Probability Theory and Related Fields (2021) 179:589–647
https://doi.org/10.1007/s00440-021-01034-8

The bead process for beta ensembles

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Received: 3 April 2019 / Revised: 5 November 2020 / Accepted: 16 February 2021 /
Published online: 13 March 2021

Abstract
The bead process introduced by Boutillier is a countable interlacing of the Sine_2 point processes. We construct the bead process for general Sine_β processes as an infinite dimensional Markov chain whose transition mechanism is explicitly described. We show that this process is the microscopic scaling limit in the bulk of the Hermite β corner process introduced by Gorin and Shkolnikov, generalizing the process of the minors of the Gaussian Unitary and Orthogonal Ensembles. In order to prove our results, we use bounds on the variance of the point counting of the circular and the Gaussian beta ensembles, proven in a companion paper (Najnudel and Virág in Some estimates on the point counting of the Circular and the Gaussian Beta Ensemble, 2019).

Mathematics Subject Classification 60B20 · 60J55 · 60F05 · 60J05

1 Introduction

In Boutillier [6], a remarkable family of point processes on $\mathbb{R} \times \mathbb{Z}$, called *bead processes*, and indexed by a parameter $\gamma \in (-1, 1)$, has been defined. They enjoy the following properties:

Interlacing	The points of two consecutive lines interlace with each other.
Invariance	The distribution of the point process is invariant and ergodic under the natural action of $\mathbb{R} \times \mathbb{Z}$ by translation.
Parameters	The expected number of points in any interval is proportional to its length. Given that $(0, 0)$ is in the process, the expected value of the first positive point on line 1 is proportional to $\arccos \gamma$.

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Gibbs property The distribution of any point X, given the other points, is uniform on the interval which is allowed by the interlacing property.

It is an open problem whether these properties determine the point process uniquely. Such uniqueness results exist for tilings, see Sheffield [17].

Existence was shown by Boutillier, who considers a determinantal process with an explicit kernel. Its restriction to a line is the standard sine-kernel process. Thus the above description proposes to be the purest probabilistic definition of the Gaudin–Mehta sine kernel process limit of the bulk eigenvalues of the Gaussian Unitary Ensemble (GUE).

Boutillier's result relies on taking limits of tilings on the torus. Since then, works starting with Johansson and Nordenstam [11] showed that the consecutive minor eigenvalues of the Gaussian Unitary Ensemble also converge to the bead process, where the tilt depends on the global location within the Wigner semicircle. These results have been refined and generalized in Adler et al. [1]. However, the corresponding questions remained open for other matrix ensembles, as the Gaussian Orthogonal Ensemble (GOE), and the Gaussian Symplectic Ensemble (GSE):

- Is there a limit of the eigenvalue minor process?
- Is there a simple characterization as for $\beta = 2$?
- Can one derive formulas related to the distribution of beads?

One of the main goals of this paper is to answer these questions positively. The limiting process is defined as an infinite-dimensional Markov chain, the transition from one line to the next being explicitly described. This transition can be viewed as a generalization of the limit, when the dimension n goes to infinity, of the random reflection walk on the unitary group $U(n)$. This walk is the unitary analogue of the random transposition walk studied, for example, in Diaconis and Shahshahani [7], Berestycki and Durrett [3] and Bormashenko [5].

The natural generalization of the transpositions to the setting of the orthogonal group corresponds to the reflections. The orthogonal matrix corresponding to the reflection across the plane with normal unit vector v is $I - 2vv^*$. To further generalize to the unitary group, we proceed as follows: given a fixed unit complex number η and a unit vector v, we define the *complex reflection* across v with angle $\arg(\eta)$ as the isometry whose matrix is given by $I + (\eta - 1)vv^*$. The random reflection walk $(Y_k)_{k\geq 1}$ on the unitary group $U(n)$ is then defined by $Y_k = X_1 \dots X_k$, where $(X_j)_{j\geq 1}$ are independent reflections for which v is chosen according to uniform measure on the complex unit sphere, and η is fixed.

Note that since the multiplicative increments of the walk are invariant under conjugation by any group element, it follows that $\bar{Y}_k$, the conjugacy class of Y_k, also follows a random walk. This, of course, is given by the eigenvalues of Y_k; the transition mechanism can be computed as follows. Assuming that the eigenvalues u_j of $\bar{Y}_k$ are distinct, the eigenvalues of Y_{k+1} are the solutions of

$$\sum_{j=0}^{n-1} i\frac{u_j + z}{u_j - z}\rho_j = i\frac{1+\eta}{1-\eta} \tag{1}$$

where for $|z| = 1$ the summands and the right-hand side are both real. The only randomness is contained in the values ρ_j, which have a Dirichlet joint distribution with all parameters equal to 1. To summarize, in order to get the evolution of $(\bar{Y}_k)_{k\geq 1}$, we pick $(\rho_j)_{1\leq j\leq n}$ from Dirichlet distribution, form the rational function given by the left-hand side of (1), and look at a particular level set to get the new eigenvalues.

This equation can be lifted to the real line. Let $(\lambda_j)_{j\in\mathbb{Z}}$ be the $(2\pi n)$-periodic set of $\lambda \in \mathbb{R}$ such that $e^{i\lambda/n} \in \{u_1, \ldots, u_n\}$, and extend the sequence $(\rho_j)_{1\leq j\leq n}$ periodically with period n to all integer indices. With $z = e^{ix/n}$, the left-hand side of (1) can be written as

$$\lim_{\ell\to\infty} \sum_{j=-\ell}^{\ell} \frac{2n\rho_j}{\lambda_j - x}.$$

and the level set of this at $i(1+\eta)/(1-\eta)$ gives the lifting of the eigenvalues at the next step. Recall that η is a complex number of modulus 1, related to the angles of the complex reflections involved in the definition of the walk $(Y_k)_{k\geq 1}$. Notice now that essentially the only role of n in the above process is given by the joint distribution of the ρ-s. These are n-periodic and Dirichlet; clearly, as $n \to \infty$ they converge, after suitable renormalization, to independent exponential variables, giving naturally an infinite-dimensional Markov chain.

In the present article, we prove the existence of this Markov chain and deduce a new construction of the bead process. By replacing the exponential variables by gamma variables with general parameter, we construct a natural generalization of the bead process, indexed by a parameter $\beta > 0$. For $\beta = 2$, this process is the bead process itself, and then it is the limit of the eigenvalues of the GUE minors when the dimension goes to infinity. For $\beta = 1$, we show that we get the limit of the eigenvalues of the GOE minors, for $\beta = 4$, we get the limit of the eigenvalues of the GSE minors, and we generalize this result to all $\beta > 0$, by considering the Hermite β corners, defined by Gorin and Shkolnikov [10], which can be informally viewed as the "eigenvalues of GβE minors".

The sequel of the present paper is organized as follows.

In Sect. 2, we give the statement of the most important results of the article, and we refer to later propositions and theorems for the proofs. Our main results involve technicalities which are also explained in the sequel of the article.

In Sect. 3, we detail the above discussion on the random reflection walk, and we deduce a property of invariance for the law of the spectrum of a Haar-distributed unitary matrix, for the transition given by the Eq. (1). We generalize this property to circular beta ensembles for any $\beta > 0$.

In Sect. 4, we generalize the notion of Stieltjes transform to a class of infinite point measures on the real line for which the series given by the usual definition is not absolutely convergent.

In Sect. 5, we construct a family of Markov chains on a space of point measures, for which the transition mechanism is obtained by taking a level set of the Stieltjes transform defined in Sect. 4.

In Sect. 6, we show how the lifting of the unit circle on the real line defined above connects the results of Sect. 3 to those of Sect. 5.

In Sect. 7, we use some bound on the variance of the number of points of the circular beta ensembles in an arc, in order to take the limit of the results in Sect. 6, when the period of the point measure goes to infinity. We show a property of invariance enjoyed by the determinantal sine-kernel process and its generalizations for all $\beta > 0$, for the Markov chain defined in Sect. 5. From this Markov chain, we deduce the construction of a stationary point process on $\mathbb{R} \times \mathbb{Z}$, for which the points of a given line follow the distribution of the Sine_β process introduced in Valkó and Virág [19].

In Sect. 8, we show, under some technical conditions, a property of continuity of the Markov chain with respect to the initial point measure and the weights.

From this result, and from a bound, proven in a companion paper [15] on the variance of the number of points of the Gaussian beta ensemble in intervals, we deduce in Sect. 9 that the generalized bead process constructed in Sect. 7 appears as a limit for the eigenvalues of the minors of Gaussian Ensembles for $\beta \in \{1, 2, 4\}$. The case $\beta = 2$ corresponds to the GUE, for which the convergence to the bead process defined by Boutillier [6] is already known from Adler et al. [1]. Combining our result with [1] then implies that our Markov chain has necessarily the same distribution as the bead process given in [6]. The case $\beta = 1$ gives the convergence of the renormalized eigenvalues of the GOE minors, and the case $\beta = 4$ gives the convergence of the renormalized eigenvalues of the GSE minors. For other values of β, we get a similar result of convergence for the renormalized points of the Hermite β corner defined in [10].

2 Statement of the main results

Our main result generalizes the bead process to any $\beta > 0$. We need the following definitions.

- Let $\mathcal{L}$ be the family of all the discrete subsets L of $\mathbb{R}$, unbounded from above and from below, and such that for $x \to \infty$, $\mathrm{Card}(L \cap [0, x]) = O(x)$, and for fixed $a, b \in \mathbb{R}$, $x \to \infty$,

$$\mathrm{Card}(L \cap [0, x + a]) - \mathrm{Card}(L \cap [-x + b, 0]) = O(x/\log^2 x).$$

  We endow the space $\mathcal{L}$ with the σ-algebra generated by the maps $L \mapsto \mathrm{Card}(L \cap B)$ for all Borel sets $B \subset \mathbb{R}$. We will use $(\lambda_j)_{j\in\mathbb{Z}}$ as the unique increasing labeling of L so that $\lambda_{-1} < 0 \leq \lambda_0$.
- Let Γ be the family of doubly infinite sequences $(\gamma_j)_{j\in\mathbb{Z}}$ satisfying the following assumptions: for k going to infinity,

$$\sum_{j=0}^{k} \gamma_j = ck + O(k/\log^2 k) \quad \text{and} \quad \sum_{j=0}^{k} \gamma_{-j} = ck + O(k/\log^2 k),$$

  where $c > 0$ is a constant. We endow Γ with the σ-algebra generated by the coordinate maps γ_j, $j \in \mathbb{Z}$.

Theorem 1 1. *There exists a map* $\mathcal{D}\colon \mathcal{L} \times \Gamma \times \mathbb{R} \to \mathcal{L}$, *defined by*

$$\mathcal{D}(L, (\gamma_j)_{j\in\mathbb{Z}}, h) = \left\{ z \in \mathbb{R}, \lim_{c\to\infty} \sum_{j\in\mathbb{Z}, \lambda_j \in L\cap[-c,c]} \frac{\gamma_j}{\lambda_j - z} = h \right\}.$$

2. *For any probability measure* Π *on* $\Gamma \times \mathbb{R}$ *and any initial condition* X_0 *in* $\mathcal{L}$, *we can define a Markov chain by as follows: let* G_k *be independent samples from* Π, *and set*

$$X_{k+1} = \mathcal{D}(X_k, G_k), \qquad k \geq 0.$$

3. *Assume that under* Π *the* $((\beta/4)\gamma_j)_{j\in\mathbb{Z}}$ *are independent Gamma random variables of shape parameter* $\beta/2$, *and* h *is a deterministic real number. Let* X_0 *be distributed as the* Sine_β*-process.*
Then X_k *is a stationary Markov chain. The* β-**bead process** *on* $\mathbb{R} \times \mathbb{N}_0$ *with level* h *is defined as the set*

$$\bigcup_{k\geq 0} (X_k \times \{k\}).$$

The bead process on $\mathbb{R} \times \mathbb{Z}$ *is the unique* $\mathbb{Z}$*-shift-invariant extension of the process on* $\mathbb{R} \times \mathbb{N}_0$.

This theorem, which defines the β-bead process, is a consequence of results proven later in the article. The fact that the map $\mathcal{D}$ is well-defined is obtained in Sect. 5, as a consequence of a discussion on the existence and the regularity in z of the limit

$$\lim_{c\to\infty} \sum_{j\in\mathbb{Z}, \lambda_j \in L\cap[-c,c]} \frac{\gamma_j}{\lambda_j - z}$$

which is made in Sect. 4, and which explains the technicalities involved in the definition of $\mathcal{L}$ and Γ. The invariance property of the distribution of the Sine_β process for the Markov chain is proven in Theorem 13. An informal definition of the β-bead process can be given as follows: a β-bead process is a countable family of Sine_β point processes, such that each of them is obtained from the previous one by putting independent Gamma$(\beta/2)$ distributed weights on the points, and by taking a given level set of the Stieltjes transform of the corresponding point measure. Notice that the point measure here is infinite and that the series defining the Stietjes transform is not absolutely convergent, which explains some of the technicalities involved in Theorem 1. Notice that the properties of the Stieltjes transform imply that two consecutive Sine_β point processes involved in a β-bead process interlace with each other.

We prove Theorem 13 by finite approximation. There is general beta version of the eigenvalue evolution of the complex reflection random walk on unitary matrices. It corresponds exactly to taking γ_i to be n-periodic with Dirichlet$(\beta/2, \ldots, \beta/2)$ distribution in Theorem 1. The periodic lifting of the circular beta ensemble points

to the real line is a stationary distribution for the corresponding Markov chain, see Theorem 12. In Theorem 13, we show that as $n \to \infty$, this sequence of Markov chains converges, and we identify the β-bead process as its limit.

The bead process introduced by Boutillier is a determinantal process, with an explicit kernel. In the present article, we do not study the question of the correlations of the β-bead process: we expect that there are no simple general formulas, since the problem of finding explicit formulas for the correlations of the Sine_β process is already unsolved for general $\beta > 0$. It may be possible to find expressions involving Pfaffians for the correlations of the β-bead process for $\beta = 1$ and $\beta = 4$.

Another natural question related to Theorem 1 is the following: is the Sine_β distribution the unique invariant measure for the Markov chain associated to i.i.d. independent $\mathrm{Gamma}(\beta/2)$ weights and independent level h? Strictly speaking, the answer is negative since we can multiply the points of the Sine_β by a non-zero constant and still get an invariant distribution for the Markov chain. If we restrict the discussion to point processes on $\mathbb{R}$ for which the number of points in $[0, x]$ and the number of points in $[-x, 0]$ are equivalent to $x/2\pi$ when x goes to infinity, we do not know if the invariant measure is unique. Symmetrically, one can also ask about the existence of other measures Π on $\Gamma \times \mathbb{R}$ for which the Sine_β distribution is invariant for the Markov chain.

The main property of the β-bead process is that it is the scaling limit of the Hermite β corner process introduced by Gorin and Shkolnikov [10], see also [8, Proposition 4.3.2]. From Definition 1.1 of [10], we have, after taking $t = 2/\beta$:

Definition 2 Let $n \geq 1$ be an integer. A Hermite β corner process with n levels is a random set of reals $(\lambda_j^{(k)})_{1 \leq j \leq k \leq n}$ subject to the interlacing conditions $\lambda_j^{(k)} \leq \lambda_j^{(k-1)} \leq \lambda_{j+1}^{(k)}$ and such that the density of its probability distribution is given by

$$\prod_{i<j} (\lambda_j^{(n)} - \lambda_i^{(n)}) \prod_{j=1}^{n} e^{-\beta \lambda_j^{(n)}/4} \prod_{k=1}^{n-1} \prod_{1 \leq i < j \leq k} (\lambda_j^{(k)} - \lambda_i^{(k)})^{2-\beta} \times \prod_{a=1}^{k} \prod_{b=1}^{k+1} |\lambda_a^{(k)} - \lambda_b^{(k+1)}|^{(\beta/2)-1}.$$

From [8, Proposition 4.3.2] (see Proposition 24) and the discussion above, we deduce that the successive levels $(\lambda^{(k)})_{1 \leq k \leq n}$ of a Hermite β corner process can be constructed as an inhomogeneous Markov chain whose transitions are explicitly written in term of level sets of Stieltjes transforms. Similar Markov chains and representations of eigenvalues of successive minors of random matrices in terms of zeros of meromorphic functions can be found in the literature: for more detail, we refer to articles by Gelfand and Naimark [9], Baryshnikov [2], Neretin [14], Okounkov and Reshetikhin [16].

This constructions implies that one can define a Hermite β corner process $(\lambda_j^{(k)})_{1 \leq j \leq k}$ with infinitely many levels, in such a way that $(\lambda_j^{(k)})_{1 \leq j \leq k \leq n}$ is a Hermite β corner process with n levels for all $n \geq 1$. Moreover, for each $n \geq 1$ the nth level $\lambda^{(n)}$ follows the Gaussian β Ensemble, i.e. its joint density, with respect to the Lebesgue

measure, is proportional to

$$e^{-\beta \sum_{k=1}^{n} \lambda_k^{(n)}/4} \prod_{j<k} |\lambda_j^{(n)} - \lambda_k^{(n)}|^{\beta}.$$

This, up to a change in the normalization, is shown in [10]. As we explain in more detail at the beginning of Sect. 9, the eigenvalues of the successive minors of a $n \times n$ matrix following the Gaussian Orthogonal Ensemble ($\beta = 1$), the Gaussian Unitary Ensemble ($\beta = 2$), or the Gaussian Symplectic Ensemble ($\beta = 4$), with a suitable normalization, have the same law as the successive levels of a Hermite β corner process with n levels. If we take the minors of an infinite matrix, we get a Hermite β corner with infinitely many levels. For this reason, for general $\beta > 0$, the Hermite β corner process can be thought as the "eigenvalues of GβE minors". In the present article, we prove that the β-bead process is the microscopic scaling limit of the Hermite β corner process. From now, if $(\Xi_n)_{n\geq 1}$, Ξ are locally finite measures on a measurable subspace of $\mathbb{R}^p$ for some $p \geq 1$ (e.g. $\mathbb{R}$ or $\mathbb{R} \times \mathbb{N}_0$), we will say that Ξ_n converges to Ξ locally weakly if and only if for all continuous functions from $\mathbb{R}^p$ to $\mathbb{R}$, compactly supported,

$$\int_{\mathbb{R}^p} \Phi \, d\Xi_n \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^p} \Phi \, d\Xi.$$

The precise statement of our result is then the following:

Theorem 3 *Let us fix $E \in (-2, 2)$. Let $(\lambda_j^{(k)})_{1\leq j\leq k}$ be a Hermite β corner process with infinitely many levels. For $n \geq 1$, we consider the point process on $\mathbb{R} \times \mathbb{Z}$ defined as the set*

$$X_n := \left\{ (\lambda_j^{(n+k)} - E\sqrt{n})\sqrt{n(4 - E^2)}, k), \ k \in \mathbb{Z} \cap [-n + 1, \infty) \right\}.$$

Then, the sum of Dirac measures at the points of X_n converges in law to the sum of Dirac measures at the points of a β-bead process on $\mathbb{R} \times \mathbb{Z}$, for the topology of locally weak convergence of measures on $\mathbb{R} \times \mathbb{Z}$, with a level h given by

$$h = -\frac{E}{\sqrt{4 - E^2}}.$$

If we restrict the point processes to $\mathbb{R} \times \mathbb{N}_0$, i.e. we take only points corresponding to $k \geq 0$, this statement corresponds to the first part of Theorem 26, the level k of the point process X_n corresponding to the point process $\Xi_n^{(k)}$. Once the result is proven for point processes on $\mathbb{R} \times \mathbb{N}_0$, a suitable shift of n gives convergence of point processes on $\mathbb{R} \times (\mathbb{Z} \cap [-m, \infty))$ for any fixed $m \in \mathbb{Z}$, and then convergence of point processes on $\mathbb{R} \times \mathbb{Z}$ since the test functions in the locally weak convergence are compactly supported. The second part of Theorem 26 gives the following property of compatibility:

Theorem 4 *For fixed $h \in \mathbb{R}$, the β-bead process on $\mathbb{R} \times \mathbb{Z}$ for $\beta = 2$ and level h has the same law as the bead process in the sense of* [6], *with parameter*

$$\gamma = -\frac{h}{\sqrt{1+h^2}}.$$

3 Random reflection chains on the unitary group

We start with a brief review of how multiplication by complex reflections changes eigenvalues. Let $U \in U(n)$ be a unitary matrix with distinct eigenvalues $u_1, \ldots, u_n$, and let v be a unit vector. Let $a_1, \ldots, a_n$ be the coefficients of v in a basis of unit eigenvectors of U, and let $\rho_j = |a_j|^2$ for $1 \leq j \leq n$: the law of $(\rho_1, \ldots, \rho_n)$ does not depend on the choice of the eigenvector basis and the sum of these numbers is equal to 1.

If $\eta \neq 1$ is a complex number of modulus 1, the complex reflection with angle $\arg \eta$ and vector v corresponds to the unitary matrix $I + (\eta - 1)vv^*$. If we multiply U by this reflection, we get a new matrix whose eigenvalues u satisfy

$$0 = \det\left(U(I + (\eta - 1)vv^*) - u\right),$$

which can be rewritten as

$$0 = \det(U - u) \det\left(I + (\eta - 1)Uvv^*(U - u)^{-1}\right)$$

when u is not an eigenvalue of U. Now, the second argument is I plus a rank-1 matrix, so its determinant equals 1 plus the trace of the rank-1 matrix. Thus the equation above reduces to

$$0 = 1 + (\eta - 1)\mathrm{tr}(Uvv^*(U - u)^{-1}) = 1 + (\eta - 1)v^*((U - u)^{-1}U)v.$$

Expanding U in the basis of its eigenvectors and eigenvalues u_j, we get

$$1 = (1 - \eta)\sum_{j=1}^{n} \rho_j \frac{u_j}{u_j - u}$$

or, after a transformation,

$$\sum_{j=1}^{n} i\rho_j \frac{u_j + u}{u_j - u} = i\frac{1+\eta}{1-\eta}. \tag{2}$$

As u moves counterclockwise on the unit circle, and on each arc between two consecutive poles, the left-hand side of (2) is continuous and strictly increasing from $-\infty$ to ∞. Hence, the matrix $U(I + (\eta - 1)vv^*)$ has exactly one eigenvalue in each arc

between eigenvalues of U: in other words, the eigenvalues of $U(I + (\eta - 1)vv^*)$ strictly interlace between those of U, and are given by the solutions u of the Eq. (2).

Consider the product of the unit sphere in $\mathbb{C}^n$ and $\mathbb{R}$, and a distribution π on this space which is invariant under permutations of the n coordinates of the sphere, and by multiplication of each of these coordinates by complex numbers of modulus one. For such a distribution, we can associate a Markov chain on unitary matrices as follows. Given $U_0, \dots, U_k$, we pick a sample $((a_1, \dots, a_n), h)$ from π independently from the past. Then, U_{k+1} is defined as the product of U_k by the reflection with parameter η so that $h = i\frac{\eta+1}{\eta-1}$, and vector $v = \sum a_j \varphi_j$, where $(\varphi_j)_{1\le j\le n}$ are unit eigenvectors of U_k (from the assumption made on π, the law of v does not depend on the choice of the phases of the eigenvectors $(\varphi_j)_{1\le j\le n}$).

From the discussion above, it is straightforward that if V_k is the spectrum of U_k, then $(V_k)_{k\ge 0}$ forms a Markov process as well; its distribution depends on the coefficients a_j only through $\rho_j = |a_j|^2$. The transition is given as follows: given V_j, $(\rho_j)_{1\le j\le n}$ and h, V_{j+1} is formed by the n solutions of (2).

When a is uniform on the unit complex sphere of $\mathbb{C}^n$, and h is independent of a, then $(\rho_j)_{1\le j\le n}$ has Dirichlet$(1, \dots, 1)$ distribution, and the corresponding reflection is independent of U_k. Thus the Markov chain reduces to a random walk: $U_j = U_0 R_1 \dots R_k$, where the reflections $(R_k)_{k\ge 1}$ are independent.

It is immediate that the Haar measure on $U(n)$ is invariant for this random walk. One deduces that if $(\rho_j)_{1\le j\le n}$ follows a Dirichlet distribution with all parameters equal to 1, if h (and then η) is independent of $(\rho_j)_{1\le j\le n}$, if the points of V_0 follow the distribution of the eigenvalues of the CUE in dimension n, and if $(V_k)_{k\ge 0}$ is the Markov chain described above, then the law of V_k does not depends of k: the CUE distribution is invariant for this Markov chain.

This invariance property can be generalized to other distributions π.

Indeed, as in Simon [18], one can associate to the point measure $\sigma := \sum_{j=1}^n \rho_j \delta_{u_j}$ a so-called *Schur function* f_σ, which is rational, and which can be defined by the equation:

$$\int_{\mathbb{U}} i\frac{v+u}{v-u} d\sigma(v) = i\frac{1+uf_\sigma(u)}{1-uf_\sigma(u)}. \tag{3}$$

Moreover, as explained in [18], by Geronimus theorem, we also have

$$f_\sigma(u) = R_{\alpha_0} \circ M_u \circ R_{\alpha_1} \circ M_u \circ R_{\alpha_2} \circ \cdots \circ R_{\alpha_{n-2}} \circ M_u(\alpha_{n-1}),$$

where M_u denotes the multiplication by u, the $(\alpha_j)_{0\le j\le n-1}$ are the Verblunsky coefficients associated to the orthogonal polynomials with respect to the measure σ, and for all $\alpha \in \mathbb{D}$, R_α is the Möbius transformation given by

$$R_\alpha(z) = \frac{\alpha + z}{1 + \overline{\alpha} z}.$$

By (3), we see that (2) is satisfied if and only if $uf_\sigma(u) = \eta$, or equivalently,

$$M_{\eta^{-1}} \circ M_u \circ R_{\alpha_0} \circ M_u \circ R_{\alpha_1} \circ \cdots \circ M_u(\alpha_{n-1}) = 1. \tag{4}$$

Now, $M_{\eta^{-1}}$ and M_u commute and for $\alpha \in \mathbb{D}$, $M_{\eta^{-1}} \circ R_\alpha = R_{\alpha\eta^{-1}} \circ M_{\eta^{-1}}$. One deduces that (4) is equivalent to

$$M_u \circ R_{\alpha_0\eta^{-1}} \circ M_u \circ R_{\alpha_1\eta^{-1}} \circ \cdots \circ M_u(\alpha_{n-1}\eta^{-1}) = 1,$$

i.e. $uf_\tau(u) = 1$, where τ is the finitely supported probability measure whose Verblunsky coefficients are $(\alpha_0\eta^{-1}, \ldots, \alpha_{n-1}\eta^{-1})$. Now, by the general construction of the Schur functions, the equation $uf_\tau(u) = 1$ is satisfied if and only if u is a point of the support of τ: in other words, this support is the set of solutions of (2). We deduce that if the distribution π and the law of $\{u_1, \ldots, u_n\}$ are chosen in such a way that $(\alpha_0\eta^{-1}, \ldots, \alpha_{n-1}\eta^{-1})$ has the same law as $(\alpha_0, \ldots, \alpha_{n-1})$, then the law of $\{u_1, \ldots, u_n\}$ is invariant for the Markov chain described above. The precise statement is the following:

Proposition 5 *Let π be a probability distribution on the product of the unit sphere of $\mathbb{C}^n$ and $\mathbb{R}$, under which the first component $(a_1, \ldots, a_n)$ is independent of the second $h = i(1+\eta)/(1-\eta)$. We suppose that the law of $(a_1, \ldots, a_n)$ is invariant by permutation of the coordinates, and by their pointwise multiplication by complex numbers of modulus* 1. *Let $\mathbb{P}$ be a probability measure of the sets of n points $\{u_1, \ldots, u_n\}$, such that under the product measure $\mathbb{P} \otimes \pi$, the sequence $(\alpha_0, \ldots, \alpha_{n-1})$ of Verblunsky coefficients associated to the measure*

$$\sigma = \sum_{1 \le j \le n} \rho_j \delta_{u_j} = \sum_{1 \le j \le n} |a_j|^2 \delta_{u_j}.$$

has a law which is invariant by multiplication by complex numbers of modulus 1. *Then, the measure $\mathbb{P}$ is invariant for the Markov chain associated to π: more precisely, under $\mathbb{P} \otimes \pi$, the law of the set of solutions of* (2) *is equal to $\mathbb{P}$.*

It is not obvious to find explicitly some measures $\mathbb{P}$ and π under which the law of the Verblunsky coefficients is invariant by rotation. An important example is obtained by considering the so-called *circular beta ensembles*. These ensembles are constructed as follows: for some parameter $\beta > 0$, one defines a probability measure $\mathbb{P}_{n,\beta}$ on the sets of n points on the unit circle, such that the corresponding n-point correlation function $r_{n,\beta}$ is given, for $z_1, \ldots z_n \in \mathbb{U}$, by

$$r_{n,\beta}(z_1, \ldots, z_n) = C_{n,\beta} \prod_{1 \le j < k \le n} |z_j - z_k|^\beta,$$

where $C_{n,\beta} > 0$ is a normalization constant. Note that, for $\beta = 2$, one obtains the distribution of the spectrum of a random $n \times n$ unitary matrix following the Haar measure. Now, let $\pi_{n,\beta}$ be any distribution on the product of the unit sphere of $\mathbb{C}^n$ and $\mathbb{R}$, such that with the notation above, h is independent of $(\rho_0, \ldots, \rho_{n-1})$, which has a Dirichlet distribution with all parameters equal to $\beta/2$. Then, under $\mathbb{P}_{n,\beta} \otimes \pi_{n,\beta}$, the distribution of the Verblunsky coefficients $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ has been computed in Killip and Nenciu [12]. One obtains the following:

- The coefficients $\alpha_0, \alpha_1, \ldots \alpha_{n-1}$ are independent random variables.

- The coefficient α_{n-1} is uniform on the unit circle.
- For $j \in \{0, 1, \ldots, n-2\}$, the law of α_j has density $(\beta/2)(n-j-1)(1-|\alpha_j|^2)^{(\beta/2)(n-j-1)-1}$ with respect to the uniform probability measure on the unit disc: note that $|\alpha_j|^2$ is then a beta variable of parameters 1 and $\beta(n-j-1)/2$.

Therefore, the law of $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$ is invariant by rotation, and one deduces the following result:

Proposition 6 *The law of the circular beta ensemble is an invariant measure for the Markov chain associated to $\pi_{n,\beta}$. More precisely, under $\mathbb{P}_{n,\beta} \otimes \pi_{n,\beta}$, the set of solutions of (2) follows the distribution $\mathbb{P}_{n,\beta}$.*

In the next sections, we will take a limit when n goes to infinity. For this purpose, we need to consider point processes on the real line instead of the unit circle, and to find an equivalent of the Eq. (2) in this setting.

4 Stieltjes transform for point measures

Let Λ be a σ-finite point measure on $\mathbb{R}$, which can be written as follows:

$$\Lambda = \sum_{\lambda \in L} \gamma_\lambda \delta_\lambda,$$

where L is a discrete subset of the real line, $\gamma_\lambda > 0$ for all $\lambda \in L$, and δ_λ is the Dirac measure at λ. The usual definition of the Stieltjes transform applied to Λ gives, for $z \in \mathbb{C} \backslash \{L\}$:

$$S_\Lambda(z) = \sum_{\lambda \in L} \frac{\gamma_\lambda}{\lambda - z}. \tag{5}$$

If the set L is finite, then $S_\Lambda(z)$ is well-defined as a rational function. If L is infinite and if the right-hand side of (5) is absolutely convergent, then this equation is still meaningful. The following result implies that under some technical assumptions, one can define S_Λ even if (5) does not apply directly:

Theorem 7 *Assume that for all $a, b \in \mathbb{R}$, $\Lambda[0, x+a] - \Lambda[-x+b, 0] = O(x/\log^2 x)$ as $x \to \infty$. Then, for all $z \in \mathbb{C} \backslash \{L\}$, there exists $S_\Lambda(z) \in \mathbb{C}$ such that*

$$\sum_{\lambda \in L \cap [-c,c]} \frac{\gamma_\lambda}{\lambda - z} \underset{c \to \infty}{\longrightarrow} S_\Lambda(z).$$

The function S_Λ defined in this way is meromorphic, with simple poles at the elements of L, and the residue at $\lambda \in L$ is equal to $-\gamma_\lambda$. The derivative of S_Λ is given by

$$S'_\Lambda(z) = \sum_{\lambda \in L} \frac{\gamma_\lambda}{(\lambda - z)^2}, \tag{6}$$

where the convergence of the series is uniform on compact sets of $\mathbb{C}\backslash\{L\}$. *For all pairs* $\{\lambda_1, \lambda_2\}$ *of consecutive points in* L, *with* $\lambda_1 < \lambda_2$, *the function* S_Λ *is a strictly increasing bijection from* (λ_1, λ_2) *to* $\mathbb{R}$. *Moreover, we have the following translation invariance: if* $y \in \mathbb{R}$ *and* Λ *satisfies the conditions above, then so does its translation* $\Lambda + y$, *and one has*

$$S_{\Lambda+y}(z+y) = S_\Lambda(z)$$

for all $z \in \mathbb{C}\backslash\{L\}$.

Remark 8 The bound $x/\log^2 x$ is not optimal (any increasing function which is negligible with respect to x and integrable against dx/x^2 at infinity would work). However, it will be sufficient for our purpose.

Proof Let $c_0 > 1$, and $z \in \mathbb{C}$ such that $|z| \leq c_0/2$. For $c > c_0$, we have:

$$\begin{aligned}
\sum_{\lambda\in L\cap([-c,-c_0]\cup[c_0,c])} \frac{\gamma_\lambda}{\lambda - z} &= \sum_{\lambda\in L\cap[c_0,c]} \gamma_\lambda \int_\lambda^\infty \frac{d\mu}{(\mu-z)^2} \\
&\quad - \sum_{\lambda\in L\cap[-c,-c_0]} \gamma_\lambda \int_{-\infty}^\lambda \frac{d\mu}{(\mu-z)^2} \\
&= \int_{c_0}^\infty \frac{\Lambda([c_0, c\wedge\mu])}{(\mu-z)^2}\, d\mu - \int_{-\infty}^{-c_0} \frac{\Lambda([(-c)\vee\mu, -c_0])}{(\mu-z)^2}\, d\mu \\
&= \int_{c_0}^\infty \left(\frac{\Lambda([c_0, c\wedge\mu])}{(\mu-z)^2} - \frac{\Lambda([-(c\wedge\mu), -c_0])}{(\mu+z)^2} \right) d\mu \\
&= \int_{c_0}^\infty \frac{\Lambda([c_0, c\wedge\mu]) - \Lambda([-(c\wedge\mu), -c_0])}{\mu^2}\, d\mu \\
&\quad + \int_{c_0}^\infty \left(\frac{(2z\mu - z^2)(\Lambda([c_0, c\wedge\mu]))}{\mu^2(\mu-z)^2} \right. \\
&\quad \left. + \frac{(2z\mu + z^2)(\Lambda([-(c\wedge\mu), -c_0]))}{\mu^2(\mu+z)^2} \right) d\mu.
\end{aligned}$$

Let F be an increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+^* := (0, \infty)$, such that $F(x)$ is equivalent to $x/\log^2 x$ when x goes to infinity. By assumption, there exists $C > 0$ such that for all $x \geq 0$, $|\Lambda([0, x]) - \Lambda([-x, 0])| \leq CF(x)$, and then, for all $\mu \geq c_0$,

$$\begin{aligned}
&|\Lambda([c_0, c\wedge\mu]) - \Lambda([-(c\wedge\mu), -c_0])| \\
&\quad \leq CF(c\wedge\mu) + \Lambda([-c_0, c_0]) \leq \left(C + \frac{\Lambda([-c_0, c_0])}{F(0)} \right) F(\mu).
\end{aligned}$$

Since $\mu \mapsto F(\mu)/\mu^2$ is integrable at infinity, one obtains, by dominated convergence,

$$\int_{c_0}^{\infty} \frac{\Lambda([c_0, c\wedge\mu]) - \Lambda([-(c\wedge\mu), -c_0])}{\mu^2}\, d\mu \xrightarrow[c\to\infty]{} \int_{c_0}^{\infty} \frac{\Lambda([c_0,\mu]) - \Lambda([-\mu, -c_0])}{\mu^2}\, d\mu,$$

where the limiting integral is absolutely convergent. Similarly, there exist $C', C'' > 0$ such that for all $x \geq 0$, $|\Lambda([0, x+1]) - \Lambda([-x, 0])| \leq C'F(x)$ and $|\Lambda([0,x]) - \Lambda([-x-1, 0])| \leq C''F(x)$, which implies that

$$\begin{aligned}\Lambda((x, x+1]) + \Lambda([-x-1, -x)) &\leq |\Lambda([0, x+1]) - \Lambda([-x, 0])| \\ &\quad + |\Lambda([0,x]) - \Lambda([-x-1, 0])| \\ &\leq (C' + C'')F(x).\end{aligned}$$

Hence, for all integers $n \geq 1$,

$$\begin{aligned}\Lambda([-n, n]) &= \Lambda(\{0\}) + \sum_{k=0}^{n-1} (\Lambda((k, k+1]) + \Lambda([-k-1, -k)) \\ &\leq \Lambda(\{0\}) + (C' + C'') \sum_{k=0}^{n-1} F(k) \leq KnF(n-1)\end{aligned}$$

where $K > 0$ is a constant, and then for all $x \geq 0$, $\Lambda([-x, x]) \leq K(1+x)F(x)$, which implies that for $\mu \geq c_0$, $\Lambda([-(c\wedge\mu), -c_0]) \leq K(1+\mu)F(\mu)$ and $\Lambda([c_0, c\wedge\mu]) \leq K(1+\mu)F(\mu)$. Moreover, since $|z| \leq c_0/2 \leq \mu/2$, one has $|\mu - z| \geq \mu/2$, $|\mu + z| \geq \mu/2$ and

$$\left|\frac{(2z\mu - z^2)}{\mu^2(\mu - z)^2}\right| + \left|\frac{(2z\mu + z^2)}{\mu^2(\mu + z)^2}\right| \leq 2\,\frac{2.5|z|\mu}{\mu^2(\mu/2)^2} = 20|z|/\mu^3 \leq 10\, c_0/\mu^3 \tag{7}$$

Since $\mu \mapsto (1+\mu)F(\mu)/\mu^3$ is integrable at infinity, one can again apply dominated convergence and obtain that

$$\int_{c_0}^{\infty} \left(\frac{(2z\mu - z^2)(\Lambda([c_0, c\wedge\mu]))}{\mu^2(\mu - z)^2} + \frac{(2z\mu + z^2)(\Lambda([-(c\wedge\mu), -c_0]))}{\mu^2(\mu + z)^2}\right) d\mu$$

tends to

$$\int_{c_0}^{\infty} \left(\frac{(2z\mu - z^2)(\Lambda([c_0, \mu]))}{\mu^2(\mu - z)^2} + \frac{(2z\mu + z^2)(\Lambda([-\mu, -c_0]))}{\mu^2(\mu + z)^2}\right) d\mu$$

when c goes to infinity. Therefore,

$$\sum_{\lambda\in L\cap([-c,-c_0]\cup[c_0,c])}\frac{\gamma_\lambda}{\lambda-z}\xrightarrow[c\to\infty]{}\int_{c_0}^{\infty}\frac{\Lambda([c_0,\mu])-\Lambda([-\mu,-c_0])}{\mu^2}\,d\mu$$
$$+\int_{c_0}^{\infty}\left(\frac{(2z\mu-z^2)(\Lambda([c_0,\mu]))}{\mu^2(\mu-z)^2}+\frac{(2z\mu+z^2)(\Lambda([-\mu,-c_0]))}{\mu^2(\mu+z)^2}\right)d\mu,$$

which proves the existence of the limit defining $S_\Lambda(z)$: explicitly, for $z\in\mathbb{C}\backslash\{L\}$ and for any $c_0>2|z|\vee 1$,

$$S_\Lambda(z)=\sum_{\lambda\in L\cap(-c_0,c_0)}\frac{\gamma_\lambda}{\lambda-z}+\int_{c_0}^{\infty}\frac{\Lambda([c_0,\mu])-\Lambda([-\mu,-c_0])}{\mu^2}\,d\mu$$
$$+\int_{c_0}^{\infty}\left(\frac{(2z\mu-z^2)(\Lambda([c_0,\mu]))}{\mu^2(\mu-z)^2}+\frac{(2z\mu+z^2)(\Lambda([-\mu,-c_0]))}{\mu^2(\mu+z)^2}\right)d\mu. \tag{8}$$

For fixed $c_0>0$, the first term of (8) is a rational function of z, the second term of (8) does not depend on z, and by dominated convergence, the third term can be differentiated in the integral if we restrict z to the set $\{|z|<c_0/2\}$. Hence, the restriction of S_Λ to the set $\{|z|<c_0/2\}$ is meromorphic, with simple poles at points $\lambda\in L\cap(-c_0/2,c_0/2)$. Since c_0 can be taken arbitrarily large, S_Λ is in fact meromorphic on $\mathbb{C}$, with poles $\lambda\in L$, the pole λ having residue $-\gamma_\lambda$. The derivative $S'_\Lambda(z)$ is given, for any $c_0>2|z|\vee 1$, by:

$$S'_\Lambda(z)=\sum_{\lambda\in L\cap(-c_0,c_0)}\frac{\gamma_\lambda}{(\lambda-z)^2}+2\int_{c_0}^{\infty}\left(\frac{(\Lambda([c_0,\mu]))}{(\mu-z)^3}+\frac{(\Lambda([-\mu,-c_0]))}{(\mu+z)^3}\right)d\mu.$$
$$=\sum_{\lambda\in L\cap(-c_0,c_0)}\frac{\gamma_\lambda}{(\lambda-z)^2}+\int_{c_0}^{\infty}\left(\sum_{\lambda\in L\cap[c_0,\mu]}\gamma_\lambda\right)\frac{2\,d\mu}{(\mu-z)^3}$$
$$+\int_{c_0}^{\infty}\left(\sum_{\lambda\in L\cap[-\mu,-c_0]}\gamma_\lambda\right)\frac{2\,d\mu}{(\mu+z)^3}$$
$$=\sum_{\lambda\in L\cap(-c_0,c_0)}\frac{\gamma_\lambda}{(\lambda-z)^2}+\sum_{\lambda\in L\cap[c_0,\infty)}\gamma_\lambda\int_{\lambda}^{\infty}\frac{2\,d\mu}{(\mu-z)^3}$$
$$+\sum_{\lambda\in L\cap(-\infty,c_0]}\gamma_\lambda\int_{-\lambda}^{\infty}\frac{2\,d\mu}{(\mu+z)^3},$$

which implies (6). Note that the implicit use of Fubini theorem in this computation is correct since all the sums and integral involved are absolutely convergent.

Now, let $\mathcal{K}$ be a compact set of $\mathbb{C}\backslash L$, let $d>0$ be the distance between $\mathcal{K}$ and L, and let $A>0$ be the maximal modulus of the elements of $\mathcal{K}$. For all $z\in\mathcal{K}$ and $\lambda\in L$,

one has, for $|\lambda| \le 2A+1$,

$$\left|\frac{\gamma_\lambda}{(\lambda-z)^2}\right| \le \frac{\gamma_\lambda}{d^2} \le \frac{1+(2A+1)^2}{d^2}\cdot\frac{\gamma_\lambda}{1+\lambda^2}$$

and for $|\lambda| \ge 2A+1$,

$$\left|\frac{\gamma_\lambda}{(\lambda-z)^2}\right| \le \frac{\gamma_\lambda}{(|\lambda|-A)^2} \le \frac{4\gamma_\lambda}{\lambda^2} \le \frac{8\gamma_\lambda}{1+\lambda^2}.$$

Hence, in order to prove the uniform convergence of (6) on compact sets, it is sufficient to check that

$$\sum_{\lambda\in L}\frac{\gamma_\lambda}{1+\lambda^2} < \infty,$$

but this convergence is directly implied by the absolute convergence of the right-hand side of (6) for any single value of $z \in \mathbb{C}\backslash L$ (say, $z = i$), which has been proven before.

The formula (6) applied to $z \in \mathbb{R}$ implies immediately that for all pairs $\{\lambda_1, \lambda_2\}$ of consecutive points in L, with $\lambda_1 < \lambda_2$, the function S_Λ is strictly increasing on the interval (λ_1, λ_2). Moreover, one has for $\lambda \in \{\lambda_1, \lambda_2\}$ and $z \to \lambda$, $S_\lambda(z) \sim \gamma_\lambda/(\lambda - z)$, which implies that $S_\Lambda(z) \to -\infty$ for $z \to \lambda_1$ and $z > \lambda_1$, and $S_\Lambda(z) \to +\infty$ for $z \to \lambda_2$ and $z < \lambda_2$. We deduce that S_Λ is a bijection from (λ_1, λ_2) to $\mathbb{R}$.

It only remains to show the invariance by translation. If we fix $y \in \mathbb{R}$, then for all $a, b \in \mathbb{R}$, and for $x \ge 0$ large enough,

$$\begin{aligned}
&(\Lambda+y)([0,x+a]) - (\Lambda+y)([-x+b,0]) \\
&\quad = \Lambda([-y,x+a-y]) - \Lambda([-x+b-y,-y]) \\
&\quad = \Lambda([0,x+a-y]) - \Lambda([-x+b-y,0]) + O(\Lambda([-|y|,|y|])) \\
&\quad = O(x/\log^2 x) + O(1) = O(x/\log^2 x),
\end{aligned}$$

and the assumptions of Theorem 7 are satisfied. One has

$$\Lambda + y = \sum_{\lambda\in L}\gamma_\lambda\delta_{\lambda+y},$$

and then for all $z \in \mathbb{C}\backslash L$,

$$S_{\Lambda+y}(z+y) = \lim_{c\to\infty}\sum_{\lambda\in(L+y)\cap[-c,c]}\frac{\gamma_{\lambda-y}}{z+y-\lambda} = \lim_{c\to\infty}\sum_{\lambda\in L\cap[-c-y,c-y]}\frac{\gamma_\lambda}{z-\lambda},$$

which is equal to $S_\Lambda(z)$, provided that we check that

$$\sum_{\lambda\in L\cap[-c-y,c-y]}\frac{\gamma_\lambda}{z-\lambda} - \sum_{\lambda\in L\cap[-c,c]}\frac{\gamma_\lambda}{z-\lambda} \underset{c\to\infty}{\longrightarrow} 0,$$

which is implied by

$$\sum_{\lambda\in L\cap[-c-|y|,-c+|y|]}\frac{\gamma_\lambda}{|z-\lambda|}+\sum_{\lambda\in L\cap[c-|y|,c+|y|]}\frac{\gamma_\lambda}{|z-\lambda|}\underset{c\to\infty}{\longrightarrow}0. \tag{9}$$

Now, for $c>|y|+|z|+1$, the left-hand side of (9) is smaller than or equal to

$$\begin{aligned}&\frac{\Lambda([-c-|y|,-c+|y|])+\Lambda([c-|y|,c+|y|])}{c-|z|-|y|}\\&\leq\frac{|\Lambda([0,c+|y|])-\Lambda([-c+|y|+1,0])|+|\Lambda([0,c-|y|-1])-\Lambda([-c-|y|,0])|}{c-|y|-|z|}=O(1/\log^2 c),\end{aligned}$$

for c tending to infinity. □

The assumption of Theorem 7 depends on the fact that the measure Λ is not too far from being symmetric with respect to a given point on the real line. The next proposition expresses this assumption in terms of the support L of Λ and the weights $(\gamma_\lambda)_{\lambda\in L}$. The following result gives a sufficient condition for Theorem 7:

Proposition 9 *Consider the measure*

$$\Lambda=\sum_{j\in\mathbb{Z}}\gamma_j\delta_{\lambda_j}$$

where $(\lambda_j)_{j\in\mathbb{Z}}$ is strictly increasing and neither bounded from above nor from below, and $\gamma_j>0$. Let L be the set $\{\lambda_j,\ j\in\mathbb{Z}\}$. Assume that for some $c>0$,

$$\sum_{j=0}^{k}\gamma_j=ck+O(k/\log^2 k)\ \text{ and }\ \sum_{j=0}^{k}\gamma_{-j}=ck+O(k/\log^2 k),$$

when $k\to\infty$. If for $x\to\infty$ one has $\mathrm{Card}(L\cap[0,x])=O(x)$ and for all $a,b\in\mathbb{R}$, $\mathrm{Card}(L\cap[0,x+a])-\mathrm{Card}(L\cap[-x+b,0])=O(x/\log^2 x)$, then the assumptions of Theorem 7 are satisfied.

Proof For $y\in\mathbb{R}$, let $N(y)$ (resp. $N(y-)$) be the largest index j such that $\lambda_j\leq y$ (resp. $\lambda_j<y$). One has, for $a,b\in\mathbb{R}$ and for x large enough,

$$\Lambda([0,x+a])=\sum_{j=N(0-)+1}^{N(x+a)}\gamma_j\ \text{ and }\ \Lambda([-x+b,0])=\sum_{j=N((-x+b)-)}^{N(0)}\gamma_j,$$

which implies that for $x\to\infty$ and then $N(x+a)\to\infty$, $N((-x+b)-)\to-\infty$:

$$\begin{aligned}\Lambda([0,x+a])-\Lambda([-x+b,0])&=c(N(x+a)-|N((-x+b)-)|)\\&+O\left(\frac{N(x+a)}{\log^2(N(x+a))}+\frac{|N((-x+b)-)|}{\log^2|N((-x+b)-)|}\right).\end{aligned}$$

Now, we have the following estimates:

$$N(x+a) = \mathrm{Card}(L \cap [0, x+a]) + O(1) = O(x+a)$$
$$+ O(1) = O(x), \frac{N(x+a)}{\log^2(N(x+a))} = O(x/\log^2 x),$$
$$N(x+a) - |N((-x+b)-)| = \mathrm{Card}(L \cap [0, x+a]) - \mathrm{Card}(L \cap [-x+b, 0])$$
$$+ O(1) = O(x/\log^2 x), |N((-x+b)-)| \leq N(x+a) + |N(x+a)$$
$$- |N((-x+b)-)|| \leq O(x) + O(x/\log^2 x) = O(x)$$

and

$$\frac{|N((-x+b)-)|}{\log^2 |N((-x+b)-)|} = O(x/\log^2 x).$$

Putting all together gives:

$$\Lambda([0, x+a]) - \Lambda([-x+b, 0]) = O(x/\log^2 x)$$

and then the assumptions of Theorem 7 are satisfied. □

As written in the statement of Theorem 7, the function S_Λ induces a bijection between each interval (λ_1, λ_2), λ_1 and λ_2 being two consecutive points of L, and the real line. It is then natural to study the inverse of this bijection, which should map each element of $\mathbb{R}$ to a set of points interlacing with L. The precise statement we obtain is the following:

Proposition 10 *Let Λ be a measure, whose support L is neither bounded from above nor from below, and satisfying the assumptions of Theorem 7. Then, for all $h \in \mathbb{R}$, the set $S_\Lambda^{-1}(h)$ of $z \in \mathbb{C}\backslash\{L\}$ such that $S_\Lambda(z) = h$ is included in $\mathbb{R}$, and interlaces with L, i.e. it contains exactly one point in each open interval between two consecutive points of L. Moreover, if Λ satisfies the assumptions of Proposition 9, then it is also the case for the set $L' := S_\Lambda^{-1}(h)$, i.e. for x going to infinity, one has* $\mathrm{Card}(L' \cap [0, x]) = O(x)$ *and for all $a, b \in \mathbb{R}$,* $\mathrm{Card}(L' \cap [0, x+a]) - \mathrm{Card}(L' \cap [-x+b, 0]) = O(x/\log^2 x)$.

Proof The interlacing property of points of $S_\Lambda^{-1}(h) \cap \mathbb{R}$ comes from the discussion above, so the first part of the proposition is proven if we check that $S_\Lambda(z) \notin \mathbb{R}$ if $z \notin \mathbb{R}$. Now, for all $z \in \mathbb{C}\backslash L$,

$$\Im\left(S_\Lambda(z)\right) = \lim_{c\to\infty} \sum_{\lambda\in L\cap[-c,c]} \Im\left(\frac{\gamma_\lambda}{\lambda - z}\right) = \lim_{c\to\infty} \sum_{\lambda\in L\cap[-c,c]} \frac{-\gamma_\lambda \Im(\lambda - z)}{\Re^2(\lambda - z) + \Im^2(\lambda - z)}$$
$$= \lim_{c\to\infty} \sum_{\lambda\in L\cap[-c,c]} \frac{\gamma_\lambda \Im(z)}{\Re^2(\lambda - z) + \Im^2(z)}.$$

If $z \notin \mathbb{R}$, each term of the last sum is nonzero and has the same sign as $\Im(z)$. One deduces that $\Im\left(S_\Lambda(z)\right)$ has the same properties, and then $S_\Lambda(z) \notin \mathbb{R}$.

Now, the interlacing property implies that for any finite interval I,

$$|\operatorname{Card}(L' \cap I) - \operatorname{Card}(L \cap I)| \leq 2.$$

If Λ satisfies the assumptions of Proposition 9, then for $a, b \in \mathbb{R}$ and for x going to infinity,

$$\operatorname{Card}(L' \cap [0, x]) = \operatorname{Card}(L \cap [0, x]) + O(1) = O(x) + O(1) = O(x)$$

and

$$\begin{aligned}\operatorname{Card}(L' \cap [0, x + a]) - \operatorname{Card}(L' \cap [-x + b, 0]) &= \operatorname{Card}(L \cap [0, x + a]) \\ - \operatorname{Card}(L \cap [-x + b, 0]) + O(1) &= O(x/\log^2 x).\end{aligned}$$

□

Proposition 10 shows that the Stieltjes transform gives a way to construct a discrete subset of $\mathbb{R}$ from another, provided that we get a family $(\gamma_j)_{j\in\mathbb{Z}}$ of weights and a parameter $h \in \mathbb{R}$. In the next section, we use and randomize this procedure in order to define a family of Markov chains satisfying some remarkable properties.

5 Stieltjes Markov chains

In order to put some randomness in the construction above, we need to define precisely a measurable space in which the point processes will be contained. The choice considered here is the following:

- We define $\mathcal{L}$ as the family of all the discrete subsets L of $\mathbb{R}$, unbounded from above and from below, and satisfying the assumptions of Proposition 9, i.e. for x going to infinity, $\operatorname{Card}(L \cap [0, x]) = O(x)$ and for all $a, b \in \mathbb{R}$, $\operatorname{Card}(L \cap [0, x + a]) - \operatorname{Card}(L \cap [-x + b, 0]) = O(x/\log^2 x)$.
- We define, on $\mathcal{L}$, the σ-algebra $\mathcal{A}$ generated by the maps $L \mapsto \operatorname{Card}(L \cap I)$ for all open, bounded intervals $I \subset \mathbb{R}$, which is also the σ-algebra generated by the maps $L \mapsto \operatorname{Card}(L \cap B)$ for all Borel sets $B \subset \mathbb{R}$.

A similar choice of measurable space has to be made for the weights $(\gamma_j)_{j\in\mathbb{Z}}$:

- We define Γ as the family of doubly infinite sequences $(\gamma_j)_{j\in\mathbb{Z}}$ satisfying the assumptions of Proposition 9, i.e. for k going to infinity,

$$\sum_{j=0}^{k} \gamma_j = ck + O(k/\log^2 k) \quad \text{and} \quad \sum_{j=0}^{k} \gamma_{-j} = ck + O(k/\log^2 k),$$

  where $c > 0$ is a constant.
- We define, on Γ, the σ-algebra $\mathcal{C}$ generated by the coordinate maps γ_j, $j \in \mathbb{Z}$.

Let $\mathcal{D}$ be the map from $\mathcal{L} \times \Gamma \times \mathbb{R}$ to $\mathcal{L}$, defined by:

$$\mathcal{D}(L, (\gamma_j)_{j\in\mathbb{Z}}, h) = S^{-1}_{\sum_{j\in\mathbb{Z}} \gamma_j \delta_{\lambda_j}}(h),$$

where λ_j is the unique increasing labeling of L so that $\lambda_{-1} < 0 \leq \lambda_0$. Proposition 10 shows that this is indeed a map to $\mathcal{L}$. It is easy to show that $\mathcal{D}$ is measurable. Now for any probability measure Π on $\Gamma \times \mathbb{R}$, it naturally defines a Markov chain $(X_k)_{k\geq 0}$ on $\mathcal{L}$. To get X_{k+1} from X_k, just take a fresh sample G_k, independent of X_k and its past, distributed according to the measure Π, and set

$$X_{k+1} = \mathcal{D}(X_k, G_k).$$

By construction, $(X_k)_{k\geq 0}$ is then a time-homogeneous Markov chain.

Clearly, if the distribution of X_0 is invariant under translations of $\mathbb{R}$, and the distribution of the $((\gamma_j)_{j\in\mathbb{Z}}, h)$ in G_k is invariant under translations of the indices j, it follows that X_1 also has a translation-invariant distribution. Here, the invariance of $((\gamma_j)_{j\in\mathbb{Z}}, h)$ by translation of the indices j is used because translating a set of points in $\mathbb{R}$ can shift the labeling of the points ($\lambda_{-1} < 0 \leq \lambda_0$).

There are two important examples of probability measures Π for which this construction applies:

- Under Π, $(\gamma_j)_{j\in\mathbb{Z}}$ is a family of i.i.d, square-integrable random variables, and h is independent of $(\gamma_j)_{j\in\mathbb{Z}}$.
- Under Π, $(\gamma_j)_{j\in\mathbb{Z}}$ is a family of random variables, n-periodic for some $n \geq 1$, such that $(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) = (\gamma_1, \ldots, \gamma_{n-1}, \gamma_0)$ in law, and h is independent of $(\gamma_j)_{j\in\mathbb{Z}}$.

The fact that $(\gamma_j)_{j\in\mathbb{Z}}$ is almost surely in Γ comes from the law of the iterated logarithm in the first example, and directly from the periodicity in the second example.

6 Periodic Stieltjes Markov chains

Consider the case when

$$\Lambda = \sum_{j\in\mathbb{Z}} \gamma_j \delta_{\lambda_j}$$

is invariant by translation by $2\pi n$, and when there are n point masses in every interval of length $2\pi n$ with total weight $2n$. In this case, Λ can be thought as $2n$ times the lifting of the measure

$$\sigma = \sum_{j=0}^{n-1} \frac{\gamma_j}{2n} \delta_{e^{i\lambda_j/n}}$$

on the unit circle $\mathbb{U}$ under a covering map. Moreover, with $u = e^{iz/n}$ the Stieltjes transform of Λ can be expressed in terms of σ by

$$S_\Lambda(z) = \sum_{j=0}^{n-1} i \frac{\gamma_j}{2n} \frac{e^{i\lambda_j/n} + u}{e^{i\lambda_j/n} - u}.$$

Indeed, periodicity implies that for $z \notin L$, we have

$$\begin{aligned} S_\Lambda(z) &= \lim_{k\to\infty} \sum_{j=-kn}^{kn-1} \frac{\gamma_j}{\lambda_j - z} = \lim_{k\to\infty} \sum_{j=0}^{n-1} \gamma_j \left(\sum_{\ell=-k}^{k-1} \frac{1}{2\pi n\ell + \lambda_j - z} \right) \\ &= \sum_{j=0}^{n-1} \gamma_j \left(\lim_{k\to\infty} \sum_{\ell=-k}^{k-1} \frac{1}{2\pi n\ell + \lambda_j - z} \right) = \frac{1}{2n} \sum_{j=0}^{n-1} \gamma_j \cot\left(\frac{\lambda_j - z}{2n} \right). \end{aligned}$$

Therefore, if we set $\rho_j := \gamma_j/2n$ and $u_j = e^{i\lambda_j/n}$, we can check that $\mathcal{D}(L, (\gamma_n)_{n\in\mathbb{Z}}, h)$ is the set of $z \in \mathbb{R}$, such that $e^{iz/n}$ satisfies (2), for $h = i(1+\eta)/(1-\eta)$.

This property shows that the lifting $u \mapsto \{z \in \mathbb{R}, e^{iz/n} = u\}$ from $\mathbb{U}$ to $\mathbb{R}$ defined above transforms the Markov chain defined in Sect. 3 to the Markov chain defined in Sect. 5. In particular, from Propositions 5 and 6, we deduce the following results:

Theorem 11 *Let Π be a probability measure on the space $(\Gamma \times \mathbb{R}, \mathcal{C} \otimes \mathcal{B}(\mathbb{R}))$, under which the following holds, for some integer $n \geq 1$:*

- *Almost surely under Π, $(\gamma_n)_{n\in\mathbb{Z}}$ is n-periodic, and $\sum_{j=0}^{n-1} \gamma_j = 2n$.*
- *The sequence $(\gamma_j)_{j\in\mathbb{Z}}$ is independent of h.*

Let $\mathbb{Q}$ be a probability on $(\mathcal{L}, \mathcal{A})$ under which almost surely, the set L is $(2n\pi)$-periodic and contains exactly n points in the interval $[0, 2\pi n)$: in this case, there exists a sequence $(u_1, \ldots, u_n)$ of elements of $\mathbb{U}$, with increasing argument in $[0, 2\pi)$, and such that

$$L = \{z \in \mathbb{R}, e^{iz/n} \in \{u_1, \ldots, u_n\}\}.$$

Under the probability $\mathbb{Q} \otimes \pi$, one can define a random probability measure σ on the unit circle by:

$$\sigma := \frac{1}{2n} \sum_{j=1}^{n} \gamma_j \delta_{u_j}.$$

Let us assume that the joint law of the Verblunsky coefficients $(\alpha_0, \ldots, \alpha_{n-1})$ of σ is invariant by rotation, i.e. for all $u \in \mathbb{U}$,

$$(\alpha_0 u, \ldots \alpha_{n-1} u) = (\alpha_0, \ldots, \alpha_{n-1})$$

in distribution. Then, the probability measure $\mathbb{Q}$ is an invariant measure for the Markov chain associated to π.

Theorem 12 *Let $\beta > 0$, $n \geq 1$, and let $\Pi_{n,\beta}$ be a probability measure under which the following holds almost surely:*

- *The sequence $(\gamma_n)_{n\in\mathbb{Z}}$ is n-periodic.*
- *The tuple $(\gamma_0/2n, \ldots, \gamma_{n-1}/2n)$ follows a Dirichlet distribution with all parameters equal to $\beta/2$.*
- *The sequence $(\gamma_j)_{j\in\mathbb{Z}}$ is independent of h.*

Let $\mathbb{Q}_{n,\beta}$ be the distribution of the set

$$\{z \in \mathbb{R}, e^{iz/n} \in V\},$$

where V is a subset of $\mathbb{U}$ following $\mathbb{P}_{n,\beta}$, i.e. a circular beta ensemble with parameter β. Then, $\mathbb{Q}_{n,\beta}$ is an invariant measure for the Markov chain associated to $\Pi_{n,\beta}$.

In the next section, we will let $n \to \infty$ and we will obtain a similar result in which the variables $(\gamma_n)_{n\geq 1}$ will be independent and identically distributed.

7 An invariant measure for independent gamma random variables

In Theorem 12, we have found an invariant measure on $\mathcal{L}$, corresponding to a measure $\Pi_{n,\beta}$ under which the sequence $(\gamma_j)_{j\in\mathbb{Z}}$ is periodic, each period forming a renormalized Dirichlet distribution. For $n \geq 1$ and $\beta > 0$ fixed, and under $\Pi_{n,\beta}$, the sequence $(\gamma_j)_{j\in\mathbb{Z}}$ can be written in function of a sequence $(g_j)_{j\in\mathbb{Z}}$ of i.i.d Gamma variables with parameter $\beta/2$, as follows:

$$\gamma_j = \frac{2ng_k}{\sum_{-n/2<\ell\leq n/2} g_\ell},$$

where $-n/2 < k \leq n/2$ and $k \equiv j$ modulo n. Here, a Gamma variable with parameter $\theta > 0$ is normalized in such a way that it has density $x \mapsto (\Gamma(\theta))^{-1}x^{\theta-1}e^{-x}$ with respect to the Lebesgue measure.

For β fixed, if we construct the sequence $(\gamma_j)_{j\in\mathbb{Z}}$ for all values of n, starting with the same sequence $(g_j)_{j\in\mathbb{Z}}$, we obtain, by the law of large numbers, that for all $j \in \mathbb{Z}$, γ_j tends almost surely to $4g_j/\beta$ when n goes to infinity. Hence, if we want to make $n \to \infty$ in Theorem 12, we should consider a measure Π_β under which $(\beta\gamma_j/4)_{j\in\mathbb{Z}}$ is a sequence of i.i.d. Gamma random variables of parameter $\beta/2$.

On the other hand, for n going to infinity, the probability $\mathbb{Q}_{n,\beta}$ converges to a limiting measure $\mathbb{Q}_\beta$, which is the distribution of the so-called Sine_β *point process*, constructed in [13,19].

Therefore, taking the limit $n \to \infty$ in Theorem 12 suggests the following result, whose proof is given below:

Theorem 13 *Let $\beta > 0$, and let Π_β be a probability measure under which the random variables h and $(\gamma_j)_{j\in\mathbb{Z}}$ are all independent, γ_j being equal to $4/\beta$ times a gamma random variable of parameter $\beta/2$. Then, the law $\mathbb{Q}_\beta$ of the Sine_β point process*

is carried by the space $\mathcal{L}$ and it is an invariant measure for the Markov chain X_k associated to Π_β.

Moreover, for the stationary Markov chains $X_{n,k}$ defined of Theorem 12 with same h, for every k, $(X_{n,0}, \dots, X_{n,k})$ converges in law to $(X_0, \dots, X_k)$ as $n \to \infty$, for the topology of locally weak convergence.

Remark 14 Since the variables $(\gamma_j)_{j\in\mathbb{Z}}$ are i.i.d. and square-integrable, we have already checked that the Markov chain associated to Π_β is well-defined, as soon as its initial distribution is fixed and carried by $\mathcal{L}$. A consequence of Proposition 16 below is that the probability measure $\mathbb{Q}_\beta$ is indeed carried by $\mathcal{L}$, which means the following: if L is the set of points corresponding to a Sine_β process, then L is unbounded from above and from below, for x going to infinity, $\mathrm{Card}(L \cap [0, x]) = O(x)$ and for all $a, b \in \mathbb{R}$, $\mathrm{Card}(L \cap [0, x + a]) - \mathrm{Card}(L \cap [-x + b, 0]) = O(x/\log^2 x)$.

In order to show the theorem just above, we will use the following results, proven in [15]:

Proposition 15 *Let L be a random set of points in $\mathbb{R}$, whose distribution is $\mathbb{Q}_{n,\beta}$ or $\mathbb{Q}_\beta$. Then, there exists $C > 0$, depending on β but not on n, such that for all $x > 0$,*

$$\mathbb{E}[(\mathrm{Card}(L \cap [0, x]) - x/2\pi)^2] \leq C \log(2 + x)$$

and

$$\mathbb{E}[(\mathrm{Card}(L \cap [-x, 0]) - x/2\pi)^2] \leq C \log(2 + x).$$

Proposition 16 *For $n \geq 1$ integer, let L_n be a random set of points in $\mathbb{R}$, whose distribution is $\mathbb{Q}_{n,\beta}$. Let L_∞ be a set of points whose distribution is $\mathbb{Q}_\beta$, and let $\alpha > 1/3$. Then, there exists a tight family $(C_n)_{n\in\{1,2,3,\dots,\infty\}}$ of random variables with values in $(0, \infty)$, such that almost surely, for all $n \in \{1, 2, 3, \dots, \infty\}$, $x \geq 0$,*

$$|\,\mathrm{Card}(L_n \cap [0, x]) - x/2\pi\,| \leq C_n(1 + x)^\alpha,$$

and

$$|\,\mathrm{Card}(L_n \cap [-x, 0]) - x/2\pi\,| \leq C_n(1 + x)^\alpha.$$

Remark 17 For finite $n \geq 1$, the periodicity of L_n implies that $|\,\mathrm{Card}(L_n \cap [0, x]) - x/2\pi\,|$ is almost surely bounded when x varies. Hence, the result above becomes trivial for n finite if we allow the family $(C_n)_{n\in\{1,2,3,\dots\}}$ not to be tight. Moreover, we expect that it remains true for any $\alpha > 0$, and not only for $\alpha > 1/3$.

Proof of Theorem 13 Let Π_β be a probability measure which satisfies the assumptions of Theorem 13, and for $n \geq 1$, let $\Pi_{n,\beta}$ be a measure satisfying the assumptions of Theorem 12, for the same value of β. We also assume that the law of h is the same under $\Pi_{n,\beta}$ and under Π_β (note that $\Pi_{n,\beta}$ and Π_β are uniquely determined by this law). By the discussion preceding the statement of Theorem 13, it is possible, by using

a unique family $(g_j)_{j\in\mathbb{Z}}$ of i.i.d. gamma variables with parameter $\beta/2$, to construct some random sequences $(\gamma_j)_{j\in\mathbb{Z}}$ and $(\gamma_j^n)_{j\in\mathbb{Z}}$ (for all $n \geq 1$) and an independent real-valued random variable h, such that the following holds:

- $((\gamma_j)_{j\in\mathbb{Z}}, h)$ follows the law Π_β.
- For all $n \geq 1$, $((\gamma_j^n)_{j\in\mathbb{Z}}, h)$ follows the law $\Pi_{n,\beta}$.
- For all $j \in \mathbb{Z}$, γ_j^n tends almost surely to γ_j when n goes to infinity.

Now, for all $n \geq 1$, let L_n be a point process following the distribution $\mathbb{Q}_{n,\beta}$, and let L be a point process following $\mathbb{Q}_\beta$. We already know that $L_n \in \mathcal{L}$ almost surely. From Proposition 16 under $\mathbb{Q}_\beta$, we immediately deduce the weaker estimates $\mathrm{Card}(L \cap [0, x]) = x/2\pi + O(x/\log^2 x)$ and $\mathrm{Card}(L \cap [-x, 0]) = x/2\pi + O(x/\log^2 x)$ for x going to infinity, which means that $L \in \mathcal{L}$ almost surely: $\mathbb{Q}_\beta$ is carried by $\mathcal{L}$.

Moreover, by [13], the measure $\mathbb{Q}_{n,\beta}$ tends to $\mathbb{Q}_\beta$ when n goes to infinity, in the following sense: for all functions f from $\mathbb{R}$ to $\mathbb{R}_+$, C^∞ and compactly supported, one has

$$\sum_{x\in L_n} f(x) \underset{n\to\infty}{\longrightarrow} \sum_{x\in L} f(x) \tag{10}$$

in distribution. By the Skorokhod representation theorem (see [4, Theorem 6.7]) one can assume that the convergence (10) holds almost surely, and one can also suppose that $(L_n)_{n\geq 1}$ and L are independent of $(\gamma_j^n)_{n\geq 1, j\in\mathbb{Z}}$, $(\gamma_j)_{j\in\mathbb{Z}}$ and h.

For $n \geq 1$, let $(\lambda_j^n)_{j\in\mathbb{Z}}$ be the strictly increasing sequence containing each point of L_n, λ_0^n being the smallest nonnegative point, and let $(\lambda_j)_{j\in\mathbb{Z}}$ be the similar sequence associated to L. One can check that the convergence (10) and the fact that $\mathbb{P}[0 \in L] = 0$ imply that for all $j \in \mathbb{Z}$, λ_j^n converges almost surely to λ_j when n goes to infinity. Indeed, for $j \geq 0$ and $\epsilon \in (0, \lambda_j/10)$, let us consider a test function f taking values in $[0, 1]$, equal to 1 on $[\epsilon, \lambda_j]$ and to 0 on $\mathbb{R}\backslash[0, \lambda_j + \epsilon]$. For ϵ small enough, L has no point in $[0, \epsilon]$ and $j+1$ points in $[0, \lambda_j]$, which implies that the sum of f at the points of L is at least $j+1$. Hence, the sum of f at the points of L_n is at least $j + (1/2)$ for n large enough, which implies that L_n has at least $j + (1/2)$ points, and then at least $j+1$ points, in the interval $[0, \lambda_j + \epsilon]$. This implies $\lambda_j^n \leq \lambda_j + \epsilon$. Similarly, if we take f in $[0, 1]$, equal to 1 on $[0, \lambda_j - 2\epsilon]$ and to 0 on $\mathbb{R}\backslash[-\epsilon, \lambda_j - \epsilon]$, the sum of f at points on L is at most j for ϵ small enough (because L has no point in $[-\epsilon, 0]$), which implies that the sum of f at points on L_n is at most $j + (1/2)$ for n large enough, and then L_n has at most j points in $[0, \lambda_j - 2\epsilon]$, i.e. $\lambda_j^n \geq \lambda_j - 2\epsilon$. The case $j < 0$ can be treated similarly.

Now, for all $c > 0$, $z \in \mathbb{C}\backslash\left(L \cup \left(\bigcup_{n\geq 1} L_n\right)\right)$, let us take the following notation:

$$S_{n,c}(z) := \sum_{j\in\mathbb{Z}} \frac{\gamma_j^n}{\lambda_j^n - z} \mathbf{1}_{|\lambda_j^n|\leq c}, \quad S_c(z) := \sum_{j\in\mathbb{Z}} \frac{\gamma_j}{\lambda_j - z} \mathbf{1}_{|\lambda_j|\leq c},$$
$$S_N(z) := \lim_{c\to\infty} S_{N,c}(z), \quad S(z) := \lim_{c\to\infty} S_c(z).$$

Almost surely, all the points of L and L_n ($n \geq 1$) are irrational. If this event occurs, then for all $c \in \mathbb{Q}_+^*$, there exists almost surely a random finite interval (possibly empty)

I_c such that $|\lambda_j| \leq c$ if and only if $j \in I_c$, and for all $n \geq 1$ large enough, $|\lambda_j^n| \leq c$ if and only if $j \in I_c$. Hence, for all $c \in \mathbb{Q}_+^*$, $z \in \mathbb{Q}$, one has almost surely

$$S_{n,c}(z) = \sum_{j \in I_c} \frac{\gamma_j^n}{\lambda_j^n - z}, \quad S_c(z) := \sum_{j \in I_c} \frac{\gamma_j}{\lambda_j - z},$$

if n is large enough. Since I_c is finite, γ_j^n tends a.s. to γ_j, and λ_j^n tends a.s. to λ_j when n goes to infinity, one deduces that almost surely, for all $c \in \mathbb{Q}_+^*$, $z \in \mathbb{Q}$,

$$S_{n,c}(z) \underset{n \to \infty}{\longrightarrow} S_c(z). \tag{11}$$

On the other hand, by (8), and by the fact that c and $-c$ are a.s. not in L or in L_n, one deduces that almost surely, for all $c \in \mathbb{Q}_+^*$, $z \in \mathbb{Q}$ such that $c > 2|z| \vee 1$, and for all $n \geq 1$,

$$\begin{aligned} S_n(z) - S_{n,c}(z) &= \int_c^\infty \frac{\Lambda_n([c,\mu]) - \Lambda_n([-\mu,-c])}{\mu^2}\, d\mu \\ &+ \int_c^\infty \left(\frac{(2z\mu - z^2)(\Lambda_n([c,\mu]))}{\mu^2(\mu - z)^2} + \frac{(2z\mu + z^2)(\Lambda_n([-\mu,-c]))}{\mu^2(\mu + z)^2} \right) d\mu \end{aligned}$$

and

$$\begin{aligned} S(z) - S_c(z) &= \int_c^\infty \frac{\Lambda([c,\mu]) - \Lambda([-\mu,-c])}{\mu^2}\, d\mu \\ &+ \int_c^\infty \left(\frac{(2z\mu - z^2)(\Lambda([c,\mu]))}{\mu^2(\mu - z)^2} + \frac{(2z\mu + z^2)(\Lambda([-\mu,-c]))}{\mu^2(\mu + z)^2} \right) d\mu, \end{aligned}$$

where $\Lambda_n := \sum_{j \in \mathbb{Z}} \gamma_j^n \delta_{\lambda_j^n}$ and $\Lambda := \sum_{j \in \mathbb{Z}} \gamma_j \delta_{\lambda_j}$. If for any bounded interval I, one defines $\Lambda_n^{(0)}(I) := \Lambda_n(I) - \mathbb{E}[\Lambda_n(I)]$ and $\Lambda^{(0)}(I) := \Lambda(I) - \mathbb{E}[\Lambda(I)]$, one has by (7), the triangle inequality, and the fact that $\mathbb{E}[\Lambda_n(I)]$ is proportional to the Lebesgue measure on I:

$$\begin{aligned} |S_n(z) - S_{n,c}(z)| &\leq \int_c^\infty \frac{|\Lambda_n^{(0)}([c,\mu])| + |\Lambda_n^{(0)}([-\mu,-c])|}{\mu^2}\, d\mu \\ &\quad + \int_c^\infty \frac{20|z|\, d\mu}{\mu^3} \left[C_1\mu + |\Lambda_n^{(0)}([c,\mu])| + |\Lambda_n^{(0)}([-\mu,-c])| \right] \\ &\leq C_2(1 + |z|) \left(\frac{1}{c} + \int_c^\infty \frac{|\Lambda_n^{(0)}([c,\mu])| + |\Lambda_n^{(0)}([-\mu,-c])|}{\mu^2}\, d\mu \right), \end{aligned}$$

where $C_1, C_2 > 0$ are universal constants. Since the distribution of L_n is invariant by translation (recall that its points are the rescaled arguments of the circular beta ensemble on the unit circle), one has

$$\mathbb{E}[|\Lambda_n^{(0)}([c,\mu])|] = \mathbb{E}[|\Lambda_n^{(0)}([-\mu,-c])|] = \mathbb{E}[|\Lambda_n^{(0)}([0,\mu-c])|]$$

and

$$\mathbb{E}[|S_n(z) - S_{n,c}(z)|] \leq C_3(1+|z|)\left(\frac{1}{c} + \int_0^\infty \frac{\mathbb{E}[|\Lambda_n^{(0)}([0,\nu])|]}{(\nu+c)^2}\,d\nu\right),$$

where $C_3 > 0$ is a universal constant. Similarly,

$$\mathbb{E}[|S(z) - S_c(z)|] \leq C_3(1+|z|)\left(\frac{1}{c} + \int_0^\infty \frac{\mathbb{E}[|\Lambda^{(0)}([0,\nu])|]}{(\nu+c)^2}\,d\nu\right).$$

Now, from Proposition 15 under $\mathbb{Q}_{n,\beta}$ and $\mathbb{Q}_\beta$, one immediately deduces that

$$\int_0^\infty \frac{\mathbb{E}[|\Lambda^{(0)}([0,\nu])|] + \sup_{n\geq 1}\mathbb{E}[|\Lambda_n^{(0)}([0,\nu])|]}{(1+\nu)^2}\,d\nu < \infty. \tag{12}$$

Hence, by dominated convergence, there exists a function ϕ from $[1,\infty)$ to $\mathbb{R}_+^*$, tending to zero at infinity, such that

$$\mathbb{E}[|S_n(z) - S_{n,c}(z)|] \leq (1+|z|)\,\phi(c)$$

and

$$\mathbb{E}[|S(z) - S_c(z)|] \leq (1+|z|)\,\phi(c).$$

We deduce that for all $c \in \mathbb{Q}_+^*$, $z \in \mathbb{Q}$ such that $c > 2|z| \vee 1$, $n \geq 1$ and $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}[|S(z) - S_n(z)| \geq \epsilon] &\leq \mathbb{P}[|S_c(z) - S_{n,c}(z)| \geq \epsilon/3] + \mathbb{P}[|S(z) - S_c(z)| \geq \epsilon/3] \\ &\quad + \mathbb{P}[|S_n(z) - S_{n,c}(z)| \geq \epsilon/3] \\ &\leq \mathbb{P}[|S_c(z) - S_{n,c}(z)| \geq \epsilon/3] + \frac{6}{\epsilon}(1+|z|)\,\phi(c).\end{aligned}$$

By the almost sure convergence (11), which implies the corresponding convergence in probability, one deduces

$$\limsup_{n\to\infty} \mathbb{P}[|S(z) - S_n(z)| \geq \epsilon] \leq \frac{6}{\epsilon}(1+|z|)\,\phi(c).$$

Now, by taking $z \in \mathbb{Q}$ fixed, $c \in \mathbb{Q}$ going to infinity and then $\epsilon \to 0$, one deduces that for all $z \in \mathbb{Q}$,

$$S_n(z) \underset{n\to\infty}{\longrightarrow} S(z)$$

in probability. By considering diagonal extraction of subsequences, one deduces that there exists a strictly increasing sequence $(n_k)_{k\geq 1}$ of integers, such that almost surely,

$$S_{n_k}(z) \underset{k\to\infty}{\longrightarrow} S(z) \tag{13}$$

for all $z \in \mathbb{Q}$.

Now, for all $j \in \mathbb{Z}, n \geq 1$, let μ_j^n (resp. μ_j) be the unique point of $\mathcal{D}(L_n, (\gamma_j^n)_{j\in\mathbb{Z}}, h)$ (resp. $\mathcal{D}(L, (\gamma_j)_{j\in\mathbb{Z}}, h)$) which lies in the interval $(\lambda_j^n, \lambda_{j+1}^n)$ (resp. $(\lambda_j, \lambda_{j+1})$). Let us fix $j \in \mathbb{Z}, \epsilon > 0$, and let us consider two random rational numbers q_1 and q_2 such that almost surely,

$$(\mu_j - \epsilon) \vee \lambda_j < q_1 < \mu_j < q_2 < (\mu_j + \epsilon) \wedge \lambda_{j+1},$$

which implies that

$$S(q_1) < h < S(q_2).$$

By (13), one deduces that almost surely, for k large enough,

$$S_{n_k}(q_1) < h < S_{n_k}(q_2),$$

which implies that $\mathcal{D}(L_{n_k}, (\gamma_j^{n_k})_{j\in\mathbb{Z}}, h)$ has at least one point in the interval (q_1, q_2). On the other hand, since λ_j^n (resp. λ_{j+1}^n) tends a.s. to λ_j (resp. λ_{j+1}) when n goes to infinity, one has almost surely, for k large enough,

$$\lambda_j^{n_k} < q_1 < q_2 < \lambda_{j+1}^{n_k}.$$

Hence, $\mathcal{D}(L_{n_k}, (\gamma_j^{n_k})_{j\in\mathbb{Z}}, h)$ has exactly one point in (q_1, q_2), and this point is necessarily $\mu_j^{n_k}$. One deduces that almost surely, $|\mu_j^{n_k} - \mu_j| \leq \epsilon$ for k large enough, which implies, by taking $\epsilon \to 0$, that $\mu_j^{n_k}$ converges almost surely to μ_j when k goes to infinity.

Now, let f be a function from $\mathbb{R}$ to $\mathbb{R}_+$, C^∞ and compactly supported. Since L is locally finite, there exists a.s. an integer $j_0 \geq 1$ such that the support of f is included in $(\lambda_{-j_0}, \lambda_{j_0})$, and then in $(\lambda_{-j_0}^{n_k}, \lambda_{j_0}^{n_k})$ for k large enough, which implies that $f(\mu_j^{n_k}) = f(\mu_j) = 0$ for $|j| > j_0$. Hence, a.s., there exists $j_0, k_0 \geq 1$, such that for $k \geq k_0$,

$$\sum_{j\in\mathbb{Z}} f(\mu_j^{n_k}) = \sum_{|j|\leq j_0} f(\mu_j^{n_k})$$

and

$$\sum_{j\in\mathbb{Z}} f(\mu_j) = \sum_{|j|\leq j_0} f(\mu_j),$$

which implies that

$$\sum_{j\in\mathbb{Z}} f(\mu_j^{n_k}) \underset{k\to\infty}{\longrightarrow} \sum_{j\in\mathbb{Z}} f(\mu_j), \tag{14}$$

since $f(\mu_j^{n_k})$ tends to $f(\mu_j)$ for each $j \in \{-j_0, -j_0+1, \ldots, j_0\}$.

The almost sure convergence (14) holds a fortiori in distribution, which implies that the law of $\mathcal{D}(L_{n_k}, (\gamma_j^{n_k})_{j\in\mathbb{Z}}, h)$ tends to the law of $\mathcal{D}(L, (\gamma_j)_{j\in\mathbb{Z}}, h)$. On the other hand, by Theorem 12, $\mathcal{D}(L_{n_k}, (\gamma_j^{n_k})_{j\in\mathbb{Z}}, h)$ has distribution $\mathbb{Q}_{n_k,\beta}$, and then $\mathcal{D}(L, (\gamma_j)_{j\in\mathbb{Z}}, h)$ follows the limit of the distribution $\mathbb{Q}_{n_k,\beta}$ for k tending to infinity, i.e. $\mathbb{Q}_\beta$. This shows the first part of Theorem 13, and the second part for $k = 2$; iterating this argument shows the general k case. □

8 Properties of continuity for the Stieltjes Markov chain

In the previous section, we have deduced the convergence of the Markov mechanism associated to $\mathbb{Q}_{n_k,\beta}$ towards the one corresponding to $\mathbb{Q}_\beta$ from the convergence of $\mathbb{Q}_{n_k,\beta}$ to $\mathbb{Q}_\beta$ itself, and the convergence of the associated weights. Later in the paper, we will prove similar results related to the Gaussian ensembles, for which the situation is more difficult to handle, in particular because of the lack of symmetry of the GβE at the macroscopic scale, when we rescale around a non-zero point of the bulk. Moreover, we will have to consider several steps of the Markov mechanism at the same time. That is why we will need a more general result, giving a property of continuity of the Markov mechanism described above, with respect to its initial data.

The main results of the present paper concern convergence in distribution of point processes. In this section, we will assume properties of strong convergence, which can be done with the help of Skorokhod's representation theorem.

The notion of convergence of holomorphic functions usually considered is the uniform convergence on compact sets. This notion cannot be directly applied to the meromorphic functions involved here, because of the poles on the real line. That is why we will need an appropriate notion of uniform convergence of meromorphic functions.

More precisely, we say that a sequence $(f_n)_{n\geq 1}$ of meromorphic functions on an open set $U \subset \mathbb{C}$ converges uniformly to a function f from U to the Riemann sphere $\mathbb{C}\cup\{\infty\}$ if and only if this convergence holds for the distance d on $\mathbb{C}\cup\{\infty\}$, given by

$$d(z_1, z_2) = \frac{|z_2 - z_1|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}}$$

for $z_1, z_2 \neq \infty$, and extended by continuity at ∞ (d corresponds to the distance of the points on the Euclidean sphere, obtained via the inverse stereographic projection). It is a classical result that the limiting function f should be meromorphic on U. One deduces the following: if a sequence $(f_n)_{n\geq 1}$ of meromorphic functions on $\mathbb{C}$ converges to a function f from $\mathbb{C}$ to $\mathbb{C}\cup\{\infty\}$, uniformly on all bounded subsets of $\mathbb{C}$, then f is meromorphic on $\mathbb{C}$. Moreover, the following lemma will be useful:

Lemma 18 *Let $(f_n)_{n\geq 1}$ (resp. $(g_n)_{n\geq 1}$) be a sequence of meromorphic functions on an open set U, uniformly convergent (for the distance d) to a function f (resp. g), necessarily meromorphic. We assume that f and g have no common pole. Then the sequence $(f_n + g_n)_{n\geq 1}$ of meromorphic functions tends uniformly to $f + g$ on all the compact sets of U.*

Remark 19 The fact that f and g have no common pole is needed in general. Indeed, if U is a neighborhood of 0, $f_n(z) = f(z) = -z^{-1}$, $g_n(z) = (z + n^{-1})^{-1}$, $g(z) = z^{-1}$, we check that f_n and g_n respectively tend to f and g, uniformly on compact sets of U, for the distance d, but $f_n + g_n$ does not uniformly converge to $f + g = 0$ in any neighborhood of 0.

Proof Let K be a compact subset of U, let $z_1, z_2, \ldots, z_p$ be the poles of f in K, and $z'_1, z'_2, \ldots, z'_q$ the poles of g in K. There exists a neighborhood V of $\{z_1, z_2, \ldots, z_p\}$ containing no pole of g, and a neighborhood W of $\{z'_1, z'_2, \ldots, z'_p\}$ containing no pole of f. If $A > 0$ is fixed, one can assume the following (by restricting V and W if it is needed):

- The infimum of $|f|$ on V is larger than $2A + 1$ and also larger than the supremum of $2|g| + 1$ on V.
- The infimum of $|g|$ on W is larger than $2A + 1$ and also larger than the supremum of $2|f| + 1$ on W.

By the assumption of uniform convergence, we deduce, for n large enough:

- The infimum of $|f_n|$ on V is larger than $2A$ and also larger than the supremum of $2|g_n|$ on V.
- The infimum of $|g_n|$ on W is larger than $2A$ and also larger than the supremum of $2|f_n|$ on W.

Now, for all $z \in V$ and n large enough, one has

$$|f_n(z) + g_n(z)| \geq |f_n(z)| - |g_n(z)| \geq |f_n(z)| - \frac{|f_n(z)|}{2} = \frac{|f_n(z)|}{2} \geq A.$$

and also

$$|f(z) + g(z)| \geq A,$$

which implies

$$d(f_n(z) + g_n(z), f(z) + g(z)) \leq 2/A.$$

Similarly, this inequality is true for $z \in W$. Moreover, there exists a compact set $L \subset K$, containing no pole of f or g, and such that K is included in $L \cup V \cup W$. Since the meromorphic functions f and g have no pole on the compact set L, they are bounded on this set. Since $(f_n)_{n\geq 1}$ (resp. $(g_n)_{n\geq 1}$) converges to f (resp. g) on L, uniformly for the distance d, and $(f_n)_{n\geq 1}$ (resp. $(g_n)_{n\geq 1}$) is uniformly bounded, the uniform convergence holds in fact for the usual distance. Hence, $(f_n + g_n)_{n\geq 1}$ tends

uniformly to $f + g$ on L for the usual distance, and a fortiori for d: by using the previous bounded obtained in V and W, one deduces, since L, V and W cover K:

$$\limsup_{n\to\infty} \sup_{z\in K} d(f_n(z) + g_n(z),\, f(z) + g(z)) \leq 2/A.$$

Since we can choose $A > 0$ arbitrarily, we are done. □

From this lemma, we deduce the following statement

Lemma 20 *Let $p \geq 1$, and let $(\lambda_k)_{1\leq k\leq p}$, $(\lambda_{n,k})_{n\geq 1, 1\leq k\leq p}$, $(\gamma_k)_{1\leq k\leq p}$, $(\gamma_{n,k})_{n\geq 1, 1\leq k\leq p}$ be some complex numbers such that all the λ_k's are distincts, all the γ_k's are nonzero, and for all $k \in \{1, \ldots, p\}$,*

$$\lambda_{n,k} \underset{n\to\infty}{\longrightarrow} \lambda_k$$

and

$$\gamma_{n,k} \underset{n\to\infty}{\longrightarrow} \gamma_k.$$

Then, one has, for n going to infinity, the convergence of the rational function

$$z \mapsto \sum_{k=1}^{p} \frac{\gamma_{n,k}}{\lambda_{n,k} - z}$$

towards the function

$$z \mapsto \sum_{k=1}^{p} \frac{\gamma_k}{\lambda_k - z},$$

uniformly on all the compact sets, for the distance d.

Proof Let us first prove the result for $p = 1$, which is implied by the following convergence

$$\frac{\gamma_{n,1}}{\lambda_{n,1} - z} \underset{n\to\infty}{\longrightarrow} \frac{\gamma_1}{\lambda_1 - z},$$

uniformly on $\mathbb{C}$ for the distance d. Let us fix $\epsilon > 0$. For n large enough, we have $|\lambda_{n,1} - \lambda_1| \leq \epsilon$ and $|\gamma_{n,1} - \gamma_1| \leq |\gamma_1|/2$. If these conditions are satisfied and if $|\lambda_1 - z| \leq 2\epsilon$, then

$$\left|\frac{\gamma_1}{\lambda_1 - z}\right| \geq \frac{|\gamma_1|}{2\epsilon}$$

and

$$\left|\frac{\gamma_{n,1}}{\lambda_{n,1}-z}\right| \geq \frac{|\gamma_1|}{6\epsilon},$$

since $|\gamma_{n,1}| \geq |\gamma_1|/2$ and

$$|\lambda_{n,1}-z| \leq |\lambda_1 - z| + |\lambda_{n,1}-\lambda_1| \leq 3\epsilon.$$

Hence, there exists $n_0 \geq 1$, independent of z satisfying $|\lambda_1 - z| \leq 2\epsilon$, such that for $n \geq n_0$,

$$d\left(\frac{\gamma_1}{\lambda_1 - z}, \frac{\gamma_{n,1}}{\lambda_{n,1}-z}\right) \leq d\left(\frac{\gamma_1}{\lambda_1 - z}, \infty\right) + d\left(\infty, \frac{\gamma_{n,1}}{\lambda_{n,1}-z}\right) \leq \frac{2\epsilon}{|\gamma_1|} + \frac{6\epsilon}{|\gamma_1|} = \frac{8\epsilon}{|\gamma_1|}.$$

Similarly, there exists $n_1 \geq 1$ such that for all $n \geq n_1$ and for all z satisfying $|\lambda_1 - z| \geq 2\epsilon$, one has:

$$|\lambda_{n,1}-z| \geq |\lambda_1 - z| - |\lambda_{n,1}-\lambda_1| \geq \epsilon.$$

This implies:

$$\begin{aligned}\left|\frac{\gamma_1}{\lambda_1 - z} - \frac{\gamma_{n,1}}{\lambda_{n,1}-z}\right| &\leq \left|\frac{\gamma_1 - \gamma_{n,1}}{\lambda_{n,1}-z}\right| + |\gamma_1|\left|\frac{1}{\lambda_1 - z} - \frac{1}{\lambda_{n,1}-z}\right| \\ &\leq \frac{|\gamma_1 - \gamma_{n,1}|}{\epsilon} + |\gamma_1|\frac{|\lambda_1 - \lambda_{n,1}|}{(2\epsilon)(\epsilon)}.\end{aligned}$$

Since this quantity does not depend on z and tends to zero at infinity, we deduce

$$\sup_{z\in\mathbb{C}, |\lambda_1 - z| \geq 2\epsilon} d\left(\frac{\gamma_1}{\lambda_1 - z}, \frac{\gamma_{n,1}}{\lambda_{n,1}-z}\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Since we know that

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{C}, |\lambda_1 - z| \leq 2\epsilon} d\left(\frac{\gamma_1}{\lambda_1 - z}, \frac{\gamma_{n,1}}{\lambda_{n,1}-z}\right) \leq \frac{8\epsilon}{|\gamma_1|},$$

we get

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{C}} d\left(\frac{\gamma_1}{\lambda_1 - z}, \frac{\gamma_{n,1}}{\lambda_{n,1}-z}\right) \leq \frac{8\epsilon}{|\gamma_1|}.$$

Now, $\epsilon > 0$ can be arbitrarily chosen, and then the lemma is proven for $p = 1$. For $p \geq 2$, let us deduce the result of the lemma, assuming that it is satisfied when p is

replaced by $p-1$. We define the meromorphic functions $(f_n)_{n\geq 1}$, f, $(g_n)_{n\geq 1}$, g by the formulas:

$$f_n(z) = \sum_{k=1}^{p-1} \frac{\gamma_{n,k}}{\lambda_{n,k} - z},$$
$$f(z) = \sum_{k=1}^{p-1} \frac{\gamma_k}{\lambda_k - z},$$
$$g_n(z) = \frac{\gamma_{n,p}}{\lambda_{n,p} - z},$$
$$g(z) = \frac{\gamma_p}{\lambda_p - z}.$$

Let $A > 0$. By the induction hypothesis, we know that f_n converges to f when n goes to infinity, uniformly on the set $\{z \in \mathbb{C}, |z| < 2A\}$ and for the distance d. Similarly, by the case $p = 1$ proven above, g_n converges to g, uniformly on the same set (in fact, uniformly on $\mathbb{C}$) and for the same distance. Moreover, the functions f and g have no common pole, since the numbers $(\lambda_k)_{1\leq k\leq p}$ are all distinct. We can then apply Lemma 18 and deduce that $f_n + g_n$ converges to $f + g$, uniformly on any compact set of $\{z \in \mathbb{C}, |z| < 2A\}$, for example $\{z \in \mathbb{C}, |z| \leq A\}$, and for the distance d. Since $A > 0$ can be arbitrarily chosen, we are done. □

We have now the ingredients needed to state the main result of this section. In this theorem, we deal with finite and infinite sequences together. So we will think of $k \mapsto \lambda_k$ as a function from $\mathbb{Z} \to \mathbb{R} \cup \{\emptyset\}$, with the convention that summation and other operations are only considered over the values that are different from $\emptyset$. We will also assume that the value $\emptyset$ is taken exactly on the complement of an interval of $\mathbb{Z}$.

The statement of the following result is long and technical, but as we will see in the next section, it will be adapted to the problem we are interested in.

Theorem 21 *Let $(\Xi_n)_{n\geq 1}$ be a sequence of discrete simple point measures on $\mathbb{R}$ (i.e. sums of Dirac masses at a locally finite set of points), converging to a simple point measure Ξ, locally weakly:*

$$\Xi_n \longrightarrow \Xi. \tag{15}$$

Let L_n denote the support of Ξ_n, and L the support of Ξ. We suppose that there exists $\alpha \in (0, 1)$, a family $(\tau_\ell)_{\ell\geq 0}$ of elements of $\mathbb{R}_+^*$, with $\tau_\ell \to 0$ as $\ell \to \infty$, such that for all $n \geq 1$, $\ell \geq 1$, we have

$$\int_{\mathbb{R}} \frac{\mathbf{1}(|\lambda| > \ell)}{|\lambda|^{1+\alpha}}\, d\Xi_n(\lambda) \leq \tau_\ell \tag{16}$$

Moreover, assume that the limits

$$h_{n,\ell} = \lim_{\ell'\to\infty} \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| < \ell')}{\lambda}\, d\Xi_n(\lambda) \tag{17}$$

exist, and so does the similar limit h_ℓ defined in terms of Ξ. Assume further that for some $h \in \mathbb{R}$, the following equalities are well-defined and satisfied:

$$\lim_{\ell\to\infty}\lim_{n\to\infty} h_{n,\ell} = h, \qquad \lim_{\ell\to\infty} h_\ell = 0, \tag{18}$$

when the limits are restricted to the condition: $\ell \notin L$ and $-\ell \notin L$.

Further, let $(\gamma_{n,k})_{k\in\mathbb{Z}}$ be a strictly positive sequence. Suppose it satisfies

$$\gamma_{n,k} \to \gamma_k > 0 \tag{19}$$

for each k, as $n \to \infty$. Also for some $\bar\gamma, c > 0$ and all $n, m \geq 1$, we assume

$$\begin{aligned}\left|\sum_{k=0}^{m-1} \gamma_{n,k} - \bar\gamma m\right| &\leq c m^{\alpha'} \\ \left|\sum_{k=-m}^{-1} \gamma_{n,k} - \bar\gamma m\right| &\leq c m^{\alpha'}\end{aligned} \tag{20}$$

with $0 < (1+\alpha)\alpha' < 1$. Let λ^* be a point outside L, and consider the weighted version Λ of Ξ where the kth point after λ^* (for $k \leq 0$, the $(1-k)$th point before λ^*) has weight γ_k. For n large enough, one has also $\lambda^* \notin L_n$: define Λ_n similarly. Then the limit

$$S_n(z) = \lim_{\ell\to\infty} \int_{[-\ell,\ell]} \frac{1}{\lambda - z}\, d\Lambda_n(\lambda)$$

exists for all $z \notin L_n$, is meromorphic with simple poles at L_n, and converges, uniformly on compacts with respect to the distance d on the Riemann sphere $\mathbb{C}\cup\{\infty\}$, to $S(z)+\bar\gamma h$, where S is a meromorphic function with simple poles at L, such that for all $z \notin L$,

$$S(z) = \lim_{\ell\to\infty} \int_{[-\ell,\ell]} \frac{1}{\lambda - z}\, d\Lambda(\lambda).$$

Moreover, for every $h' \in \mathbb{R}$, the sum of delta masses Ξ'_n at $S_n^{-1}(h' + \bar\gamma h)$ converges locally weakly to the sum of delta masses Ξ' at $S^{-1}(h')$, and $(\Xi'_n)_{n\geq 1}$, and Ξ'_n, Ξ' satisfy assumptions equivalent to (15)–(18), i.e.

$$\Xi'_n \longrightarrow \Xi', \tag{21}$$

$$\int_{\mathbb{R}} \frac{\mathbf{1}(|\lambda| > \ell)}{|\lambda|^{1+\alpha}}\, d\Xi'_n(\lambda) \leq \tau'_\ell \tag{22}$$

for a family $(\tau'_\ell)_{\ell\geq 0}$ of elements of $\mathbb{R}_+^*$, with $\tau'_\ell \to 0$ as $\ell \to \infty$,

$$\lim_{\ell\to\infty}\lim_{n\to\infty} h'_{n,\ell} = h, \qquad \lim_{\ell\to\infty} h'_\ell = 0, \tag{23}$$

where

$$h'_{n,\ell} = \lim_{\ell' \to \infty} \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| < \ell')}{\lambda} \, d\Xi'_n(\lambda), \quad h'_\ell = \lim_{\ell' \to \infty} \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| < \ell')}{\lambda} \, d\Xi'(\lambda) \tag{24}$$

and the limits in ℓ and n are restricted to the condition: $\ell \notin L'$ and $-\ell \notin L'$, L' being the support of Ξ'.

Remark 22 After a suitable translation of Ξ_n and Ξ, one can assume that $0 \notin L$ and then one can take $\lambda^* = 0$. The choice of λ^* does not change the structure of the proof of the theorem: replacing λ^* by 0 may slightly simplify its reading. In (18), the condition $\ell \notin L$, $-\ell \notin L$ ensures that the boundary terms in the integral defining $h_{n,\ell}$ does not perturb the existence of the limit of $h_{n,\ell}$ when $n \to \infty$. It may be possible to avoid this technicality by considering upper and lower limits in n.

Proof We have to show the following:

- The existence of the limits defining S_n and S, and the fact that they are meromorphic, with simple poles at L_n and L, respectively.
- The convergence of S_n towards $S + \bar{\gamma} h$, uniformly in compacts, for the distance d.
- The assumptions (21)–(24), which should be satisfied by Ξ'_n and Ξ'.

We will successively show these three statements: the most difficult one is the convergence of S_n towards $S + \bar{\gamma} h$.

Existence and properties of the limits defining S_n and S: Let $n \geq 1$, large enough in order to ensure that $\lambda^* \notin L_n$, $1 < \ell_0 < \ell \leq \infty$, and let z be a complex number with modulus smaller than $\ell_0/2$. Let k_{n,ℓ_0} be the smallest index k (if it exists) such that $\lambda_{n,k} > \ell_0$, where $\lambda_{n,k}$ is (if it exists) the kth point of L_n after λ^* for $k \geq 1$, and the $(1-k)$th point of L_n before λ^* for $k \leq 0$. Similarly, let $K_{n,\ell} - 1$ be the largest index k (if it exists) such that $\lambda_{n,k} \leq \ell$. For $\ell = \infty$ and L_n bounded from above, we get that $K_{n,\infty} - 1$ is the largest index k (if it exists) such that $\lambda_{n,k}$ exists, i.e. the index of the largest point of L_n. For $\ell = \infty$ and L_n not bounded from above, we have $K_{n,\infty} = \infty$. If for $m \in \mathbb{Z}$,

$$\Delta_{n,m} := \mathbf{1}_{m \geq 0} \sum_{k=0}^{m-1} \gamma_{n,k} - \mathbf{1}_{m<0} \sum_{k=m}^{-1} \gamma_{n,k} - \bar{\gamma}\, m,$$

then, in the case where k_{n,ℓ_0} and $K_{n,\ell}$ are well-defined and $k_{n,\ell_0} < K_{n,\ell}$:

$$\int_{(\ell_0,\ell]} \frac{1}{\lambda - z} \, d\Lambda_n(\lambda) = \sum_{k_{n,\ell_0} \leq k < K_{n,\ell}} \frac{\gamma_{n,k}}{\lambda_{n,k} - z} = \bar{\gamma} \sum_{k_{n,\ell_0} \leq k < K_{n,\ell}} \frac{1}{\lambda_{n,k} - z}$$

$$+\sum_{k_{n,\ell_0}\le k<K_{n,\ell}}\frac{\Delta_{n,k+1}-\Delta_{n,k}}{\lambda_{n,k}-z}=\bar\gamma\left(\sum_{k_{n,\ell_0}\le k<K_{n,\ell}}\frac{1}{\lambda_{n,k}-z}\right)+\frac{\Delta_{n,K_{n,\ell}}}{\lambda_{n,K_{n,\ell}-1}-z}$$
$$-\frac{\Delta_{n,k_{n,\ell_0}}}{\lambda_{n,k_{n,\ell_0}}-z}+\sum_{k_{n,\ell_0}+1\le k<K_{n,\ell}}\Delta_{n,k}\left(\frac{1}{\lambda_{n,k-1}-z}-\frac{1}{\lambda_{n,k}-z}\right),$$

which implies

$$\int_{(\ell_0,\ell]}\frac{1}{\lambda-z}\,d\Lambda_n(\lambda)-\bar\gamma\int_{(\ell_0,\ell]}\frac{d\Xi_n(\lambda)}{\lambda}$$
$$=\bar\gamma z\left(\sum_{k_{n,\ell_0}\le k<K_{n,\ell}}\frac{1}{\lambda_{n,k}(\lambda_{n,k}-z)}\right)+\frac{\Delta_{n,K_{n,\ell}}}{\lambda_{n,K_{n,\ell}-1}-z}$$
$$-\frac{\Delta_{n,k_{n,\ell_0}}}{\lambda_{n,k_{n,\ell_0}}-z}+\sum_{k_{n,\ell_0}+1\le k<K_{n,\ell}}\Delta_{n,k}\left(\frac{\lambda_{n,k}-\lambda_{n,k-1}}{(\lambda_{n,k-1}-z)(\lambda_{n,k}-z)}\right).$$

Note that in case where k_{n,ℓ_0} or $K_{n,\ell}$ is not well-defined, and in case where $k_{n,\ell_0}\ge K_{n,\ell}$, the left-hand side is zero, since L_n has no point in the interval $(\ell_0,\ell]$. Let us now check that for ℓ going to infinity, this quantity converges, uniformly in $\{z\in\mathbb{C},\,|z|<\ell_0/2\}$, to the function T_{n,ℓ_0}, holomorphic on this open set, and given by

$$T_{n,\ell_0}(z)=\bar\gamma z\left(\sum_{k\ge k_{n,\ell_0}}\frac{1}{\lambda_{n,k}(\lambda_{n,k}-z)}\right)+\frac{\Delta_{n,K_{n,\infty}}}{\lambda_{n,K_{n,\infty}-1}-z}-\frac{\Delta_{n,k_{n,\ell_0}}}{\lambda_{n,k_{n,\ell_0}}-z}$$
$$+\sum_{k\ge k_{n,\ell_0}+1}\Delta_{n,k}\left(\frac{\lambda_{n,k}-\lambda_{n,k-1}}{(\lambda_{n,k-1}-z)(\lambda_{n,k}-z)}\right),\tag{25}$$

if L_n has at least one point in (ℓ_0,∞), and $T_{n,\ell_0}(z)=0$ otherwise. In the formula above, when L_n is not bounded from above, and then $K_{n,\infty}=\infty$, we let, by convention:

$$\frac{\Delta_{n,K_{n,\infty}}}{\lambda_{n,K_{n,\infty}-1}-z}:=0.$$

In order to prove this convergence, it is sufficient to check, in the case $L_n\cap(\ell_0,\infty)\neq\emptyset$, the uniform convergence

$$\frac{\Delta_{n,K_{n,\ell}}}{\lambda_{n,K_{n,\ell}-1}-z}\underset{\ell\to\infty}{\longrightarrow}\frac{\Delta_{n,K_{n,\infty}}}{\lambda_{n,K_{n,\infty}-1}-z},$$

for $|z| \leq \ell_0/2$, and the fact that

$$\sup_{z\in\mathbb{C},|z|\leq\ell_0/2}\left(\sum_{k\geq k_{n,\ell_0}}\frac{1}{|\lambda_{n,k}||\lambda_{n,k}-z|}+\sum_{k\geq k_{n,\ell_0}+1}|\Delta_{n,k}|\left(\frac{|\lambda_{n,k}-\lambda_{n,k-1}|}{|\lambda_{n,k-1}-z|\,|\lambda_{n,k}-z|}\right)\right)<\infty. \quad (26)$$

The first statement is immediate if L_n is bounded from above. If L_n is unbounded from above, let us remark that for $k \geq k_{n,\ell_0}$, $|z| \leq \ell_0/2$, one has $|\lambda_{n,k} - z| \geq \lambda_{n,k}/2$, and then it is sufficient to show:

$$\frac{\Delta_{n,K_{n,\ell}}}{\lambda_{n,K_{n,\ell}-1}} \underset{\ell\to\infty}{\longrightarrow} 0. \quad (27)$$

Similarly, the statement (26) is implied by:

$$\sum_{k\geq k_{n,\ell_0}}\frac{1}{\lambda_{n,k}^2}+\sum_{k\geq k_{n,\ell_0}+1}\frac{|\Delta_{n,k}|(\lambda_{n,k}-\lambda_{n,k-1})}{\lambda_{n,k-1}\,\lambda_{n,k}}<\infty. \quad (28)$$

In order to prove (27), let us first use the majorization (16), which implies, for all $\ell > 2$,

$$\Xi_n([2,\ell]) \leq \ell^{1+\alpha}\int_2^\ell \frac{d\,\Xi_n(\lambda)}{\lambda^{1+\alpha}} \leq \ell^{1+\alpha}\int_{\mathbb{R}}\frac{\mathbf{1}(|\lambda|>1)}{|\lambda|^{1+\alpha}}\,d\,\Xi_n(\lambda) \leq \tau_1\,\ell^{1+\alpha},$$

and then

$$\Xi_n([\lambda^*,\ell]) \leq \tau\,\ell^{1+\alpha}$$

where

$$\tau := \tau_1 + \Xi_n([\lambda^* \wedge 2, 2]). \quad (29)$$

We deduce, for $k \geq 1$ large enough in order to insure that $\lambda_{n,k} > 2$,

$$k = \Xi_n([\lambda^*,\lambda_{n,k}]) \leq \tau\,\lambda_{n,k}^{1+\alpha}$$

and then

$$\lambda_{n,k} \geq (k/\tau)^{1/(1+\alpha)}. \quad (30)$$

By using (20), this inequality implies:

$$\frac{\Delta_{n,k+1}}{\lambda_{n,k}} \leq c(k+1)^{\alpha'}(k/\tau)^{-1/(1+\alpha)},$$

which tends to zero when k goes to infinity, since $\alpha' < 1/(1+\alpha)$ by assumption. Therefore, we have (27).

Moreover, the left-hand side of (28) is given by

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\mathbf{1}(|\lambda| > \ell_0)}{|\lambda|^2}\, d\Xi_n(\lambda) + \sum_{k \geq k_{n,\ell_0}+1} |\Delta_{n,k}| \left(\frac{1}{\lambda_{n,k-1}} - \frac{1}{\lambda_{n,k}} \right) \\
&\leq \int_{\mathbb{R}} \frac{\mathbf{1}(|\lambda| > \ell_0)}{|\lambda|^{1+\alpha}}\, d\Xi_n(\lambda) + c \sum_{k \geq k_{n,\ell_0}+1} |k|^{\alpha'} \left(\frac{1}{\lambda_{n,k-1}} - \frac{1}{\lambda_{n,k}} \right) \\
&\leq \tau_{\ell_0} + c \left(\frac{|k_{n,\ell_0}+1|^{\alpha'}}{\lambda_{n,k_{n,\ell_0}}} + \sum_{k \geq k_{n,\ell_0}+1} \frac{(|k+1|^{\alpha'} - |k|^{\alpha'})}{\lambda_{n,k}} \right).
\end{aligned}$$

If L_n is bounded from above, the finiteness of this quantity is obvious. Otherwise, we know that for k large enough, $(|k+1|^{\alpha'} - |k|^{\alpha'})$ is bounded by a constant times $k^{\alpha'-1}$, and $\lambda_{k,n}$ dominates $k^{1/(1+\alpha)}$. Hence, it is sufficient to check the finiteness of the following expression:

$$\sum_{k=1}^{\infty} k^{\alpha'-1} k^{-1/(1+\alpha)},$$

which is satisfied since by assumption,

$$\alpha' - 1 - \frac{1}{1+\alpha} < -1.$$

We have now proven:

$$\int_{(\ell_0,\ell]} \frac{1}{\lambda - z}\, d\Lambda_n(\lambda) - \bar{\gamma} \int_{(\ell_0,\ell]} \frac{d\Xi_n(\lambda)}{\lambda} \underset{\ell\to\infty}{\longrightarrow} T_{n,\ell_0}(z), \tag{31}$$

uniformly on the set $\{z \in \mathbb{C}, |z| < \ell_0/2\}$, where the holomorphic function T_{n,ℓ_0} is given by the formula (25).

Similarly, there exists an holomorphic function U_{n,ℓ_0} on $\{z \in \mathbb{C}, |z| < \ell_0/2\}$, such that uniformly on this set,

$$\int_{[-\ell,-\ell_0)} \frac{1}{\lambda - z}\, d\Lambda_n(\lambda) - \bar{\gamma} \int_{[-\ell,-\ell_0)} \frac{d\Xi_n(\lambda)}{\lambda} \underset{\ell\to\infty}{\longrightarrow} U_{n,\ell_0}(z). \tag{32}$$

The function U_{n,ℓ_0} can be explicitly described by a formula similar to (25) (we omit the detail of this formula). By combining (17), (31) and (32), one deduces the following uniform convergence on $\{z \in \mathbb{C}, |z| < \ell_0/2\}$:

$$\int_{[-\ell,\ell]\setminus[-\ell_0,\ell_0]} \frac{1}{\lambda - z}\, d\Lambda_n(\lambda) \underset{\ell\to\infty}{\longrightarrow} T_{n,\ell_0}(z) + U_{n,\ell_0}(z) + \bar{\gamma} h_{n,\ell_0}.$$

One deduces, by using Lemma 18, that

$$\int_{[-\ell,\ell]} \frac{1}{\lambda - z}\, d\Lambda_n(\lambda) \underset{\ell\to\infty}{\longrightarrow} T_{n,\ell_0}(z) + U_{n,\ell_0}(z) + \bar{\gamma} h_{n,\ell_0} + \int_{[-\ell_0,\ell_0]} \frac{1}{\lambda - z}\, d\Lambda_n(\lambda) =: S_{n,\ell_0}(z),$$

uniformly on any compact subset of $\{z \in \mathbb{C}, |z| < \ell_0/2\}$, for the distance d on the Riemann sphere. One checks immediately that the poles of S_{n,ℓ_0} with modulus smaller than or equal to $\ell_0/2$ are exactly the points of L_n satisfying the same condition. Moreover, the convergence just above implies that for $\ell_1 > \ell_0 > 1$, the meromorphic functions S_{n,ℓ_0} and S_{n,ℓ_1} coincide on $\{z \in \mathbb{C}, |z| < \ell_0/2\}$: hence, there exists a meromorphic function S_n on $\mathbb{C}$, such that for all $\ell_0 > 1$, the restriction of S_n to $\{z \in \mathbb{C}, |z| < \ell_0/2\}$ is equal to S_{n,ℓ_0}. The poles of S_n are exactly the points of L_n, and one has, uniformly on all compact sets of $\mathbb{C}$ and for the distance d,

$$\int_{[-\ell,\ell]} \frac{1}{\lambda - z}\, d\Lambda_n(\lambda) \underset{\ell\to\infty}{\longrightarrow} S_n(z).$$

In particular, the convergence holds pointwise for all $z \notin L_n$.

In an exactly similar way, one can prove that uniformly on compact sets of $\mathbb{C}$, for the distance d,

$$\int_{[-\ell,\ell]} \frac{1}{\lambda - z}\, d\Lambda(\lambda) \underset{\ell\to\infty}{\longrightarrow} S(z)$$

where for all $\ell_0 > 1$,

$$S(z) := T_{\ell_0}(z) + U_{\ell_0}(z) + \bar{\gamma} h_{\ell_0} + \int_{[-\ell_0,\ell_0]} \frac{1}{\lambda - z}\, d\Lambda(\lambda),$$

on the set $\{z \in \mathbb{C}, |z| < \ell_0/2\}$, T_{ℓ_0} and U_{ℓ_0} being defined by the same formulas as T_{n,ℓ_0} and U_{n,ℓ_0}, except than one removes all the indices n. In order to show this convergence, it is sufficient to check that the assumptions (16) and (20) are satisfied if the indices n are removed. For (20), it is an immediate consequence of the convergence (19), since the constant c does not depend on n. For (16), let us first observe that for all $\ell > 1$, and for any continuous function Φ with compact support, such that for all $\lambda \in \mathbb{R}$,

$$\Phi(\lambda) \le \frac{\mathbf{1}(|\lambda| > \ell)}{|\lambda|^{1+\alpha}},$$

one has for all $n \ge 1$,

$$\int_{\mathbb{R}} \Phi(\lambda) d\Xi_n(\lambda) \le \int_{\mathbb{R}} \frac{\mathbf{1}(|\lambda| > \ell)}{|\lambda|^{1+\alpha}}\, d\Xi_n(\lambda) \le \tau_\ell.$$

Since Ξ_n converges weakly to Ξ when n goes to infinity, one deduces:

$$\int_{\mathbb{R}} \Phi(\lambda) d\Xi(\lambda) \leq \tau_\ell.$$

By taking Φ increasing to

$$\lambda \mapsto \mathbf{1}(|\lambda| > \ell)/|\lambda|^{1+\alpha},$$

one obtains

$$\int_{\mathbb{R}} \frac{\mathbf{1}(|\lambda| > \ell)}{|\lambda|^{1+\alpha}} d\Xi(\lambda) \leq \tau_\ell,$$

i.e. the equivalent of (16) for the measure Ξ.

Convergence of S_n towards $S + \bar{\gamma} h$: Once the existence of the functions S_n and S is ensured, it remains to prove the convergence of S_n towards $S + \bar{\gamma} h$, uniformly on compact sets for the distance d. In order to check this convergence, it is sufficient to prove that for all $\ell > 1$, there exists $\ell_0 > \ell$ such that uniformly on any compact set of $\{z \in \mathbb{C}, |z| < \ell_0/2\}$,

$$\begin{aligned} &T_{n,\ell_0}(z) + U_{n,\ell_0}(z) + \bar{\gamma} h_{n,\ell_0} + \int_{[-\ell_0,\ell_0]} \frac{1}{\lambda - z} d\Lambda_n(\lambda) \\ &\underset{n\to\infty}{\longrightarrow} T_{\ell_0}(z) + U_{\ell_0}(z) + \bar{\gamma}(h_{\ell_0} + h) + \int_{[-\ell_0,\ell_0]} \frac{1}{\lambda - z} d\Lambda(\lambda). \end{aligned}$$

In fact, we will prove this convergence for any $\ell_0 > 2$ such that ℓ_0 and $-\ell_0$ are not in L, and then not in L_n for n large enough. By Lemma 18, it is sufficient to check for such an ℓ_0:

$$h_{n,\ell_0} \underset{n\to\infty}{\longrightarrow} h_{\ell_0} + h, \tag{33}$$

$$\int_{[-\ell_0,\ell_0]} \frac{1}{\lambda - z} d\Lambda_n(\lambda) \underset{n\to\infty}{\longrightarrow} \int_{[-\ell_0,\ell_0]} \frac{1}{\lambda - z} d\Lambda(\lambda), \tag{34}$$

uniformly on $\{z \in \mathbb{C}, |z| < \ell_0/2\}$ for the distance d,

$$T_{n,\ell_0}(z) \underset{n\to\infty}{\longrightarrow} T_{\ell_0}(z), \tag{35}$$

uniformly on $\{z \in \mathbb{C}, |z| < \ell_0/2\}$, and

$$U_{n,\ell_0}(z) \underset{n\to\infty}{\longrightarrow} U_{\ell_0}(z), \tag{36}$$

also uniformly on $\{z \in \mathbb{C}, |z| < \ell_0/2\}$. Since the proof of (36) is exactly similar to the proof of (35), we will omit it and we will then show successively (33)–(35). □

Proof of 33 For all $\ell_1 > \ell_0$ such that $-\ell_1$ and ℓ_1 are not in L, one has

$$h_{n,\ell_0} - h_{n,\ell_1} = \int_{\mathbb{R}} \frac{\mathbf{1}(\ell_0 < |\lambda| \leq \ell_1)}{\lambda} \, d\Xi_n(\lambda)$$

and

$$h_{\ell_0} - h_{\ell_1} = \int_{\mathbb{R}} \frac{\mathbf{1}(\ell_0 < |\lambda| \leq \ell_1)}{\lambda} \, d\Xi(\lambda).$$

Now, $-\ell_1, -\ell_0, \ell_0, \ell_1$ are not in the support of Ξ, and since Ξ is a discrete measure, there is a neighborhood of $\{-\ell_1, -\ell_0, \ell_0, \ell_1\}$ which does not charge Ξ. One deduces that there exist two functions Φ and Ψ from $\mathbb{R}$ to $\mathbb{R}_+$, continuous with compact support, such that for all $\lambda \in \mathbb{R}$,

$$\Phi(\lambda) \leq \frac{\mathbf{1}(\ell_0 < |\lambda| \leq \ell_1)}{\lambda} \leq \Psi(\lambda)$$

and

$$\int_{\mathbb{R}} \Phi(\lambda) \, d\Xi(\lambda) = h_{\ell_0} - h_{\ell_1} = \int_{\mathbb{R}} \Psi(\lambda) \, d\Xi(\lambda).$$

Since Ξ_n tends weakly to Ξ when n goes to infinity, one deduces that

$$\int_{\mathbb{R}} \Phi(\lambda) \, d\Xi_n(\lambda) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}} \Phi(\lambda) \, d\Xi(\lambda) = h_{\ell_0} - h_{\ell_1}$$

and similarly,

$$\int_{\mathbb{R}} \Psi(\lambda) \, d\Xi_n(\lambda) \underset{n\to\infty}{\longrightarrow} h_{\ell_0} - h_{\ell_1}.$$

By the squeeze theorem, one deduces

$$h_{n,\ell_0} - h_{n,\ell_1} = \int_{\mathbb{R}} \frac{\mathbf{1}(\ell_0 < |\lambda| \leq \ell_1)}{\lambda} \, d\Xi_n(\lambda) \underset{n\to\infty}{\longrightarrow} h_{\ell_0} - h_{\ell_1}.$$

Hence,

$$\lim_{n\to\infty} h_{n,\ell_0} - \lim_{n\to\infty} h_{n,\ell_1} = h_{\ell_0} - h_{\ell_1}.$$

where, by assumption, the two limits in the left-hand side are well-defined. By (18), one deduces, by taking $\ell_1 \to \infty$,

$$\lim_{n\to\infty} h_{n,\ell_0} - h = h_{\ell_0},$$

which proves (33). □

Proof of 34 Let us now check the following properties, available for all $k \in \mathbb{Z}$:

- If λ_k is well-defined, then $\lambda_{n,k}$ is well-defined for all n large enough and tends to λ_k when n goes to infinity.
- If λ_k is not well-defined, then for all $A > 0$, there are finitely many indices n such that $\lambda_{n,k}$ is well-defined and in the interval $[-A, A]$.

By symmetry, we can assume that $k \geq 1$. We know that λ^* is not in L, and then for $\epsilon > 0$ small enough,

$$L \cap [\lambda^* - 3\epsilon, \lambda^* + 3\epsilon] = \emptyset, \tag{37}$$

Let us fix $\epsilon > 0$ satisfying this property. Since Ξ_n tends locally weakly to Ξ, we deduce that for n large enough,

$$L_n \cap [\lambda^* - 2\epsilon, \lambda^* + 2\epsilon] = \emptyset, \tag{38}$$

which implies that $\lambda_k \geq \lambda_1 > \lambda^* + 2\epsilon$. Now, let Φ and Ψ be two continuous functions with compact support, such that:

- For $\lambda \leq \lambda^* - \epsilon$,

$$\Phi(\lambda) = \Psi(\lambda) = 0.$$

- For $\lambda^* - \epsilon \leq \lambda \leq \lambda^* + \epsilon$,

$$0 \leq \Phi(\lambda) = \Psi(\lambda) \leq 1.$$

- For $\lambda^* + \epsilon \leq \lambda \leq \lambda_k - \epsilon$,

$$\Phi(\lambda) = \Psi(\lambda) = 1$$

  (recall that $\lambda^* + \epsilon < \lambda_k - \epsilon$).
- For $\lambda_k - \epsilon \leq \lambda \leq \lambda_k$,

$$0 \leq \Phi(\lambda) \leq \Psi(\lambda) = 1.$$

- For $\lambda_k \leq \lambda \leq \lambda_k + \epsilon$,

$$0 = \Phi(\lambda) \leq \Psi(\lambda) \leq 1.$$

- For $\lambda \geq \lambda_k + \epsilon$,

$$\Phi(\lambda) = \Psi(\lambda) = 0.$$

By using (37), we deduce

$$\int_{\mathbb{R}} \Phi(\lambda) d\,\Xi(\lambda) \leq \Xi([\lambda^* - \epsilon, \lambda_k)) = \Xi([\lambda^*, \lambda_k)) = k - 1$$

and

$$\int_{\mathbb{R}} \Psi(\lambda) d\,\Xi(\lambda) \geq \Xi([\lambda^* + \epsilon, \lambda_k]) = \Xi([\lambda^*, \lambda_k]) = k.$$

Hence, for n large enough,

$$\int_{\mathbb{R}} \Phi(\lambda) d\,\Xi_n(\lambda) \leq k - 1/2$$

and

$$\int_{\mathbb{R}} \Psi(\lambda) d\,\Xi_n(\lambda) \geq k - 1/2,$$

which implies

$$\Xi_n([\lambda^*, \lambda_k - \epsilon)) = \Xi_n([\lambda^* + \epsilon, \lambda_k - \epsilon)) \leq \int_{\mathbb{R}} \Phi(\lambda) d\,\Xi_n(\lambda) \leq k - 1/2$$

and

$$\Xi_n([\lambda^*, \lambda_k + \epsilon)) = \Xi_n([\lambda^* - \epsilon, \lambda_k + \epsilon]) \geq \int_{\mathbb{R}} \Psi(\lambda) d\,\Xi_n(\lambda) \geq k - 1/2.$$

Therefore, for n large enough the point $\lambda_{n,k}$ is well-defined and between $\lambda_k - \epsilon$ and $\lambda_k + \epsilon$. Since ϵ and be taken arbitrarily small, we have proven the convergence claimed above in the case where λ_k is well-defined. If λ_k is not well-defined, let us choose $\epsilon > 0$ satisfying (37), and $A > |\lambda^*|$. Let Φ be a continuous function with compact support, such that:

- For all $\lambda \in \mathbb{R}$, $\Phi(\lambda) \in [0, 1]$.
- For all $\lambda \in [\lambda^*, A]$, $\Phi(\lambda) = 1$.
- For all $\lambda \notin (\lambda^* - \epsilon, A + \epsilon)$, $\Phi(\lambda) = 0$.

Since λ_k is not well-defined,

$$\int_{\mathbb{R}} \Phi(\lambda) d\,\Xi(\lambda) \leq \Xi([\lambda^* - \epsilon, A + \epsilon]) = \Xi([\lambda^*, A + \epsilon]) \leq \Xi([\lambda^*, \infty)) \leq k - 1,$$

and then for n large enough,

$$\Xi_n([\lambda^*, A]) \leq \int_{\mathbb{R}} \Phi(\lambda) d\,\Xi_n(\lambda) \leq k - 1/2,$$

which implies that $\lambda_{n,k}$ cannot be well-defined and smaller than or equal to A. This proves the second claim. Let us now go back to the proof of (34). If $L \cap [-\ell_0, \ell_0] = \emptyset$, then $L \cap [-\ell_0 - \epsilon, \ell_0 + \epsilon] = \emptyset$ for some $\epsilon > 0$. Hence, there exists a nonnegative, continuous function with compact support Φ such that $\Phi(\lambda) = 1$ for all $\lambda \in [-\ell_0, \ell_0]$, and

$$\int_{\mathbb{R}} \Phi(\lambda) d\Xi(\lambda) = 0,$$

which implies, for n large enough,

$$\Xi([-\ell_0, \ell_0]) \leq \int_{\mathbb{R}} \Phi(\lambda) d\Xi_n(\lambda) \leq 1/2,$$

i.e. $L_n \cap [-\ell_0, \ell_0] = \emptyset$. Hence, for n large enough, the two expressions involved in (34) are identically zero. If $L \cap [-\ell_0, \ell_0] \neq \emptyset$, let k_1 and k_2 be the smallest and the largest indices k such that $\lambda_k \in (-\ell_0, \ell_0)$. Since λ_{n,k_1} and λ_{n,k_2} converge respectively to λ_{k_1} and λ_{k_2} when n goes to infinity, one has λ_{n,k_1} and λ_{n,k_2} in the interval $(-\ell_0, \ell_0)$ for n large enough. On the other hand, λ_{k_2+1} is either strictly larger than ℓ_0 (strictly because by assumption, $\ell_0 \notin L$), or not well-defined. In both cases, there are only finitely many indices n such that $\lambda_{n,k_2+1} \leq \ell_0$. Similarly, by using the fact that $-\ell_0 \notin L$, one checks that there are finitely many indices n such that $\lambda_{n,k_1-1} \geq -\ell_0$. Hence, for n large enough, the indices k such that $\lambda_{n,k} \in [-\ell_0, \ell_0]$ are exactly the integers between k_1 and k_2, which implies

$$\int_{[-\ell_0,\ell_0]} \frac{1}{\lambda - z} d\Lambda_n(\lambda) = \sum_{k=k_1}^{k_2} \frac{\gamma_{n,k}}{\lambda_{n,k} - z},$$

whereas

$$\int_{[-\ell_0,\ell_0]} \frac{1}{\lambda - z} d\Lambda(\lambda) = \sum_{k=k_1}^{k_2} \frac{\gamma_k}{\lambda_k - z}.$$

We have shown that for all k between k_1 and k_2, $\lambda_{n,k}$ tends to λ_k when n goes to infinity and by assumption, $\gamma_{n,k}$ tends to γ_k. Moreover, the numbers λ_k are all distincts, and by assumption, $\gamma_k \neq 0$ for all k. Hence, one can apply Lemma 20 to deduce (34). □

Proof of 35 If $L \cap (\ell_0, \infty) = \emptyset$, this statement can be deduced from the following convergences, uniformly on $\{z \in \mathbb{C}, |z| < \ell_0/2\}$:

$$\mathbf{1}_{L_n \cap (\ell_0,\infty) \neq \emptyset} \sum_{k \geq k_{n,\ell_0}} \frac{1}{\lambda_{n,k}(\lambda_{n,k} - z)} \underset{n\to\infty}{\longrightarrow} 0, \tag{39}$$

$$\mathbf{1}_{L_n \cap (\ell_0,\infty) \neq \emptyset} \frac{\Delta_{n,K_{n,\infty}}}{\lambda_{n,K_{n,\infty}-1} - z} \underset{n\to\infty}{\longrightarrow} 0, \tag{40}$$

$$\mathbf{1}_{L_n \cap (\ell_0, \infty) \neq \emptyset} \frac{\Delta_{n,k_{n,\ell_0}}}{\lambda_{n,k_{n,\ell_0}} - z} \underset{n\to\infty}{\longrightarrow} 0 \tag{41}$$

and

$$\mathbf{1}_{L_n \cap (\ell_0, \infty) \neq \emptyset} \sum_{k \geq k_{n,\ell_0}+1} \Delta_{n,k} \left(\frac{\lambda_{n,k} - \lambda_{n,k-1}}{(\lambda_{n,k-1} - z)(\lambda_{n,k} - z)} \right) \underset{n\to\infty}{\longrightarrow} 0. \tag{42}$$

If $L \cap (\ell_0, \infty) \neq \emptyset$, then we have proven previously that $\lambda_{n,k_{\ell_0}}$ is well-defined for n large enough and converges to $\lambda_{k_{\ell_0}} > \ell_0$ when n goes to infinity: in particular, $\lambda_{n,k_{\ell_0}} > \ell_0$ for n large enough. Moreover, one of the two following cases occurs:

- If $\lambda_{k_{\ell_0}-1}$ is well-defined, then it is strictly smaller than ℓ_0 (strictly because ℓ_0 is, by assumption, not in L), and then $\lambda_{n,k_{\ell_0}-1}$ is, for n large enough, well-defined and strictly smaller than ℓ_0.
- If $\lambda_{k_{\ell_0}-1}$ is not well-defined, and if $A > 0$, then for n large enough, $\lambda_{n,k_{\ell_0}-1}$ is not well-defined or has an absolute value strictly greater than A. By taking $A = \lambda_{k_{\ell_0}} + 1$, one deduces that for n large enough, $\lambda_{n,k_{\ell_0}-1}$ is not-well defined, strictly smaller than $-\lambda_{k_{\ell_0}} - 1$ or strictly larger than $\lambda_{k_{\ell_0}} + 1$. This last case is impossible for n large enough, since $\lambda_{n,k_{\ell_0}-1}$ is smaller than $\lambda_{n,k_{\ell_0}}$, which tends to $\lambda_{k_{\ell_0}}$. Hence, there are finitely many indices n such that $\lambda_{n,k_{\ell_0}-1}$ is well-defined and larger than $-\lambda_{k_{\ell_0}} - 1$, and a fortiori, larger than or equal to ℓ_0.

All this discussion implies easily that for n large enough, $k_{n,\ell_0} = k_{\ell_0}$, and then it is sufficient to prove the uniform convergences on $\{z \in \mathbb{C}, |z| < \ell_0/2\}$:

$$\sum_{k \geq k_{\ell_0}} \frac{1}{\lambda_{n,k}(\lambda_{n,k} - z)} \underset{n\to\infty}{\longrightarrow} \sum_{k \geq k_{\ell_0}} \frac{1}{\lambda_k(\lambda_k - z)}, \tag{43}$$

$$\frac{\Delta_{n,k_{\ell_0}}}{\lambda_{n,k_{\ell_0}} - z} \underset{n\to\infty}{\longrightarrow} \frac{\Delta_{k_{\ell_0}}}{\lambda_{k_{\ell_0}} - z} \tag{44}$$

and

$$\begin{aligned} &\frac{\Delta_{n,K_{n,\infty}}}{\lambda_{n,K_{n,\infty}-1} - z} + \sum_{k \geq k_{\ell_0}+1} \Delta_{n,k} \left(\frac{\lambda_{n,k} - \lambda_{n,k-1}}{(\lambda_{n,k-1} - z)(\lambda_{n,k} - z)} \right) \\ &\underset{n\to\infty}{\longrightarrow} \frac{\Delta_{K_\infty}}{\lambda_{K_\infty-1} - z} + \sum_{k \geq k_{\ell_0}+1} \Delta_k \left(\frac{\lambda_k - \lambda_{k-1}}{(\lambda_{k-1} - z)(\lambda_k - z)} \right), \end{aligned} \tag{45}$$

with obvious notation.

Let us first prove (39). If $L \cap (\ell_0, \infty) = \emptyset$, then $L \cap (\ell_0 - \epsilon, \infty) = \emptyset$ for some $\epsilon > 0$ (recall that $\ell_0 \notin L$). Hence, for all $A > \ell_0$, and n large enough depending on

A, $L_n \cap (\ell_0, A] = \emptyset$, which implies, for $|z| \leq \ell_0/2$,

$$\begin{aligned}\left|\mathbf{1}_{L_n\cap(\ell_0,\infty)\neq\emptyset}\sum_{k\geq k_{n,\ell_0}}\frac{1}{\lambda_{n,k}(\lambda_{n,k}-z)}\right| &\leq 2\int_{\mathbb{R}}\frac{\mathbf{1}(\lambda>\ell_0)}{\lambda^2}\,d\Xi_n(\lambda)\\ &= 2\int_{\mathbb{R}}\frac{\mathbf{1}(\lambda>A)}{\lambda^2}\,d\Xi_n(\lambda)\\ &\leq 2\int_{\mathbb{R}}\frac{\mathbf{1}(\lambda>A)}{\lambda^{1+\alpha}}\,d\Xi_n(\lambda)\leq 2\tau_A.\end{aligned}$$

By letting $n\to\infty$ and then $A\to\infty$, one deduces (39).

Let us prove (40) and (41). By using the estimates (29) and (30) proven above, one deduces that for

$$\tilde{\tau} := \tau_1 + \Xi([\lambda^*\wedge 2, 2]) + \sup_{n\geq 1}\Xi_n([\lambda^*\wedge 2, 2]),$$

one has, for any $k\geq 1$,

$$\lambda_k \geq (k/\tilde{\tau})^{1/(1+\alpha)}, \tag{46}$$

if $\lambda_k > 2$, and uniformly in n,

$$\lambda_{k,n} \geq (k/\tilde{\tau})^{1/(1+\alpha)}, \tag{47}$$

if $\lambda_{k,n} > 2$. Now, let us assume that $L\cap(\ell_0,\infty)=\emptyset$ and $L_n\cap(\ell_0,\infty)\neq\emptyset$. If n is large enough, then for any index k such that $\lambda_{n,k}>\ell_0$, one has also $\lambda_{n,k}>\lambda^*\vee 2$, since $L\cap(\ell_0-\epsilon,(\lambda^*\vee 2)+1)=\emptyset$ for some $\epsilon>0$, and $\Xi_n\to\Xi$. Hence, $k\geq 1$ and (47) is satisfied. By using this inequality and (20), one deduces, for $|z|\leq\ell_0/2$,

$$\begin{aligned}&\left|\mathbf{1}_{L_n\cap(\ell_0,\infty)\neq\emptyset}\frac{\Delta_{n,K_{n,\infty}}}{\lambda_{n,K_{n,\infty}-1}-z}\right| + \left|\mathbf{1}_{L_n\cap(\ell_0,\infty)\neq\emptyset}\frac{\Delta_{n,k_{n,\ell_0}}}{\lambda_{n,k_{n,\ell_0}}-z}\right|\\ &\leq \sup_{k\geq 1}\frac{2c(k+1)^{\alpha'}}{(k/\tilde{\tau})^{1/(1+\alpha)}\vee\lambda_{n,k_{n,\ell_0}}} + \sup_{k\geq 1}\frac{2ck^{\alpha'}}{(k/\tilde{\tau})^{1/(1+\alpha)}\vee\lambda_{n,k_{n,\ell_0}}}\\ &\leq 4c(1+2^{\alpha'})(1+\tilde{\tau})^{1/(1+\alpha)}\sup_{k\geq 1}\frac{k^{\alpha'}}{k^{1/(1+\alpha)}\vee\lambda_{n,k_{n,\ell_0}}}\\ &= 4c(1+2^{\alpha'})(1+\tilde{\tau})^{1/(1+\alpha)}\sup_{k\geq 1}\frac{(k^{1/(1+\alpha)})^{\alpha'(1+\alpha)}}{k^{1/(1+\alpha)}\vee\lambda_{n,k_{n,\ell_0}}}\\ &\leq 4c(1+2^{\alpha'})(1+\tilde{\tau})^{1/(1+\alpha)}\sup_{k\geq 1}(k^{1/(1+\alpha)}\vee\lambda_{n,k_{n,\ell_0}})^{\alpha'(1+\alpha)-1}\\ &\leq 4c(1+2^{\alpha'})(1+\tilde{\tau})^{1/(1+\alpha)}\lambda_{n,k_{n,\ell_0}}^{\alpha'(1+\alpha)-1},\end{aligned}$$

where $\lambda_{n,k_{n,\ell_0}}$ is taken equal to ∞ for $L_n \cap (\ell_0, \infty) = \emptyset$. Note that in the previous computation, the last inequality is a consequence of the inequality $\alpha'(1+\alpha) - 1 < 0$. Now, $\lambda_{n,k_{n,\ell_0}}$ tends to infinity with n, since for all $A > \ell_0$, one has $L_n \cap (\ell_0, A] = \emptyset$ for n large enough. Hence, we get (40) and (41).

Moreover, in case where $L \cap (\ell_0, \infty) = \emptyset$, $L_n \cap (\ell_0, \infty) \neq \emptyset$, n is large enough, and $|z| < \ell_0/2$, the left-hand side of (42) is smaller than or equal to:

$$
\begin{aligned}
&4c \sum_{k \geq k_{n,\ell_0}+1} |k|^{\alpha'} \left(\frac{\lambda_{n,k} - \lambda_{n,k-1}}{\lambda_{n,k-1}\lambda_{n,k}} \right) = 4c \sum_{k \geq k_{n,\ell_0}+1} |k|^{\alpha'} \left(\frac{1}{\lambda_{n,k-1}} - \frac{1}{\lambda_{n,k}} \right) \\
&= 4c \left(\frac{|k_{n,\ell_0}+1|^{\alpha'}}{\lambda_{n,k_{n,\ell_0}}} + \sum_{k \geq k_{n,\ell_0}+1} \frac{|k+1|^{\alpha'} - |k|^{\alpha'}}{\lambda_{n,k}} \right) \\
&\leq 4c \left(\frac{|k_{n,\ell_0}+1|^{\alpha'}}{\lambda_{n,k_{n,\ell_0}} \vee (|k_{n,\ell_0}|/\tilde{\tau})^{1/(1+\alpha)}} + \sum_{k \geq 1} \frac{(k+1)^{\alpha'} - k^{\alpha'}}{\lambda_{n,k_{n,\ell_0}} \vee (k/\tilde{\tau})^{1/(1+\alpha)}} \right),
\end{aligned}
$$

when $k_{n,\ell_0} \geq 1$, which occurs for n large enough. The first term of the last quantity is dominated by

$$
(\lambda_{n,k_{n,\ell_0}} \vee (k_{n,\ell_0}/\tilde{\tau})^{1/(1+\alpha)})^{\alpha'(1+\alpha)-1} \leq (\lambda_{n,k_{n,\ell_0}})^{\alpha'(1+\alpha)-1},
$$

which tends to zero when n goes to infinity, since $\lambda_{n,k_{n,\ell_0}}$ goes to infinity and $\alpha'(1+\alpha) - 1 < 0$. Similarly,

$$
\sum_{k \geq 1} \frac{(k+1)^{\alpha'} - k^{\alpha'}}{\lambda_{n,k_{n,\ell_0}} \vee (k/\tilde{\tau})^{1/(1+\alpha)}} \underset{n \to \infty}{\longrightarrow} 0,
$$

by dominated convergence. Hence, we get (42).

We can now assume $L \cap (\ell_0, \infty) \neq \emptyset$ and it remains to prove (43)–(45).

For $k \geq k_{\ell_0}$, let us define $\lambda_{n,k}$ and λ_k as ∞ if these numbers are not well-defined: this does not change the quantities involved in (43). Moreover, for all $k \geq k_{\ell_0}$:

- If λ_k is well-defined as a finite quantity, then $\lambda_{n,k}$ is also well-defined for n large enough and tends to λ_k when n goes to infinity.
- If $\lambda_k = \infty$, then for all $A > 0$, and for n large enough, one has $\lambda_{n,k} \notin [-A, A]$. Since for n large enough,

$$
\lambda_{n,k} \geq \lambda_{n,k_{\ell_0}} > \lambda_{k_{\ell_0}} - 1 > \ell_0 - 1 > 0,
$$

  one has $\lambda_{n,k} > A$: in other words, $\lambda_{n,k}$ tends to infinity with n.

We have just checked that with the convention made here, one has always $\lambda_{n,k}$ converging to λ_k when n goes to infinity, for all $k \geq k_{\ell_0}$. Hence, (43) is a consequence of

the dominated convergence theorem and the majorization:

$$\sum_{k\geq k_{\ell_0}} (\lambda_k \wedge \inf_{n\geq n_0} \lambda_{n,k})^{-2} < \infty,$$

for some $n_0 \geq 1$. Now, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, one has $k_{n,\ell_0} = k_{\ell_0}$, and then for all $k \geq 1 \vee k_{\ell_0}$, $\lambda_k > \ell_0 > 2$, $\lambda_{n,k} > 2$ and $k \geq 1$, which implies the minorizations (46) and (47). Hence one gets (43), since

$$\sum_{k\geq 1} (k/\tilde{\tau})^{-2/(1+\alpha)} < \infty.$$

Since (44) is easy to check, it remains to show (45), which can be rewritten as follows:

$$\sum_{k\geq k_{\ell_0}+1} \Delta_{n,k} \left(\frac{1}{\lambda_{n,k-1} - z} - \frac{1}{\lambda_{n,k} - z} \right) \underset{n\to\infty}{\longrightarrow} \sum_{k\geq k_{\ell_0}+1} \Delta_k \left(\frac{1}{\lambda_{k-1} - z} - \frac{1}{\lambda_k - z} \right),$$

where for $k \geq K_{n,\infty}$ (resp. $k \geq K_\infty$), one defines $\lambda_{k,n} := \infty$ (resp. $\lambda_k := \infty$). Note that with this convention, $\lambda_{n,k}$ tends to λ_k when n goes to infinity, for all $k \geq k_{\ell_0}$. Note that each term of the left-hand side of this last convergence converges uniformly on $\{z \in \mathbb{C}, |z| < \ell_0/2\}$ towards the corresponding term in the right-hand side. Indeed, for n large enough, for all $k \geq k_{\ell_0}$, for $|z| < \ell_0/2$, and for λ_k, $\lambda_{n,k}$ finite,

$$\begin{aligned} \left| \frac{1}{\lambda_{n,k} - z} - \frac{1}{\lambda_k - z} \right| &= \frac{|\lambda_k - \lambda_{n,k}|}{|\lambda_{n,k} - z||\lambda_k - z|} \leq \frac{4|\lambda_k - \lambda_{n,k}|}{\lambda_{n,k}\lambda_k} \\ &\leq 4 \left| \frac{1}{\lambda_{n,k}} - \frac{1}{\lambda_k} \right| \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

this convergence, uniform in z, being in fact also true if $\lambda_{n,k}$ or λ_k is infinite. Hence, one has, for all $k' > k_{\ell_0} + 1$, the uniform convergence:

$$\sum_{k_{\ell_0}+1\leq k\leq k'} \Delta_{n,k} \left(\frac{1}{\lambda_{n,k-1} - z} - \frac{1}{\lambda_{n,k} - z} \right) \underset{n\to\infty}{\longrightarrow} \sum_{k_{\ell_0}+1\leq k\leq k'} \Delta_k \left(\frac{1}{\lambda_{k-1} - z} - \frac{1}{\lambda_k - z} \right),$$

Hence, it is sufficient to check, for $n_0 \geq 1$ such that $k_{n,\ell_0} = k_{\ell_0}$ if $n \geq n_0$, that

$$\sup_{n\geq n_0, |z|<\ell_0/2} \left| \sum_{k>k'} \Delta_{n,k} \left(\frac{1}{\lambda_{n,k-1} - z} - \frac{1}{\lambda_{n,k} - z} \right) \right| \underset{k'\to\infty}{\longrightarrow} 0 \tag{48}$$

and

$$\sup_{|z|<\ell_0/2} \left| \sum_{k>k'} \Delta_k \left(\frac{1}{\lambda_{k-1} - z} - \frac{1}{\lambda_k - z} \right) \right| \underset{k'\to\infty}{\longrightarrow} 0. \tag{49}$$

Now, for $k' \geq 1 \vee (k_{\ell_0} + 1)$, $n \geq n_0$ and $|z| < \ell_0/2$, one has

$$\begin{aligned}
\left| \sum_{k>k'} \Delta_{n,k} \left(\frac{1}{\lambda_{n,k-1} - z} - \frac{1}{\lambda_{n,k} - z} \right) \right| &\leq \sum_{k>k'} |\Delta_{n,k}| \left| \frac{1}{\lambda_{n,k-1} - z} - \frac{1}{\lambda_{n,k} - z} \right| \\
&\leq 4c \sum_{k>k'} k^{\alpha'} \left(\frac{1}{\lambda_{n,k-1}} - \frac{1}{\lambda_{n,k}} \right) \\
&= 4c \left(\frac{(k'+1)^{\alpha'}}{\lambda_{n,k'}} + \sum_{k>k'} \frac{(k+1)^{\alpha'} - k^{\alpha'}}{\lambda_{n,k}} \right) \\
&\leq 4c \left(\frac{(k'+1)^{\alpha'}}{(k'/\tilde{\tau})^{1/(1+\alpha)}} + \sum_{k>k'} \frac{(k+1)^{\alpha'} - k^{\alpha'}}{(k/\tilde{\tau})^{1/(1+\alpha)}} \right) \\
&\underset{k' \to \infty}{\longrightarrow} 0,
\end{aligned}$$

which proves (48). One shows (49) in an exactly similar way, which finishes the proof of the convergence of S_n towards $S + \bar{\gamma} h$, uniformly on compact sets for the distance d. □

Proof of the formulas (21)–(24) By observing the sign of the imaginary parts $\Im(S_n)$ and $\Im(S)$, we deduce that the sets Ξ'_n and Ξ' are included in $\mathbb{R}$. Moreover, the derivatives S'_n and S' are strictly positive, respectively on $\mathbb{R} \backslash L_n$ and $\mathbb{R} \backslash L$, and all the left (resp. right) limits of S_n and S at their poles are equal to $+\infty$ (resp. $-\infty$). We deduce that the support of Ξ'_n (resp. Ξ') strictly interlaces with the points in L_n (resp. L).

The Eq. (21) is a direct consequence of the convergence of S_n towards $S + \bar{\gamma} h$, as written in the statement of Theorem (21). More precisely, for two points a and b ($a < b$) not in the support of Ξ' and such that $\Xi'((a, b)) = k$, there exist real numbers $a = q_0 < q_1 < q_2 < \cdots < q_{2k} < b = q_{2k+1}$ such that $-\infty < S(q_{2j-1}) < h' < S(q_{2j}) < \infty$ for $j \in \{1, \ldots, k\}$, which implies that these inequalities are also satisfied for $S_n - \bar{\gamma} h$ instead of S if n is large enough: one has $\Xi'_n((a, b)) \geq k$. On the other hand, since the support of Ξ' has exactly one point on each interval $[q_{2j-1}, q_{2j}]$ ($1 \leq j \leq k$) and no point on the intervals $[q_{2j}, q_{2j+1}]$ ($0 \leq j \leq k$), one deduces that S is bounded on the intervals $[q_{2j-1}, q_{2j}]$ and bounded away from h' on the intervals $[q_{2j}, q_{2j+1}]$. These properties remain true for $S_n - \bar{\gamma} h$ if n is large enough, and one easily deduces that $\Xi'_n((a, b)) \leq k$.

The properties (22)–(24) can be deduced from the property of interlacing. More precisely, for $\ell \geq 1$,

$$\int_{\mathbb{R}} \frac{\mathbf{1}(\lambda > \ell)}{\lambda^{1+\alpha}} \, d\Xi'_n(\lambda) = \sum_{\lambda \in L'_n \cap (\ell, \infty)} \frac{1}{\lambda^{1+\alpha}},$$

where L'_n is the support of Ξ'_n. By the interlacing property, if $L'_n \cap (\ell, \infty)$ is not empty and if its smallest element is $\lambda' > \ell$, then it is possible to define an injection between $(L'_n \cap (\ell, \infty)) \backslash \{\lambda'\}$ and $L_n \cap (\ell, \infty)$, such that the image of each point is smaller than

this point. One deduces

$$\int_{\mathbb{R}} \frac{\mathbf{1}(\lambda > \ell)}{\lambda^{1+\alpha}}\, d\Xi_n'(\lambda) \leq \frac{1}{\lambda'} + \sum_{\lambda \in L_n \cap (\ell,\infty)} \frac{1}{\lambda^{1+\alpha}} \leq \frac{1}{\ell} + \int_{\mathbb{R}} \frac{\mathbf{1}(\lambda > \ell)}{\lambda^{1+\alpha}}\, d\Xi_n(\lambda).$$

By looking similarly at the integral for $\lambda < -\ell$, one deduces

$$\int_{\mathbb{R}} \frac{\mathbf{1}(|\lambda| > \ell)}{|\lambda|^{1+\alpha}}\, d\Xi_n'(\lambda) \leq \frac{2}{\ell} + \int_{\mathbb{R}} \frac{\mathbf{1}(|\lambda| > \ell)}{|\lambda|^{1+\alpha}}\, d\Xi_n(\lambda) \leq \frac{2}{\ell} + \tau_\ell \underset{\ell\to\infty}{\longrightarrow} 0,$$

which proves (22) for the measure Ξ_n'.

By a similar argument, for $\ell'' > \ell' > \ell \geq 1$,

$$\int_{\mathbb{R}} \frac{\mathbf{1}(\ell' \leq \lambda < \ell'')}{\lambda}\, d\Xi_n'(\lambda) \leq \int_{\mathbb{R}} \frac{\mathbf{1}(\ell' \leq \lambda < \ell'')}{\lambda}\, d\Xi_n(\lambda) + \frac{1}{\ell'},$$

and one has the similar inequalities obtained by exchanging Ξ_n and Ξ_n', and by changing the sign of λ. Hence,

$$\left| \int_{\mathbb{R}} \frac{\mathbf{1}(\ell' \leq |\lambda| < \ell'')}{|\lambda|}\, d\Xi_n'(\lambda) - \int_{\mathbb{R}} \frac{\mathbf{1}(\ell' \leq |\lambda| < \ell'')}{|\lambda|}\, d\Xi_n(\lambda) \right| \leq \frac{4}{\ell'}.$$

This inequality and the existence of the limit $h_{n,\ell}$ for the measure Ξ_n implies that

$$\limsup_{\ell' \wedge \ell'' \to \infty} \left| \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| < \ell'')}{|\lambda|}\, d\Xi_n'(\lambda) - \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| < \ell')}{|\lambda|}\, d\Xi_n'(\lambda) \right| = 0,$$

and then the limit $h_{n,\ell}'$ given by (24) exists, and we get the existence of h_ℓ' in a similar way. Moreover, for all $\ell \geq 1$, one gets the majorization:

$$\left| \lim_{\ell' \to \infty} \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| < \ell')}{|\lambda|}\, d\Xi_n'(\lambda) - \lim_{\ell' \to \infty} \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| < \ell')}{|\lambda|}\, d\Xi_n(\lambda) \right| \leq \frac{4}{\ell}, \tag{50}$$

and a similar inequality without the index n. Hence, we have

$$|h_{n,\ell}' - h_{n,\ell}| = O(\ell^{-1}),\ |h_\ell' - h_\ell| = O(\ell^{-1}).$$

We then deduce (23) from (18) (with the same value of h), provided that we check the existence of the limit of $h_{n,\ell}'$ when $n \to \infty$, which is involved in (23):

$$\lim_{n\to\infty} \lim_{\ell' \to \infty} \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| < \ell')}{|\lambda|}\, d\Xi_n'(\lambda) \tag{51}$$

for each ℓ such that ℓ and $-\ell$ are not in the support of Ξ'. Let us first assume that ℓ and $-\ell$ are also not in the support of Ξ. We have, for all $\ell'' > \ell$,

$$\lim_{\ell' \to \infty} \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| < \ell')}{|\lambda|} \, d\Xi'_n(\lambda) = \int_{\mathbb{R}} \frac{\mathbf{1}(\ell < |\lambda| \le \ell'')}{|\lambda|} \, d\Xi'_n(\lambda) + \lim_{\ell' \to \infty} \int_{\mathbb{R}} \frac{\mathbf{1}(\ell'' < |\lambda| < \ell')}{|\lambda|} \, d\Xi'_n(\lambda), \tag{52}$$

and a similar equality with Ξ'_n replaced by Ξ_n. Since ℓ and $-\ell$ are not in the support of Ξ or Ξ', the convergences of Ξ_n towards Ξ and of Ξ'_n towards Ξ' imply that the lower and upper limits (when n goes to infinity) of the first term of (52) (both with Ξ'_n and with Ξ_n) differ by $O(1/\ell'')$. For the second term, the difference between the lower and upper limits should change only by $O(1/\ell'')$ when we replace (52) by the same equation with Ξ_n, thanks to (50). Hence, this observation is also true for the sum of the two terms. On the other hand, the existence of the limit of $h_{n,\ell}$ when n goes to infinity (for Ξ_n) implies that in (52) with Ξ'_n replaced by Ξ_n, the difference between the upper and lower limit is zero. Therefore, the difference is $O(1/\ell'')$ without replacement of Ξ'_n by Ξ_n : letting $\ell'' \to 0$ gives the existence of the limit (51) for $-\ell, \ell$ not in the support of Ξ and Ξ'. If $-\ell$ or ℓ is in the support of Ξ (but not in the support of Ξ'), we observe that for some $\epsilon > 0$ and n large enough, there is no point in the supports of Ξ'_n and Ξ' in the intervals $\pm\ell + (-\epsilon, \epsilon)$, which implies that the integral involved in (51) does not change if we change ℓ by less than ϵ. By suitably moving ℓ, we can then also avoid the support of Ξ. □

9 Convergence of Hermite corners towards the bead process

In this section, we consider, for all $\beta > 0$, the Gaussian β Ensemble, defined as a set of n points $(\lambda_j)_{1 \le j \le n}$ whose joint density, with respect to the Lebesgue measure is proportional to

$$e^{-\beta \sum_{k=1}^n \lambda_k/4} \prod_{j<k} |\lambda_j - \lambda_k|^\beta.$$

We will use the following crucial estimate, proven in [15]:

Theorem 23 *For $-\infty \le \Lambda_1 < \Lambda_2 \le \infty$, let $N(\Lambda_1, \Lambda_2)$ be the number of points, between Λ_1 and Λ_2, of a Gaussian beta ensemble with n points, and let $N_{sc}(\Lambda_1, \Lambda_2)$ be n times the measure of (Λ_1, Λ_2) with respect to the semi-circle distribution on the interval $[-2\sqrt{n}, 2\sqrt{n}]$:*

$$N_{sc}(\Lambda_1, \Lambda_2) := \frac{n}{2\pi} \int_{\Lambda_1/\sqrt{n}}^{\Lambda_2/\sqrt{n}} \sqrt{(4 - x^2)_+} \, dx.$$

Then,

$$\mathbb{E}[(N(\Lambda_1, \Lambda_2) - N_{sc}(\Lambda_1, \Lambda_2))^2] = O(\log(2 + (\sqrt{n}(\Lambda_2 - \Lambda_1) \wedge n))).$$

For $\beta \in \{1, 2, 4\}$, the Gaussian β Ensemble can be represented by the eigenvalues of real symmetric (for $\beta = 1$), complex Hermitian (for $\beta = 2$), or quaternionic Hermitian (for $\beta = 4$) Gaussian matrices. The law of the entries of these matrices, corresponding respectively to the Gaussian Orthogonal Ensemble, the Gaussian Unitary Ensemble and the Gaussian Symplectic Ensemble, are given as follows:

- The diagonal entries are real-valued, centered, Gaussian with variance $2/\beta$.
- The entries above the diagonal are real-valued for $\beta = 1$, complex-valued for $\beta = 2$, quaternion-valued for $\beta = 4$, with independent parts, centered, Gaussian with variance $1/\beta$.
- All the entries involved in the previous items are independent.

By considering the top-left minors A_n of an infinite random matrix A following the law described just above, and their eigenvalues, we get a family of sets of points, the nth set following the GβE of order n. Conditionally on the matrix A_n, whose eigenvalues are denoted $(\lambda_1, \ldots, \lambda_n)$, supposed to be distinct (this holds almost surely), the law of the eigenvalues of A_{n+1} can be deduced by diagonalizing A_n inside A_{n+1}, which gives a matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & g_1 \\ 0 & \lambda_2 & \cdots & 0 & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n & g_n \\ \overline{g_1} & \overline{g_2} & \cdots & \overline{g_n} & g \end{pmatrix},$$

where $g_1, \ldots g_n, g$ are independent, centered Gaussian, g being real-valued with variance $2/\beta$, $g_1, \ldots, g_n$ being real-valued of variance 1 for $\beta = 1$, complex-valued with independent real and imaginary parts of variance $1/2$ for $\beta = 2$, quaternion-valued with independent parts of variance $1/4$ for $\beta = 4$. Expanding the characteristic polynomial and dividing by the product of $\lambda_j - z$ for $1 \leq j \leq \lambda_n$, we see that the eigenvalues of A_{n+1} are the solutions of the equation:

$$g - z - \sum_{j=1}^{n} \frac{|g_j|^2}{\lambda_j - z} = 0.$$

Hence, if for $n \geq 1$, we consider the eigenvalues of the matrices $(A_{n+k})_{k \geq 0}$, we get an inhomogeneous Markov chain defined as follows:

- The first set corresponds to the $G\beta E$ with n points.
- Conditionally on the sets of points indexed by $0, 1, \ldots, k$, the set indexed by k containing the distinct points $\lambda_1, \ldots, \lambda_{n+k}$, the set indexed by $k + 1$ contains the

zeros of

$$g - z - \sum_{j=1}^{n+k} \frac{(2/\beta)\gamma_j}{\lambda_j - z},$$

g being centered, Gaussian of variance $2/\beta$, γ_j being a Gamma variable of parameter $\beta/2$, all these variables being independent.

Similar expressions of eigenvalues of successive minors of random matrices in terms of zeros of meromorphic functions can be found in the literature: for more detail, we refer to articles by Gelfand and Naimark [9], Baryshnikov [2], Neretin [14], Okounkov and Reshetikhin [16].

The Markov chain above can be generalized to all $\beta > 0$: this can be viewed as the "eigenvalues of the GβE minors". In fact, what we obtain is equivalent (with suitable scaling) to the Hermite β corners introduced by Gorin and Shkolnikov [10]. This fact is due to the following result, proven (up to scaling) in [8, Proposition 4.3.2]:

Proposition 24 *The density of transition probability from the set* $(\lambda_1, \dots, \lambda_n)$ *to the set* $(\mu_1, \dots, \mu_{n+1})$*, subject to the interlacement property*

$$\mu_1 < \lambda_1 < \mu_2 < \cdots < \mu_n < \lambda_n < \mu_{n+1},$$

is proportional to

$$\prod_{1 \le p < q \le n+1} (\mu_q - \mu_p) \prod_{1 \le p < q \le n} (\lambda_q - \lambda_p)^{1-\beta}$$
$$\prod_{1 \le p \le n, 1 \le q \le n+1} |\mu_q - \lambda_p|^{\beta/2-1} e^{-\frac{\beta}{4}\left(\sum_{1 \le q \le n+1} \mu_q^2 - \sum_{1 \le p \le n} \lambda_p^2\right)}.$$

As explained in [10], the marginals of the Hermite β corner correspond to the Gaussian β Ensemble, which implies the following:

Proposition 25 *For all* $\beta > 0$*, the set of* $n + k$ *points corresponding to the step* k *of the Markov chain just above has the distribution of the Gaussian* β *Ensemble of dimension* $n + k$*, if the initial distribution is the Gaussian* β *Ensemble of dimension* n*. In particular, if we take* $n = 1$*, we get a coupling of the G*β*E in all dimensions.*

Now, we show that a suitable scaling limit of this Markov chain is the β-bead process introduced in the paper.

We choose $E \in (-2, 2)$ (this corresponds to the bulk of the spectrum), $n \ge 1$, and we center the spectrum around the level $E\sqrt{n}$. The expected density of eigenvalues around this level is approximated by $\sqrt{n}\rho_{sc}(E)$, where ρ_{sc} is the density of the semi-circular distribution. In order to get an average spacing of 2π, we should then scale the eigenvalues by a factor $2\pi\sqrt{n}\rho_{sc}(E) = \sqrt{n(4 - E^2)}$. For $k \ge 0$, we then consider the simple point measure $\Xi_n^{(k)}$ given by putting Dirac masses at the points $(\lambda_j^{(n,k)} - E\sqrt{n})\sqrt{n(4 - E^2)}$, where $(\lambda_j^{(n,k)})_{1 \le j \le n+k}$ is the set of $n + k$ points obtained at the

step k of the Markov chain above. The sequence of measures $\Xi_n^{(k)}$ can be recovered as follows:

- For $k = 0$, $\Xi_n^{(0)}$ corresponds to the point measure associated with the suitably rescaled GβE point process, with n points.
- Conditionally on $\Xi_n^{(k)}$, $\Xi_n^{(k+1)}$ is obtained by taking the zeros of

$$-\frac{E}{\sqrt{4-E^2}} + \frac{g^{(k)}}{\sqrt{n(4-E^2)}} - \frac{z}{n(4-E^2)} - \int \frac{1}{\lambda - z} d\Lambda_n^{(k)}(\lambda),$$

where $g^{(k)}$ is a centered Gaussian variable of variance $2/\beta$, and $\Lambda_n^{(k)}$ is the weighted version of $\Xi_n^{(k)}$, the weights being i.i.d. with distribution corresponding to $2/\beta$ times a Gamma variable of parameter $\beta/2$.

We are now able to prove the following result:

Theorem 26 *The Markov chain $(\Xi_n^{(k)})_{k\geq 0}$ converges in law to the Markov chain defined in Theorem 13, for the topology of locally weak convergence of locally finite measures on $\mathbb{R} \times \mathbb{N}_0$ (i.e. convergence of the integrals against continuous functions with compact support), and for the level*

$$h = -\frac{E}{\sqrt{4-E^2}}.$$

For $\beta = 2$ and $h \in \mathbb{R}$ fixed, the law of the Markov chain of Theorem 13 corresponds (after dividing the points by 2) to the bead process introduced by Boutillier, with parameter

$$\gamma = -\frac{h}{\sqrt{1+h^2}},$$

if we take the notation of [6].

Proof By the result of Valkó and Virág, $\Xi_n^{(0)}$ converges in distribution to the Sine$_\beta$ point process.

Hence, the family, indexed by n, of the distributions of $(\Xi_n^{(0)})_{n\geq 1}$, is tight in the space of probability measures on $\mathcal{M}(\mathbb{R})$, $\mathcal{M}(\mathbb{R})$ being the space of locally finite measures on the Borel sets of $\mathbb{R}$, endowed with the topology of locally weak convergence. Hence, for $\epsilon > 0$, there exists $(C_K)_{K\in\mathbb{N}}$ such that with probability at least $1-\epsilon$, the number of points in $[-K, K]$ of $\Xi_n^{(0)}$ is at most C_K for all $K \in \mathbb{N}$, independently of n. Since the points of $\Xi_n^{(k)}$ interlace with those of $\Xi_n^{(k-1)}$, the condition just above is satisfied with $\Xi_n^{(k)}$ instead of $\Xi_n^{(0)}$. Hence, the family, indexed by n, of the laws of $(\Xi_n^{(k)})_{k\geq 0}$ is tight in the space of probability measures on $\mathcal{M}(\mathbb{R} \times \mathbb{N}_0)$, $\mathcal{M}(\mathbb{R} \times \mathbb{N}_0)$ being the space of locally finite measures on $\mathbb{R} \times \mathbb{N}_0$, again endowed with the topology of locally weak convergence. Recall that this property of tightness means that for all $\epsilon > 0$, there exists a compact subset $\mathcal{K}_\epsilon$ of $\mathcal{M}(\mathbb{R} \times \mathbb{N}_0)$ such that the locally finite measure on

$\mathbb{R} \times \mathbb{N}_0$ corresponding to $\Xi_n^{(0)}$ is in $\mathcal{K}_\epsilon$ with probability at least $1-\epsilon$. For all $\epsilon > 0$, the compact set $\mathcal{K}_\epsilon$ exists because we can find $(C_{K,k})_{K\in\mathbb{N},k\in\mathbb{N}_0}$ such that with probability at least $1-\epsilon$, $\Xi_n^{(k)}$ has at most $C_{K,k}$ points in the interval $[-K, K]$, for all k, K and independently of n.

From the tightness and Prokhorov's theorem, it is enough to prove that the law of the Markov chain of Theorem 13 is the only possible limit for a subsequence of the laws of $(\Xi_n^{(k)})_{k\geq 1}$. Let us consider such a subsequence which converges in law. We define the following random variable

$$Y_n := \sup_{\ell\geq 0}(1+\ell)^{-3/4}(|\widetilde{\Xi}_n^{(0)}([0,\ell])| + |\widetilde{\Xi}_n^{(0)}([-\ell,0])|)$$

where

$$\widetilde{\Xi}_n^{(0)}([a,b]) := \Xi_n^{(0)}([a,b]) - N_{sc}\left(\left[E\sqrt{n} + \frac{a}{\sqrt{n(4-E^2)}}, E\sqrt{n} + \frac{b}{\sqrt{n(4-E^2)}}\right]\right),$$

for

$$N_{sc}(\Lambda_1,\Lambda_2) := \frac{n}{2\pi}\int_{\Lambda_1/\sqrt{n}}^{\Lambda_2/\sqrt{n}} \sqrt{(4-x^2)_+}dx.$$

The family $(Y_n)_{n\geq 0}$ is tight. Indeed, by Theorem 23,

$$\begin{aligned}&\mathbb{E}[(1+\ell)^{-3/2}(|\widetilde{\Xi}_n^{(0)}([0,\ell])|^2 + |\widetilde{\Xi}_n^{(0)}([-\ell,0])|^2)]\\ &= O\left((1+\ell)^{-3/2}\log\left(2+\sqrt{n}\frac{\ell}{\sqrt{n(4-E^2)}}\right)\right),\end{aligned}$$

which shows that

$$\mathbb{E}\left[\sum_{\ell=0}^{\infty}(1+\ell)^{-3/2}(|\widetilde{\Xi}_n^{(0)}([0,\ell])| + |\widetilde{\Xi}_n^{(0)}([-\ell,0])|)^2\right] \leq C_{E,\beta},$$

where $C_{E,\beta} < \infty$ depends only on E and β (in particular, not on n). This implies that $(Y_n)_{n\geq 1}$ is tight.

The point processes $\Xi_n^{(k)}$, $k \geq 0$, are constructed from $\Xi_n^{(0)}$, and families $\gamma_{n,k,k'}$ of weights, $\gamma_{n,k,k'}$ being the weight, involved in the construction of $\Xi_n^{(k+1)}$, of the (k')th nonnegative point of $\Xi_n^{(k)}$ if $k' > 0$, the $(1-k')$th negative point of $\Xi_n^{(k)}$ if $k' \leq 0$. Notice that in this discussion, for all infinite families of real-valued random variables, we can consider them as single random variables on $\mathbb{R}^I$ for a suitable infinite set I: in this case, the σ-algebra taken on $\mathbb{R}^I$ is the Borel σ-algebra associated to the topology of the pointwise convergence of the coordinates. All the variables $\gamma_{n,k,k'}$ are i.i.d.,

distributed like $2/\beta$ times a Gamma variable of parameter $\beta/2$, and independent of $\Xi_n^{(0)}$. We can consider the variables

$$Z_{n,k} = \sup_{m\geq 1} m^{-0.51}\left|m - \sum_{k'=0}^{m-1}\gamma_{n,k,k'}\right| + \sup_{m\geq 1} m^{-0.51}\left|m - \sum_{k'=-m}^{-1}\gamma_{n,k,k'}\right|.$$

By classical tail estimates of the Gamma variables, or by the law of iterated logarithm, $Z_{n,k} < \infty$ almost surely, and since its law does not depend on n and k, $(Z_{n,k})_{n\geq 1,k\geq 0}$ is a tight family of random variables.

Hence, $(Z_n := (Z_{n,k})_{k\geq 0})_{n\geq 1}$ is a tight family of random variables on $\mathbb{R}^{\mathbb{N}_0}$, endowed with the σ-algebra generated by the sets $\{(z_k)_{k\geq 0}, z_0 \in A_0, z_1 \in A_1, \ldots, z_p \in A_p\}$ for $p \geq 0$ and $A_j \in \mathcal{B}(\mathbb{R})$.

Let us go back to our subsequence of $(\Xi_n^{(k)})_{k\geq 0}$ which converges in law. If we join $(\gamma_{n,k,k'})_{k\geq 0,k'\in\mathbb{Z}}$, Y_n and Z_n, we still get a tight family of probability measures on a suitable probability space. Hence, we can find a sub-subsequence for which the family of random variables $((\Xi_n^{(k)})_{k\geq 0}, (\gamma_{n,k,k'})_{k\geq 0,k'\in\mathbb{Z}}, Y_n, Z_n)$ converges in law, and a fortiori $(\Xi_n^{(0)}, (\gamma_{n,k,k'})_{k\geq 0,k'\in\mathbb{Z}}, Y_n, Z_n)$ converges in law. We can now apply Skorokhod representation theorem (see [4, Theorem 6.7]). Indeed, the random variables take values in the product space $\mathcal{M}(\mathbb{R}\times\mathbb{N}_0)\times\mathbb{R}^I$ for a suitable countable set I. The space $\mathbb{R}^I$ is separable since we consider the topology of pointwise convergence. The space of probability measures on the separable metric space $\mathbb{R}\times\mathbb{N}_0$, endowed with the topology of the weak convergence, is separable. We deduce that it is also the case for the space of finite measures on $\mathbb{R}\times\mathbb{N}_0$, after multiplying the probability measures of a dense sequence by all positive rational numbers. Since the finite measures on $\mathbb{R}\times\mathbb{N}_0$ are dense in the space of locally finite measures on $\mathbb{R}\times\mathbb{N}_0$, for the topology of locally weak convergence (because the test functions have compact support), we deduce that $\mathcal{M}(\mathbb{R}\times\mathbb{N}_0)$ is a separable space. By Skorokhod representation theorem, the family of random variables $(\Xi_n^{(0)}, (\gamma_{n,k,k'})_{k\geq 0,k'\in\mathbb{Z}}, Y_n, Z_n)$ has the same law as some family $(\Xi_n^{'(0)}, (\gamma'_{n,k,k'})_{k\geq 0,k'\in\mathbb{Z}}, Y'_n, Z'_n)$ which converges almost surely along the same sub-subsequence as the one for which the family $((\Xi_n^{(k)})_{k\geq 0}, (\gamma_{n,k,k'})_{k\geq 0,k'\in\mathbb{Z}}, Y_n, Z_n)$ converges in law. Note that Y'_n is function of $\Xi_n^{'(0)}$ and Z'_n is function of the weights $\gamma'_{n,k,k'}$. Since we know that $\Xi_n^{'(0)}$ converges in law to a Sine_β process, its almost sure limit is a simple point measure.

From the boundedness of Y'_n along our sub-subsequence, and by Proposition 16, we deduce that the part of Theorem 21 concerning $\Xi_n^{'(0)}$ is satisfied, with

$$h = \lim_{\ell\to\infty}\lim_{n\to\infty} h_{n,\ell}^{sc},$$

for

$$h_{n,\ell}^{sc} = \int_{(-\infty,-\ell]\cup[\ell,\infty)} \frac{1}{\lambda} dN_{sc}\left(E\sqrt{n} + \frac{\lambda}{\sqrt{n(4-E^2)}}\right).$$

Indeed, the convergence of $\Xi_n^{'(0)}$ to a Sine$_\beta$ process $\Xi^{'(0)}$ gives (15). Moreover,

$$\Xi_n^{'(0)}([-\ell,\ell]) = O(1+\ell+Y_n'(1+\ell)^{3/4}) = O(Y_n'(1+\ell))$$

by the boundedness of the semi-circle density and the definition of Y_n'. We easily deduce (16) for any $\alpha \in (0,1)$ (with τ_ℓ decaying like $\ell^{-\alpha}$). From now, we choose α such that $0 < \alpha < 49/51$.

The integral involved in the definition of $h_{n,\ell}$ can be written as the sum of an integral with respect to the semi-circle distribution and an integral with respect to $d\widetilde{\Xi}_n^{'(0)}$, where $\widetilde{\Xi}_n^{'(0)}$ is the signed measure involved in the definition of Y_n':

$$\widetilde{\Xi}_n^{'(0)}([a,b]) := \Xi_n^{'(0)}([a,b]) - N_{sc}\left(\left[E\sqrt{n}+\frac{a}{\sqrt{n(4-E^2)}}, E\sqrt{n}+\frac{b}{\sqrt{n(4-E^2)}}\right]\right).$$

The integral with respect to the semi-circle distribution tends to $h_{n,\ell}^{sc}$ when $\ell' \to \infty$, which itself tends to h when $n \to \infty$ and then $\ell \to \infty$. The integral with respect to $d\widetilde{\Xi}_n^{'(0)}$ can be transformed via an integration by part, as follows. If we restrict the integral to $\lambda > 0$, we get

$$\int_{\mathbb{R}} \frac{\mathbf{1}(\ell<\lambda<\ell')}{\lambda} d\widetilde{\Xi}_n^{'(0)} = \left[\frac{\widetilde{\Xi}_n^{'(0)}([0,\lambda])}{\lambda}\right]_\ell^{(\ell')-} - \int_\ell^{\ell'} \frac{\widetilde{\Xi}_n^{'(0)}([0,\lambda])}{\lambda^2} d\lambda.$$

Since $\widetilde{\Xi}_n^{'(0)}([0,\lambda])$ is dominated by $(1+\lambda)^{3/4}$ because of the boundedness of Y_n', we can let $\ell' \to \infty$ and we get the limit

$$\left[\frac{\widetilde{\Xi}_n^{'(0)}([0,\lambda])}{\lambda}\right]_\ell^{\infty} - \int_\ell^\infty \frac{\widetilde{\Xi}_n^{'(0)}([0,\lambda])}{\lambda^2} d\lambda.$$

From (15) and dominated convergence, due to the fact that Y_n' is bounded along the subsequence we consider, we get that the last expression converges to

$$\left[\frac{\widetilde{\Xi}^{'(0)}([0,\lambda])}{\lambda}\right]_\ell^{\infty} - \int_\ell^\infty \frac{\widetilde{\Xi}^{'(0)}([0,\lambda])}{\lambda^2} d\lambda. \tag{53}$$

when

$$\begin{aligned}\widetilde{\Xi}^{'(0)}[0,\lambda] &= \Xi^{'(0)}[0,\lambda] - \lim_{n\to\infty} N_{sc}\left(\left[E\sqrt{n}, E\sqrt{n}+\frac{\lambda}{\sqrt{n(4-E^2)}}\right]\right)\\ &= \Xi^{'(0)}[0,\lambda] - \frac{\lambda}{2\pi}.\end{aligned}$$

Notice that there is no problem of discontinuity for the bracket at ℓ, because we take limits for ℓ outside the support of $\Xi^{'(0)}$. Now, it is clear that (53) tends to zero when $\ell \to \infty$. Similarly

$$\int_{\mathbb{R}} \frac{\mathbf{1}(-\ell' < \lambda < -\ell)}{\lambda} d\widetilde{\Xi}_n^{'(0)}$$

tends to zero, after taking successive limits $\ell' \to \infty$, $n \to \infty$, $\ell \to \infty$. We then deduce (17) and (18) for $\Xi_n^{'(0)}$, by adding the limits of the integral with respect to the semi-circle distribution and the integral with respect to $d\widetilde{\Xi}_n^{'(0)}$. For $\Xi^{'(0)}$, the method is the same: we use Proposition 16 in order to estimate the point counting of the Sine_β process, and we replace the semi-circle distribution by $1/2\pi$ times the Lebesgue measure, which ensures the vanishing limit in (18).

We can now compute h explicitly. We have:

$$dN_{sc}\left(E\sqrt{n} + \frac{\lambda}{\sqrt{n(4-E^2)}}\right) = \frac{n}{2\pi} d\left(\int_{-\infty}^{E+\left(\lambda/(n\sqrt{4-E^2})\right)} \sqrt{(4-x^2)_+}\, dx\right)$$
$$= \frac{1}{2\pi\sqrt{4-E^2}} \sqrt{\left[4 - \left(E + \left(\lambda/(n\sqrt{4-E^2})\right)\right)^2\right]_+}\, d\lambda$$

If we do a change of variable $\lambda = \mu n\sqrt{4-E^2}$, we get

$$h_{n,\ell}^{sc} = \int_{(-\infty,-\ell/(n\sqrt{4-E^2})]\cup[\ell/(n\sqrt{4-E^2}),\infty)} \frac{1}{2\pi\sqrt{4-E^2}} \sqrt{[4-(E+\mu)^2]_+}\, \frac{d\mu}{\mu}.$$

Taking $n \to \infty$, we get a quantity independent of ℓ, given by

$$h = \frac{1}{2\pi\sqrt{4-E^2}} \int_{\mathbb{R}} \sqrt{(4-y^2)_+}\, \frac{dy}{y-E},$$

the integral in the neighborhood of E being understood as a principal value. From the value of the Stieltjes transform of the semi-circle law, we deduce

$$h = -\frac{E}{2\sqrt{4-E^2}}.$$

From the boundedness of $Z'_{n,0}$, we deduce that the part of Theorem 21 concerning the weights is also satisfied for $\alpha' = 0.51$, since by the assumption $0 < \alpha < 49/51$ made before, we have $0 < \alpha'(1+\alpha) < 1$.

Finally, in this theorem, it is almost surely possible to take $\lambda^* = 0$, by the absolutely continuity of the densities of the ensembles which are considered.

All the assumptions of the theorem are satisfied. If we denote by $\Lambda_n^{'(0)}$ the measure constructed from $\Xi_n^{'(0)}$ and the weights $\gamma'_{n,0,k'}$ ($k' \in \mathbb{Z}$), and $\Lambda^{'(0)}$ the measure

constructed from the a.s. limits of these points and weights, we deduce that for an independent standard Gaussian variable $g^{(0)}$, the function

$$z \mapsto -\frac{E}{\sqrt{4-E^2}} - \int \frac{1}{\lambda - z} d\Lambda_n^{'(0)}(\lambda),$$

and then also the function

$$z \mapsto -\frac{E}{\sqrt{4-E^2}} + \frac{g^{(0)}}{\sqrt{n(4-E^2)}} - \frac{z}{n(4-E^2)} - \int \frac{1}{\lambda - z} d\Lambda_n^{'(0)}(\lambda),$$

converges uniformly on compact sets, for the topology of the Riemann sphere given in Theorem 21, to the function

$$z \mapsto -\frac{E}{\sqrt{4-E^2}} - \int \frac{1}{\lambda - z} d\Lambda^{'(0)}(\lambda) - h = -\frac{E}{2\sqrt{4-E^2}} - \int \frac{1}{\lambda - z} d\Lambda^{'(0)}(\lambda).$$

As in the proof of Theorem 21, we deduce that the point process $\Xi_n^{'(1)}$ given by

$$-\frac{E}{\sqrt{4-E^2}} + \frac{g^{(0)}}{\sqrt{n(4-E^2)}} - \frac{z}{n(4-E^2)} - \int \frac{1}{\lambda - z} d\Lambda_n^{'(0)}(\lambda) = 0,$$

locally weakly converges to the point process $\Xi^{'(1)}$ given by

$$\lim_{\ell \to \infty} \int_{[-\ell,\ell]} \frac{1}{\lambda - z} d\Lambda^{'(0)}(\lambda) = -\frac{E}{2\sqrt{4-E^2}}.$$

The points of $\Xi_n^{'(1)}$ and satisfy the assumptions of Theorem 21, since they interlace with those of $\Xi_n^{'(0)}$. It is also the same for the weights $\gamma'_{n,1,k'}$ ($k' \in \mathbb{Z}$), by the boundedness of Z'_n. We then deduce that for an independent Gaussian variable $g^{(1)}$, the point process $\Xi_n^{'(2)}$ given by

$$-\frac{E}{\sqrt{4-E^2}} + \frac{g^{(1)}}{\sqrt{n(4-E^2)}} - \frac{z}{n(4-E^2)} - \int \frac{1}{\lambda - z} d\Lambda_n^{'(1)}(\lambda) = 0$$

locally weakly converges to the process $\Xi^{'(2)}$ given by

$$\lim_{\ell \to \infty} \int_{[-\ell,\ell]} \frac{1}{\lambda - z} d\Lambda^{'(1)}(\lambda) = -\frac{E}{2\sqrt{4-E^2}},$$

where $\Lambda_n^{'(1)}$ is given by $\Xi_n^{'(0)}$ and the weights $\gamma'_{n,1,k'}$ and $\Lambda^{'(1)}$ are given by their limits. We can then iterate the construction, which gives a family of point processes $\Xi_n^{'(k)}$ ($k \geq 0$), converging to $\Xi^{'(k)}$. From the way we do this construction, we check that $(\Xi_n^{'(k)})_{k \geq 0}$

has the same law as $(\Xi_n^{(k)})_{k\geq 0}$, and that $\Xi'^{(k)}$ has the same law as the generalized bead process introduced in the present paper (with level lines at $-E/2\sqrt{4-E^2}$). Hence, any subsequence of $((\Xi_n^{(k)})_{k\geq 0})_{n\geq 1}$ converging in law has a sub-subsequence tending in law to the generalized bead process. By tightness, we deduce the convergence of the whole sequence $((\Xi_n^{(k)})_{k\geq 0})_{n\geq 1}$. This gives the first part of the theorem, after doubling the weights and the value of h. The second part is deduced by using the convergence of the GUE minors towards the bead process introduced by Boutillier, proven in [1]. The factor 2 is due to the fact that the average density of points is $1/\pi$ in [6] and $1/2\pi$ here. The value of the parameter γ in [6] (a in [1]) corresponds to $E/2$ (the bulk corresponds to the interval $(-1, 1)$ in [1] and to $(-2, 2)$ in the present paper). We then have

$$h = -\frac{E}{\sqrt{4-E^2}} = -\frac{\gamma}{\sqrt{1-\gamma^2}},$$

and finally

$$\gamma = -\frac{h}{\sqrt{1+h^2}}.$$

□

Acknowledgements B.V. was supported by the Canada Research Chair program, the NSERC Discovery Accelerator Grant, the MTA Momentum Random Spectra research group, and the ERC consolidator Grant 648017 (Abert).

Data availability This manuscript has no associated data.

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