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The Behavior of Superconducting Thin Films in the Presence of Magnetic Fields and Currents*)

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The general Gor'kov equation in the presence of arbitrary vector potential $\boldsymbol{A}(\boldsymbol{x})$ is solved for a superconducting thin film on the following two assumptions; 1) the ordering parameter Δ is constant all over the specimen, 2) $l/\xi_0 \ll 1$, where l is the mean free path of an electron and ξ_0 the coherence length of an electron pair.

It is shown that the phase transition due to magnetic field is always of the second order. The critical field is determined from the condition $\Delta=0$.

§ 1. Introduction

The behavior of a superconducting small specimen in a large field is usually discussed by the use of the phenomenological Ginzburg-Landau equation[1] which is valid only in the small temperature region close to the transition temperature.

In a large field one must take account of the effects of nonlinear terms in a consistent theory. The effects are roughly classified as the effects appearing through the change of the ordering parameter Δ and the others appearing explicitly say in the expression of a current.[2] The importance of the former has been pointed out by Pippard.[3] Recently Nambu and Tuan[4] has obtained a field dependent expression of Δ up to the second order of a vector potential A with arbitrary wave number.

Bardeen[5] has treated semiphenomenologically the thermodynamical behavior of small specimens by the use of the expression of Δ calculated to the second order of fields and currents.

In the present paper we investigate the behavior of superconducting small specimen in the presence of fields and currents, using the general Gor'kov equation. The method employed here is the direct extension of that developed by the present author for study of persistent currents in a superconducting alloy.[6] In the following we assume that Δ is constant all over the specimen and that $\tau\Delta \ll 1$, where τ is the collision time. The above assumptions are rather plausible for small specimens where the electronic mean free path is limited

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by the boundary scattering of the fine crystalline structure.

In § 2 the free energy is expressed in terms of Δ and the field strength and it is shown that the phase transition in the magnetic field is always of the second order. In § 3 a general consideration of the transformation of $\boldsymbol{A}$ is made in connection with possible states carrying a persistent current. In § 4 the ordering parameter Δ and the gap in the energy spectrum ω_0 are calculated. In § 5 the current is obtained at finite temperature.

§ 2. General Gor'kov equation in the presence of fields

We start with the following Gor'kov equation[7] in the presence of random scattering centers:

$$\left(i\frac{\partial}{\partial t}-\frac{1}{2m}(\boldsymbol{p}-e\boldsymbol{A})^2+\mu+V(\boldsymbol{x})\right)G(x, x')$$
$$-igF(x, x)F^+(x, x')=\delta(x-x'),$$
$$\left(i\frac{\partial}{\partial t}+\frac{1}{2m}(\boldsymbol{p}+e\boldsymbol{A})^2-\mu-V(\boldsymbol{x})\right)F^+(x, x')$$
$$+igF^+(x, x)G(x, x')=0, \tag{1}$$

where μ is the chemical potential, $\boldsymbol{A}$ the vector potential, $V(\boldsymbol{x})=\sum_a u(\boldsymbol{x}-\boldsymbol{x}_a)$ the interaction between an electron and scattering centers and G and F are the two Green's functions defined as

$$\delta_{\alpha\beta}\, G(x, x')=-i\langle T(\psi_\alpha(x), \psi_\beta{}^+(x'))\rangle,$$
$$\hat{I}_{\alpha\beta}\, F^+(x, x')=\langle T(\psi_\alpha{}^+(x), \psi_\beta{}^+(x'))\rangle,\ \hat{I}^2=-1, \tag{2}$$

where $\langle T(\quad)\rangle$ means the ensemble average of the time ordered product.

Let us first consider the case where no scattering is present. In this case the above equation is solved formally for the Fourier transform of G and F;

$$G(p)=\left(\omega+\xi+\frac{e}{m}\boldsymbol{p}\cdot\boldsymbol{A}\right)\Big/\Sigma(p), \tag{3}$$
$$F^+(p)=i\Delta/\Sigma(p),$$

where

$$\Sigma(p)=\left(\omega+\frac{e}{m}\boldsymbol{p}\cdot\boldsymbol{A}\right)^2-\xi^2-\Delta^2-\left[\xi, \frac{e}{m}\boldsymbol{p}\cdot\boldsymbol{A}\right], \tag{4}$$
$$\xi=p^2/2m+e^2|\boldsymbol{A}|^2/2m-\mu,$$

and

$$\Delta=|g|F^+(x, x). \tag{5}$$

In deriving Eqs. (3) and (4) we made use of the assumption that Δ is constant (i.e. $[\xi, \Delta]=0$, etc.).

In the presence of scattering, according to the standard treatment of impurity scattering in metals,[8] we obtain the following expressions for G and F:

$$G(p)=\left(\tilde{\omega}+\frac{e}{m}\boldsymbol{p}\cdot\boldsymbol{A}+\xi\right)\Big/\tilde{\Sigma}(p),$$
$$F^{+}(p)=i\tilde{\Delta}/\tilde{\Sigma}(p), \tag{6}$$

where

$$\tilde{\Sigma}(p)=\left(\tilde{\omega}+\frac{e}{m}\boldsymbol{p}\cdot\boldsymbol{A}\right)^2-\xi^2-\tilde{\Delta}^2-[\xi, \frac{e}{m}\boldsymbol{p}\cdot\boldsymbol{A}]. \tag{7}$$

In the above expression G and F generally have non-vanishing off-diagonal elements.

We determine $\tilde{\omega}$ and $\tilde{\Delta}$ from the following equations:

$$\tilde{\omega}=\omega-\frac{n}{(2\pi)^3}\int|u(p-p')|^2G(p')d^3p',$$
$$\tilde{\Delta}=\Delta-\frac{n}{i(2\pi)^3}\int|u(p-p')|^2F^{+}(p')d^3p', \tag{8}$$

where n is the number of scattering centers in a unit volume.

We assume here that the scattering is isotropic. Substituting Eqs. (6) and (7) in the integrand of Eq. (8) and expanding the integrand in the power of (e/m) $\boldsymbol{p}\cdot\boldsymbol{A}$, we obtain

$$\omega=\tilde{\omega}-\frac{\tilde{\omega}}{2\tau\sqrt{\tilde{\Delta}^2-\tilde{\omega}^2}}\left(1+\frac{3}{8}\cdot\frac{4\tilde{\Delta}^2\alpha-\beta}{(\tilde{\Delta}^2-\tilde{\omega}^2)^2}+I_{\tilde{\omega}}\right),$$
$$\Delta=\tilde{\Delta}-\frac{\tilde{\Delta}}{2\tau\sqrt{\tilde{\Delta}^2-\tilde{\omega}^2}}\left(1+\frac{(8\tilde{\omega}^2+4\tilde{\Delta}^2)\alpha-3\beta}{8(\tilde{\Delta}^2-\tilde{\omega}^2)^2}+I_{\tilde{\Delta}}\right), \tag{9}$$

where

$$\tau^{-1}=\frac{nmp_0}{(2\pi)^2}\int|u(\theta)|^2d\Omega,$$

$$\alpha=(e/m)^2\langle(\boldsymbol{p}\cdot\boldsymbol{A})^2\rangle=\frac{1}{3}(ep_0/m)^2\frac{1}{V}\int_V|A(x)|^2d^3x,$$

and

$$\beta=(e/m)^2\langle(\boldsymbol{q}\cdot\boldsymbol{v}_0)^2(\boldsymbol{p}\cdot\boldsymbol{A})^2\rangle, \tag{10}$$

with v_0 the fermi velocity.

In the following we are interested in the case where $\tau\Delta\ll1$. In this limit it is not difficult to show that $I_{\tilde{\omega}}$ and $I_{\tilde{\Delta}}$ are of the order of $(\tau\Delta)^2$ and we neglect those terms hereafter.

From Eq. (9) we obtain

$$\frac{\omega}{\Delta} = u\left(1 - \zeta \frac{1}{\sqrt{1-u^2}}\right), \tag{11}$$

where

$$u = \tilde{\omega}/\tilde{\Delta} \quad \text{and} \quad \zeta = \frac{2\tau\alpha}{\Delta}. \tag{12}$$

More precise treatment of scattering shows that τ in Eq. (12) should be replaced by the transport collision time τ_{tr} given as

$$\tau_{tr}^{-1} = \frac{nmp_0}{(2\pi)^2} \int |u(\theta)|^2 (1 - \cos\theta) d\Omega.$$

A relation similar to Eq. (11) has been first obtained by Abrikosov and Gor'kov[9] in their study of the effect of paramagnetic impurity on superconductivity. We can show quite generally that this relation holds in the presence of any interaction Hamiltonian H_1 which breaks the symmetry in time reversal operation in the case where $\tau\Delta \ll 1$, if we use ζ defined as $\zeta = \tau\langle H_1^2 \rangle/\Delta$.

At $T = 0$°K the ordering parameter Δ and the current j are easily calculated;[6),9)]

$$\begin{aligned} \ln\frac{\Delta}{\Delta_{00}} &= -\frac{\pi}{4}\zeta, \qquad \text{for } \zeta \leq 1, \\ &= -\text{arccosh}\,\zeta - \frac{1}{2}(\zeta \arcsin \zeta^{-1} - \sqrt{1-\zeta^{-2}}), \\ &\qquad \text{for } \zeta > 1, \end{aligned} \tag{13}$$

where Δ_{00} is the gap in the energy spectrum at $T = 0$°K and $\zeta = 0$,

$$\begin{aligned} \boldsymbol{j}_q &= -\frac{2e^2N}{m}\tau\Delta\left(\frac{\pi}{2} - \frac{2}{3}\zeta\right)\boldsymbol{A}_q, \qquad \text{for } \zeta \leq 1, \\ &= -\frac{2e^2N}{m}\tau\Delta\left\{\arcsin\zeta^{-1} - \frac{2}{3}(\zeta - \sqrt{\zeta^2 - 1}) \right. \\ &\qquad \left. + \frac{1}{3}\zeta^{-1}\sqrt{1-\zeta^{-2}}\right\}\boldsymbol{A}_q, \qquad \text{for } \zeta > 1. \end{aligned} \tag{14}$$

We note here that the gap in the energy spectrum is not Δ but given from Eq. (11) as $\omega_0 = \Delta(1 - \zeta^{2/3})^{3/2}$, for $\zeta \leq 1$ and 0 for $\zeta > 1$.

Equation (14) is the generalization of the London equation in which nonlinear terms appear explicitly.

The free energy is obtained from[7)]

$$F_s - F_n = \int_0^{|g|} \delta\left(\frac{1}{|g|}\right) \Delta^2, \tag{15}$$

which gives at $T=0$°K

$$-\frac{mp_0}{4\pi^2}\Delta^2\left(1-\frac{\pi}{2}\zeta+\frac{2}{3}\zeta^2\right), \qquad \text{for } \zeta \leq 1,$$

and

$$-\frac{mp_0}{4\pi^2}\Delta^2\{1-\zeta \arcsin \zeta^{-1}+\zeta^2(1-\sqrt{1-\zeta^{-2}})$$

$$-\frac{1}{3}\zeta^2(1-(1-\zeta^{-2})^{3/2})\}, \quad \text{for } \zeta > 1, \tag{16}$$

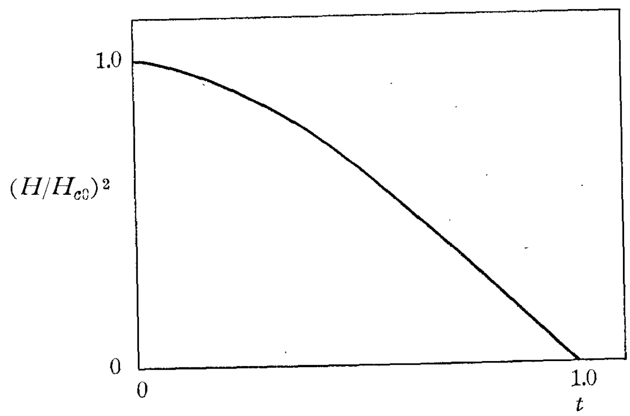

Fig. 1. The square of the reduced critical field $(H/H_{c0})^2$ is plotted against the reduced temperature $t=T/T_{c0}$.

where we made use of the relation $\delta(1/|g|) = -mp_0/2\pi^2 \cdot \delta\Delta/\Delta$ and Eq. (13).

From Eq. (15) one sees F_s is smaller than F_n as long as Δ does not vanish and the phase transition is always of the second order.

The above conclusion is always valid independent of the details of the assumptions made on the electronic mean free path, if the magnetic field is homogeneous in the specimen. Many years ago Pippard[3] obtained the same results on the phenomenological consideration.

In the homogeneous magnetic field $\boldsymbol{H}$ the vector potential is written as $\boldsymbol{A}(\boldsymbol{x}) = [\boldsymbol{x}\times\boldsymbol{H}]/2$ and ζ is found to be $(2\tau/3\Delta)(ep_0/m)^2(aH)^2$, where a is the characteristic length of a sample and can be calculated if the geometrical configuration of the sample is given.

The critical fields are determined from the condition $\Delta=0$.

At finite temperatures we obtain[10] (See also § 4.)

$$-\ln\frac{T}{T_{c0}} = \psi\left(\frac{1}{2}+\frac{\rho}{2}\right)-\psi\left(\frac{1}{2}\right), \tag{17}$$

where T_{c0} is the critical temperature for $H=0$,

$$\psi(z) = \Gamma'(z)/\Gamma(z) \quad \text{and} \quad \rho = \frac{\Delta}{\pi T}\zeta.$$

In Fig. 1 we plotted the reduced field $h=H/H_{c0}$ against the reduced temperature $t=T/T_{c0}$ where H_{c0} is the critical field at $T=0$°K.

§ 3. Transformation property of $A(x)$ and persistent currents

In this section we explore the physical consequence of some transformations of vector potential $\boldsymbol{A}(\boldsymbol{x})$ which are found useful to introduce a persistent current into the system.

A gauge transformation is described by a set of transformations;

$$\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) + \text{grad } \chi(\boldsymbol{x}),$$
$$\psi(\boldsymbol{x}) \to \exp(-ie\chi(\boldsymbol{x}))\psi(\boldsymbol{x}). \tag{18}$$

It is well known that a vector potential is uniquely decomposed as

$$\boldsymbol{A}(\boldsymbol{x}) = \text{rot}\,\boldsymbol{\omega} + \text{grad}\,\phi. \tag{19}$$

And even if we fix the gauge by the condition that div $\boldsymbol{A}(\boldsymbol{x}) = 0$, we can choose an arbitrary function for ϕ which satisfies $\nabla^2\phi = 0$. This freedom is closely connected with the possible existence of physically different states carrying a persistent current, flow velocity of which is determined by a velocity potential $e\phi/m$.*)

For a closed system we can determine ϕ uniquely besides a constant term, if the fluxoids threading each hole in the specimen are given. Especially we have $\phi = \text{const}$ for a simply connected specimen.

For an open system ϕ depends on an external current also. Inserting Eq. (19) in the expression of ζ we obtain

$$\zeta = \frac{2\tau}{3\Delta}\left(\frac{ep_0}{m}\right)^2 \left\{\frac{1}{V}\int_V (\text{rot}\,\boldsymbol{\omega})^2 d^3x + \frac{1}{V}\int_V (\text{grad}\,\phi)^2 d^3x\right\}, \tag{20}$$
$$= \zeta_m + \zeta_c$$

due to the orthogonality between rot$\boldsymbol{\omega}$ and grad ϕ.

From Eq. (20) one sees the effects of a magnetic field and a current appear additively in the expression of ζ.

In order to describe a state carrying a uniform flow we put $\phi(\boldsymbol{x}) = m/e\, \boldsymbol{v}_s \cdot \boldsymbol{x}$, where $\boldsymbol{v}_s$ is the velocity of a condensed pair and we obtain

$$\zeta_c = \frac{2\tau}{3\Delta}\left(\frac{ep_0}{m}\right)^2 \frac{1}{V}\int_V (\nabla\phi)^2 d^3x = \frac{\tau}{3\Delta}(p_0 v_s)^2. \tag{21}$$

In the presence of both a homogeneous magnetic field $\boldsymbol{H}$ and a uniform current we obtain

$$\zeta = \frac{2\tau}{3\Delta}(p_0)^2\left\{\left(\frac{ea}{m}\right)^2 (H)^2 + (v_s)^2\right\}. \tag{22}$$

*) Such a transformation was used first in the study of quantization of flux in superconductors. See K. Maki and T. Tsuneto Prog. Theor. Phys. **27** (1962), 228, and J. M. Blatt, Prog. Theor. Phys. **26** (1961), 761.

The first term in the bracket is just the mean square of the Larmor velocity of an electron pair.

Using this expression we can discuss the behavior of the system in the presence of both field and current in a quite general form.

As an example we determine the critical current in the presence of a field at $T=0°K$. The critical current is determined from the maximization condition for the current;

$$\left(\frac{\partial j_s}{\partial v_s}\right)_H=0, \tag{23}$$

where

$$j_s=2eN(\tau\Delta)\left(\frac{\pi}{2}-\frac{2}{3}\zeta\right)v_s, \qquad \text{for } \zeta\leq 1,$$

$$=2eN(\tau\Delta)\left\{\arcsin\zeta^{-1}-\frac{2}{3}\left(\zeta-\sqrt{\zeta^2-1}\right)+\frac{1}{3}\zeta^{-1}\sqrt{1-\zeta^{-2}}\right\}v_s, \qquad \text{for } \zeta>1, \tag{24}$$

and ζ is given in Eq. (22).

When $\zeta\leq 1$ or $\boldsymbol{H}$ is not so large, Eq. (23) is reduced to the following algebraic equation:

$$\frac{\pi^2}{4}(\zeta-\zeta_m)-\left(1-\frac{\pi}{4}\zeta\right)\left(\frac{\pi}{2}-2\zeta+\frac{4}{3}\zeta_m\right)=0, \tag{25}$$

where

$$\zeta_m=\frac{2\tau}{3\Delta}\left(\frac{ep_0}{m}\right)^2(aH)^2.$$

The solution is given as

$$\zeta=\left(\frac{3\pi}{8}+\frac{2}{\pi}+\frac{1}{3}\zeta_m\right)-\sqrt{\frac{9\pi^2}{64}+\frac{1}{2}+\frac{4}{\pi^2}-\zeta_m\left(\frac{\pi}{4}+\frac{4}{3\pi}\right)+\frac{1}{9}\zeta_m^2}. \tag{26}$$

Substituting this expression in Eq. (24) we obtain the critical current in the presence of fields. It is interesting to note that when

$$\zeta_m\geq\left(\frac{3\pi^2}{8}-\pi+2\right)\Big/\left(\frac{\pi^2}{4}-\frac{\pi}{3}+\frac{4}{3}\right)=0.895,$$

the current attains its maximum for $\zeta>1$.

In the region $\zeta>1$ the gap in the energy spectrum vanishes while the ordering characteristic of a superconductor still persists. We may conclude that the possibility of superflow depends essentially on the existence of nonvanishing Δ, since the state in the region is stable against any microscopic excitation.

One must bear in mind, however, that when the current varies slowly in this region, it is accompanied by a small dissipation of energy which is described by a conductivity $\sigma(1-\zeta^{-2})$, where $\sigma=Ne^2\tau_{tr}/m$ the conductivity for the normal state.

§ 4. Field dependence of the ordering parameter $\varDelta$ and the gap in the energy spectrum ω_0

In this section we derive explicit expressions of $\varDelta$ and ω_0 at finite temperatures. The ordering parameter $\varDelta$ is determined from the equation

$$\varDelta=|g|T\sum_n\frac{1}{(2\pi)^3}\int d^3pF^+(p), \tag{27}$$

where the summation is made over u_n which is defined from

$$\frac{\omega_n}{\varDelta}=u_n\left(1-\zeta\frac{1}{\sqrt{u_n^2+1}}\right), \tag{28}$$

and $\qquad\omega_n=2\pi(n+1/2)T.$

T is the temperature and n the integer. Equation (26) is rewritten as

$$\ln\frac{\varDelta}{\varDelta_{00}}=\lim_{\omega_D\to\infty}\left\{\frac{2\pi T}{\varDelta}\sum^{\omega_D\to\infty}\frac{1}{\sqrt{u_n^2+1}}-\int_0^{\omega_D}\frac{d\omega}{\sqrt{\omega^2+\varDelta^2}}\right\}. \tag{29}$$

When T is small ($\varDelta/T\gg1$) we have

$$\begin{aligned}\ln(\varDelta/\varDelta_{00})&=\int_0^\infty\frac{d\omega}{\varDelta}\left(\frac{1}{\sqrt{u^2+1}}-\frac{1}{\sqrt{(\omega/\varDelta)^2+1}}\right)\\&\quad-2\sum_{n=1}^\infty(-1)^{n+1}\int_{\omega_0}^\infty e^{-n\omega/T}\,\mathrm{Im}\left\{\frac{1}{\sqrt{1-u^2}}\right\}\frac{d\omega}{\varDelta}\\&\cong-\frac{\pi}{4}\zeta-\zeta^{-2/3}(1-\zeta^{2/3})^{1/4}\sqrt{2\pi/3}\left(\frac{T}{\varDelta}\right)^{3/2}e^{-\omega_0/T}\\&\qquad\qquad\text{for }\zeta\le1,\\&=-\frac{\pi}{4}-\frac{\sqrt{3}}{4}\Gamma(5/3)\zeta(5/3)(1-2^{-2/3})\left(\frac{2T}{\varDelta}\right)^{5/3}\\&\qquad\qquad\text{for }\zeta=1,\\&=-\operatorname{arccosh}\zeta-\frac{1}{2}(\zeta\arcsin\zeta^{-1}-\sqrt{1-\zeta^{-2}})\\&\quad-\frac{\pi^2}{6}\zeta^{-2}(1-\zeta^{-2})^{-1/2}\left(\frac{T}{\varDelta}\right)^2\\&\qquad\qquad\text{for }\zeta>1,\end{aligned} \tag{30}$$

where $\omega_0 = \Delta(1-\zeta^{2/3})^{3/2}$ for $\zeta<1$, $\omega_0=0$ for $\zeta\geq 1$.

On the other hand in the case where $\Delta/T \ll 1$ we obtain

$$-\ln t = \psi\left(\frac{1}{2}+\frac{\rho}{2}\right)-\psi\left(\frac{1}{2}\right)$$
$$+\frac{\eta}{2^3}\left(\sum_{n=0}^{\infty}\frac{1}{(n+(1+\rho)/2)^3}-\frac{\rho}{2}\sum_{n=0}^{\infty}\frac{1}{(n+(1+\rho)/2)^4}\right)$$
$$-\frac{1}{2^5}\frac{3}{4}\eta^2\left(\sum_{n=0}^{\infty}\frac{1}{(n+(1+\rho)/2)^5}-\frac{3}{2}\rho\sum_{n=0}^{\infty}\frac{1}{(n+(1+\rho)/2)^6}\right), \qquad (31)$$

where

$$\rho=\Delta\zeta/\pi T,\ \eta=(\Delta/\pi T)^2,\ t=T/T_{c0} \text{ and } \psi(z)=\Gamma'(z)/\Gamma(z).$$

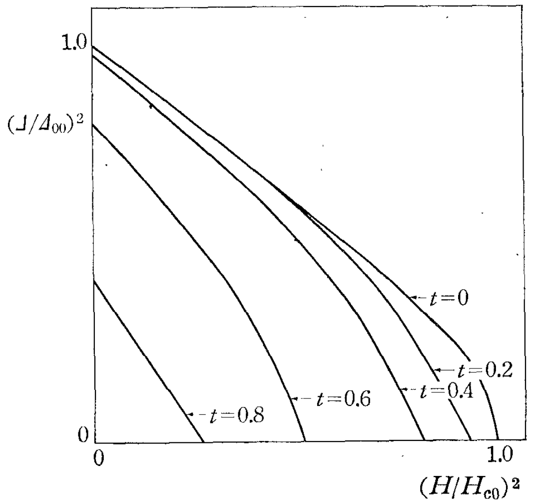

Fig. 2. Change in the ordering parameter Δ with the square of field H at various temperatures.

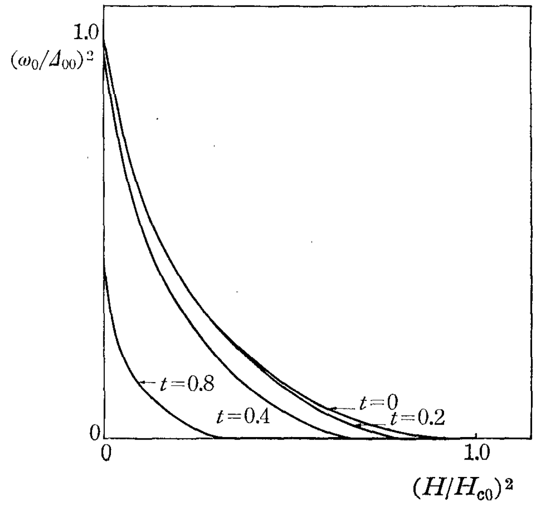

Fig. 3. Change in the gap parameter ω_0 with the square of field H.

Using the above expressions we calculate the ordering parameter Δ and the gap in the energy spectrum as the function of magnetic field H at various temperatures, which are depicted in Figs. 2 and 3.

One sees from these graphs the change of ω_0 is very steep compared with that of Δ. Equation (17) in § 2 is easily obtained from Eq. (31) simply putting $\Delta=0$.

§ 5. Current and penetration depth

The current density is evaluated from the expression

$$j=(e/m)T\sum\frac{1}{(2\pi)^3}\int d^3p(\boldsymbol{p}-e\boldsymbol{A})G(p), \qquad (32)$$

which is rewritten as

$$\boldsymbol{j}_q = -Q\boldsymbol{A}_q, \tag{33}$$

where

$$Q = \frac{e^2 N}{m}(\tau 2\pi T)\sum \frac{1}{u_n^2+1}\,. \tag{34}$$

In the limit where $\Delta/T \gg 1$, we obtain

$$Q = \frac{2e^2N}{m}(\tau\Delta)\left\{\int_0^\infty \frac{d\omega/\Delta}{1+u^2} - 2\sum_{n=1}^\infty (-1)^{n+1}\int_{\omega_0}^\infty e^{-n\omega/T}\,\mathrm{Im}\left\{\frac{1}{1-u^2}\right\}\frac{d\omega}{\Delta}\right\}$$

$$\cong 2\frac{e^2N}{m}(\tau\Delta)\left\{\frac{\pi}{2} - \frac{2}{3}\zeta - 2\sqrt{2\pi/3}\,\zeta^{-1}(1-\zeta^{2/3})^{1/4}\left(\frac{T}{\Delta}\right)^{3/2} e^{-\omega_0/T}\right\}$$

$$\text{for } \zeta < 1\,,$$

$$= 2\frac{e^2N}{m}(\tau\Delta)\left\{\frac{\pi}{2} - \frac{2}{3} - \frac{\sqrt{3}}{2}\Gamma(5/3)\zeta(5/3)(1-2^{-2/3})\left(\frac{2T}{\Delta}\right)^{5/3}\right.$$

$$\text{for } \zeta = 1,$$

$$= 2\frac{e^2N}{m}(\tau\Delta)\left\{\arcsin\zeta^{-1} - \frac{2}{3}(\zeta - \sqrt{\zeta^2-1}) + \frac{1}{3}\zeta^{-1}\sqrt{1-\zeta^{-2}}\right.$$

$$\left. - \frac{\pi^2}{3}\zeta^{-3}(1-\zeta^{-2})^{-1/2}\left(\frac{T}{\Delta}\right)^2\right\}$$

$$\text{for } \zeta > 1. \tag{35}$$

In the vicinity of T_c where $\eta \ll 1$, we obtain

$$Q = \frac{e^2N}{m}(\tau\pi T)\eta\left\{\sum_{n=0}^\infty \frac{1}{(n+(1+\rho)/2)^2}\right.$$

$$\left. - \frac{\eta}{2^3}\left(\sum_{n=0}^\infty \frac{1}{(n+(1+\rho)/2)^4} - \frac{\rho}{2}\sum_{n=0}^\infty \frac{1}{(n+(1+\rho)/2)^5}\right)\right\}. \tag{36}$$

The penetration depth is expressed by the use of Q as

$$\delta = \sqrt{c/4\pi Q}\,.$$

§ 6. Concluding remarks

As we have seen above the general Gor'kov equation is much simplified and solved explicitly for small specimens on the assumptions that Δ is constant and the electronic mean free path is short. Using the above Green's functions we can discuss the thermodynamical as well as the electromagnetic properties of the system in a quite general way.

We emphasize here that the gap in the excitation spectrum vanishes when the magnetic field is sufficiently large (at $T=0\,°K$, $H \geqq 0.95 H_{c0}$ where H_{c0} is the critical field). The most important prediction of the present theory is the drastic change of the gap in the energy spectrum as the function of the fields. When $T \ll T_{c0}$ this effect appears explicitly not only in the exponential factor of the specific heat but also in the various transport coefficients. The ordering parameter Δ, however, seems to have no direct connection with the physical observables and the determination of it seems very difficult experimentally.

On the other hand when $T_{c0}-T \ll T_{c0}$ the present theory predicts the same behaviors of Δ and the current j as the Ginzburg-Landau theory does. In this region the transport coefficients may be determined by the gross structure of the spectrum function rather than the detailed shape near the threshold, since the variation of $f(E)=(1+e^{E/T})^{-1}$ is comparatively gradual. And it is possible that Δ is a more important parameter than ω_0 in this region.

Giaever and Megerle[10] have made the direct measurement of the gap of aluminium film in the magnetic fields using techniques of tunnelling at temperatures close to T_{c0}. It is difficult to derive any definite conclusion as to the behavior of ω_0 from their results, since the expression of the tunnelling current in this temperature region is complicated. Such measurements at much lower temperatures are desired.

If one considers a larger sample, one must generalize the whole expression so as to include the effect of the local variation of Δ. In this case the important expansion parameter is $l\xi_0 \nabla^2$ and we obtain an equation similar to the Ginzburg-Landau's if we retain only the first two terms in the expansion.

The validity of such an expansion and the possibility of the mixed state at lower temperatures will be discussed in a future paper.

In conclusion the author wishes to thank Dr. T. Tsuneto for valuable discussions.

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Note added in proof: In recent papers by Y. Nambu and S. F. Tuan, Phys. Rev. Letters **11** (1963) 119 and Phys. Rev. **133** (1964), A1, they study the field dependence of the ordering parameter in superconducting thin films on the model of the discrete quantization in momentum space and obtain similar results.