varepsilon'}$ and $\langle v, z'\rangle \in K_{\theta_2'} \,\&\, K_{\gamma_2'} \subseteq$ $\subseteq A \,\&\, \langle\theta_2', \gamma_2'\rangle \in W_{\varepsilon'}$. Hence, there are ϱ_1, ϱ_2 such that $\langle\varrho_1, \gamma_1'\rangle \in W_{\varepsilon(v)} \,\&\, K_{\varrho_1} = \{z\} \,\&$ $\&\, z = \varphi_{fd(\gamma_1')}(v)$, and $\langle\varrho_2, \gamma_2'\rangle \in W_{\varepsilon(v)} \,\&\, K_{\varrho_2} = \{z'\} \,\&\, z' = \varphi_{fd(\gamma_2')}(v)$. Since A is *sv*, and Seq (γ_1') and Seq (γ_2'), we have that $K_{\gamma_1'} \subseteq K_{\gamma_2'}$ or $K_{\gamma_2'} \subseteq K_{\gamma_1'}$. We may assume that $K_{\gamma_1'} \subseteq K_{\gamma_2'}$. Thus, $\varphi_{fd(\gamma_1')}(v) = \varphi_{fd(\gamma_2')}(v)$; that is, $z = z'$. Q.E.D.

LEMMA 2. – *If* A *is* sv *and* dom (K_θ) *has a largest element then* $K_\theta \subseteq \Phi_{\varepsilon'}(A) \leftrightarrow$ $(E\gamma)[\langle\theta, \gamma\rangle \in W_{\varepsilon'} \,\&\, K_\gamma \subseteq A]$.

PROOF. – Let A and K_θ be as in the hypothesis and suppose that $K_\theta \subseteq \Phi_{\varepsilon'}(A)$. Let v be the largest element of dom (K_θ) and $\langle v, z\rangle \in K_\theta$. Thus $(E\theta')(E\gamma')[\langle v, z\rangle \in$ $\in K_{\theta'} \,\&\, \langle\theta', \gamma'\rangle \in W_{\varepsilon'} \,\&\, K_{\gamma'} \subseteq A]$. So, for suitable ϱ_v, γ_v $\langle\varrho_v, \gamma_v\rangle \in W_{\varepsilon(v)} \,\&\, K_{\gamma_v} \subseteq A \,\&$ $\&\, K_{\varrho_v} = \{z\} \,\&\, z = \varphi_{fd(\gamma_v)}(v)$. Consider any $u \leq v$, $u \in \text{dom}\,(K_\theta)$. Then as above for v, if $\langle u, z'\rangle \in K_\theta$, for suitable ϱ_u, γ_u we have that $\langle\varrho_u, \gamma_u\rangle \in W_{\varepsilon(u)} \,\&\, K_{\gamma_u} \subseteq A \,\&\, K_{\varrho_u} =$ $= \{z'\} \,\&\, z' = \varphi_{fd(\gamma_u)}(u)$. But since $u \leq v$, by definition of $W_{\varepsilon(v)}$, $\varphi_{fd(\gamma_v)}(u)\downarrow$. As A is *sv*, Seq (γ_u), and Seq (γ_v), we have that $K_{\gamma_u} \subseteq K_{\gamma_v}$ or $K_{\gamma_v} \subseteq K_{\gamma_u}$. Of course if $K_{\gamma_u} \subseteq K_{\gamma_v}$, then $\varphi_{fd(\gamma_u)}(u) = \varphi_{fd(\gamma_v)}(u)$. Assume $K_{\gamma_v} \subset K_{\gamma_u}$. Then again $\varphi_{fd(\gamma_v)}(u) = \varphi_{fd(\gamma_u)}(u)$, as $\varphi_{fd(\gamma_v)}(u)\downarrow$. Noting that for all $\nu \in B$ and all u, $h(\nu, u) \in D$, we thus see that $(u \leq v)[u \in \text{dom}\,(K_\theta) \to$ $\to (E\varrho)[\langle\varrho, \gamma_v\rangle \in W_{\varepsilon(u)}] \,\&\, K_\theta \subseteq \varphi_{fd(\gamma_v)}$. Our conclusion is that if A is sv and $K_\theta \subseteq \Phi_{\varepsilon'}(A)$ and dom (K_θ) has a largest member v, then $\langle\theta, \gamma_v\rangle \in W_{\varepsilon'} \,\&\, K_{\gamma_v} \subseteq A$. Q.E.D.

LEMMA 3. – *If* A *is* sv, *and* dom $(\Phi_\varepsilon(A))$ *has a largest member or* dom $(\Phi_\varepsilon(A))$ *is unbounded, then for all* θ, $K_\theta \subseteq \Phi_\varepsilon(A) \leftrightarrow (E\gamma)[\langle\theta, \gamma\rangle \in W_\varepsilon \,\&\, K_\theta \subseteq A]$.

PROOF. – The direction from right to left is immediate. Now, first of all, let us observe that for *all* sets $A \in 2^\alpha$, $\Phi_\varepsilon(A) = \Phi_{\varepsilon'}(A)$. Let A be sv and dom $(\Phi_\varepsilon(A))$ be as in the hypothesis, and let $K_\theta \subseteq \Phi_\varepsilon(A)$. If dom (K_θ) has a largest member, then by Lemma 2 and the fact that $W_{\varepsilon'} \subset W_\varepsilon$, we deduce that $(E\gamma)[\langle\theta, \gamma\rangle \in W_\varepsilon \,\&\, K_\gamma \subseteq A]$. Suppose dom (K_θ) does not have a largest member. Then since by Lemma 1 K_θ is sv, there is a $v \in \operatorname{dom}(\Phi_\varepsilon(A))$, $v \geq \sup(\operatorname{dom}(K_\theta))$, and $\langle v, \Phi_\varepsilon(A)(v)\rangle \in \Phi_\varepsilon(A)$, $\langle v, \Phi_\varepsilon(A)(v)\rangle \notin K_\theta$. Consider the α-finite set $K_{\theta_1} = K_\theta \cup \{\langle v, \Phi_\varepsilon(A)(v)\rangle\}$. $K_{\theta_1} \subseteq \Phi_\varepsilon(A)$ and dom (K_{θ_1}) has a largest member. Since $\Phi_\varepsilon(A) = \Phi_{\varepsilon'}(A)$, by Lemma 2, $(E\gamma)[\langle\theta_1, \gamma\rangle \in W_{\varepsilon'} \,\&\, K_\gamma \subseteq A]$. For each $u < v$, we define $K_{\theta_1}u = \{\langle w, K_{\theta_1}(w)\rangle | w \leq u\}$. Then $K_{\theta_1}u \subseteq K_\theta$ and by definition of $W_{\varepsilon'}$, $\langle\theta_1^u, \gamma\rangle \in W_{\varepsilon'}$. Therefore,

$$(u)[u \in \operatorname{dom}(K_\theta) \to (E\theta')(E\gamma')[u \in \operatorname{dom}(K_{\theta'}) \,\&\, K_{\theta'} \subseteq K_\theta \,\&\, K_{\gamma'} \subseteq K_\gamma \,\& \\ \& \langle\theta', \gamma'\rangle \in W_{\varepsilon'}] \,\&\, K_\theta \text{ is } sv \,\&\, \operatorname{Seq}(\gamma)] .$$

That is, $\langle\theta, \gamma\rangle \in W_\varepsilon$. So $(E\gamma)[\langle\theta, \gamma\rangle \in W_\varepsilon \,\&\, K_\gamma \subseteq A]$. Q.E.D.

We thus see that the mapping Φ_ε satisfies condition $(***)$. Let F_ε be the limited α-recursive operator defined by Φ_ε. Let $\varphi_\delta \in \mathfrak{Q}$. Let $\langle v, z\rangle \in \Phi_\varepsilon(\varphi_\delta)$. Then $\langle v, z\rangle \in \Phi_{\varepsilon'}(\varphi_\delta)$. By virtue of the density of B and $(**)_v$, $\langle v, z\rangle \in \mathcal{F}(\varphi_\delta) = \varphi_{f(\delta)}$. Now, let $\langle v, z\rangle \in \mathcal{F}(\varphi_\delta)$. Then $(u \leq v)(Ey)T(h(\delta, u), h(\delta, u), y)$. Thus, for suitable $\nu \in B$ and $\bar{y}$, using $(**)_v$, we have that $\langle v, z\rangle \in K_\theta$, $\langle\theta, \gamma\rangle \in W_{\varepsilon'}$, with $K_\theta = \varphi_{f(\delta)}\restriction v + 1 = \varphi_{f(\nu)}\restriction v + 1$ and $K_\gamma = \varphi_\delta\restriction\bar{y} = \varphi_\nu\restriction\bar{y}$. So, $\langle\theta, \gamma\rangle \in W_\varepsilon$, and $\langle v, z\rangle \in \Phi_\varepsilon(\varphi_\delta)$. Q.E.D.

OBSERVATION 7. – One may ask: why not define W_ε in one of the more obvious ways which are at least more direct than the definition we have chosen? For example, let us put

$$\langle\theta, \gamma\rangle \in W_\varepsilon \leftrightarrow \operatorname{Seq}(\gamma) \,\&\, (E\nu)\Big[\nu \in B \,\&\, (v)\Big[v \in \operatorname{dom}(K_\theta) \to \\ \to (Ey)\big[T(h(\nu, v), h(\nu, v), y) \,\&\, (x < y)[\varphi_\nu(x) = K_\gamma(x)] \,\& \\ \& \, y \subseteq \operatorname{dom}(K_\gamma)\big]\Big]\Big] \,\&\, K_\theta \text{ is sv} \,\&\, K_\theta \subseteq \varphi_{fd(\gamma)} .$$

Suppose A is sv and $K_\mu \subseteq \Phi_\varepsilon(A)$. With each $w \in K_\mu$ there is associated a θ^w and γ^w such that $\langle\theta^w, \gamma^w\rangle \in W_\varepsilon \,\&\, w \in K_{\theta^w} \,\&\, K_{\gamma^w} \subseteq A$. Even under the assumption that dom (K_μ) has a largest member v, may we conclude, as above, that $\langle\mu, \gamma_v^w\rangle \in W_\varepsilon$ with $K_{\gamma_v^w} \subseteq A$, $v = \pi_1 w$? If $u \in \operatorname{dom}(K_\mu)$, $u < v$, $w' \in K_{\theta^{w'}}$, and $u = \pi_1 w'$, then certainly $K_{\gamma_u^{w'}} \subseteq K_{\gamma_v^w}$ or $K_{\gamma_v^w} \subset K_{\gamma_u^{w'}}$; but if the latter holds, how are we to guarantee that $\varphi_{fd(\gamma_v^w)}(u)\downarrow$? Failing this, of course $\bigcup_w \{K_{\gamma_w} | w \in K_\mu\} \subseteq A$, but why need this union be α-finite for arbitrary A? These considerations justify our definition of W_ε given in the proof.

OBSERVATION 8. – We may define Φ_ε to be *an α-enumeration operator for the α-re sets* for every sv α-re set A, $\Phi_\varepsilon(A) \leq_{\alpha e} A$. Consider an sv α-re set A and the α-re sets $W_{\varepsilon'}$ and W_ε defined in the proof of Theorem 9. Thus, $\Phi_\varepsilon(A)$ is sv. Given a typical $\langle v, z\rangle \in \Phi_\varepsilon(A)$, $(E\theta_v, \gamma_v)[\langle\theta_v, \gamma_v\rangle \in W_\varepsilon \,\&\, K_{\gamma_v} \subseteq A]$, $\langle v, z\rangle \in K_{\theta_v}$. Since such a K_{γ_v} is an α-finite *sequence*, A is sv and α-re, it follows that either $A = \varphi_\delta$ where φ_δ is α-recursive or $\Phi_\varepsilon(A) = \Phi_\varepsilon(\varphi_\delta \restriction \beta)$ for some $\beta < \alpha$. Let $K_\theta \subseteq \Phi_\varepsilon(A)$. Now, $(v)[v \in \operatorname{dom}(K_\theta) \to (E\theta_v, \gamma_v)[\langle\theta_v, \gamma_v\rangle \in W_{\varepsilon'} \,\&\, K_{\gamma_v} \subseteq A]]$, by definition of W_ε. Let $K = \bigcup\{K_{\gamma_v}\}$. Clearly, we may assume that dom(K) is bounded. Since $K \subseteq A$, K is the restriction of φ_δ to some ordinal λ. Hence, K is α-finite, $K = K_\gamma$. Let $K_{\mu_v} = K_{\theta_v} \cap K_\theta$. Then $\langle\mu_v, \gamma_v\rangle \in W_{\varepsilon'} \,\&\, K_{\gamma_v} \subseteq A$. Therefore, since $\bigcup\{K_{\mu_v}\} = K_\theta$, $\langle\theta, \gamma_v\rangle \in W_{\varepsilon'} \,\&\, K_\gamma \subseteq A$. Obviously, $F_\varepsilon(\varphi_\delta) = \mathcal{F}(\varphi_\delta)$ as before, where F_ε is the operator defined by Φ_ε. This proves the easy and unsurprising theorem

COROLLARY 7. – *If $\mathcal{F}$ is an f.e.o. on R, then there is a functional operator F_ε defined by an α-enumeration operator for the α-re sets such that $F_\varepsilon(\varphi_\delta) = \mathcal{F}(\varphi_\delta)$ for all $\varphi_\delta \in \mathfrak{Q}$.*

THEOREM 11. – *Let $\mathcal{F}$ be a f.e.o. that is total on $\mathcal{R}$: $\mathcal{F}(\varphi_x) = \varphi_{f(x)}$, $\varphi_x \in \mathcal{R}$, $\varphi_{f(x)} \in \mathcal{T}\mathcal{R}$. Then there is a weak α-recursive operator F such that $\mathcal{F}(\varphi_\delta)(x) = F(\varphi_\delta; x)$ for all α-recursive φ_δ and x such that $\mathcal{F}(\varphi_\delta)(x)\downarrow$.*

PROOF. – As when $\alpha = \omega$.

Total effective operations on $\mathcal{R}$ have occurred in the theory of complexity of computation. The principal instance of such an occurrence is the Operator Gap Theorem. To render our description intelligible, we need several definitions.

An *α-complexity measure Γ* is an enumeration $\{\Gamma_\varepsilon | \varepsilon < \alpha^*\}$ of the α-step counting functions Γ_ε associated with the partial α-recursive function φ_ε in a standard enumeration $\{\varphi_\varepsilon | \varepsilon < \alpha^*\}$ of the partial α-recursive functions such that

(1) for each ε, Γ_ε is a partial α-recursive function and $\operatorname{dom}(\Gamma_\varepsilon) = \operatorname{dom}(\varphi_\varepsilon)$; and

(2) for each ε, the graph of Γ_ε is α-recursive.

Let f be α-recursive. The computational complexity class for f relative to Γ is the class $C_f = \{\varphi_\varepsilon | \varphi_\varepsilon$ is α-recursive $\& \,\Gamma_\varepsilon(\beta) \le f(\beta)$ for all but an α-finite set of $\beta\}$. For $\alpha = \omega$, the gap theorem of Borodin answers in the negative the question as to whether the bound f on complexity classes C_f can always be increased in a uniform effective manner so that enlarged complexity classes result. However, the proofs of the naming theorem of McCreight and Meyer and the compression theorem of Blum define a mapping $\mathcal{F}$ from the class $\mathcal{R}$ into $\mathcal{R}$ such that for all $f \in \mathcal{R}$, $C_f \subset C_{\mathcal{F}(f)}$. The question thus arises: can $\mathcal{F}$ be a total effective operation on $\mathcal{R}$? Constable answered this question in the negative in [1], with his operator gap theorem. JACOBS, who lifted many of the elementary notions and results of computational complexity to level α, including the gap, naming, and compression theorems, asked in [7] whether there were admissible α, measures Γ, and total effective operations $\mathcal{F}$ such that $C_f \subset C_{\mathcal{F}(f)}$; that is, whether for suitable α, Constables result does not lift. This was also answered in the negative by DI PAOLA [3] and by YANG DONG-PING [18], who independently proved the

THEOREM 12. – (α-*Operator Gap Theorem.*) *For all* Γ *and all total effective operations* $\mathcal{F}$ *there are arbitrarily large increasing* α-*recursive functions* b *such that for all* ε *if* $b(\beta) \le \Gamma_\varepsilon(\beta) \le F(b;\beta)$ *for* β *without bound, then* $F(b;\gamma) < \Gamma_\varepsilon(\gamma)$ *for* γ *without bound. Thus, there is no* α-*recursive* φ_ε *in* $C_{F(b)} - C_b$. *Here* F *is a weak* α-*recursive operator that agrees with* $\mathcal{F}$ *on* $\mathfrak{R}$.

PROOF. – The proof proceeds along the main lines of Constable's, but several non-trivial departures are required (cf. [3], pp. 124, 128, footnotes 2 and 4). We refer the reader to [3] or [18]. Also, at the time of the writing of [3], our present development of operators and functionals was not yet formulated. This explains any differences in the statement of the theorem in [3] and above.

7. – Limit functionals.

In this section we use the notation «$[f]$» to denote the canonical index of an α-finite function f. A functional F is *total on* $\mathfrak{R}$ if $\mathfrak{R} \subseteq \mathrm{dom}\,(F)$.

Let F be a functional that is total on $\mathfrak{R}$. F is a *limit functional* if there is a partial α-recursive function φ such that

(1) $\varphi([f])$ is defined for all α-finite sequences f;

(2) $\lim\limits_{\beta\to\alpha} \varphi([f\restriction\beta])$ exists and equals $F(f)$ for all α-recursive functions f.

A functional F total on $\mathfrak{R}$ is a *Banach-Mazur functional* if, for every binary α-recursive function f, there is an α-recursive g such that for all β, $F(\lambda\gamma f(\beta,\gamma)) = g(\beta)$.

Our purpose in this section, beyond setting down the definitions, is to record some theorems about limit functionals that are true for arbitrary α, and to discuss the proof at level α of Friedberg's theorem about Banach-Mazur functionals. This latter result has already been treated by us in another journal [4], but we shall discuss it in outline fashion here because it furnishes an example, about functionals to boot, of the special difficulties one encounters in lifting a really difficult theorem to level α. Among other things, its proof exemplifies use of the projectum α^* and Σ_2-cofinality λ in a non-trivial context.

Among the theorems establishing the more basic relationships that hold among the classes of α-recursive, limit, and Banach-Mazur functionals are theorems XXXIII, XXXIV, and XXXV, § 15.3, pp. 364-365 of ROGERS' text [13]. They remain true for all α, and their proofs are virtual carbon-copies of the ω-proof. We therefore collect them under one heading:

THEOREM FOR ALL α. – (1) *the restriction to* $\mathfrak{R}$ *of any* α-*recursive functional total on* $\mathfrak{R}$ *is a Banach-Mazur functional*; (2) *every Banach-Mazur functional is a limit functional*; (3) *there is a limit functional which is not a Banach-Mazur functional.*

In [10], footnote 4, KREISEL, LACOMBE, and SHOENFIELD had asked whether there is a Banach-Mazur functional that is the restriction to $\mathfrak{R}$ of no recursive functional. FRIEDBERG in [6] demonstrated the existence of such a BM (Banach-Mazur) functional. In broad outline, he accomplishes this by showing that the two classes of functionals in question have inequivalent defining predicates. That is, let us define the sets $A_1 = \{u|\varphi_u$ defines a limit functional which is BM$\}$ and $A_2 = \{u|\varphi_u$ defines a limit functional which coincides on $\mathfrak{R}$ with a recursive functional whose domain includes $\mathfrak{R}\}$. Friedberg proves that A_2 is Σ_4-complete, in fact, many-one complete for Σ_4-sets, and that A_1 is a Π_4-set.

The proof that A_2 is Σ_4-complete involves a priority argument. Also, the quantificational structure of Σ_4-predicates enters into the design of the construction in an interesting way. Given an arbitrary Σ_4-predicate P, Friedberg works with a derived Σ_4-predicate P'. The priority scheme is combined with the quantificational structure of P' in such a way as to take advantage of properties peculiar to P'. Implicitly defined by the entire construction is a recursive function r that reduces P to the defining predicate of A_2.

Let us examine Friedberg's construction and argument more closely. Consider the predicates: (1) *φ_u defines a limit functional which coincides on $\mathfrak{R}$ with a recursive functional that is total on $\mathfrak{R}$*; (2) *φ_u defines a limit functional which is Banach-Mazur.* (1) can be expressed as a Σ_4-predicate, (2) as a Π_4-predicate. Now, consider a recursive predicate R such that (2) $\leftrightarrow (x)(Ey)(z)(Ew)R(u, x, y, z, w)$. Since Friedberg proves that (1) is many-one complete for the class of Σ_4-predicates, we obtain from R a recursive function r such that

$$(Ex)(y)(Ez)(w)\sim R(u, x, y, z, w) \leftrightarrow (Ex)(y)(Ez)(w)S\big(r(u), x, y, z, w\big)\,,$$

where S is a recursive predicate such that

(1) $\leftrightarrow (Ex)(y)(Ez)(w)S(u, x, y, z, w)$. By the recursion theorem there is a number u_0 such that $\varphi_{u_0} = \varphi_{r(u_0)}$. Consequently, ($\varphi_{u_0}$ defines a limit functional that coincides on $\mathfrak{R}$ with a recursive functional that is total on $\mathfrak{R}$) $\leftarrow\!/\!\rightarrow$ (φ_{u_0} defines a limit functional that is BM). The left side of this inequivalence implies the right, as it is not hard to see. Hence the left side is false and the right side is true. (We note in passing that one could obtain the crucial inequivalence without use of the recursion theorem, using the hierarchy theorem instead.)

In establishing that the predicate (1) above is many one complete for the class of Σ_4-predicates, Friedberg first trades in the recursive predicate R for a recursive M defined by

$$M(u, e, s, n, a) \leftrightarrow (x < e)(Ey < s)(z < n)(Ew < a)R(u, x, y, z, u)$$

and observes that

$$(x)(Ey)(z)(Ew)R(u, x, y, z, w) \leftrightarrow (e)(Es)(n)(Ea)M(u, e, s, n, a).$$

The recursive function r that many-one reduces the negation of $(e)(Es)(n)(Ea)M(u, e, s, n, a)$, and hence $\sim (x)(Ey)(z)(Ew)R(u, x, y, z, w)$ to the predicate (1) is defined by specifying for each pair $\langle e, s\rangle$ of natural numbers the value of $\varphi_{r(u)}$ on all finite extensions of $\langle e, s\rangle$. This specification is carried out in an infinite sequence of stages with the help of a non-decreasing recursive function $t_{e,s}$. The general scheme is that just prior to execution of stage a of the construction, $\varphi_{r(u)}$ has been assigned the value 0 on all finite extensions of $\langle e, s, t\rangle$, $t > t_{e,s}(a-1)$, provided that $t_{e,s}(a-1) = t_{e,s}(a')$ for all $a' > a-1$. In addition, at stage a the attempt is continued to define a recursive function $f_{e,s,t}$ with $t = t_{e,s}(a-1)$ such that $f_{e,s,t}(0) = e$, $f_{e,s,t}(1) = s$, $f_{e,s,t}(2) = t$. The construction is designed so that if s is the least number satisfying $(n)(Ea)M(u, e, s, n, a)$, then for some t, $f_{e,s,t}$ is totally defined, and hence a recursive function, and F_e, the recursive functional with index e, is either undefined at $f_{e,s,t}$ or differs at $f_{e,s,t}$ from the limit functional $\varphi^*_{r(u)}$ defined by $\varphi_{r(u)}$. To effect this design, Friedberg divides each step $a > 0$ of the construction into five cases, to be considered in order. If one of these, Case 3, applies at stage a, requiring $M(u, e, s-1, t, a)$, $t = t_{e,s}(a-1)$, then one lets $t_{e,s}(a) = a$, defines $f_{e,s,a}$ at 0, 1, and 2, and abandons the further specification of the function $f_{e,s,t}$ $t = t_{e,s}(a-1)$. To understand why, let us notice that for all s_1, s_2 if $s_1 < s_2$, then $M(u, e, s_1, t, a) \rightarrow M(u, e, s_2, t, a)$ Thus, the fact that $M(e, u, s-1, t, a)$ holds presents us with the possibility that for some $s' < s$, $(n)(Ea)M(u, e, s', n, a)$; in this situation precedence is given to the pair $\langle e, s-1\rangle$ over the pair $\langle e, s\rangle$, and thus to the definition of $f_{e,s-1,t}$ over $f_{e,s,t}$. For the pair $\langle e, s\rangle$ at the next stage, the attempt commences to define suitably the function $f_{e,s,a}$.

Assume that $(e)(Es)(n)(Ea)M(u, e, s, n, a)$. We wish to conclude that $\varphi^*_{r(u)}$ differs from every recursive functional. Consider such a functional F_e. Take the least s such that $(n)(Ea)M(u, e, s, n, a)$. Our aims could be frustrated if for this pair $\langle e, s\rangle$, Case 3 were to occur infinitely often. But, for this pair, we have $(Et)(a)\sim M(u, e, s-1, t, a)$ but $(n)(Ea)M(u, e, s, n, a)$. Hence, by monotonicity properties of M and the growth of t as Case 3 occurs, Case 3 can occur but a finite number of times for $\langle e, s\rangle$. Other consequences follow which show that for suitable t, $F_e(f_{e,s,t}) \neq \varphi^*_{r(u)}(f_{e,s,t})$.

If, on the other hand, $(Ee)(s)(En)(a)\sim M(u, e, s, n, a)$, let $e_0 = \min_e (s)(En)(a) \sim M(u, e, s, n, a)$. For each $e < e_0$, let $s(e) = \min_s (n)(Ea)M(u, e, s, n, a)$. The set $\{\langle e, s(e)\rangle | e < e_0\}$ is finite, and using this and some additional facts, Friedberg defines a recursive functional F that agrees with $\varphi^*_{r(u)}$ on every recursive function and that is total on $\mathfrak{R}$. Thus, he proves that the predicate (1) above is complete for the class of Σ_4-predicates with respect to many-one reducibility.

In [4], we proved the following lift of Friedberg's theorem

THEOREM 13. – *If $\lambda < \alpha^*$; then there is a Banach-Mazur functional that coincides with no (weak) α-recursive functional that is total on $\mathfrak{R}$; if $\alpha = \omega_1^{CK}$, there is a Banach-Mazur metafunctional that coincides with no (weak) metarecursive functional that is total on $\mathfrak{R}$.*

DISCUSSION OF PROOF. – First we remark that the construction in [4] also applies for $\alpha = \omega$. Assume $\alpha > \omega_1^{CK}$. Let us suppose that in lifting Friedberg's construction and argument we have made the modifications and changes necessary to overcome stages of the construction defined by limit ordinals. Also, we are working with the predicate $(\varepsilon < \alpha^*)(E\sigma < \alpha^*)(n)(E\beta)M(u, \varepsilon, \sigma, n, \beta)$. That we here need consider only ε and σ less than α^* follows easily from the fact that, as in [4], we have replaced the predicate (1) discussed above with (1′): if φ_u *defines a limit functional, then* φ_u *agrees on* $\mathfrak{R}$ *with an* α*-recursive functional that is total on* $\mathfrak{R}$, a replacement that serves as a convenience in several respects; (1′) is shown in [4] to be a Σ_4-predicate that is many-one complete for all Σ_4-predicates.

Suppose that one of the conditions defining an occurrence of Case 3 relative to the pair ε, σ is now $(E\sigma' < \sigma)M(u, \varepsilon, \sigma', t, \beta)$. Assume $(\varepsilon < \alpha^*)(E\sigma < \alpha^*)(n)(E\beta)M(u, \varepsilon, \sigma, n, \beta)$. Let F_ε be a weak α-recursive functional with index $\varepsilon < \alpha^*$. If we now take the least $\sigma < \sigma^*$ such that $(n)(E\beta)M(u, \varepsilon, \sigma, n, \beta)$, then for all $\sigma' < \sigma(Et)(\beta) \sim \sim M(u, \varepsilon, \sigma', t, \beta)$. But is the function $t(\sigma') = \min_t (\beta) \sim M(u, \varepsilon, \sigma', t, \beta)$ bounded? How are we to guarantee that Case 3 occurs at only an α-finite number of stages β? If this set of stages is not α-finite, the construction collapses. We overcome this obstacle by a suitable partition of α^* into λ pieces or blocks, adapting a technique used originally by SHORE [16], as follows. Let A be a $1-1$ α-recursive function from α into α^*. Let Γ be a Σ_2-function from λ with range unbounded in α^*. For each $\nu < \lambda$, we define $\Delta(\nu) = \bigcup \{\Gamma(\mu) | \mu < \nu\}$; Δ is Σ_2. Observe that $0 < \Delta(0) \leq \ldots \leq \leq \Delta(\nu) \leq \Delta(\nu + 1) \leq \ldots < \alpha^*$; and for each $\varepsilon < \alpha^*$, $\Delta(\nu) \leq \varepsilon < \Delta(\nu + 1)$ for an unique $\nu < \lambda$. Using the fact that Γ is Σ_2, we let H be an α-recursive function from $\alpha < \lambda$ into α^* such that $\Gamma(\nu) = \lim_\beta H(\beta, \nu)$ for all $\nu < \lambda$. Define $\Pi(\beta, \nu) = \bigcup \{H(\beta, \mu) | \mu < < \nu\}$. Then $\Pi(\beta, \nu)$ is α-recursive and $\Delta(\nu) = \lim_\beta \Pi(\beta, \nu)$. Also,

$$(\nu < \lambda)(E\beta)(\beta' \geq \beta)(\mu \leq \nu)[\Pi(\beta', \mu) = \Delta(\mu)], \quad \text{and for each } \beta < \alpha,$$

$$0 = \Pi(\beta, 0) \leq \ldots \leq \Pi(\beta, \nu) \leq \Pi(\beta, \nu + 1) \leq \ldots < \alpha^*.$$

Among the defining conditions for an occurrence of Case 3 we now have $(E\mu)(E\sigma')[\mu < < \nu \;\&\; \sigma' \leq \Pi(\beta, \mu) \;\&\; M(u, \varepsilon, \sigma', t, \beta)]$. Now, assume as before that $(\varepsilon < \alpha^*)(E\sigma < < \alpha^*)(n)(E\beta)M(u, \varepsilon, \sigma, n, \beta)$ and F_ε^w is a weak α-recursive functional. Again take the least $\sigma < \alpha^*$ such that $(n)M(E\beta)(u, \varepsilon, \sigma, n, \beta)$. Then for all $\sigma' < \sigma$, $(Et)(\beta) \sim M(u, \varepsilon, \sigma', t, \beta)$. Let ν_0 be the least $\nu < \lambda$ such that $\sigma \leq \Delta(\nu)$. Let β_0 be so big that for all $\beta \geq \beta_0$ and all $\nu \leq \nu_0$, $\Pi(\beta, \nu) = \Delta(\nu)$, and note that Π is α-recursive. Consequently, for all $\nu < \nu_0$, $\beta \geq \beta_0$ and all σ' such that $\sigma' \leq \Pi(\beta, \nu)$, we have $(Et)(\beta) \sim M(u, \varepsilon, \sigma', t, \beta)$. Next, define $t(\nu) = \min_t [(\beta) \sim M(u, \varepsilon, \Pi(\beta_0, \nu), t, \beta)]$, where $\nu < \nu_0$, and hence $\Pi(\beta_0, \nu) < \sigma$. Then t is a Σ_2-function from ν_0 into α, and by definition of λ, t is bounded in α, say by $\bar{t}$. From this it follows that there is a stage β beyond which Case 3 does not occur for the pair $\langle \varepsilon, \nu_0 \rangle$. Hence, Case 3 occurs only α-finitely often. One is now able to conclude that for suitable t, $f_{\varepsilon, \nu_0, t}$ is completely defined and either

$F^w_\varepsilon(f_{\varepsilon,\nu_0,t})\uparrow$ or $F^w_\varepsilon(f_{\varepsilon,\nu_0 t}) \neq \overset{*}{\varphi}_{r(u)}(f_{\varepsilon,\nu_0,t})$. As the reader perhaps has observed, we are no longer so interested in all pairs $\langle\varepsilon, \sigma\rangle$ but especially in pairs $\langle\varepsilon, \nu\rangle$, $\varepsilon < \alpha^*$, $\nu < \lambda$.

The function Δ partitions α^* into λ intervals or blocks, each of size less than α^*. In our case, the intervals of particular interest are the initial segments of α^* defined by each $\Delta(\nu)$, $\nu < \lambda$. The priority ordering has been shifted to the ν's less than λ from the σ's less than α^*, all σ's that fall in the interval $\Delta(\nu)$ $(\sigma \leq \Delta(\nu))$ being treated on a par. More specifically, the construction is now designed with the following objective: if ν_0 is the least $\nu < \lambda$ for which there is a $\sigma < \alpha^*$ such that $(n)(E\beta)M(u, \varepsilon, \sigma, n, \beta)$, then for some t, $f_{\varepsilon,\nu_0,t}$ is an α-recursive function and F^w_ε, the weak α-recursive functional with index ε, is either undefined at $f_{\varepsilon,\nu_0,t}$ or $\overset{*}{\varphi}_{r(u)}(f_{\varepsilon,\nu_0,t}) \neq$ $\neq F^w_\varepsilon(f_{\varepsilon,\nu_0,t})$. Of course, we actually work with a suitable α-recursive approximation Π to Δ, which exists by the admissibility of α. Accordingly, the pertinent defining condition of Case 3 now reads

$$(E\mu)(E\sigma')[\mu < \nu \;\&\; \sigma' \leq \Pi(\beta, \mu) \;\&\; M(u, \varepsilon, \sigma', t, \beta)].$$

If this condition holds, there is the possibility that for some $\mu < \nu$ there is a $\sigma' \leq \Delta(\mu)$ such that $(n)(E\beta)M(u, \varepsilon, \sigma', t, \beta)$. Having received this signal, we give precedence to the pair $\langle\varepsilon, \mu\rangle$ over the pair $\langle\varepsilon, \nu\rangle$, and abandon the attempt to further define the function $f_{\varepsilon,\nu,t}$. That Case 3 can occur but α-finitely often follows from the fact the function $t(\nu)$ defined above is a Σ_2-function from $\nu_0 < \lambda$ into α, and hence is bounded in α.

Now, suppose $(E\varepsilon < \alpha^*)(\sigma < \alpha^*)(En)(\beta) \sim M(u, \varepsilon, \sigma, n, \beta)$. Let $\varepsilon_0 = \min_\varepsilon (\sigma < \alpha^*) \cdot$ $\cdot (En)(\beta) \sim M(u, \varepsilon, \sigma, n, \beta)$. If we now define $\sigma(\varepsilon) = \min_\sigma (n)(E\beta)M(u, \varepsilon, \sigma, n, \beta)$ for each $\varepsilon < \varepsilon_0$, we certainly cannot conclude that the set $\{\langle\varepsilon, \sigma(\varepsilon)\rangle | \varepsilon < \varepsilon_0\}$ is α-finite. But suppose we instead take the set $S = \{\tau(\varepsilon, \nu) | \varepsilon < \varepsilon_0 \;\&\; \nu < \lambda\}$, where τ is a rudimentary pairing function uniform for all admissible ordinals, so that if $\lambda < \alpha^*$, $\tau(\varepsilon, \nu) < \alpha^*$. We then define E to be those members of S satisfying various other α-re conditions natural to the construction, as in [4]. Since by hypothesis $\lambda < \alpha^*$, E is an α-re set bounded below α^*, and hence α-finite. (It is here, and only here, that the special hypothesis $\lambda < \alpha^*$ is used in [4].) One is then able to define a weak α-recursive enumeration operator, and thence a weak α-recursive functional F^w_ζ total on $\mathfrak{R}$ such that $F^w_\zeta(\varphi) = \overset{*}{\varphi}_{r(u)}(\varphi)$ for all $\varphi \in \mathfrak{R}$.

For the metarecursive case of course $\lambda = \alpha^* = \omega$. But in this case, we are able to dispense with use of λ completely. One is able to define the set B mentioned above so that it constitutes a bona fide finite set of finite ordinals. Of course, in all cases deduction of the theorem relative to α-recursive functionals from the version relative to weak α-recursive functionals is a triviality.

In [4], there are a number of other significant departures from the construction as given by Friedberg. For example, in Friedberg's construction at a given stage certain instructions prescribe that the function $\varphi_{r(u)}$ be defined on all finite extensions of some triple $\langle e, s, t\rangle$ while in [4] at a given stage the corresponding (or analogous) instructions require that in extending $\varphi_{r(u)}$ we *refrain* from defining it on some α-finite extensions of a triple $\langle \varepsilon, \nu, t\rangle$. Also, in [4] each of Cases 3, 4, and 5 undergoes some material change. As a result of these changes it becomes less clear than in [6] that the function $\varphi_{r(u)}$, which is to define the limit functional $\varphi^*_{r(u)}$, is defined on all α-finite sequences. Accordingly, a proof of this fact is included in [4].

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