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THE L^p BEHAVIOR OF EIGENFUNCTION EXPANSIONS

BY

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ABSTRACT. We investigate the extent to which the eigenfunction expansions arising from a large class of two-point boundary value problems behave like Fourier series expansions in the norm of $L^p(0, 1)$, $1 < p < \infty$. We obtain our results by relating Green's function to the Hilbert transform.

1. **Introduction.** If f is in $L^p(0, 1)$, $1 < p < \infty$, and if $s_N f$ denotes the Nth partial sum of the Fourier series of f, then it is known [1, p. 100] that there exists a constant $K_p > 0$ such that

$$\|s_N f\|_p \le K_p \|f\|_p, \tag{1.1}$$

and in addition,

$$\lim_{N\to\infty} \|f - s_N f\|_p = 0. \tag{1.2}$$

In this paper, we investigate the extent to which these relations remain true for the eigenfunction expansions arising from a large class of two-point boundary value problems. Let $T: L^p \longrightarrow L^p$ denote a linear differential operator arising from such a boundary value problem. (A detailed definition of T will be given below.) Let $S_N f$ denote the Nth partial sum of the expansion of f in eigenfunctions of T. For each integer $m \geq 1$, let B^p_m denote the subspace of $L^p(0, 1)$ consisting of functions f such that $f^{(m)}$ exists and is in L^p, and such that $f^{(k)}(0) = f^{(k)}(1) = 0$ for $0 \le k \le m-1$. Let $B^p_0 = L^p$. Our basic result is that each operator T determines a smallest interger $m \geq 0$ such that if f is in B^p_m, then

$$\lim_{N\to\infty} \|f - S_N f\|_p = 0, \tag{1.3}$$

and there is a constant $K_p = K_p(T) \geq 0$ such that

$$\|S_N f\|_p \le K_p[\|f\|_p + \|f^{(1)}\|_p + \cdots + \|f^{(m)}\|_p]. \tag{1.4}$$

For *Birkhoff regular* [3, p. 49] boundary conditions $m = 0$. Since $B^p_0 = L^p$, we see that (1.3) and (1.4) then are direct analogs of (1.1) and (1.2). Thus our results include the results of Rutovitz [11], Smart [5], and Turner [12] for special cases of Birkhoff regular problems. For $m \geq 1$, it is no longer possible to expand

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all functions in L^p. Referring to Example 2 in [4, p. 23], it is merely a matter of computation to see that for the function $f(x) = 1$, $0 \leq x \leq 1$, $\|S_{N+1}f - S_N f\|_2$ does not converge to zero as $N \to \infty$. It is still an open problem to determine if B^p_m is the largest class of functions for which L^p convergence occurs. We note however that for each m, the set of eigenfunctions of T is complete in L^p, $1 \leq p < \infty$ [13].

One method for comparing general eigenfunction expansions with Fourier series involves using Paley-Wiener theorems on perturbation of bases [14]. A recent discussion of such ideas can be found in [15], and applications to eigenfunction expansions can be found in [16] and [17]. Referring again to Example 2 in [4], we see that our results apply to eigenfunctions which do not come close to satisfying any extant conditions for Paley-Wiener theorems.

For $p = 2$, the results (1.3) and (1.4), with $m = 0$, are known [2, Chapter XIX] for the class of *spectral* differential operators which arise from Birkhoff regular boundary conditions. In fact, in this case one can carry the analogy with Fourier series even further, since the eigenfunction expansions of spectral operators are unconditionally convergent, as are Fourier series in $L^2(0, 1)$. For $p = 2$ and $m \geq 1$, we have examples [4] which show that unconditional convergence is not preserved. For $p \neq 2$, even if $m = 0$, there is no longer unconditional convergence in general, since Fourier series arise as a special case [5, p. 82]. We will not pursue any further in this paper the question of rearrangements of eigenfunction expansions.

It is our hope that by discovering the extent to which general eigenfunction expansions reflect the properties of Fourier series, we will be able to predict these results from general principles.

Let τ denote the nth order linear differential expression defined for appropriate functions u by

$$\tau(u) = u^{(n)} + a_{n-1}(x)u^{(n-1)} + \cdots + a_0(x)u,$$

where the coefficients a_j are bounded measurable functions for $0 \leq x \leq 1$, $j = 0, 1, \ldots, n-1$. Let M, N denote $n \times n$ matrices of complex constants with n linearly independent columns between them, and let $\hat{u}(x)$ denote the column vector with components $(u(x), u^{(1)}(x), \cdots, u^{(n-1)}(x))$. Let U stand for the boundary expression

$$U(u) = M\hat{u}(0) + N\hat{u}(1).$$

For fixed p, $1 < p < \infty$, let $\Delta = \Delta_p$ denote the subspace of $L^p(0, 1)$ consisting of all functions u of class $C^{n-1}[0, 1]$ such that $u^{(n-1)}$ is absolutely continuous, $u^{(n)}$ is of class $L^p(0, 1)$, and $U(u) = 0$. Then the linear operator $T: L^p \to L^p$ is defined on Δ by $Tu = \tau(u)$. Since Δ contains all functions u of class $C^n[0, 1]$ such that $u^{(k)}(0) = u^{(k)}(1) = 0$, $0 \leq k \leq n-1$, we see that Δ

is dense in $L^p(0, 1)$.

For any complex number ρ, the differential equation $\tau(u) = -\rho^n u$ has n linearly independent solutions u such that u is in $C^{n-1}[0, 1]$ and u is absolutely continuous [6, p. 703]. Thus u is in $L^p(0, 1)$, and consequently, $\tau(u)$ is in $L^p(0, 1)$. Therefore, if $\Phi(x, \rho)$ is any fundamental matrix for $\tau(u) = -\rho^n u$, $-\rho^n$ is an eigenvalue of T if and only if

$$D(\rho) = \det [M\Phi(0, \rho) + N\Phi(1, \rho)]$$

is zero. If $D(\rho) \neq 0$, then Green's function $G(x, t, \rho)$ is bounded in x and t, so that

$$u(x, \rho) = \int_0^1 G(x, t, \rho) f(t)\, dt \tag{1.5}$$

is in $L^p(0, 1)$ if f is in $L^p(0, 1)$. Additionally, u is in Δ and $Tu = -\rho^n u + f$. Thus the Green's function for the problem is the kernel of the resolvent operator $[T + \rho^n I]^{-1}$, and the partial sums S_N of eigenfunction expansions corresponding to T are obtained as the sums of residues of (1.5). Since it is more convenient to work with the complex parameter ρ than with $\lambda = -\rho^n$, we shall integrate over circular arcs subtending angles $2\pi/n$ in the ρ-plane, rather than over circles in the λ plane.

In §2 of this paper we discuss the asymptotic properties of solutions to $\tau(u) = -\rho^n u$, and we define the class of boundary conditions we shall consider. This material is basically a convenient reformulation of known material, with some generality added. In §3 we will investigate the Green's function for T and we shall show how to associate the integer m to a particular problem. In §4 we will use the Hilbert transform to estimate the norms of a family of integral operators, and in §5 we shall obtain the results (1.3), (1.4).

2. Boundary conditions and solutions to $\tau(u) = -\rho^n u$.

Theorem 2.1. *Let* $\tau(u) = u^{(n)} + a_{n-2}(x)u^{(n-2)} + \cdots + a_0(x)u$. *Let* $\nu \geq 0$ *be a fixed integer. For* $\max(n - \nu - 2, 0) \leq j \leq n - 2$, *assume* $a_j^{(\nu+2+j-n)}$ *is of class* $L^\infty[0, 1]$. *For* $0 \leq j \leq n - \nu - 2$, *assume* a_j *is of class* $L^\infty[0, 1]$. *Then for each sector* $k\pi/n \leq \arg \rho \leq (k + 1)\pi/n$, *and for* $|\rho|$ *sufficiently large, the equation* $\tau(u) = -\rho^n u$ *has an analytic fundamental matrix* $\Phi(x, \rho)$ *such that* (*in the notation of* [7, p. 482])

$$\Phi(x, \rho) = T(\rho)[I + (1/\rho)A_1(x) + \cdots + (1/\rho)^\nu A_\nu(x) + (1/\rho)^{\nu+1}A(x, \rho)]E(x, \rho), \tag{2.1}$$

where $A(x, \rho) = O(1)$ *uniformly in* x, $0 \leq x \leq 1$, *as* $|\rho| \to \infty$, *and* $A_i^{(\nu-i)}$ *is of class* $L^\infty[0, 1]$ *for* $1 \leq i \leq \nu$.

Remark. Theorem 2.1 can be proved by translating into matrix notation the computations given in [3, p. 46]. An indication of this technique is given in [8, Theorem 2.2].

Lemma 2.1. *Let* $A(x, \rho) = I + \sum_{i=1}^{\nu} (1/\rho)^i A_i(x) + (1/\rho)^{\nu+1} A_{\nu+1}(x, \rho)$, *where* ρ *varies in an unbounded region* S *of the* ρ*-plane*, $0 \le x \le 1$, $A_i^{(\nu-i)}$ *is of class* $L^\infty[0, 1]$ *for* $1 \le i \le \nu$, *and* $A_{\nu+1}(x, \rho) = O(1)$ *uniformly in* x *for* $0 \le x \le 1$. *Then for* $|\rho|$ *sufficiently large in* S_j,

$$A^{-1}(x, \rho) = I + \sum_{i=1}^{\nu} (1/\rho)^i B_i(x) + (1/\rho)^{\nu+1} B_{\nu+1}(x, \rho),$$

where $B_{\nu+1}(x, \rho) = O(1)$ *uniformly in* x, $0 \le x \le 1$, *and* $B_i^{(\nu-i)}$ *is of class* $L^\infty[0, 1]$ *for* $1 \le i \le \nu$.

Proof. Let $B_0(x) = I$. It is easily verified that $B_k = -\sum_{l=1}^{k} A_l(x) B_{k-l}(x)$ for $1 \le k \le \nu$, and

$$B_{\nu+1}(x, \rho) = -A^{-1}(x, \rho) A_{\nu+1}(x, \rho)\left[I + \sum_{i=1}^{\nu} (1/\rho)^i B_i(x)\right].$$

The B_i's clearly have the required properties, $1 \le i \le \nu + 1$.

Corollary. $\Phi(x, \rho)$ *has the properties obtained in Theorem* 2.1, *then for* $|\rho|$ *sufficiently large*, $k\pi/n \le \arg \rho \le (k+1)\pi/n$,

$$\begin{aligned}\Phi^{-1}(x, \rho) = E^{-1}(x, \rho)[I + (1/\rho)B_1(x) + \cdots + (1/\rho)^\nu B_\nu(x) \\ + (1/\rho)^{\nu+1} B(x, \rho)]T^{-1}(\rho),\end{aligned} \tag{2.2}$$

where $B(x, \rho) = O(1)$ *uniformly in* x, $0 \le x \le 1$, *and* $B_i^{(\nu-i)}$ *is of class* $L^\infty[0, 1]$, $1 \le i \le \nu$.

Corollary. *If the components of the last column of* $\Phi^{-1}(x, \rho)$ *are denoted by* $\nu_j(x, \rho)$, $j = 1, \cdots, n$, *then*

$$\begin{aligned}\nu_j(x, \rho) = (\omega_j/n\rho^{n-1})e^{-\rho\omega_j x}[1 + (1/\rho)B_{1j}(x) + \cdots \\ + (1/\rho)^\nu B_{\nu j}(x) + (1/\rho)^{\nu+1} B_j(x, \rho)],\end{aligned} \tag{2.3}$$

where $B_j(x, \rho) = O(1)$ *uniformly in* x, $0 \le x \le 1$, *and* $B_{ij}^{(\nu-i)}$ *is of class* $L^\infty[0, 1]$ *for* $1 \le i \le \nu$.

Proof. The only difficulty here is to express $T^{-1}(\rho)$. It is easy to verify that the last column of $T^{-1}(\rho)$ is $(-1/n\rho^{n-1})[\omega_1 \cdots \omega_n]$.

The case that $a_{n-1}(x) \ne 0$ can be reduced to the case considered above, provided $a_{n-1}^{(n-1)}$ exists. Then the substitution $u = qv$ is permissible, where $q(x) = \exp[(-1/n)\int_0^x a_{n-1}]$, and then $\tau(u) = q\tau'(v)$, where τ' is an nth order differential expression with no $(n-1)$th derivative term. If the coefficients of τ' are to satisfy the conditions of Theorem 2.1, it suffices to assume that the coefficients $a_0, \cdots, a_{n-2}$ of τ satisfy the conditions given there, and in addition that $a_{n-1}^{(k)}$ be of class $L^\infty[0, 1]$, where $k = \max(n-1, \nu+1)$.

In the case that τ has no $(n-1)$th derivative term, the boundary expression U and the coefficients a_j are to be selected so that the resulting boundary value problem is Stone regular [7, p. 487]. As is pointed out in [7, p. 494], there are boundary value problems which are Stone regular for all choices of coefficients a_j, $j = 0, 1, \cdots, n-2$, provided certain linear combinations of determinants formed from M and N are not zero.

If there is an $(n-1)$th derivative term in τ, the influence of the substitution $u = qv$ is easily described. Since

$$u^{(j)}(x) = \sum_{l=0}^{j} \binom{j}{l} v^{(l)}(x) q^{(j-l)}(x),$$

we see that $\hat{u}(x) = Z(x)\hat{v}(x)$, where

$$Z(x) = \begin{bmatrix} q(x) & & & \\ q^{(1)}(x) & q(x) & & 0 \\ \vdots & \vdots & \ddots & \\ q^{(n-1)}(x) & (n-1)q^{(n-2)}(x) & \cdots & q(x) \end{bmatrix}$$

Thus $Uu = U'v$, where $U'v = MZ(0)\hat{v}(0) + NZ(1)\hat{v}(1) = M'\hat{v}(0) + N'\hat{v}(0)$. Using primes to denote the transformed problem, we have, in the notation of [7, p. 481], $\mathfrak{D}(\rho) = \mathfrak{D}'(\rho)$, $P(\rho) = P'(\rho)$, $Q(\rho) = Q'(\rho)$, so for any ρ not a zero of $D(\rho)$,

$$G(x, t, \rho) = q(x)G'(x, t, \rho)q^{-1}(t). \tag{2.4}$$

Although the notation in [7] is for problems in which $a_{n-1} \equiv 0$, the expressions (2.2)–(2.5) apply also to the case that $a_{n-1} \neq 0$. Let $A: L^p(0, 1) \to L^p(0, 1)$ be defined by $(Af)(x) = q(x)f(x)$. Then from (2.4) and the definition of the partial sums S_N, we have for any f in $L^p(0, 1)$,

$$S_N f = (AS'_N A^{-1})f.$$

Thus the L^p convergence of the sequence $S_N f$ is exactly the same as the convergence of the sequence $S'_N f$.

From now on we shall assume that $a_{n-1} \equiv 0$ and that the problem (τ, U) is Stone regular. We shall no longer use the prime notation.

3. **Green's function.** In this section we will rely heavily on the notation and equations of [7].

The expression

$$(S_N f)(x) = \int_0^1 (1/2\pi i) \int_{C_R} n\rho^{n-1} G(x, t, \rho)\, d\rho\, f(t)\, dt \tag{3.1}$$

denotes the Nth partial sum of the expansion of f in eigenfunctions of T, where C_R is a circular arc of radius R subtending an angle of $2\pi/n$, and the sum includes the invariant subspaces corresponding to the N eigenvalues $\lambda = -\rho^n$ for which $|\rho| < R$. If $G_0(x, t, \rho)$ denotes the Green's function for the operator corresponding to $\tau(u) = u^{(n)}$, $M = I = -N$, then the corresponding eigenfunction expansion coincides with the Fourier series expansion. Thus to compare a general eigenfunction expansion with the Fourier series expansion, one considers the expression

$$\int_0^1 (1/2\pi i)\int_{C_R} n\rho^{n-1}[G(x, t, \rho) - G_0(x, t, \rho)]\,d\rho\, f(t)\,dt.$$

As is indicated in [7, p. 479], it is convenient to introduce a function $\Gamma(x, t, \rho)$ related to G [7, p. 483], and the analogous function Γ_0 for G_0, and then to write $G - G_0 = (G - \Gamma) + (\Gamma - \Gamma_0) + (\Gamma_0 - G_0)$. (An indication of the intrinsic meaning of Γ is given in [8].) Let

$$F_R(x) = \int_0^1 (1/2\pi i)\int_{C_R} n\rho^{n-1}[\Gamma_0(x, t, \rho) - G_0(x, t, \rho)]\,d\rho\, f(t)\,dt, \tag{3.2}$$

$$H_R(x) = \int_0^1 (1/2\pi i)\int_{C_R} n\rho^{n-1}[\Gamma(x, t, \rho) - \Gamma_0(x, t, \rho)]\,d\rho\, f(t)\,dt. \tag{3.3}$$

Lemma 3.1. *For any f in $L^p(0, 1)$, $1 < p < \infty$,*

$$\lim_{R\to\infty} \|F_R\|_p = 0, \qquad \lim_{R\to\infty} \|H_R\|_p = 0,$$

and there exists a constant $K_p \geq 0$ such that

$$\|F_R\|_p \leq K_p\|f\|_p, \qquad \|H_R\|_p \leq K_p\|f\|_p.$$

Proof. For F_R, we note that

$$(1/2\pi i)\int_{C_R} n\rho^{n-1}\Gamma_0(x, t, \rho)\,d\rho = [\sin R(x - t)]/[\pi(x - t)] = D(x, t, R)$$

which is the Dirichlet kernel for the Fourier integral, and

$$\int_0^1 (1/2\pi i)\int_{C_R} n\rho^{n-1}G_0(x, t, \rho)\,d\rho\, f(t)\,dt = \sum_{|k|\leq N} (f, e_k)e_k(x)$$

where $e_k(x) = e^{2k\pi ix}$. For any f of class $L^p(-\infty, \infty)$ which is of compact support, we have [10, p. 143],

$$\left\|\int_0^1 D(x, t, R)f(t)\,dt\right\|_p \leq K_p\|f\|_p$$

for R sufficiently large. Since the same is true of the partial sums of the Fourier series expansion of f, we see that the transformation $f \to F_R$ is uniformly bounded. By Theorem 2.3 in [7, p. 486], we see that $\|F_R\| \to 0$ on a dense subset of $L^p(0, 1)$, so by the uniform boundedness, $\|F_R\| \to 0$ on all of $L^p(0, 1)$. (The argument in this paragraph sets the pattern for the more involved arguments in §§4 and 5.)

For H_R, using equations (2.9) and (2.11) in [7, p. 484], we can easily verify that the kernel of (3.3) is uniformly bounded and, by the argument following Theorem 2.3 in [7, p. 486], that for any f in $L^1(0, 1)$, $H_R(x)$ goes to zero uniformly as R gets large.

Note that the properties of Γ are independent of the assumption of Stone regularity. This assumption is needed to obtain for $G - \Gamma$ results analogous to those above.

Referring to the expression for $G - \Gamma$ given in the proof of Theorem 4.1 in [7, p. 492], let m denote the largest of the integers $l_{kj} - l_i$. Having determined the integer m in this way, we now assume that the differential expression τ satisfies the conditions of Theorem 2.1, for $\nu = m$. Then since

$$v_j(t, \rho) = [-\omega_j/n\rho^{n-1}] \cdot [1 + (1/\rho)B_{1j}(t) + \cdots + (1/\rho)^m B_{mj}(t) + \cdots + (1/\rho^{m+1})B_{m+1}(t, \rho)]e^{-\rho\omega_j t},$$

where $B_{ij}^{(m-i)}$ is of class $L^\infty[0, 1]$ for $1 \le i \le m$ and $B_{m+1\ j} = O(1)$, and

$$u_k(x, \rho) = [1 + (1/\rho)A_k(x, \rho)]e^{\rho\omega_k x}$$

where $A_k(x, \rho) = O(1)$, we have

$$n\rho^{n-1}[G(x, t, \rho) - \Gamma(x, t, \rho)] = \sum_{k=1}^{n} \sum_{j=1}^{n} \Lambda_{kj}(x, t, \rho),$$

where

$$\Lambda_{kj}(x, t, \rho) = \rho^m b_{kj}(x, \rho)[1 + (1/\rho)B_{1j}(t) + \cdots + (1/\rho)^m B_{mj}(t) + (1/\rho)^{m+1}B_{m+1\ j}(t, \rho)]e^{\rho\gamma_{kj}(x, t)}. \tag{3.4}$$

In (3.4), b_{kj} stands for a uniformly bounded function of all indicated variables and the exponent $\gamma_{kj}(x, t)$ is as defined in the proof of Theorem 4.1 in [7, p. 492]. We will suppress the subscripts k and j whenever possible.

Lemma 3.2. *For f in B_m^p,*

$$\int_0^1 (1/2\pi i)\int_{C_R} \Lambda(x, t, \rho)\,d\rho\, f(t)\,dt = \sum_{i=0}^{m} \int_0^1 \left[\int_{C_R} b_i(x, \rho)e^{\rho\gamma(x, t)}d\rho\right][B_i(t)f(t)]^{(m-i)}dt + \int_0^1 \int_{C_R} (1/\rho)b_{m+1}(x, t, \rho)e^{\rho\gamma(x, t)}d\rho f(t)\,dt, \tag{3.5}$$

where the b_i's are uniformly bounded in all indicated variables.

Proof. The proof is an immediate consequence of integration by parts, noting the simple form of the exponents $\gamma(x, t)$, and the fact that the boundary values of $f^{(i)}$ are zero for $0 \le i \le m - 1$.

The value of (3.5) is that each of the positive powers of ρ appearing in (3.4) has been eliminated. In addition, each of the first $m + 1$ terms on the right of (3.5) is of the form

$$\int_0^1 K(x, t, R)[B_i(t)f(t)]^{(m-i)}dt \tag{3.6}$$

where

$$K(x, t, R) = \int_{C_R} b(x, \rho)e^{\rho\gamma(x, t)}d\rho, \tag{3.7}$$

while the last term in (3.5) is of the form

$$\int_0^1 K(x, t, R) f(t)\, dt \tag{3.8}$$

where

$$K(x, t, R) = \int_{C_R} (1/\rho) b(x, t, \rho) e^{\rho\gamma(x, t)} d\rho. \tag{3.9}$$

We consider these integral operators in the next section.

4. **A family of integral operators.** Let g be of class $L^p(0, 1)$, $1 < p < \infty$, and consider the transformation

$$G_R(x) = \int_0^1 K(x, t, R) g(t)\, dt \tag{4.1}$$

where

$$K(x, t, R) = \int_{C_R} b(x, t, \rho) e^{\rho(x+t)} d\rho. \tag{4.2}$$

In this section of the paper, C_R is the arc $\rho = Re^{i\theta}$, $\pi/2 \leq \theta \leq 3\pi/2$.

Lemma 4.1. *$K(x, t, R) = O(1/(x + t))$, for R sufficiently large.*

Proof. From (4.2), $|K(x, t, R)| \leq MR \int_{\pi/2}^{3\pi/2} e^{R(x+t)\cos\theta} d\theta$, for some constant M. Using the relations $\cos\theta \leq 1 - 2\theta/\pi$ for $\pi/2 \leq \theta \leq \pi$, $\cos\theta \leq -3 + 2\theta/\pi$ for $\pi \leq \theta \leq 3\pi/2$, we have

$$|K(x, t, R)| \leq M[1 - e^{-R(x+t)}]/[x + t] \leq M/(x + t).$$

Theorem (M. Riesz) [9, p. 132]. *If $g \in L^p(-\infty, \infty)$, $1 < p < \infty$, then*

$$H_g(x) = \text{C.P.V.} \int_{-\infty}^{\infty} g(t)/(x - t)\, dt$$

exists almost everywhere. In addition, there exists a constant $K_p \geq 0$ such that $\|H_g\|_p \leq K_p \|g\|_p$.

Remark. The function H_g is called the *Hilbert transform* of g.

Theorem 4.1. *There exists a constant $K_p \geq 0$ such that for G_R as defined in* (4.1), (4.2), *and for R sufficiently large*, $\|G_R\|_p \leq K_p \|g\|_p$.

Proof. From Lemma 4.1, for R sufficiently large,

$$|G_R(x)| \leq M \int_0^1 |g(t)|/(x + t)\, dt.$$

We make the change of variable $t = -s$ and define $g(-s)$ to be zero outside of $[-1, 0]$. Thus

$$|G_R(x)| \leq M \int_{-\infty}^{\infty} |g(-s)|/(x - s)\, ds.$$

If $0 < x < 1$, then

$$\text{C.P.V.} \int_{-\infty}^{\infty} |g(-s)|/(x - s)\, ds = \int_{-\infty}^{\infty} |g(-s)|/(x - s)\, ds,$$

since $g(-s)$ is zero for s near x. Thus using the theorem of M. Riesz, and restricting back to functions in $L^p(0, 1)$, $\|G_R\|_p \leq K_p \|g\|_p$.

Lemma 4.2. *If* $K(x, t, R) = \int_{C_R} (1/\rho) h(x, t, \rho) e^{\rho(x+t)} d\rho$, *then for any function* g *of class* $L^1(0, 1)$ *which is zero for* $0 \le t \le \delta$, $\delta > 0$, $G_R(x) = O(1/R)$, *uniformly in* x, $0 \le x \le 1$.

Proof. Repeating the estimates in the proof of Lemma 4.1, we now have

$$|K(x, t, R)| \le M[1 - e^{-R(x+t)}]/[R(x+t)] \le M/R\delta$$

for $0 \le x \le 1$. Thus $|G_R(x)| \le M \|g\|_1 / R\delta$.

Corollary. *If* $K(x, t, R) = \int_{C_R} h(x, \rho) e^{\rho(x+t)} d\rho$, *and if* g *is absolutely continuous on* $[0, 1]$ *and vanishes for* $0 \le t \le \delta$, $\delta > 0$, *then* $G_R(x) = O(1/R)$ *uniformly in* x, $0 \le x \le 1$.

Proof. We use integration by parts:

$$\begin{aligned} G_R(x) &= \int_{C_R} h(x, \rho) e^{\rho x} \int_0^1 e^{\rho t} g(t)\, dt\, d\rho \\ &= \int_{C_R} (1/\rho) h(x, \rho) e^{\rho x} \left[e^{\rho} g(1) - e^{\rho\delta} g(\delta) - \int_\delta^1 e^{\rho t} g^{(1)}(t)\, dt \right] d\rho \\ &= \int_{C_R} (1/\rho) h(x, \rho) g(1) e^{\rho(x+1)} d\rho - \int_{C_R} (1/\rho) h(x, \rho) g(\delta) e^{\rho(x+\delta)} d\rho \\ &\quad - \int_\delta^1 \int_{C_R} (1/\rho) h(x, \rho) e^{\rho(x+t)} d\rho\, g^{(1)}(t)\, dt. \end{aligned}$$

Each of these three terms is $O(1/R)$ by Lemma 4.2 and its corollary.

Theorem 4.2. *If* $K(x, t, R)$ *is as in Lemma* 4.2 *or its corollary, then for each* g *in* $L^p(0, 1)$,

$$\lim_{R \to \infty} \|G_R\|_p = 0.$$

Proof. The class of functions g which are in $C'[0, 1]$ and which vanish in $[0, \delta]$ (for all $\delta > 0$) is dense in $L^p(0, 1)$. By Lemma 4.2 and its corollary, $\|G_R\|_p = O(1/R)$ for such functions. But by Theorem 4.1, the transformation $g \to G_R$ is uniformly bounded on all of $L^p(0, 1)$. Thus using the density, the result follows.

If the expression $x + t$ is replaced by any of $1 - x + t$, $1 + x - t$, or $2 - x - t$, Theorems 4.1 and 4.2 remain true. For the density arguments using $1 + x - t$ and $2 - x - t$, it would be necessary to consider intervals $[0, 1 - \delta]$.

5. **The expansion theorem.** We return to the expressions (3.6) – (3.9). The exponents $\gamma(x, t) = \gamma_{kj}(x, t)$ are given explicitly in [7, p. 392]. Let σ, τ be as defined in [7, p. 485], and let the sets of indices A_i, B_i and the sectors S_i be as defined in [7, p. 483].

Lemma 5.1. *For* ρ *in* S_i,

$$e^{\rho\gamma_{kj}(x,t)} = e^{\rho\sigma(1-x+t)}b_1(x,t,\rho) + e^{\rho\tau(1-x+t)}b_2(x,t,\rho), \quad k \text{ in } A_i,\ j \text{ in } A_i,$$
$$= e^{\rho\sigma(2-x-t)}b_1(x,t,\rho) + e^{\rho\tau(2-x-t)}b_2(x,t,\rho), \quad k \text{ in } A_i,\ j \text{ in } B_i,$$
$$= e^{\rho\sigma(x+t)}b_1(x,t,\rho) + e^{\rho\tau(x+t)}b_2(x,t,\rho), \quad k \text{ in } B_i,\ j \text{ in } A_i,$$
$$= e^{\rho\sigma(1+x-t)}b_1(x,t,\rho) + e^{\rho\tau(1+x-t)}b_2(x,t,\rho), \quad k \text{ in } B_i,\ j \text{ in } B_i,$$

where the b_i's are uniformly bounded in all indicated variables.

Proof. Note that σ and τ depend upon i. Consider the case k in A_i, j in A_i. Then

$$e^{\rho[\omega_k(x-1)-\omega_j t]} = e^{\rho\sigma(1-x+t)}e^{-\rho[(1-x)(\sigma+\omega_k)+t(\sigma+\omega_j)]},$$
$$= e^{\rho\tau(1-x+t)}e^{-\rho[(1-x)(\tau+\omega_k)+t(\tau+\omega_j)]}.$$

For given ρ in S_i, the exponent of at least one of the two right-hand factors will have nonpositive real part.

The other possibilities for k and j are handled similarly.

Lemma 5.2. *If $K(x, t, R)$ is as in (3.7), and if for g in $L^p(0, 1)$,*

$$G_R(x) = \int_0^1 K(x, t, R)g(t)\,dt, \tag{5.1}$$

then

$$\lim_{R\to\infty} \|G_R\|_p = 0. \tag{5.2}$$

Proof. Using the simple form of each of the exponents $\gamma(x, t)$, we see that Lemma 4.2 and its corollary hold when $x + t$ is replaced by $\gamma(x, t)$, except that it may be necessary to use the interval $[0, 1-\delta]$ rather than $[\delta, 1]$. By Lemma 5.1 and Theorem 4.1, we see that the transformation (5.1) is uniformly bounded. Thus (5.2) follows by a density argument.

Theorem 5.1. *Let T be a differential operator arising from a Stone regular boundary value problem. Let m be the integer associated with T, and assume the coefficients of T satisfy the conditions of Theorem 2.1, for $\nu = m$. Then for f in B^p_m,*

$$\lim_{N\to\infty} \|f - S_N f\|_p = 0 \tag{5.3}$$

and there is a constant $K_p = K_p(T) \geq 0$ such that for all N,

$$\|S_N f\|_p \leq K_p[\|f\|_p + \|f^{(1)}\|_p + \cdots + \|f^{(m)}\|_p]. \tag{5.4}$$

Proof. Referring to (3.5), let

$$F_i(x, R) = \int_0^1 \left[\int_{C_R} b_i(x, \rho)e^{\rho\gamma(x,t)}d\rho\right][B_i(t)f(t)]^{(m-i)}dt$$

for $0 \leq i \leq m$, and let

$$F_{m+1}(x, R) = \int_0^1 \left[\int_{C_R} (1/\rho)b(x, t, \rho)e^{\rho\gamma(x,t)}d\rho\right]f(t)\,dt.$$

It suffices to prove that

$$\lim_{R\to\infty} \|F_i(\cdot, R)\|_p = 0 \tag{5.5}$$

and

$$\|F_i(\cdot, R)\|_p \le K_p[\|f\|_p + \|f^{(1)}\|_p + \cdots + \|f^{(m)}\|_p], \tag{5.6}$$

for R sufficiently large, $0 \le i \le m+1$. For $i = m+1$, replace $e^{\rho\gamma(x,t)}$ by the appropriate expression from Lemma 5.1. Then (5.5) is obtained by using Theorem 4.2, and from Theorem 4.1 we obtain

$$\|F_{m+1}(\cdot, R)\|_p \le K_p\|f\|_p.$$

For $0 \le i \le m$, the result (5.5) follows from Lemma 5.2. To obtain (5.6), replace $e^{\rho\gamma(x,t)}$ by the appropriate expression from Lemma 5.1. Then by Theorem 4.1,

$$\|F_i(\cdot, R)\|_p \le \tilde{K}_p\|[B_i f]^{(m-i)}\|_p \le K_p[\|f\|_p + \cdots + \|f^{(m-i)}\|_p].$$

Note that we have used the fact that each B_i is determined only by T, and in addition each B_i is in $L^\infty[0, 1]$.

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