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THE BEHAVIOR OF DS-DIVISORS OF POSITIVE INTEGERS

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Abstract: We study the behavior of DS-divisors of positive integers. Here *"DS"* stands for *"divisor-squared."* For an integer n, a positive divisor q of n is called a DS-divisor if $q^2 \mid n - q$. Such a pair (n, q) is called a DS-pair. Using a table generated for DS-pairs, we examine the existence and the numbers of positive DS-divisors of prime powers, products of two prime powers, and other cases represented by primary factorization. We also investigate patterns and structures of DS-divisors derived from our observations of the table. In addition, we study relationships between the numbers of DS-divisors and the values of Euler function. This research is related to the Primality Test problem of positive integers.

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1. Introduction

Our problem was originally derived in one of the second author's number theory classes. It then developed into a research project explored by a student, Ms. Nicole Robichaux. In her work, positive integer pairs (q, n) were considered, where $q^2 \mid n - q$. The properties of such pairs were examined and several results were obtained [6]. Similar sequences were also explored by Li and Unnithan [4].The project expands further research on Ms. Robichaux's work. In

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this paper, we attempt to identify the DS-divisors of a given positive integer, determine how many there are, and study patterns and distributions of DS-divisors. We also investigate a relationship between the number of DS-divisors and Euler's Totient.

Definition 1. Consider a pair (q, n) of positive integers. We call this integer pair a DS-pair if it satisfies $q^2 \mid n - q$. The integer q is called a DS-divisor of n, and naturally, the integer n is called a DS-multiple of q. We define D-divisibility as "$\underset{D}{|}$" and furthermore, denote $q \underset{D}{|} n$ when q is a DS-divisor of n.

Remarks 2. Based on the definition stated above, we note the following:

1. It is obvious that if q is a DS-divisor of n, then q is a divisor of n.
2. Every positive integer has two trivial DS-divisors, 1 or itself.

Lemma 3. *DS-divisibility satisfies the following:*

1. *If $n = ab$, then $a \underset{D}{|} n \iff a \mid b - 1$.*
2. *DS-divisibility is a transitive relation. That is, if $n_1 \underset{D}{|} n_2$ and $n_2 \underset{D}{|} n_3$, then $n_1 \underset{D}{|} n_3$.*
3. *If $q > 1$ and $q \underset{D}{|} n$, then $q < \sqrt{n-1}$.*

It is natural to ask the following questions.

Questions 4. Consider a positive integer $n > 1$.

1. Does n have non-trivial DS-divisors?
2. How many DS-divisors does n have and what are they?
3. Which pairs of positive integers are DS-pairs?

To answer these questions, we constructed a table for DS-pairs. A portion of this table is provided in Appendix A and is referenced as Table 5. After examining the table, one observation that we made was that powers of primes only occur in the first column and first row. If an integer n is the product of 2 and odd prime, then 2 is a DS-divisor. We also examine other integers represented by primary factorization of n. We noticed that every prime power has only the trivial DS-divisors and if n is a product of two primes, the bigger prime cannot be a DS-divisor.

A closer inspection of table 1 reveals several patterns and interesting sequences. Through the use of the least non-negative residue property [2], our investigation revealed a periodical sequence for the ending numbers in the columns and rows. Finally, we recalled the Euler function $\phi(n)$, $\sigma(n)$, $\tau(n)$, the formula $\sum_{d \mid n} \phi(d) = n$ and define similar arithmetic functions for DS-divisors. We constructed a table for them as well. A portion of this table is given in Appendix A and is referenced as Table 5. Using the properties of Euler's totient [5], our new definitions and in conjunction with our main theorem, Theorem 5, we were able to provide formulas for the sum of the DS-divisors, the number of DS-divisors and a relationship with the Euler function. Additionally, we provide a bound for certain cases.

2. Non-Trivial DS-Divisors

In this section, we explore non-trivial DS-divisors of prime powers, product of two powers of primes, and furthermore, positive integers which have more than two prime divisors.

Theorem 5. *Consider a positive integer $n = p_1^{e_1} \cdots p_k^{e_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes, $p_1 < p_2 < \cdots < p_k$, and $e_j \geq 1$ for each j.*

1. *If $k = 1$, then n has no non-trivial DS-divisors.*
2. *If $k = 2$ then n has at most one non-trivial DS-divisor, $p_1^{e_1}$ or $p_2^{e_2}$.*
3. *If $k = 2$ and $e_1 = e_2 = 1$, then $p_2 \underset{D}{\nmid} n$. Furthermore, $p_1 \underset{D}{\mid} n \iff p_1 \mid p_2 - 1$. In particular, if $p_1 = 2$, then 2 must be a non-trivial DS-divisor of n.*
4. *Let d be a positive divisor of n and write $d = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$.*
   - *(a) If $\exists\, j$ such that $1 \leq i_j \leq e_j - 1$, then $d \underset{D}{\nmid} n$.*
   - *(b) The set $\{n/p_i^{e_i} \mid i = 1, 2, \ldots, k\}$ contains only one DS-divisor of n.*

Proof. 1. We only need to consider the case $e_1 > 1$. Consider a proper divisor p_1^j of n, where $0 < j < e_1$. Write $n = p_1^j p_1^{e_1 - j}$. By Lemma 3, $p_1^j \underset{D}{\nmid} n$ because $p_1 \nmid p_1^{e_1 - j} - 1$.

2. Assume $p_1^i p_2^j$ is a divisor of n, where $0 < i < e_1$ or $0 < j < e_2$. Then $n = (p_1^i p_2^j)(p_1^{e_1-i} p_2^{e_2-j})$ and $p_1^i p_2^j \nmid (p_1^{e_1-i} p_2^{e_2-j}) - 1$. Thus $p_1^i p_2^j \underset{D}{\nmid} n$ by Lemma 3. Therefore, the only possible non-trivial DS-divisors are $p_1^{e_1}$ or $p_2^{e_2}$. Now we show that they cannot both be DS-divisors of n. Assume $p_1^{e_1} \underset{D}{\mid} n$ and $p_2^{e_2} \underset{D}{\mid} n$. Then $p_1^{e_1} \mid p_2^{e_2} - 1$ and $p_2^{e_2} \mid p_1^{e_1} - 1 \Rightarrow p_2^{e_2} \leq p_1^{e_1} - 1 \leq (p_2^{e_2} - 1) - 1$. So, $p_2^{e_2} \leq p_2^{e_2} - 2$, a contradiction.

3. It is straightforward from Lemma 3.

4. (a) By the assumption, $p_j \mid d$ and $p_j \mid n/d$. Thus, $p_j^{i_j} \nmid (n/d) - 1$. Thus by Lemma 3, $d \underset{D}{\nmid} n$.

   (b) It is equivalent to show for $i \neq j$, $n/p_i^{e_i}$ and $n/p_j^{e_j}$ cannot be both DS-divisors of n. Without loss of generality, let $d_k = (p_1^{e_1} \cdot p_2^{e_2} \cdots p_{k-1}^{e_{k-1}})$, $d_1 = (p_2^{e_2} \cdot p_3^{e_3} \cdots p_k^{e_k})$. Then $n = d_k p_k^{e_k} = d_1 p_1^{e_1}$.

   Assume d_k and d_1 are both DS-divisors of c. Then, by Lemma 3, $p_1^{e_1} \mid p_k^{e_k} - 1$ and $p_k^{e_k} \mid p_1^{e_1} - 1 \Rightarrow p_1^{e_1} \leq p_1^{e_1} - 2$, a contradiction.

□

Corollary 6. *Let $n = pq$, where $2 < p < q$ and $q - 1 < 2p$. Then $p \underset{D}{\nmid} n$.*

Proof. Assume $p \underset{D}{\mid} n$. Then $p \mid q - 1$ which is even. It implies $p \mid (q-1)/2$ and so $2p \leq q - 1$, a contradiction. Therefore, $p \underset{D}{\nmid} n$. □

3. A Table of DS-Pairs and its Patterns

We explore another area which involves studying the table of DS-pairs and the sequences generated by the rows and columns of the table. A portion of the table of DS-pairs is provided in Appendix A and is referenced as Table 5.

In Table 5, the far left column represents values for q. The top row represents the values for r and are used to generate the DS-multiples for that specific q. The interior portion of the table consists of values of n that are DS-multiples of the q to its far left. The values are generated using the property $n = rq^2 + q$. We can observe from this table, the patterns which are the outcome of the Theorem 5.

Additionally, the On-Line Encyclopedia of Integer Sequences [7], contains sequences which are similar to the row sequences from our table. For instance, the sequences from rows $q = 2$ and $q = 3$ are identical with A016825 and A017197, respectively.

Now, let's compare two positive integers whose difference is divisible by a perfect square. We notice that they share a common DS-divisor. For example, compare 30 and 12. The difference is 18, which is divisible by a perfect square, 3^2. We can see from the table that 30 and 12 share a common DS-divisor, 3. Similarly, the same happens for 84 and 36. The difference is 48, which is divisible by a perfect square, 4^2. Again, from the table we can see that 84 and 36 share a common DS-divisor, 4. We summarize and confirm these observations as the following property.

Property 7. *Assume* $q^2 \mid n_1 - n_2$. *Then* $q \underset{D}{\mid} n_1 \iff q \underset{D}{\mid} n_2$.

Next, we study the columns and rows of the table which gives consequence to the following.

Definition 8. Consider positive integers q and r. Define the number in the q^{th} row and r^{th} column to be $n(q, r)$. That is, $n(q, r) = rq^2 + q$. Denote $b(q, r)$ as the least non-negative residue of $n(q, r) \pmod{10}$.

Given this, the following proposition develops.

Proposition 9. *For each fixed* $r > 0$, $b(q, r)$ *forms a periodical sequence with respect to* q *of period 5 when* r *is odd, or 10 when* r *is even.*

The first period of the sequence is given as follows:

$$b(q, 1) = 2, 6, 2, 0, 0 \qquad b(q, 2) = 3, 0, 1, 6, 5, 8, 5, 6, 1$$

$$b(q, 3) = 4, 4, 0, 2, 0 \qquad b(q, 4) = 5, 8, 9, 8, 5, 0, 3, 4, 3$$

$$b(q, 5) = 6, 2, 8, 4, 0 \qquad b(q, 6) = 7, 6, 7, 0, 5, 2, 1, 2, 5$$

$$b(q, 7) = 8, 0, 6, 6, 0 \qquad b(q, 8) = 9, 4, 5, 2, 5, 4, 9, 0, 7$$

$$b(q, 9) = 0, 8, 4, 8, 0 \qquad b(q, 10) = 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

Idea of the proof. Since $b(q, r) \equiv q^2 r + q \pmod{10}$, we evaluate $b(q, r)$ in order of $q = 1, 2, \ldots, 9, 0 \pmod{10}$. For instance, for $r = 5$, by evaluating $b(q, r) \equiv q^2 \cdot 5 + q \pmod{10}$, we obtain $6, 2, 8, 4, 0, 6, 2, 8, 4, 0$. The proof for each r is similar.

Similarly, the results of our observation of the rows are as follows, we skip the proof.

Proposition 10. *For each fixed $q > 0$, $b(q, r)$ forms a periodical sequence with respect to r of period 5 when q is even, or 10 when q is odd.*

The first period of the sequence is given as follows:

$$b(1,r) = 1,2,3,4,5,6,7,8,9,0 \qquad b(2,r) = 2,6,0,4,8$$
$$b(3,r) = 3,2,1,0,9,8,7,6,5,4 \qquad b(4,r) = 4,0,6,2,8$$
$$b(5,r) = 5,0,5,0,5,0,5,0,5,0 \qquad b(6,r) = 6,2,8,4,0$$
$$b(7,r) = 7,6,5,4,3,2,1,0,9,8 \qquad b(8,r) = 8,2,6,0,4$$
$$b(9,r) = 9,0,1,2,3,4,5,6,7,8 \qquad b(10,r) = 0,0,0,0,0.$$

Based on these propositions, for any given DS-divisor q we can evaluate and produce the last digits of positive integers which is D-divisible by q. For example, for $q = 78$, the possible last digits are $b(78, r) \equiv b(8, r) \in \{8, 2, 6, 0, 4\}$. Since $r \equiv 5 \pmod{10}$, $b(78, 345) = 8$.

4. The Relationship Between the Number of DS-divisors and the Euler Totients

Next, we investigate and observe patterns in the number of positive DS-divisors. Consider a positive integer n. We recall the Euler's ϕ-Function, $\phi(n)$, the sum of positive divisors $\sigma(n)$, and the number of divisors $\tau(n)$. It is known that $\sum_{d \mid n} \phi(d) = n$. We define $\sigma_{\mathrm{D}}(n)$ and $\tau_{\mathrm{D}}(n)$ similarly as follows.

Definition 11. Let n be a positive integer. We denote $\sigma_{\mathrm{D}}(n)$ as the sum of the DS-divisors of n and $\tau_{\mathrm{D}}(n)$ as the number of positive DS-divisors of n.

Lemma 12. *Let $n = m_1 m_2$, where $1 < m_1 < m_2 < n$. Assume m_1, m_2 are relatively prime. Then m_2 is not a DS-divisor of n.*

Proof. If $m_2 \underset{D}{\mid} n$, then $n = m_2^2 q_2 + m_2$ for some $q_2 \in \mathbb{Z}^+$. This implies $m_1 = m_2 q_2 + 1 > m_2$, a contradiction. Therefore, m_2 is not a DS-divisor of n. □

Now consider the sum $\sum_{d \underset{D}{\mid} n} \phi(d)$. Certainly

$$\sum_{d \underset{D}{\mid} n} \phi(d) \leq \sum_{d \mid n} \phi(d) = n.$$

We organized a separate table for $\tau(n)$, $\tau_{\mathrm{D}}(n)$, and $\sum_{d \underset{D}{|} n} \phi(d)$. A portion of this table is provided in Appendix A and is referenced as Table 5. We can immediately recognize the following facts from the properties of Euler's totient and Theorem 5.

Theorem 13. *Let p, q be two prime numbers, where $p < q$, and s, t be two positive integers.*

1. *If $n = p^s$, then $\sigma_D(n) = 1 + n$ and $\tau_{\mathrm{D}}(n) = 2$.*
2. *If $n = 2q$, then $\sigma_D(n) = 3 + n$ and $\tau_{\mathrm{D}}(n) = 3$.*
3. *If $n = pq$, where $2 < p$, then $\sigma_D(n) \leq 1 + p + n$ and $\tau_{\mathrm{D}}(n) \leq 3$.*
4. *If $n = p^s q^t$, then $\sigma_D(n) \leq 1 + n + \max\{p^s, q^t\}$ and $\tau_{\mathrm{D}}(n) \leq 3$.*
5. *If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes, $e_j \geq 1$ and $k > 2$, then $\tau_{\mathrm{D}}(n) \leq 2^k - k + 1$. In particular, if $k \geq 4$, $\tau_{\mathrm{D}}(n) \leq k + 3$.*

Proof.

1.-3. These are straightforward from the Theorem 5.

4. By Theorem 5, all possible DS-divisors are from the combinations of $p_1^{e_1}, p_2^{e_2}, \ldots, p_k^{e_k}$, thus there are at most 2^k many DS-divisors. Furthermore, there are k divisors of n in the form of $n/p_i^{e_i} \underset{D}{|} n$. But only one of these k-divisors can be a DS-divisor of n. Therefore, $\tau_{\mathrm{D}}(n) \leq 2^k - k + 1$. We can further reduce the bound by Lemma 12. For each $r = 2, 3, \ldots, k-1$, we choose r prime power factors of n such as $m_1 = (p_i^{e_{i_1}} \cdots p_{i_r}^{e_{i_r}})$, $m_2 = (p_{i_{r+1}}^{e_{i_{r+1}}} \cdots p_{i_k}^{e_{i_k}})$ and $n = m_1 m_2$. By Lemma 12, either m_1 or m_2 is not a DS-divisor of n. There are $\displaystyle\sum_{r=2}^{k-1} \binom{k}{r}$ many such non-DS-divisors. While $\displaystyle\sum_{r=2}^{k-1} \binom{k}{r} = \sum_{r=0}^{k} \binom{k}{r} - 2\binom{k}{0} - 2\binom{k}{1} = 2^k - 2 - 2k$, thus $\tau_{\mathrm{D}}(n) \leq 2^k - k + 1 - (2^k - 2 - 2k) = k + 3$.

□

We show two examples below. One has two prime factors and the other has five.

Example 14. For $n = 10$, $k = 2$ and the DS-divisors of 10 are 1, 2 and 10. Therefore,

$$\sigma_{\mathrm{D}}(10) = 1 + 2 + 10 = 13 \quad \text{and} \quad \tau_{\mathrm{D}}(10) = 3 \leq 2^k - k + 1 = 4.$$

Example 15. While the number $n = 2310 = 2 \cdot 5 \cdot 3 \cdot 7 \cdot 11$ has seven DS-divisors: $1, 2, 6, 7, 10, 11, 2310$ and $k = 5$. Thus,

$$\sigma_{\mathrm{D}}(2310) = 1 + 11 + 2 + 7 + 6 + 2310 = 2337$$

and $\tau_{\mathrm{D}}(2310) = 7 \leq k + 3 = 8$.

Remarks 16. Let n be a positive integer. Then,

1. If n has only trivial the DS-divisors, then $\sum_{d \underset{D}{|} n} \phi(d) = 1 + \phi(n)$.
2. If n is prime, then $\sum_{d \underset{D}{|} n} \phi(d) = n$ and $\tau_{\mathrm{D}}(n) = 2$.
3. In general, $\sum_{d \underset{D}{|} n} \phi(d) \leq n$.

By the Theorem 5, we can claim the following:

Theorem 17. *Let p, q be two prime numbers, where $p < q$, and s, t be two positive integers.*

1. *If $n = p^s$, where $s \geq 1$, then $\sum_{d \underset{D}{|} n} \phi(d) = p^s - p^{s-1} + 1$;*
2. *If $n = 2q$, where $q > 2$, then $\sum_{d \underset{D}{|} n} \phi(d) = q + 1$;*
3. *If $n = pq$, where $2 < p < q$, then*

$$\sum_{d \underset{D}{|} n} \phi(d) = \begin{cases} n - q + 1 & \text{if } p \underset{D}{|} n, \\ n - p - q + 2 & \text{if } p \underset{D}{\nmid} n. \end{cases}$$

4. *If $n = p^s q^t$, $(s, t) \neq (1, 1)$, then*

$$\sum_{d \underset{D}{|} n} \phi(d) \leq 1 + \max\{ p^s - p^{s-1} , q^t - q^{t-1} \} + n \cdot \frac{pq - p - q + 1}{pq}.$$

Proof. 1. It is obvious when referring to Theorem 5.

2. The DS-divisors of n are $1, 2$ and n. Therefore, $\sum_{d \underset{D}{|} n} \phi(d) = \phi(1)+\phi(2)+\phi(n) = 1 + 1 + (q-1) = q + 1$.

3. Assume $n = pq$. Then q does not divide n. If $p \underset{D}{|} n$, then $\sum_{d \underset{D}{|} n} \phi(d) = \phi(1)+\phi(p)+\phi(n) = 1+(p-1)+(p-1)(q-1) = p+pq-p-q+1 = n-q+1$. If $p \underset{D}{\nmid} n$, then $\sum_{d \underset{D}{|} n} \phi(d) = \phi(1)+\phi(n) = 1+(p-1)(q-1) = n-p-q+2$.

4. By Theorem 5(4), n has at most one non-trivial DS-divisor which is either p^s or q^t. Thus,

$$\sum_{d \underset{D}{|} n} \phi(d) \leq 1 + \max\{\, \phi(p^s)\, ,\, \phi(q^t)\,\} + \phi(n)$$
$$= 1 + \max\{\, p^s - p^{s-1}\, ,\, q^t - q^{t-1}\,\} + n \cdot \tfrac{pq-p-q+1}{pq}.$$

□

5. Conclusions and Future Research

We have identified the DS-divisors of a given positive integer, recognized how many there are, and, in addition, discovered patterns and structures among the DS-divisors. We also investigated a relationship between the number of DS-divisors and Euler's totient. In the future, we will further investigate connections with Giuga Numbers as well as the sequences in [7] by comparing them to our sequences. In addition, we will examine better bounds for $\sigma_{\mathrm{D}}(n)$ and $\tau_{\mathrm{D}}(n)$ and explore the possibilities of an "ϕ-Function" with respect to DS-Divisors. Additionally, we will continue to observe our table for more patterns with a particular focus on the minor diagonals of the table.

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Appendix A: Tables

$r =$	0	1	2	3	4	5	6	7	8
$q \downarrow$									
1	1	2	3	4	5	6	7	8	9
2	2	6	10	14	18	22	26	30	34
3	3	12	21	30	39	48	57	66	75
4	4	20	36	52	68	84	100	116	132
5	5	30	55	80	105	130	155	180	205
6	6	42	78	114	150	186	222	258	294
7	7	56	105	154	203	252	301	350	399
8	8	72	136	200	264	328	392	456	520
9	9	90	171	252	333	414	495	576	657
10	10	110	210	310	410	510	610	710	810
11	11	132	253	374	495	616	737	858	979
12	12	156	300	444	588	732	876	1020	1164
13	13	182	351	520	689	858	1027	1196	1365
14	14	210	406	602	798	994	1190	1386	1582
15	15	240	465	690	915	1140	1365	1590	1815
16	16	272	528	784	1040	1296	1552	1808	2064
17	17	306	595	884	1173	1462	1751	2040	2329
18	18	342	666	990	1314	1638	1962	2286	2610
19	19	380	741	1102	1463	1824	2185	2546	2907
20	20	420	820	1220	1620	2020	2420	2820	3220

Table 1: $n = n(q, r) = q^2 r + q$

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$\tau(n)$	2	2	3	2	4	2	4	3	4	2	6	2	4
$\tau_{\mathrm{D}}(n)$	2	2	2	2	3	2	2	2	3	2	3	2	3
$\sum_{d \underset{D}{\mid} n} \phi(d)$	2	3	3	5	4	7	5	7	6	11	7	13	8

Table 2: Numbers of DS-divisors and their sums

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