The Behavior of Gravitational Systems

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Except when $n = 2$ or $n = 3$, little information is available about the behavior of Newtonian gravitational systems of n mass particles as the time t becomes infinite, even on the *à priori* assumption that no catastrophe occurs during the motion. The purpose of this paper is to obtain some precise statements in the general case. Although the methods used are primitive, the results appear to be new.

Two assumptions are made throughout. The first is a matter of convenience; the second is forced upon us by the total ignorance concerning the possible singularities which can occur. We assume, first, that *the center of mass* 0 is fixed; this means, of course, that we are concerned only with the internal motion of the system. Secondly, we suppose that *the coordinates of its particles in some* (and hence each) *rectangular coordinate system centered at* 0 *encounter no singularity for positive time t.*

Except for our use of the letter I, our notation is the customary one. The vector $\mathbf{r}_k$ represents the position of the k^{th} particle relative to 0, $\mathbf{v}_k$ its velocity, m_k its mass, r_k its distance from 0, v_k its speed, r_{jk} its distance from the j^{th} particle, and v_{jk} its speed relative to the j^{th} particle. The (internal) kinetic energy T is defined to be $\frac{1}{2} \sum_k m_k v_k^2$.

The symbol $\sum^*$ represents the double sum $\sum_{1 \leq j < k \leq n}$ and G the gravitational constant. The self-potential U is defined by

$$U = \sum{}^* G m_j m_k / r_{jk} .$$

The second assumption made above is equivalent to the requirement that U remains finite for all positive time, and implies, among other things, that no collisions occur. It is, of course, unknown whether the only singularities are collisions.

According to the principle of conservation of energy the quantities T and U are connected by the relation

$$T = U + h, \tag{1}$$

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where h is the constant total energy. A second important principle is the source of most of our knowledge of the n-body problem. It concerns the quantity I, defined here to be one-half the moment of inertia:

$$I = \tfrac{1}{2} \sum_k m_k r_k^2 . \tag{2}$$

The principle, due to Lagrange and Jacobi, tells us that

$$\ddot{I} = 2T - U.$$

In view of (1) this can be written in the equivalent forms

$$\ddot{I} = U + 2h \quad \textit{and} \quad \ddot{I} = T + h. \tag{3}$$

Because the center of mass is fixed, the quantities I and T can be written in the alternate forms

$$I = \frac{1}{2M} \sum{}^* m_j m_k r_{jk}^2 , \qquad T = \frac{1}{2M} \sum{}^* m_j m_k v_{jk}^2 , \tag{4}$$

where M is the total mass of the system, $\sum_k m_k$.

All these matters are standard, and can be found, for example in Chapter II of [2].

We follow the customary use of the $o - O$ symbols, and employ the letters $A, B, \cdots$ to denote quantities *independent of the time*; they need not be the same at each occurrence.

The following inequalities are basic and can be found in the reference just given. Denote by r and R respectively the quantities

$$r = \min_{j,k} r_{jk} , \qquad R = \max_{j,k} r_{jk} .$$

Then

$$0 < B \leqq rU \leqq A \tag{5}$$

and

$$0 < BR^2 \leqq I \leqq AR^2. \tag{6}$$

In a sense then, the quantity U^{-1} is a measure of the minimum spacing between particles at each instant of time, and $I^{1/2}$, of the maximum spacing.

1. Growth of the system. We begin with a review of the basic facts about the behavior of r. They are undoubtedly well-known, but an explicit statement is hard to find.

Theorem 1.1. *If $h < 0$, then r remains bounded. If $h = 0$ then $r = 0(t^{2/3})$, $t \to \infty$. If $h > 0$, then $r = O(t)$, $t \to \infty$.*

To prove this suppose, first, that $h < 0$. Then, by (1), $T = U - |h|$. Since $T \geqq 0$, $U \geqq |h|$. Therefore, by (5), r is bounded.

The remaining cases depend on a fundamental inequality. According to the definition of U we have

$$\dot{U} = -\sum{}^{*} Gm_j m_k \dot{r}_{jk}/r_{jk}^2 .$$

Since $r \leqq r_{jk}$ it follows that (remember our use of letters like A) that

$$|\dot{U}| \leqq Ar^{-2} \sum{}^{*} (m_j m_k)^{1/2} |\dot{r}_{jk}|,$$

or, by (5) once again,

$$|\dot{U}| \leqq AU^2 \sum{}^{*} (m_j m_k)^{1/2} |\dot{r}_{jk}|.$$

By the Cauchy inequality

$$|\dot{U}|^2 \leqq AU^4 \sum{}^{*} m_j m_k \dot{r}_{jk}^2 \leqq AU^4 \sum{}^{*} m_j m_k v_{jk}^2 .$$

In view of (4) we obtain

$$|\dot{U}| \leqq AU^2(T)^{1/2}. \tag{1.1}$$

Now let $\rho = U^{-1}$. Since $T = U + h$, the equation (1.1) becomes

$$|\dot{\rho}| \leqq A\left(\frac{1}{\rho} + h\right)^{1/2}. \tag{1.2}$$

If $h = 0$, then $\rho^{1/2} |\dot{\rho}| \leqq A$. Integration yields $\rho^{3/2} \leqq At$, or $U^{-3/2} \leqq At$. According to (5), this in turn implies $r = O(t^{2/3})$.

There remains the case $h > 0$. Define $p = \rho$ if $\rho \geqq 1$ and $p = 1$ if $\rho < 1$. In case $\rho \geqq 1$ the inequality (1.2) shows that $\dot{p}$ is bounded. Also on each interval where $\rho < 1$ the function p is constant, so that $\dot{p}$ is zero. Then $\dot{p}$ is bounded everywhere. Therefore $p = O(t)$. Since $\rho \leqq p$ it follows that $\rho = O(t)$, $U^{-1} = O(t)$, and by (5), that $r = O(t)$.

This theorem furnishes upper bounds to the growth of r. It is reasonable to ask for lower bounds. Nothing seems to be known for $n > 3$, not even so mild an assertion as, say $r \geqq Ae^{-e^t}$, $A > 0$, and we have no contribution to make. As to the quantity R, the problem of *upper* bounds is untouched, and no statement like $R = O(e^{e^t})$, for example, has ever been established.

Some primitive information about *lower* bounds for R is readily available if $h \geqq 0$. First, if $h > 0$, we can apply (3) to obtain $\ddot{I} \geqq 2h$, or after integration twice, $I \geqq At^2$, $A > 0$. Then (6) yields $R \geqq At$, $A > 0$. Secondly, if $h = 0$. Theorem 1.1 and (5) tell us that $U \geqq At^{-2/3}$, so that by (3), $\ddot{I} \geqq At^{-2/3}$, $I \geqq At^{4/3}$; and by (6), $R \geqq At^{2/3}$. We have established

Theorem 1.2. *If $h > 0$ then $R \geqq At$; if $h = 0$, then $R \geqq At^{2/3}$. In each case $A > 0$.*

The method just used is useless if $h < 0$. For then all that (3) yields is $\ddot{I} \geqq -|h|$, which is not very helpful. In view of Theorem 1.1 it is natural to conjecture that Theorem 1.2 ought to conclude with the assertion: if $h < 0$, then $R \geqq A > 0$. But not even such an inequality as

$$R \geqq Ae^{-e^{t^t}}$$

is known. On the other hand, it is familiar that $R \to 0$, $t \to \infty$, is impossible [2, p. 43].
A minor contribution to this question is offered in the final section of this paper.

Apart from a considerable sharpening of Theorem 1.2 in the sequel, our main object is to provide connections between the growth properties of r and R. As an example of such a connection we quote without proof the following theorem [2, p. 46].

Theorem 1.3. *If r is bounded away from zero, then $R = O(t)$.*

At one time the author conjectured that the limitation $R = O(t)$ is a property of all systems, no counterexample being known to him. On the other hand in all known examples it appears that r *is* bounded below, so the conjecture loses much of its force.

2. Relations between r and R. It is reasonable to ask whether the converse of Theorem 1.3 is true. This, too, is unanswered, but the following can be proved.

Theorem 2.1. *If $R = O(t)$, $t \to \infty$, then on the average r is bounded below, that is,*

$$\frac{1}{t}\int_0^t r(u)\,du \geqq A > 0$$

for all $t > 0$.

We start with Taylor's formula with remainder to obtain

$$I(2t) = I(t) + t\dot{I}(t) + (t^2/2)\ddot{I}(\xi),$$

where $t < \xi < 2t$. Since $I \geqq 0$, and since, by (3), $\ddot{I} \geqq 2h$ we can conclude that

$$I(2t) \geqq t\dot{I} + ht^2. \tag{2.1}$$

By hypothesis $R = O(t)$, so, according to (6), $I = O(t^2)$. From this and (2.1) we deduce that $t\dot{I} \leqq At^2$, or $\dot{I} \leqq At$. Now, according to (3),

$$\dot{I} = \int_0^t U(u)\,du + 2ht + \text{constant}.$$

Since $\dot{I} \leqq At$, this leads us to $\int_0^t U(u)\,du \leqq At$, or, from (5), to

$$\int_0^t \frac{du}{r(u)} \leqq At. \tag{2.2}$$

By the Schwarz inequality and (2.2) we have

$$t^2 = \left[\int_0^t (r(u))^{1/2}\frac{du}{(r(u))^{1/2}}\right]^2 \leqq \int_0^t r(u)\,du \int_0^t \frac{du}{r(u)} \leqq At\int_0^t r(u)\,du.$$

This establishes the theorem.

Further progress depends upon a new inequality whose proof is deferred to the next section.

Theorem 2.2. *There is a time t_0, depending on the initial conditions, such that*

$$I^{1/2} < t(T^{1/2} + U^{1/2}), \qquad t \geqq t_0 . \tag{2.3}$$

As a consequence we can prove.

Theorem 2.3. *As $t \to \infty$, $R/t \to \infty$ if and only if $r \to 0$.*

Suppose first that $r \to 0$. Then $U \to \infty$, and by (3), $\ddot{I} \to \infty$. For each $M > 0$ there is t_1 such that $\ddot{I} \geqq M$, $t \geqq t_1$. Integrate twice and divide by t^2. Then $\liminf I/t^2 \geqq \frac{1}{2}M$. Since M can be made arbitrarily large $I/t^2 \to \infty$, and so, by (6) $R/t \to \infty$.

Conversely, suppose $R/t \to \infty$. Then, by (6) again, $I^{\frac{1}{2}}/t \to \infty$, and, by (2.3), $T^{1/2} + U^{1/2} \to \infty$. Since $T = U + h$ we get $U \to \infty$, or $r \to 0$.

As an addendum to this theorem, it should be observed that the catastrophe $R/t \to \infty$, cannot be caused by the rapid explosion of *every* distance r_{jk}. Stated precisely: it is impossible that $r_{jk}/t \to \infty$ for every pair of particles. For this implies that $t/r_{jk} \to 0$ for all pairs (j, k) so that $tU \to 0$, $r/t \to \infty$. This contradicts Theorem 1.1. On the other hand it is *not* ruled out that $\limsup r_{jk}/t = \infty$ for every pair of particles.

It is known that $n \leqq 3$, the event $r \to 0$ cannot occur at all, [1], and hence it is impossible that $R/t \to \infty$. But for $n = 3$ the possibility that $\limsup R/t = \infty$ occurs still exists, and for $n > 3$ nothing is known about the possible occurrence of either of these events.

3. Proof of the main inequality. We turn to the proof of (2.3). First we establish Sundman's inequality (3.1). It is stronger than we need for the present purpose, but the proof seems to be the simplest available. Let θ_k denote the angle between $\mathbf{r}_k$ and $\mathbf{v}_k$. Since, according to (2), $I = \frac{1}{2} \sum m_k (\mathbf{r}_k \cdot \mathbf{r}_k)$ it follows that $\dot{I} = \sum m_k(\mathbf{r}_k \cdot \mathbf{v}_k)$, or

$$|\dot{I}| \leqq \sum m_k r_k v_k \,|\cos \theta_k|.$$

By the Cauchy inequality

$$\dot{I}^2 \leqq \sum m_k r_k^2 \sum m_k v_k^2 \cos^2 \theta_k .$$

Let $\bar{\mathbf{c}}$ denote the constant angular momentum. Then [2, p. 42]

$$\mathbf{c} = \sum m_k(\bar{\mathbf{r}}_k \times \bar{\mathbf{v}}_k),$$

so that

$$c \leqq \sum m_k r_k v_k \sin \theta_k ,$$

where $c = |\mathbf{c}|$. Hence

$$c^2 \leqq \sum m_k r_k^2 \sum m_k v_k^2 \sin^2 \theta_k .$$

Therefore

$$\dot{I}^2 + c^2 \leqq \sum m_k r_k^2 \sum m_k v_k^2 ,$$

or

$$\dot{I}^2 + c^2 < 4IT. \tag{3.1}$$

Next we observe that if I_0 is the initial value of I the equation

$$I - t\dot{I} = I_0 - \int_0^t uU(u)\, du = ht^2 \tag{3.2}$$

is correct for $t = 0$. That it is correct for $t > 0$ follows on differentiating both sides and comparing with (3).

Now because $U^{-1} \leqq Ar$ the inequalities of Theorem 1.1. imply that the integral in (3.2) diverges as $t \to \infty$. Hence (3.2) enables us to conclude that from some time t_0 on

$$I - t\dot{I} < -ht^2, \qquad t \geqq t_0 . \tag{3.3}$$

But according to (3.1) we have $\dot{I} \leqq 2(IT)^{1/2}$. Therefore

$$I - 2t(IT)^{1/2} < -ht^2, \qquad t \geqq t_0 .$$

Add t^2T to both sides of this last inequality. Since $T - h = U$ we find that

$$(I^{1/2} - tT^{1/2})^2 < t^2U, \qquad t \geqq t_0 ,$$

which implies (2.3).

4. Growth of the system, resumed. At the close of §1 there occurred the conjecture that always $R = O(t)$, or what is equivalent, $R^3 = O(t^3)$. If this were true it would imply that

$$R^2r = O(t^3), \qquad t \to \infty, \tag{4.1}$$

since $r \leqq R$. Actually this weaker assertion *is* true.

Theorem 4.1. *The estimate* (4.1) *always holds. In fact, unless both $h > 0$ and r is unbounded, it is even true that*

$$R^2r = O(t^2), \qquad t \to \infty . \tag{4.2}$$

According to (2.3) and (1) we have

$$R \leqq At[(U + h)^{1/2} + U^{1/2}]$$

for $t \geqq t_0$. By (5)

$$R \leqq At\left[\left(\frac{B}{r} + h\right)^{1/2} + \left(\frac{B}{r}\right)^{1/2}\right], \tag{4.3}$$

so that

$$R(r)^{1/2} \leqq At[(B + hr)^{1/2} + B^{1/2}]. \tag{4.4}$$

But according to Theorem 1.1 it is true that $r = O(t)$, so the right-hand side of (4.4) is $O(t^{3/2})$. This establishes (4.1).

If $h \leqq 0$ or r is bounded the expression in parentheses on the right-hand side of (4.4) is bounded. Hence $R(r)^{1/2} \leqq At$, which is (4.2).

We turn to the task of sharpening Theorem 1.2. This is accomplished in the next two sections.

5. Systems of zero energy. If h $= 0$ there exist precise results only for $n = 2$, $n = 3$. In the first case, that of parabolic motion, the particles separate at a rate asymptotic to $At^{2/3}$, $A > 0$, [2, p. 13]. If $n = 3$ Chazy has shown [1] that there are two possibilities: either $r \sim At^{2/3}$, $R \sim At^{2/3}$, or $r = O(1)$, $R \sim At$. We shall establish the following estimates which, unfortunately, when $n = 3$ are not precise as those of Chazy.

Theorem 5.1. *If $h = 0$ then either*

$$U \sim \alpha t^{-2/3} \quad \textit{and} \quad I \sim \tfrac{9}{4}\alpha t^{4/3}$$

for some positive constant α, or

$$r = o(t^{2/3}) \quad \text{and} \quad It^{-4/3} \to \infty .$$

The two cases arise this way. Since $\ddot{I} = T$ when $h = 0$, the derivative of $\dot{I}I^{-1/4}$ is $(4IT - \dot{I}^2)/4I^{5/2}$, which is non-negative according to the inequality (3.1). Therefore $\dot{I}I^{-1/4} \to L$, where $-\infty < L < \infty$, or $L = \infty$.

Suppose first that L is finite. Since $\dot{I}I^{-1/4} \to L$, integration yields $1^{3/4} \sim At$, or $I \sim At^{4/3}$. According to Theorem 1.2, $A > 0$. Since also $\dot{I} \sim LI^{1/4}$, we conclude that $\dot{I} \sim Bt^{1/3}$. Moreover $B \neq 0$ since otherwise $I = o(t^{4/3})$. But $\ddot{I} = U$ because $h = 0$, so that $\dot{I} = \int_0^t U(u)\, du +$ constant. Hence

$$\int_0^t U(u)\, du \sim Bt^{1/3}. \tag{5.1}$$

But also, since $h = 0$, (1.1) becomes

$$|\dot{U}| \leqq AU^{5/2}. \tag{5.2}$$

According to a Tauberian theorem [3, Theorem 4] the condition (5.2) permits us to differentiate both sides of (5.1) to conclude that $U \sim \alpha t^{-2/3}$ for some $\alpha > 0$. Clearly, because $\ddot{I} = U$, this in turn implies $I \sim (9/4)\alpha t^{4/3}$.

We turn to the case $L = \infty$. Since $\dot{I}I^{-1/4} \to \infty$. Then for each M there is a time t_1 such that

$$\dot{I}I^{-1/4} \geqq M, \qquad t \geqq t_1 .$$

Integration yields lim sup $I^{3/4}/t \to \infty$, or $(1/t^{2/3})^{1/2} \to \infty$. We can now apply (2.3) with $T = U$ to conclude that $t^{1/3}(U)^{1/2} \to \infty$, or $(t^{2/3}U)^{-1} \to 0$. By (5) this means $r = o(t^{2/3})$. This completes the proof.

6. Systems of positive energy. If $h > 0$ the basic results are due to Chazy [1]. He shows that the limit

$$\lim \frac{r}{R} = L \tag{6.1}$$

always exists and is finite. Moreover, if $L = 0$

$$r/R \leqq At^{-1/3}. \tag{6.2}$$

for $L \neq 0$ he obtains a complete and precise theory of the behavior of the system, in particular that $U \sim A/t$, $I \sim ht^2$. But if $L = 0$ his methods restrict him to $n = 3$, in which case he concludes from (6.2) that when $L = 0$

$$r = O(t^{2/3}), \qquad t \to \infty. \tag{6.3}$$

Our present methods enable us to conclude that (6.3) holds when $L = 0$, even if $n \neq 3$. For if we multiply the inequalities (4.3) and (6.2) the result is

$$r \leqq At^{2/3}\left[\left(\frac{B}{r} + h\right)^{\frac{1}{2}} + \left(\frac{B}{r}\right)^{1/2}\right].$$

Now if $r \geqq 1$ the expression in parentheses on the right is bounded. Therefore $r \leqq \alpha t^{2/3}$ for some constant α. In addition, if $r \leqq 1$ then automatically $r \leqq \alpha t^{2/3}$ for large enough t. This establishes (6.3).

We propose now to present an alternative treatment of the case $h > 0$ without working Chazy's results. Part of his technique is borrowed, however, in the proof of

Lemma 6.1. Whatever the sign of the energy the limit

$$\lim 1/tU = l \tag{6.4}$$

always exists. If $l = 0$ then (6.3) *holds.*

Observe that if $h \leqq 0$ then equation (6.4) with $l = 0$ already follows from Theorem 1.1 and contributes nothing new. On the other hand our proof, unlike that of (6.1), does not require that $h > 0$.

To prove the lemma we begin with the definition of U to obtain the formula

$$\frac{d}{dt}(tU) = \sum{}^{*} Gm_i m_k (r_{ik} - t\dot{r}_{ik})/r_{ik}^2 .$$

By mimicking the analysis leading to (1.1) we obtain

$$\left|\frac{d}{dt}(tU)\right| \leqq AU^2 \sum{}^{*} (m_i m_k)^{1/2} \left|r_{ik} - t\dot{r}_{ik}\right|,$$

and hence

$$\left|\frac{d}{dt}(tU)\right|^2 \leqq AU^4 \sum{}^{*} m_i m_k (r_{ik} - t\dot{r}_{ik})^2.$$

Expand the square on the right. Using the fact that $|\dot{r}_{ik}| \leqq |v_{ik}|$ we find from (4) that

$$\left|\frac{d}{dt}(tU)\right|^2 \leqq AU^4(I - t\dot{I} + t^2T).$$

Since $T = U + h$ this and (3.3) lead to

$$\left|\frac{d}{dt}(tU)\right|^2 \leqq AU^4(Ut^2),$$

or

$$(tU)^{-5/2}\left|\frac{d}{dt}(tU)\right| \leqq At^{-3/2}.$$

For simplicity let $f(t) = tU$. Integration of the preceding line between t_1 and t_2 shows that

$$|(f(t_1))^{-3/2} - (f(t_2))^{-3/2}| \leqq A(t_1^{-1/2} - t_2^{-1/2}) \tag{6.5}$$

Let $t_1, t_2 \to \infty$. The right-hand side approaches zero, carrying the left-hand side with it. By Cauchy's criterion for the existence of a limit the function $(f(t))^{-3/2}$ has a limit as $t \to \infty$. This establishes (6.4).

Suppose the limit in question is zero. Let $t_2 \to \infty$ in (6.5) and drop the subscript from t_1. We get $(f(t))^{-3/2} \leqq At^{-1/2}$, which after adjustment of the notation, is (6.3).

This lemma furnishes the first part of

Theorem 6.1. *If $h > 0$ then either $U \sim \alpha/t$ for some positive constant α,* or *$r = O(t^{2/3})$. In the former case*

$$I = ht^2 + \alpha t \log t + o(t \log t), \qquad t \to \infty,$$

in the latter $I > ht^2 + At^{4/3}$.

The second part of the theorem follows from the first on two integrations of the equation $\ddot{I} = U + 2h$.

7. The problem of escape. It is sometimes stated that if $h > 0$ a particle must escape from the system, that is, that $r_k \to \infty$ for some value of k. The proof rests on the (correct) fact that if $h > 0$ then $I \to \infty$. By (2) this means that $\sum m_k r_k^2 \to \infty$. It is then (incorrectly) supposed that because the sum becomes infinite at least one term of the sum must also. This argument can be countered by the sum $e^t \cos^2 t + e^t \sin^2 t$. A correct conclusion that can be drawn is that for some value of k, $\limsup r_k = \infty$, and that is not the same thing as $r_k \to \infty$. Observe also that if the argument is correct, it is equally valid for $h = 0$, although this fact appears to have escaped notice.

We offer a less ambitious assertion.

Theorem 7.1. If $h > 0$ and $\int_0^\infty U^2(u)\, du < \infty$ then at least one particle escapes.

From the differential equations of motion

$$\ddot{\mathbf{r}}_k = \sum_{j \neq k} \frac{Gm_j}{r_{jk}^2} \frac{\mathbf{r}_j - \mathbf{r}_k}{r_{jk}}$$

we deduce that

$$|\ddot{\mathbf{r}}_k| \leqq \sum_{j \neq k} \frac{1}{r_{jk}^2} \leqq \frac{A}{r^2} \leqq AU^2.$$

Therefore

$$|\mathbf{v}_k(t_1) - \mathbf{v}_k(t_2)| \leqq \int_{t_1}^{t_2} U^2(u)\, du.$$

Since the right-hand side vanishes as t_1 , $t_2 \to \infty$ so does the left. Therefore $\lim_{t\to\infty} \mathbf{v}_k = \mathbf{l}_k$ exists for each k. Because $\mathbf{v}_k = \dot{\mathbf{r}}_k$ we conclude that $\mathbf{r}_k/t \to \mathbf{l}_k$ for each k, or $r_k/t \to l_k$ (The symbol l_k denotes the length $|\mathbf{l}_k|$.)

To complete the proof we need only show that at least one of the l_k is not zero. If they are all zero then each $v_k \to 0$, so that $T \to 0$. But $T = U + h$, so that $U \to -h$. This is impossible because U is positive, while $-h$ is a negative constant.

Theorem 7.2. If $h > 0$, then in case $U \sim \alpha/t$, $\alpha > 0$, at least $(n - 1)$ of the particles escape.

To prove this observe that the first paragraph of the preceding proof stands unaltered. It remains only to show that at most one of the l_k can be zero. For if two are zero, then

$$\mathbf{r}_i/t \to 0, \qquad \mathbf{r}_j/t \to 0$$

for some pair of integers i, j. Therefore $r_{ij}/t \to 0$. But $r \leqq r_{ij}$ so that $r/t \to 0$, $(tU)^{-1} \to 0$, contradicting the assumption $U \sim \alpha/t$.

8. A generalized virial theorem. The classical virial theorem states that if the system remains bounded, or, what comes to the same thing, if I and T remain bounded for all time, then the time averages

$$\hat{T} = \lim_{t\to\infty} \frac{1}{t} \int_0^t T(u)\, du, \qquad \hat{U} = \lim_{t\to\infty} \frac{1}{t} \int_0^t U(u)\, du$$

exist and satisfy the three (equivalent) relations

$$2\hat{\hat{T}} = U, \qquad \hat{T} = -h, \qquad \hat{U} = -2h. \tag{8.1}$$

The equivalence comes, of course, from the fact that $T = U + h$, whence $\hat{T} = \hat{U} + h$.

I have shown [2, p. 44] that the conditions imposed above on I and T are irrelevant to the conclusion (8.1) which is in fact implied by and implies the condition $I = o(t^2)$. In view of the interest in expanding clusters we shall state a generalization. The proof is omitted because it follows the same lines as the special case $\alpha = 0$.

Theorem 8.1. *The condition*

$$\frac{I}{t^2} \to \alpha$$

is necessary and sufficient that $\hat{T}$, $\hat{U}$ exist and satisfy

$$\hat{T} = 2\alpha - h, \qquad \hat{U} = 2(\alpha - h).$$

If $\alpha \neq h$ this conclusion is equivalent to

$$2\hat{T} = k\hat{U}, \qquad k = (2\alpha - h)/(\alpha - h).$$

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