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The Beltrami Spectrum for Incompressible Fluid Flows

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Abstract. Recently V. Yakhot, S. Orszag, and their co-workers have suggested that turbulent flows in various regions of space organize into a coherent hierarchy of weakly interacting superimposed approximate Beltrami flows. A mathematical framework is developed here to study organized Beltrami hierarchies in a systematic fashion. This framework is applied to several important classes of examples with universal Beltrami hierarchies. An analysis of the persistence of such Beltrami hierarchies is also presented for general solutions of the Navier-Stokes equations.

Introduction

Recently V. Yakhot, S. Orszag, and their co-workers [3, 4] have suggested that in various regions of space, turbulent flows organize into a coherent hierarchy of weakly interacting superimposed approximate Beltrami flows. Their evidence for such behavior is based on detailed numerical experiments for channel flows and decaying homogeneous turbulence utilizing spectral methods; however the mechanisms for the existence of such a hierarchy are not understood.

In this paper we develop a mathematical framework to study organized Beltrami hierarchies and then we analyze this structure in solutions of the Navier-Stokes equations. We advocate the theoretical framework presented in detail in Sect. 1 as a readily implemented new diagnostic for further numerical tests with spectral codes for the existence of weakly interacting Beltrami hierarchies. In Sect. 1 we also describe the concept of *Beltrami spectrum* as an effective measure of the extent to which a given incompressible velocity field is an organized superposition of weakly interacting Beltrami flows. Expressions yielding the same numerical value as the Beltrami spectrum were mentioned in works on helical

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turbulence, under the names "degree of helicity at higher wave numbers" [13] and "relative helicity" [14].

In Sect. 2, we exhibit some general families of exact solutions of the 3-D Navier-Stokes equations which are weakly interacting but include solutions with enormous enstrophy production at high Reynolds numbers. Nevertheless, the Beltrami spectrum associated with these flows is *universal* - it is invariant with time, Reynolds number, and also for appropriate external forces. The Beltrami spectrum for these flows is computed by explicit formulae in Sect. 2. We also analyze the physical pancake eddies observed in the numerical experiments in [4] and prove that appropriate nearby configurations exhibit no local enstrophy production at moderately large Reynolds numbers. This theorem provides rigorous justification that appropriate solutions of the Navier-Stokes equations near such pancake eddies are indeed weakly interacting. Finally in Sect. 3, we study the evolution of the Beltrami spectrum for general solutions of the Navier-Stokes equations. We establish a direct link between changes in the Beltrami spectrum and the depletion of energy in a given energy shell. Then we use this link together with some straightforward estimates to obtain a general quantitative estimate for time intervals where the Beltrami spectrum has negligible change; the estimates for this time interval are non-dimensional and depend on the Reynolds number.

Next we present some background information which provides a link between some of the ideas presented in [4] with the quantitative developments in the remaining sections of this paper.

The three-dimensional Navier-Stokes equations in rotation form are

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - u\times\omega = -\nabla\left(p+\frac{|u|^2}{2}\right) + \nu\Delta u + F(x,t), \qquad t\geqq 0 \\ \operatorname{div} u = 0\,, \end{aligned}\right\} \tag{0.1}$$

where $u={}^t(u_1,u_2,u_3)$ is the fluid velocity, p is the scalar pressure, $\omega=\operatorname{curl}u$ is the vorticity while $F(x,t)$ is a prescribed external force and $\nu\geqq 0$ is the viscosity. Here for simplicity in exposition we primarily discuss the case of periodic fluid flows with period L so that with $x=(x_1,x_2,x_3)$, $u(t,x+e_iL)=u(t,x)$ with $\{e_i\}_{i=1}^3$ the standard unit basis vectors. For L-periodic flows, any square integrable incompressible velocity field, u, admits the Fourier expansion

$$u(x)=\sum_{\lambda\in\Lambda}\left(\sum_{|k|^2=\lambda} u_k e^{\frac{2\pi i}{L}\langle x,k\rangle}\right) \tag{0.2}$$

with Fourier coefficients u_k for $k\in\mathbb{Z}^3$ satisfying

$$\left.\begin{aligned} &\text{i) } u_{-k}=\overline{u_k} &&\text{(to guarantee real valued velocities)} \\ &\text{ii) } \langle u_k,k\rangle=0 &&\text{(to guarantee incompressibility).} \end{aligned}\right\} \tag{0.3}$$

Here we use the notation $\langle w,\xi\rangle=\sum_{i=1}^3 w_i\xi_i$, for $w,\xi\in\mathbb{C}^3$. In (0.2), we have grouped the Fourier coefficients into energy shells. Thus, Λ is the countable set of numbers so that λ belongs to Λ if and only if λ is a sum of squares of three integers and the inner sum in (0.2) involves all Fourier modes on the energy shell associated with a given λ.

Beltrami flows are exact steady solutions of the inviscid Euler equations [defined by setting $v \equiv 0$ in (0.1) with external forces $F \equiv 0$] so that the velocity and vorticity are collinear everywhere, i.e., there is a smooth function $\alpha(x)$ so that

$$\operatorname{curl} u = \alpha(x) u, \qquad \operatorname{div} u = 0. \tag{0.4}$$

An obvious compatibility condition for the existence of a Beltrami flow is that the function $\alpha(x)$ satisfies

$$(u\nabla)\alpha = 0, \tag{0.5}$$

i.e., $\alpha(x)$ is constant on streamlines. Clearly Beltrami flows are non-interacting exact solutions of the Euler equations since the nonlinear terms on the left-hand side of (0.1) vanish. Periodic Beltrami flows are readily generated through Fourier series [7]. If we consider a fixed energy shell defined by $\lambda \in \Lambda$ and prescribe arbitrary coefficients u_k satisfying (0.3) then the functions

$$u_\lambda^\pm = \sum_{|k|^2=\lambda} \frac{1}{2}\left(u_k \pm \frac{ik}{|k|} \times u_k\right) e^{\frac{2\pi i}{L}\langle x, k\rangle} \tag{0.6}$$

define Beltrami flows since

$$\operatorname{curl} u_\lambda^\pm = \pm \frac{2\pi}{L} \sqrt{\lambda}\, u_\lambda^\pm. \tag{0.7}$$

We also remark that $\exp\left(-\lambda v \frac{4\pi^2}{L^2} t\right) u_\lambda^\pm(x)$ generates a solution of the Navier-Stokes equations with initial data corresponding to the Beltrami flow defined in (0.6). Our main result in Sect. 1 is a proof that any incompressible velocity field u not only has the Fourier expansion in (0.2) but also the refined orthogonal decomposition

$$u = \sum_{\lambda \in \Lambda} u_\lambda^+ + u_\lambda^- \tag{0.8}$$

with $u_\lambda^\pm$ defined via the Fourier coefficients of u through the formulae in (0.6). All functions on the right-hand side in (0.8) are mutually L^2-orthogonal; in particular, on a fixed energy shell, $(u_\lambda^+, u_\lambda^-) = 0$ for the L^2 inner product, (u, v), given by

$$(u, v) = \int_{Q_L} \langle u(x), \overline{v(x)}\rangle\, dx = L^3 \sum_{k \in \mathbb{Z}^3} \langle u_k, v_{-k}\rangle, \tag{0.9}$$

with $|u|^2 = (u, u)$ and $Q_L = \left[-\frac{L}{2}, \frac{L}{2}\right]^3$.

The implications of (0.6)–(0.8) for fluid dynamics is that *every incompressible fluid flow* is a *superposition* of *Beltrami flows*. Furthermore, by writing the Navier-Stokes equations in (0.1) as nonlocal equations for the corresponding Fourier coefficients and utilizing the refined decomposition in (0.6)–(0.8), we see that *every solution of the Navier-Stokes equations* is a *superposition* of *interacting Beltrami flows*. Next, we compute the nonlinear interaction terms in the Navier-Stokes equations from (0.1) with the expansion in (0.8),

$$u \times \omega = \sum_{\lambda, \mu \in \Lambda} \frac{2\pi}{L} \sqrt{\mu} (u_\lambda^+ + u_\lambda^-) \times (u_\mu^+ - u_\mu^-). \tag{0.10}$$

The local contribution on a fixed energy shell to the nonlinear interaction terms in (0.10) do not contribute to the dynamic evolution of u for a given λ if and only if

$$u_\lambda^+ \times u_\lambda^- = \nabla q_\lambda \tag{0.11}$$

for some function q_λ. As discussed by Yakhot et al. in [4], there are two simple ways in which the nonlinear interactions in wave number space can vanish. The first possibility occurs when there is nontrivial pressure balancing so that there is $q_\lambda \neq 0$ satisfying (0.11); this occurs in steady rotating eddy solutions of the 2-dimensional Navier-Stokes equations [8]. The second possibility is that either u_λ^+ or u_λ^- is identically zero on a given energy shell. This discussion leads to the following

Definition 1. An incompressible velocity field u is a *generalized Beltrami function* provided that for each $\lambda \in \Lambda$, either u_λ^+ or u_λ^- in the decomposition in (0.8) vanishes identically.

Thus, if a solution of the Navier-Stokes equations is a generalized Beltrami function at a given instant in time, it is weakly interacting at that time since all local Fourier interactions in (0.10) vanish identically. The numerical experiments described in [4] give several tests which indicate regions in wave number space where classes of turbulent fluid flows behave approximately like generalized Beltrami functions for appropriate bands of wave numbers – one configuration typically observed in the numerical experiments from [4] are the pancake eddies which we analyze at the end of Sect. 2. The concept of Beltrami spectrum which we describe at the end of Sect. 1 is one effective and simple measure of the extent to which a given incompressible velocity field behaves like a generalized Beltrami function for a regime of wave numbers; therefore a corresponding solution of the Navier-Stokes equations with this velocity at a given instant in time is weakly interacting in this band of wave numbers.

Definition 2. With the expansion in (0.8) for an arbitrary incompressible velocity field, u, the *Beltrami spectrum* of u, $\{\beta_\lambda(u)\}_{\lambda \in \Lambda}$, is the sequence of numbers defined by

$$\beta_\lambda(u) = \frac{|u_\lambda^+|^2 - |u_\lambda^-|^2}{|u_\lambda^+|^2 + |u_\lambda^-|^2}. \tag{0.12}$$

Of course, $\beta_\lambda(u)$ is only defined provided there is any energy at all on the λ-shell, i.e., $|u_\lambda^+|^2 + |u_\lambda^-|^2 > 0$. The following elementary properties of the Beltrami spectrum are discussed at the end of Sect. 1:

i) $-1 \leqq \beta_\lambda(u) \leqq 1$ for all λ.

ii) $|\beta_\lambda(u)| = 1$ for all $\lambda \in \Lambda$, if and only if u is a generalized Beltrami flow. (0.13)

iii) For any purely two-dimensional flow u, $\beta_\lambda(u) = 0$ for all λ.

In particular, with property ii) and the above discussion, one effective way to quantify the study of weakly interacting Beltrami hierarchies as proposed in [4] is to find solutions of the fluid equations with energy shells prescribed by $\lambda \in \Lambda$, where the Beltrami numbers $\beta_\lambda(u)$ satisfy

$$(1 - \beta_\lambda^2(u))^{1/2} \leqq q - 1 \tag{0.14}$$

with $q-1>0$, a prescribed small number. We also remark that purely two-dimensional flows are very far from Beltrami flows and indeed property iii) guarantees that such flows have a Beltrami spectrum which has the largest possible deviation from generalized Beltrami functions. Some useful refinements of the Beltrami spectrum are presented at the end of Sect. 1. With these concepts, Sect. 2 is devoted to non-trivial examples of classes of fluid flows with remarkably rigid Beltrami spectra and also satisfying (0.14) for a broad band of wave numbers, while Sect. 3 contains an analysis of the evolution of the Beltrami spectrum for general solutions of the Navier-Stokes equations.

We end this introduction with a few general comments. Beltrami flows are exact solutions of the inviscid Euler equations with an extremely simple structure in wave number space, but such flows can exhibit tremendous complexity in physical space. Even simple examples such as the Arnold-Beltrami-Childress flows can have everywhere dense particle trajectories [6]; such flows are difficult to represent economically by a direct local description in physical space such as computational vortex methods. Earlier in this introduction, we have described the fashion in which the Navier-Stokes equations can be rewritten as the evolution of interacting Beltrami flows. In contrast, as Chorin has remarked in 1973 [1], vortex methods represent solutions of the Navier-Stokes equations as interacting superpositions of steady rotating eddies which are exact solutions to the 2-D inviscid Euler equations – the physical space description of a rotating eddy is extremely simple but the Fourier description is rather complex and not illuminating. Similar remarks apply for 3-D vortex methods. It is conceivable that turbulent flows may exhibit both regimes of weak local interaction and strong local interaction (due for instance to vortex stretching) between their Beltrami components. These facts suggest the possibility of a "nonlinear uncertainty principle" for turbulent fluid flows which mediates between regions of effective description of the flow field in physical space and regions of effective description of the flow field in wave number space. The discovery of such an uncertainty principle might be a major breakthrough in the understanding of turbulent fluid flows.

1. The Beltrami Decomposition and Beltrami Spectrum of Arbitrary Divergence Free Functions

Let u be a smooth periodic function with period L,

$$u: Q_L \to \mathbb{R}^3$$

with $Q_L = \left[-\frac{L}{2}, \frac{L}{2}\right]^3$. The function u is recovered from its Fourier series by

$$u(x) = \sum_{k \in \mathbb{Z}^3} u_k \exp\left(\frac{2\pi i}{L}\langle x, k\rangle\right). \tag{1.1}$$

We consider real incompressible velocity fields u, with mean zero – this last condition is imposed only for simplicity in exposition. Thus, the Fourier coefficients, u_k, satisfy the conditions in (0.3) and also $u_0 \equiv 0$. Degrees of smoothness of the function u are measured by various Sobolev norms.

The Sobolev spaces H^s are defined by

$$H^s=\left\{u=\sum_{k\in\mathbb{Z}^3} u_k \exp\left(\frac{2\pi i}{L}\langle x,k\rangle\right)\middle|\, u_k \quad \text{satisfy (0.3)} \right.$$
$$\left. \text{and} \quad \sum_{k\in\mathbb{Z}^3}(1+|k|^2)^s|u_k|^2<\infty\right\}.$$

In particular the L^2 space corresponding to $s=0$ will be denoted by H. The scalar product and norm in H are denoted by (,) and $|\cdot|$ and are given by the standard formulae already presented in (0.9).

We are going to describe a canonical orthogonal decomposition of any periodic, divergence free function as a sum of Beltrami functions, i.e., the special exact solutions of the 3-D equations already discussed in (0.4)–(0.7). In order to describe this decomposition we define, for each $k\in\mathbb{Z}^3\backslash\{0\}$ the projections $P_k^{\pm}:\mathbb{C}^3\to\mathbb{C}^3$ by

$$P_k^{\pm}(z)=\frac{1}{2}\left(z\pm\frac{ik}{|k|}\times z\right). \tag{1.2}$$

We gather together a few facts about the maps $P_k^{\pm}$ in the following

Proposition 1.1. *Let $k\neq 0$ be fixed. Then*

(i) $P_{-k}^{\pm}=P_k^{\mp}$.

(ii) $z=P_k^{+}(z)+P_k^{-}(z)$ *for all* $z\in\mathbb{C}^3$.

(iii) $\overline{P_k^{\pm}(z)}=P_k^{\mp}(\bar{z})$ *for all* $z\in\mathbb{C}^3$.

(iv) $P_k^{\pm}(P_k^{\pm}(z))=P_k^{\pm}(z)$ *for all* $z\in\mathbb{C}^3$ *satisfying* $\langle z,k\rangle=0$.

(v) $P_k^{\pm}(P_k^{\mp}(z))=0$ *for all* $z\in\mathbb{C}^3$ *satisfying* $\langle z,k\rangle=0$.

(vi) $\langle P_k^{\pm}(z),\zeta\rangle=\langle z,P_k^{\mp}(\zeta)\rangle$ *for all* $z,\zeta\in\mathbb{C}^3$.

(vii) $i\frac{k}{|k|}\times P_k^{\pm}(z)=\pm P_k^{\pm}(z)$ *for all* $z\in\mathbb{C}^3$ *satisfying* $\langle z,k\rangle=0$.

Proof. The first three properties are obvious. Properties (iv), (v), and (vii) follow from the identity

$$i\frac{k}{|k|}\times\left(\frac{ik}{|k|}\times z\right)=z, \tag{1.3}$$

valid for all $z\in\mathbb{C}^3$ satisfying $\langle z,k\rangle=0$. We use the notation $\langle a,b,c\rangle$ to denote the determinant of the complex matrix (a,b,c) formed by the column vectors $a,b,c\in\mathbb{C}^3$. The property (vi) follows from the observations

$$\left\langle\frac{ik}{|k|}\times z,\zeta\right\rangle=\left\langle\frac{ik}{|k|},z,\zeta\right\rangle \qquad \text{for all} \quad z,\zeta\in\mathbb{C}^3, \tag{1.4}$$

and

$$\left\langle\frac{ik}{|k|},z,\zeta\right\rangle=-\left\langle\frac{ik}{|k|},\zeta,z\right\rangle, \quad \text{for} \quad z,\zeta\in\mathbb{C}^3. \tag{1.5}$$

The above algebraic properties readily yield the important analytic fact

Lemma 1. *Consider a fixed wave number* $k \neq 0$:

1) *For arbitrary complex vectors* $u_k, v_k \in \mathbb{C}^3$, *with* $\langle u_k, k\rangle = 0$, *the two functions* $P_k^+(u_k)\exp\left(\frac{2\pi i}{L}\langle x,k\rangle\right)$ *and* $P_k^-(v_k)\exp\left(\frac{2\pi i}{L}\langle x,k\rangle\right)$ *are always orthogonal in* L^2.

2) *For any vector* $u_k \in \mathbb{C}^3$ *with* $\langle u_k, k\rangle = 0$, $u_k = P_k^+(u_k) + P_k^-(u_k)$ *and the two functions* $\mathrm{Re}\left(P_k^{\pm}(u_k)\exp\left(\frac{2\pi i}{L}\langle x,k\rangle\right)\right)$ *are simple single mode Beltrami flows satisfying* (0.4) *as well as being orthogonal in* H.

The proof of this lemma is a straightforward exercise utilizing the identities in Proposition 1.1 and we leave this for the reader. Nevertheless, the above lemma indicates that any incompressible velocity with a single mode decomposes into a unique L^2-orthogonal sum of Beltrami flows. This is the crucial observation for the results which we present next.

First, we give a precise notation for the grouping of Fourier coefficients along energy shells which we have already described in (0.2). We recall that Λ is the countable set of numbers so that λ belongs to Λ if and only if λ is the sum of squares of three integers. If $\lambda \geqq 0$ is an arbitrary number, we denote by R_λ the spectral projector of the Stokes operator onto the eigenspace corresponding to the eigenvalue $\frac{4\pi^2}{L^2}\lambda$:

$$R_\lambda(u) = \sum_{\substack{k\in\mathbb{Z}^3\setminus 0\\ |k|^2=\lambda}} u_k \exp\left(\frac{2\pi i}{L}\langle\cdot,k\rangle\right). \tag{1.6}$$

For $\lambda \in \Lambda$, $R_\lambda(u)$ is the projection on the corresponding energy shell. Of course, if λ does not belong to Λ then $\frac{4\pi^2}{L^2}\lambda$ is not an eigenvalue; we set R_λ equal to zero for such λ in order to unify notation. R_λ is a projector in H and also simultaneously in all H^s, $s \geqq 0$. This means that $R_\lambda^* = R_\lambda$ and $R_\lambda^2 = R_\lambda$, where we denote by T^* the adjoint operator of the bounded linear operator T. Now we define a decomposition of R_λ into a sum of two projectors P_λ^+ and P_λ^- corresponding to eigenvalues $+1$ and -1 of the operator $(-\Delta)^{-1/2}(\nabla\times\cdot)$ restricted to $R_\lambda H$. These projections yield the Beltrami decomposition described earlier in the introduction. Thus, we define $P_\lambda^{\pm}(u)$ by

$$P_\lambda^{\pm}(u) = \sum_{\substack{k\in\mathbb{Z}^3\\ |k|^2=\lambda}} P_k^{\pm}(u_k)\exp\left(\frac{2\pi i}{L}\langle x,k\rangle\right). \tag{1.7}$$

We will also use the notation

$$P_\lambda^{\pm}(u) = u_\lambda^{\pm}\,. \tag{1.8}$$

First we remark that by virtue properties (i) and (iii) of Proposition 1.1, if u is a real valued function so is $P_\lambda^{\pm}(u)$. Actually the operators $P_\lambda^{\pm}$ are linear bounded operators from H into H. They commute with R_λ and consequently with any power of the Stokes operator. They are actually bounded in all the Sobolev spaces H^s. Some of the most important properties of the operators $P_\lambda^\pm$ are collected in the following proposition:

Proposition 1.2. *Let $\lambda \geqq 0$ belong to Λ. Then*

(i) $R_\lambda = P_\lambda^+ + P_\lambda^-$.

(ii) $(P_\lambda^\pm)^* = P_\lambda^\pm$.

(iii) $P_\lambda^\pm P_\mu^\pm = P_\lambda^\pm P_\mu^\mp = 0$ *if* $\lambda \neq \mu$.

(iv) $P_\lambda^+ P_\lambda^- = 0$.

(v) $(P_\lambda^\pm)^2 = P_\lambda^\pm$.

Proof. Properties (i) and (iii) are obvious. Property (ii) follows from properties (vi), (iii), and (i) of Proposition 1.1 and the formula (0.9) defining the scalar product. Properties (iv) and (v) follow from properties (v) and (iv) of Proposition 1.1.

Now we can define the decomposition of arbitrary divergence free L^2 functions into Beltrami flows. We will set $P_\lambda^\pm = 0$ if λ does not belong to Λ.

Definition. Let $u \in H$ (i.e., u is an L^2 divergence free function). We call the orthogonal expansion

$$u = \sum_{\lambda \in \Lambda} (u_\lambda^+ + u_\lambda^-) \tag{1.9}$$

the Beltrami decomposition of u.

We note that the convergence of the series in (1.9) takes place in H. If the function u belongs to H^s then all the functions $u_\lambda^\pm$ belong to H^s and the convergence of (1.13) is in H^s. In particular, if $s > 3/2$ there is uniform pointwise convergence. We list the most important properties of the Beltrami decomposition in the following.

Theorem 1.1. *Let $u \in H^1$ with $u = \sum_{\lambda \in \Lambda} (u_\lambda^+ + u_\lambda^-)$ its Beltrami decomposition. Then*

(i) *For each $\lambda \in \Lambda$, u_λ^+ and u_λ^- are Beltrami flows:*

$$\nabla \times u_\lambda^\pm = \pm \left(\frac{2\pi}{L} \sqrt{\lambda} \right) u_\lambda^\pm . \tag{1.10}$$

(ii) *The functions $u_\lambda^\pm$ are mutually orthogonal in both H and H^1.*

(iii) *The Beltrami decomposition of the* curl $\nabla \times u$ *is*

$$\nabla \times u = \frac{2\pi}{L} \sum_{\lambda \in \Lambda} \sqrt{\lambda} (u_\lambda^+ - u_\lambda^-) . \tag{1.11}$$

(iv) *The energy, enstrophy and helicity of u are given respectively by*

$$|u|^2 = \sum_{\lambda \in \Lambda} (|u_\lambda^+|^2 + |u_\lambda^-|^2), \tag{1.12}$$

$$|\nabla \times u|^2 = \frac{4\pi^2}{L^2} \sum_{\lambda \in \Lambda} \lambda (|u_\lambda^+|^2 + |u_\lambda^-|^2), \tag{1.13}$$

$$(\nabla \times u, u) = \frac{2\pi}{L} \sum_{\lambda \in \Lambda} \sqrt{\lambda} (|u_\lambda^+|^2 - |u_\lambda^-|^2) . \tag{1.14}$$

The proof is straightforward from Propositions 1.1 and 1.2. Let us note that if $u \in H^1$, then the equality in (1.11) takes place in L^2 sense. However, if the function u belongs to H^s with $s > \frac{3}{2} + 1$, then the functions $u_\lambda^\pm$ are continuously differentiable and (1.11) is a pointwise identity. In general, $u_\lambda^\pm$ are always infinitely differentiable and are mutually orthogonal in all the H^s spaces.

Now we are going to introduce the notion of Beltrami spectrum of a divergence free function. First we define *a generalized Beltrami* function to be one for which all the restrictions $R_\lambda(u)$ are Beltrami functions. Clearly if in the Beltrami decomposition of a function, for each λ at least one of u_λ^+ and u_λ^- is zero then u is a generalized Beltrami function. The converse is also true. Indeed, assume that $R_\lambda(u) = v$ is a Beltrami function. Since the function v belongs to the λ energy shell, it follows that v satisfies the elliptic system

$$\begin{cases} -\Delta v = \dfrac{4\pi^2}{L^2}\lambda v \\ \operatorname{div} v = 0 . \end{cases}$$

Also according to (0.4) $\nabla \times v = \alpha v$ for some function $\alpha(x)$. We also require that $\alpha(x)$ is continuously differentiable. Taking the curl of the last relation we get

$$\frac{4\pi^2}{L^2}\lambda v = \nabla \times (\alpha v) = \alpha(\nabla \times v) + \nabla\alpha \times v .$$

We form the scalar product, at fixed x with $v(x)$ to get

$$\frac{4\pi^2}{L^2}\lambda |v(x)|^2 = (\alpha(x))^2 \, |v(x)|^2 .$$

It follows that $|\alpha(x)| = \frac{2\pi}{L}\sqrt{\lambda}$ for all x for which $|v(x)| \neq 0$. Because v is not identically zero and is a solution of an elliptic system, v is real analytic and thus $|v(x)| \neq 0$ almost everywhere. Since $\alpha(x)$ is continuous, it follows that $\alpha(x) = \pm\frac{2\pi}{L}\sqrt{\lambda}$ and consequently $R_\lambda(u) = u_\lambda^\pm$. The Beltrami spectrum of a function provides a quantitative description of the extent to which the function is or is not a generalized Beltrami function: it measures the angles between $\nabla \times R_\lambda(u)$ and $R_\lambda(u)$.

Definition. Let $u \in H$. For each $\lambda \geqq 0$ for which $R_\lambda(u) \neq 0$ we define the Beltrami numbers $\beta_\lambda(u)$ by

$$\beta_\lambda(u) = \frac{|u_\lambda^+|^2 - |u_\lambda^-|^2}{|u_\lambda^+|^2 + |u_\lambda^-|^2} . \tag{1.15}$$

The sequence $(\beta_\lambda(u))_{\lambda \in \Lambda}$ is called the Beltrami spectrum of u.

First we observe that the Beltrami numbers satisfy

$$|\beta_\lambda(u)| \leqq 1$$

for all u, λ, by definition. Clearly, u is a generalized Beltrami function if and only if $|\beta_\lambda(u)|=1$ for all λ. Also we note that

$$\beta_\lambda(u)=\frac{(R_\lambda(u), \nabla\times R_\lambda(u))}{|R_\lambda(u)|\cdot|\nabla\times R_\lambda(u)|}. \tag{1.16}$$

This means that $\beta_\lambda(u)$ is the cosine of the angle between $R_\lambda(u)$ and $\nabla\times R_\lambda(u)$ in H. Because $P_\lambda^\pm, R_\lambda$ are spectral projections it follows that the cosine of the angle between $R_\lambda(u)$ and $\nabla\times R_\lambda(u)$ is the same in all the Sobolev spaces H^s. From (1.16) it follows obviously that two dimensional functions have trivial Beltrami spectrum.

That is, if $u=u(x_1,x_2)=\begin{pmatrix}u_1\\u_2\\0\end{pmatrix}$ then $\beta_\lambda(u)=0$ for all λ. Clearly also

$$\beta_\lambda(u)\in\left[\inf_x\cos\theta_\lambda(x), \sup_x\cos\theta_\lambda(x)\right],$$

where

$$\cos\theta_\lambda(x)=\frac{\langle R_\lambda(u)(x), (\nabla\times R_\lambda(u))(x)\rangle}{|R_\lambda(u)(x)|\,|(\nabla\times R_\lambda(u))(x)|}.$$

Refined Beltrami Decompositions. Finally we mention that one can decompose each $u_\lambda^\pm$ further into sums of Beltrami functions. It is enough to take any partition of each energy shell $|k|^2=\lambda$ with $\lambda\in\Lambda$ into disjoint sets (closed under the operation $k\to -k$, in order to be able to get real valued functions). The finest such partition is formed by the sets $\{k, -k\}$. Corresponding to it one has the functions

$$P_k^\pm(u_k)\exp\left(\frac{2\pi i}{L}\langle x,k\rangle\right)+P_{-k}^\pm(u_{-k})\exp\left(\frac{2\pi i}{L}\langle x,-k\rangle\right)=u_{\{k,-k\}}^\pm$$

which are Beltrami functions. By Lemma 1 the functions $u_{\{k,-k\}}^\pm$ are orthogonal. One can associate to this decomposition a Beltrami spectrum indexed by the projective integers $P\mathbb{Z}^3=\{\{k,-k\}; k\neq 0, k\in\mathbb{Z}^3\}$. For $\{k,-k\}\in P\mathbb{Z}^3$ one defines

$$\beta_{\{k,-k\}}(u)=\frac{|u_{\{k,-k\}}^+|^2-|u_{\{k,-k\}}^-|^2}{|u_{\{k,-k\}}^+|^2+|u_{\{k,-k\}}^-|^2}.$$

These numbers measure the Beltrami spectrum for the finest partition. Obviously, we can use Lemma 1 to generate the Beltrami spectrum for other partitions of the energy shell into groups of Fourier modes in a similar fashion.

2. Classes of Fluid Flows with Universal Beltrami Spectrum

Here we compute the Beltrami spectrum explicitly for a class of interesting time dependent exact solutions. We also study configurations which are perturbations of the "pancake eddies" observed in the numerical experiments from [4].

A) A Universal Beltrami Spectrum for a Class of Quasi-Two-Dimensional Flows

We let $v={}^t(v_1(t,x_1,x_2), v_2(t,x_1,x_2))$ be the velocity field for any solution of the 2-*D* Navier-Stokes equations

$$\left.\begin{aligned}&\frac{\partial v}{\partial t}-\nu\Delta v+(v\cdot\nabla)v=\nabla p+f\\&\operatorname{div}v=0\end{aligned}\right\}\tag{2.1}$$

with periodic boundary conditions, i.e., the function v satisfies $v(t,x_1+L,x_2)=v(t,x_1,x_2+L)=v(t,x_1,x_2)$. There is a procedure to generate general solutions of the 3-*D* Navier-Stokes equations which depend only on the two spatial coordinates (x_1,x_2). The reader can readily verify that if $v_3(t,x_1,x_2)$ satisfies the scalar diffusion equation,

$$\left.\begin{aligned}&\frac{\partial v_3}{\partial t}+(v\cdot\nabla)v_3-\nu\Delta v_3=g(t,x_1,x_2),\qquad t>0\\&v_3|_{t=0}=v_3{}^0(x_1,x_2),\end{aligned}\right\}\tag{2.2}$$

then the velocity field

$$u={}^t(v_1(t,x_1,x_2), v_2(t,x_1,x_2), v_3(t,x_1,x_2))$$

is a solution of the 3-*D* Navier-Stokes equations in (0.1) with external force $F={}^t(f_1, f_2, g)$. Such exact solutions are called "quasi-two-dimensional" flows in the literature although the authors regard the terminology as unfortunate since such fluid flows can exhibit tremendous vorticity growth in time (see the discussion below). Special flows of this sort are important in understanding secondary instabilities in shear layers (see [2, 8]).

Here we compute the Beltrami spectrum for a class of quasi-two-dimensional flows. To define these flows, first we assume that the external forces $f={}^t(f_1(x_1,x_2), f_2(x_1,x_2))$ in (2.1) are smooth, periodic, and time independent. The scalar vorticity, $\omega=\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}$, of the 2-*D* flow in (2.1) satisfies the scalar diffusion equation,

$$\frac{\partial\omega}{\partial t}-\nu\Delta\omega+(v\cdot\nabla)\omega=\operatorname{curl}f\equiv g\,.\tag{2.3}$$

Thus, from (2.2) and (2.3), we see that the velocity,

$$u(t,x_1,x_2)=\begin{pmatrix}v_1(t,x_1,x_2)\\v_2(t,x_1,x_2)\\\varepsilon\omega(t,x_1,x_2)\end{pmatrix}\tag{2.4}$$

defines a special class of quasi two-dimensional fluid flows for any constant ε. The solution in (2.4) always has non-zero helicity for $\varepsilon\neq 0$ and can display very strong enstrophy growth. Indeed simple computations show that

$$\int_{Q_L} |\nabla \times u|^2 \, dx = L\left(\varepsilon^2 \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} |\nabla \omega|^2 \, dx_1 \, dx_2 + \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} |\omega|^2 \, dx_1 \, dx_2 \right), \quad (2.5)$$

$$\int_{Q_L} (u \cdot \nabla \times u) \, dx = L\varepsilon \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} |\omega|^2 \, dx_1 \, dx_2 \, . \quad (2.6)$$

The first relation shows that the enstrophy of u is related directly to the gradient of the vorticity of the 2-*D* flow. It is well known that in 2-*D* flow at high Reynolds numbers the gradients of the vorticity can become very large; they are the quantities governing 2-*D* turbulence (see [5]). The second relation shows that the helicity of u is a multiple of the enstrophy of v. One sees from the above relations and from the construction of u that a purely 2-*D* fluid flow corresponds to $\varepsilon = 0$ and that at $\varepsilon \neq 0$ one gets a way of injecting 2-*D* turbulence into 3-*D*. We compute the Beltrami spectrum for the 3-*D* flows defined in (2.4) in the following

Theorem 2.1. *Let v be any solution of the 2-D periodic Navier-Stokes equation* (2.1). *Let $\omega = \operatorname{curl} v$ and let*

$$u = \begin{pmatrix} v_1 \\ v_2 \\ \frac{L}{2\pi} \delta \omega \end{pmatrix},$$

where L is the period and δ is an arbitrary nondimensional parameter. Then u solves the 3-D Navier-Stokes equation. The Beltrami spectrum of u is independent of time, viscosity, initial data and body forces and is given by the universal formula

$$\beta_\lambda(u) = \frac{2\delta \sqrt{\lambda}}{1 + \delta^2 \lambda}.$$

With $|k|$ identified with $\sqrt{\lambda}$, the same universal formula also applies to the Beltrami numbers, $\beta_{\{-k,k\}}(u)$, associated with the finest decomposition described at the end of Sect. 1.

We see that the Beltrami spectrum of this class of quasi-two-dimensional flows is *time independent*, *universal* and *nontrivial*. By universal we mean independent of the viscosity ν, body forces b and initial data. By nontrivial we mean not only that $\beta_\lambda(u)$ is not identically zero but also that it exhibits a broad band of wave numbers with a weakly interacting Beltrami hierarchy. That is, if $\delta = \frac{2\pi\varepsilon}{L}$ is nonzero, say positive, then $\beta_\lambda(u) \geqq 1 - \eta$ for $\lambda \in \left[\frac{c_1}{\delta^2}, \frac{c_2}{\delta^2} \right]$ for some $c_1 < 1 < c_2$ depending on η only. The length of the interval where $\beta_\lambda(u) \geqq 1 - \eta$ gets larger as $\delta \to 0$. This is the behavior of a singular perturbation because, as we noted earlier, at $\delta = 0$ the fluid flow is purely 2-*D* so the spectrum $\beta_\lambda(u)$ is trivial, $\beta_\lambda(u) = 0$. Of course, as $\delta \to 0$ the location of the broad band with a significant Beltrami hierarchy moves far to the right along the λ axis to regions where viscous effects eventually are overwhelming at a fixed Reynolds number.

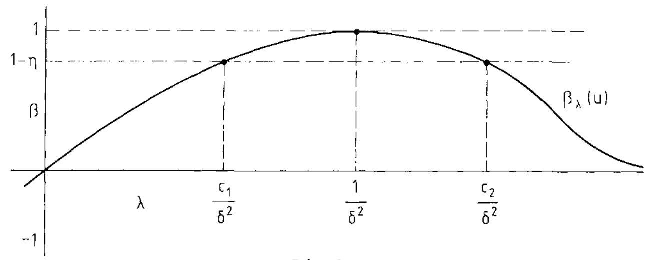

Fig. 2.1

To prove the theorem we compute the Beltrami spectrum for this class of quasi-two-dimensional flows. First we represent the vorticity ω:

$$\omega = \sum_{k \in \mathbb{Z}^2 \setminus 0} w_k \left(\exp\left(\frac{2\pi i}{L} \langle x, k \rangle \right) \right) \tag{2.7}$$

with $w_k \in \mathbb{C}$, $\bar{w}_k = w_{-k}$. We identify $\mathbb{Z}^2$ with the subset $\{(k_1, k_2, 0) | k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}\}$ of $\mathbb{Z}^3$. The function v has the representation

$$v = \frac{L}{2\pi} \sum_{k \in \mathbb{Z}^2 \setminus 0} \frac{w_k}{|k|} \begin{pmatrix} \frac{ik_2}{|k|} \\ \frac{-ik_1}{|k|} \end{pmatrix} \exp\left(\frac{2\pi i}{L} \langle x, k \rangle \right). \tag{2.8}$$

Therefore, the Fourier series of u is

$$u = \frac{L}{2\pi} \sum_{k \in \mathbb{Z}^3 \setminus 0} \frac{w_k}{|k|} \begin{pmatrix} \frac{ik_2}{|k|} \\ \frac{-ik_1}{|k|} \\ \frac{2\pi\varepsilon}{L} |k| \end{pmatrix} \exp\left(\frac{2\pi i}{L} \langle x, k \rangle \right). \tag{2.9}$$

We make a few remarks about dimensions. Clearly, if the function u defined in (2.4) is to have the dimensions of velocity then ε should have the dimension of length. Thus

$$\delta = \frac{2\pi\varepsilon}{L} \tag{2.10}$$

is a nondimensional parameter. We recall that w_k, ω have the dimension $(\text{time})^{-1}$ and u, u_k have the correct dimension $(\text{length}) \times (\text{time})^{-1}$. We denote by $\gamma(|k|^2)$

$$\gamma = \gamma(|k|^2) = \frac{2\pi\varepsilon}{L} |k| = \delta |k| . \tag{2.11}$$

Then, with these notations from (2.9) we have

$$u_k = \frac{L}{2\pi}\frac{w_k}{|k|}\begin{pmatrix} \dfrac{ik_2}{|k|} \\ \dfrac{-ik_1}{|k|} \\ \gamma(|k|^2) \end{pmatrix} \quad \text{for} \quad k = \begin{pmatrix} k_1 \\ k_2 \\ 0 \end{pmatrix}. \tag{2.12}$$

We note that

$$\begin{pmatrix} \dfrac{ik_1}{|k|} \\ \dfrac{ik_2}{|k|} \\ 0 \end{pmatrix} \times \begin{pmatrix} \dfrac{ik_2}{|k|} \\ \dfrac{-ik_1}{|k|} \\ \gamma \end{pmatrix} = \begin{pmatrix} \gamma\left(\dfrac{ik_2}{|k|}\right) \\ \gamma\left(\dfrac{-ik_1}{|k|}\right) \\ 1 \end{pmatrix}. \tag{2.13}$$

Then, computing $P_k^{\pm}(u_k)$ we get

$$P_k^{\pm}(u_k) = \frac{L}{4\pi}\frac{w_k}{|k|}\begin{pmatrix} (1\pm\gamma)\left(\dfrac{ik_2}{|k|}\right) \\ (1\pm\gamma)\left(\dfrac{-ik_1}{|k|}\right) \\ \pm(1\pm\gamma) \end{pmatrix} \tag{2.14}$$

and

$$|P_k^{\pm}(u_k)|^2 = 2\left(\frac{L}{4\pi}\right)^2 \frac{|w_k|^2}{|k|^2}(1\pm\gamma)^2. \tag{2.15}$$

Denoting by $R_\lambda\omega = \sum\limits_{|k|^2=\lambda} w_k\left[\exp\left(\dfrac{2\pi i}{L}\langle x,k\rangle\right)\right]$, we have

$$|P_\lambda^{\pm}(u)|^2 = \frac{2}{\lambda}\left(\frac{L}{4\pi}\right)^2 |R_\lambda\omega|^2(1\pm\gamma(\lambda))^2. \tag{2.16}$$

Thus, in the expression of the Beltrami numbers $\beta_\lambda(u)$ the common factors $\frac{2}{\lambda}\left(\frac{L}{4\pi}\right)^2 |R_\lambda\omega|^2$ cancel and we obtain the formula claimed in Theorem 2.1,

$$\beta_\lambda(u) = \frac{2\gamma(\lambda)}{1+(\gamma(\lambda))^2} = \frac{2\delta\sqrt{\lambda}}{1+(\delta\sqrt{\lambda})^2}. \tag{2.17}$$

Obviously, from (2.15) the same calculation applies for the refined Beltrami spectrum, $\{\beta_{\{k,-k\}}(u)\}$.

We make a simple remark on the preceding calculations. First the formula

$$\beta_\lambda(u) = \frac{2\gamma(\lambda)}{1+(\gamma(\lambda))^2} \tag{2.18}$$

is valid for any function u whose Fourier coefficients have the structure

$$u_k = \frac{L}{2\pi}\frac{w_k}{|k|}\begin{pmatrix} \frac{ik_2}{|k|} \\ \frac{-ik_1}{|k|} \\ \gamma(|k|^2) \end{pmatrix} \quad \text{at} \quad k = \begin{pmatrix} k_1 \\ k_2 \\ 0 \end{pmatrix},$$

for any function $\gamma(\lambda)$. Of course, if the function $\gamma(\lambda)$ is not a multiple of $\sqrt{\lambda}$ then this kind of structure is not preserved in time by the 3-D Navier-Stokes equations. However, this observation implies that any sequence of numbers can be realized as the Beltrami spectrum of an incompressible velocity field depending on two space variables.

B) "Pancake Eddies" and Enstrophy Production

First we consider an incompressible initial velocity field u_0 which has Fourier coefficients supported on a straight line in $\mathbb{Z}^3$; i.e., there is $k_0 \in \mathbb{Z}^3 \setminus \{0\}$ so that u_0 has the Fourier expansion

$$u_0 = \sum_{l=-\infty}^{\infty} u_{lk_0} \exp\left(\frac{2\pi i}{L} l\langle x, k_0\rangle\right) \tag{2.19}$$

with Fourier coefficients u_{lk_0} satisfying the conditions in (0.3). It is well known that for such initial data, the Navier-Stokes equations reduce to linear equations. The Fourier coefficients of the exact solution $u(t,x)$ of the Navier-Stokes equations with this initial data satisfy the linear equations

$$\begin{cases} \dot{u}_k + \frac{4\pi^2}{L^2}\nu|k|^2 u_k = 0, \qquad \langle u_k, k\rangle = 0, \\ u_k|_{t=0} = \begin{cases} u_{lk_0}, & k = lk_0 \\ 0, & k \neq lk_0 . \end{cases} \end{cases} \tag{2.20}$$

Thus, the solution $u(t,x)$ is given by

$$u(t,x) = \sum_{l=-\infty}^{\infty} u_{lk_0} \exp\left(\frac{-4\pi^2}{L^2} l^2 |k_0|^2 \nu t\right) \exp\left(\frac{2\pi i}{L} l\langle x, k_0\rangle\right). \tag{2.21}$$

These exact solutions have the form,

$$u = v_1(\langle x, k_0\rangle, t)e_1 + v_2(\langle x, k_0\rangle, t)e_2, \tag{2.22}$$

where e_i, $i=1,2$ are two orthonormal vectors with $\langle k_0, e_i\rangle = 0$, $i=1,2$. Such flows provide other examples of the quasi-two-dimensional flows described earlier in (2.1), (2.2). While such flows are very special, as reported in [4] the more general fluid flows computed in [4] have regions of both physical and Fourier space with a local structure approximately given by the simple flows in (2.19)–(2.22). The term, "pancake eddy" is used in [4] and can be motivated by the form of these exact solutions given in (2.22). It follows from (2.21) that the Beltrami spectrum is invariant with time for these exact solutions. In particular, if the initial data is a generalized Beltrami function, the solution is a generalized Beltrami function at any later time. Also, since all nonlinear interaction terms vanish in these exact solutions, the enstrophy always decays.

In the remainder of this section, we prove that if a solution of the Navier-Stokes equations has a velocity with Fourier coefficients concentrated in a suitably narrow cone (depending on Reynolds number) at some instant in time, then the enstrophy necessarily decays at that time. This result yields a rigorous proof of the assertion that suitable perturbed flows near the pancake eddy configurations in wave number space are weakly interacting. More precisely we prove the following fact:

Theorem 2.2. *Assume that $u(t,x)$ is a solution of the Navier-Stokes equation. Consider the nondimensional Reynolds number R defined at a given time t by*

$$R = L^{1/2}\frac{|\nabla \times u(t)|}{\nu} \tag{2.23}$$

with $|\ |$ the L^2-norm. Then there is a fixed constant c_3 so that if δ satisfies

$$\delta \leqq c_3 R^{-1}, \tag{2.24}$$

and if the Fourier coefficients $u_j(t)$ for $u(t,x)$ vanish for those j satisfying both

$$\left|\frac{j}{|j|} \pm \frac{k_0}{|k_0|}\right| > \frac{\delta}{2} \quad \text{for some} \quad k_0 \neq 0, \tag{2.25}$$

then there is no enstrophy production at time t, i.e., $\frac{d}{dt}|\nabla \times u|^2 < 0$.

Before proving the theorem, we make a few remarks. First, it is well known (see [10–12]) that if the number R defined in (2.23) is sufficiently small at time $t=0$, then there is no enstrophy production for any later time. Theorem 2.2 implies that if the Reynolds number R is arbitrary but the Fourier transform of the solution is concentrated in a narrow cone of aperture roughly $1/R$, then again there is no enstrophy production. Of course, it is not necessary that the Fourier coefficients vanish identically outside the cone; the proof below shows that these coefficients only need to be suitably small [see (2.28) below].

For the proof of Theorem 2.2, first we write down the equation for evolution of the enstrophy for a solution of 3-D Navier-Stokes,

$$\frac{1}{2}\frac{d}{dt}|\nabla \times u|^2 + \nu|\Delta u|^2 + \left(\frac{2\pi}{L}\right)^3 i \sum_{\substack{j+k+l=0 \\ j,k,l\in\mathbb{Z}^3\setminus 0}} \langle u_j, k\rangle \langle u_k, u_l\rangle |l|^2 = 0. \tag{2.26}$$

Next we split the sum into two parts: One in which $\left|\frac{j}{|j|} - \frac{k}{|k|}\right| < \delta$ or $\left|\frac{j}{|j|} + \frac{k}{|k|}\right| < \delta$ and the rest. Because of the fact that $\langle u_j, j\rangle = 0$, we write

$$\langle u_j, k\rangle = \left\langle u_j, k \pm \frac{|k|}{|j|} j\right\rangle .$$

If we are in one of the situations $\left|\frac{j}{|j|} \pm \frac{k}{|k|}\right| \leqq \delta$ this implies that

$$|\langle u_j, k\rangle| \leqq \delta |u_j|\,|k|\,.$$

Therefore we get the estimate

$$\begin{aligned}\Bigg|\sum_{\substack{j+k+l=0\\ \left|\frac{j}{|j|} \pm \frac{k}{|k|}\right| \leqq \delta}} \langle u_j, k\rangle \langle u_k, u_l\rangle |l|^2\Bigg| &\leqq \delta \sum_{j+k+l=0} |u_j|\,|u_k|\,|u_l|\,|l|^2\,|k| \\ &\leqq c\delta \Big(\sum_{k\in\mathbb{Z}^3} |u_j|^2\,|j|^2\Big)^{3/4}\Big(\sum_{j\in\mathbb{Z}^3} |u_j|^2\,|j|^4\Big)^{3/4}.\end{aligned} \tag{2.27}$$

Now we take a cone Γ of aperture $\delta/2$ and denote u_Γ by

$$u_\Gamma = \sum_{j\in\Gamma} u_j \exp\left(\frac{2\pi i}{L}\langle x, j\rangle\right),$$

and $u_{\Gamma'}$ by

$$u_{\Gamma'} = \sum_{j\notin\Gamma} u_j \exp\left(\frac{2\pi i}{L}\langle x, j\rangle\right).$$

Since either $j\in\Gamma$ and $k\in\Gamma$ or at least one of j and k is not in Γ by combining (2.26) and (2.27) obtain the bound

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla\times u|^2 + \nu|\Delta u|^2 &\leqq c_0\delta|\nabla\times u|^{3/2}\,|\Delta u|^{3/2} \\ &\quad + c_1|\nabla\times u_{\Gamma'}|^{1/2}\,|\nabla\times u|\,|\Delta u|^{3/2} \\ &\quad + c_1|\nabla\times u|^{1/2}\,|\nabla\times u_{\Gamma'}|\,|\Delta u|^{3/2}.\end{aligned}$$

After using Poincaré's inequality

$$|\nabla\times u| \leqq c_2 L|\Delta u|,$$

finally we get

$$\begin{aligned}0 \geqq &\frac{1}{2}\frac{d}{dt}|\nabla\times u|^2 + |\Delta u|^{3/2}\,|\nabla\times u|^{1/2} \\ &\times\left((c_2L)^{-1/2}\nu - c_0\delta|\nabla\times u| - c_1|\nabla\times u_{\Gamma'}|^{1/2}\left(|\nabla\times u|^{1/2} + |\nabla\times u_{\Gamma'}|^{1/2}\right)\right).\end{aligned} \tag{2.28}$$

It follows that, if the enstrophy of $u_{\Gamma'}$, i.e., the part of the enstrophy which is produced outside the cone Γ, is small, and if $\delta > 0$ is small enough then there is no enstrophy production, $\frac{d}{dt}|\nabla\times u|^2 < 0$. In particular for functions u whose Fourier transform is supported in Γ there is no enstrophy production if

$$\delta \leqq c_3 R^{-1},$$

where $R = L^{1/2}\frac{|\nabla\times u|}{\nu}$ is defined in (2.23). This completes the proof.

3. Time Evolution of the Beltrami Spectrum of General Flows

Here we consider an arbitrary solution of the periodic 3-D Navier-Stokes equation

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u &= \nabla p \\ \operatorname{div} u &= 0 \end{aligned}\right\} \tag{3.1}$$

with smooth initial data

$$u(0,x) = u_0(x) \tag{3.2}$$

and L periodic boundary conditions. We use the notations

$$\|u\|^2 = \int_{Q_L} |\nabla \times u|^2 \, dx\,, \tag{3.3}$$

$$b(u,v,w) = \int_{Q_L} (u \cdot \nabla v) \cdot w \, dx \tag{3.4}$$

for real, smooth, divergence free, L periodic functions u, v, w.

With the examples which we just presented in Sect. 2, it is obviously interesting to understand the evolution of the Beltrami spectrum for general solutions of the Navier-Stokes equations. Our main result in this section is a proof of the fact that as long as there is no complete depletion of the λ-energy level, there is no significant change in the Beltrami numbers, $\beta_\lambda(u)$. We also give some simple general quantitative bounds for this depletion time which are non-dimensional and depend on Reynolds number. Before stating this theorem, we recall from (1.6) that for $\lambda \in \Lambda$, $R_\lambda(u)$ is the projection of u onto the energy shell corresponding to λ, i.e.,

$$R_\lambda(u) = \sum_{|k|^2 = \lambda} u_k \exp\left(\frac{2\pi i}{L} \langle \cdot, k \rangle\right).$$

The energy in the λ-shell at time t is measured by $|R_\lambda(u(t))|^2$ while a natural measure of the Beltrami number for the λ-shell is $\sqrt{1-(\beta_\lambda(u(t)))^2}$. Below we prove the following

Theorem 3.1. *Let $u(t)$ solve the L periodic 3-D Navier-Stokes equation in* (3.1). *We fix $q > 1$ (with $q-1$ small typically) and $s > 3/2$. Then on any interval of time where the bound given below in* (3.13) *is satisfied, simultaneously both of the estimates*

$$|R_\lambda(u(t))| \exp\left(\frac{4\pi^2}{L^2} \nu \lambda t\right) \geqq \frac{1}{q} |R_\lambda(u_0)|\,,$$

and

$$\left|\sqrt{1-(\beta_\lambda(u(t)))^2} - \sqrt{1-(\beta_\lambda(u_0))^2}\right| \leqq q - 1$$

are true. In particular, both bounds are true for t with $0 \leqq t \leqq T_\lambda$ with T_λ given in terms of the initial data by the a priori estimate

$$\sqrt{\frac{\nu T_\lambda}{L^2}} \exp\left(4\pi^2(\lambda-1)\frac{\nu T_\lambda}{L^2}\right) = \left(\sqrt{2}\, c_s \lambda^{s/2} \frac{|u_0|}{\nu L^{1/2}}\right)^{-1} \frac{|R_\lambda u_0|}{|u_0|} \frac{q-1}{q}$$

(see (3.18) *below).*

The first bound measures the energy depletion in the λ-energy shell while the second bound measures the change in the Beltrami spectrum -- both are controlled simultaneously at least for a time interval T_λ given by the nondimensional formula stated above. The constant c_s is a non-dimensional embedding constant.

We begin the proof of Theorem 3.1 by deriving an equation for the evolution of the Beltrami numbers, β_λ.

Proposition 3.1. *Let $\lambda \geqq 0$ be a sum of three squares of integers. Then, as long as $|R_\lambda(u(t))| \neq 0$, the evolution of the Beltrami numbers $\beta_\lambda(u(t))$ of the solution $u(t)$ of* (3.1) *is given by the equation*

$$\frac{1}{2}\frac{d}{dt}\beta_\lambda(u(t)) + \frac{1}{|R_\lambda(u(t))|^2} b(u(t), u(t), (1-\beta_\lambda)u_\lambda^+(t) - (1+\beta_\lambda)u_\lambda^-(t)) = 0, \quad (3.5)$$

where $\beta_\lambda = \beta_\lambda(u(t))$ and $u_\lambda^\pm(t) = P_\lambda^\pm(u(t))$.

We note the fact that the viscosity is not explicitly present in the equation.

Proof. The evolution of $|u_\lambda^\pm(t)|^2$ is given by

$$\frac{1}{2}\frac{d}{dt}|u_\lambda^\pm(t)|^2 + 4\pi^2\frac{v\lambda}{L^2}|u_\lambda^\pm(t)|^2 + b(u(t), u(t), u^\pm(t)) = 0. \quad (3.6)$$

Using the definition

$$\beta_\lambda(u(t)) = \frac{|u_\lambda^+(t)|^2 - |u_\lambda^-(t)|^2}{|u_\lambda^+(t)|^2 + |u_\lambda^-(t)|^2},$$

we obtain (3.5) by a straightforward computation.

We compute that the norm of

$$(1-\beta_\lambda)u_\lambda^+ - (1+\beta_\lambda)u_\lambda^-$$

is given by

$$\begin{aligned}|(1-\beta_\lambda)u_\lambda^+ - (1+\beta_\lambda)u_\lambda^-|^2 &= (1-\beta_\lambda)^2|u_\lambda^+|^2 + (1+\beta_\lambda)^2|u_\lambda^-|^2 \\ &= (1+\beta_\lambda^2)|R_\lambda(u)|^2 - 2\beta_\lambda(|u_\lambda^+|^2 - |u_\lambda^-|^2) = (1-\beta_\lambda^2)|R_\lambda(u)|^2.\end{aligned}$$

Thus

$$|(1-\beta_\lambda)u_\lambda^+ - (1+\beta_\lambda)u_\lambda^-| = \sqrt{1-\beta_\lambda(u)^2}\,|R_\lambda(u)|. \quad (3.7)$$

By using the Sobolev imbedding $H^s \subset L^\infty$ for $s > 3/2$ and the fact that $(1-\beta_\lambda)u_\lambda^+ - (1+\beta_\lambda)u_\lambda^-$ lies in the eigenspace $R_\lambda H$ of the Stokes operator we get the inequality

$$\|(1-\beta_\lambda)u_\lambda^+ - (1+\beta_\lambda)u_\lambda^-\|_{L^\infty} \leqq c_s\lambda^{s/2}\sqrt{1-\beta_\lambda^2}\,L^{-3/2}|R_\lambda(u)| \quad (3.8)$$

with a nondimensional constant c_s. Using (3.8) we obtain from (3.5) the bound

$$\left|\frac{1}{2}\frac{d}{dt}\beta_\lambda(u(t))\right| \leqq c_s\lambda^{s/2}L^{-3/2}\frac{|u(t)|\,\|u(t)\|}{|R_\lambda(u(t))|}\sqrt{1-(\beta_\lambda(u(t)))^2} \quad (3.9)$$

for every t, where $|R_\lambda(u(t))| \neq 0$, $s > 3/2$. Because $|\beta_\lambda(u(t))| \leqq 1$, by dividing we get

$$\left|\frac{d}{dt}\sqrt{1-(\beta_\lambda(u(t)))^2}\right| \leqq 2c_s\lambda^{s/2}L^{-3/2}\frac{|u(t)|\,\|u(t)\|}{|R_\lambda(u(t))|}. \quad (3.10)$$

On the other hand, from (3.6) we obtain the bound

$$\left|\frac{d}{dt}|R_\lambda(u(t))| + 4\pi^2\frac{\nu\lambda}{L^2}|R_\lambda(u(t))|\right| \leqq 2c_s\lambda^{s/2}L^{-3/2}|u(t)|\,\|u(t)\|. \tag{3.11}$$

Therefore we have

$$\left|\exp\left(\frac{4\pi^2}{L^2}\nu\lambda t\right)|R_\lambda(u(t))| - |R_\lambda u_0|\right| \leqq 2c_s\lambda^{s/2}L^{-3/2}\int_0^t \exp\left(\frac{4\pi^2}{L^2}\nu\lambda\tau\right)|u(\tau)|\,\|u(\tau)\|\,d\tau. \tag{3.12}$$

In order to control the evolution of $\beta_\lambda(u(t))$ we see from (3.10) that we need to make sure that $|R_\lambda(u(t))|$ is bounded from below. Let $q>1$ be arbitrary. We see from (3.12) that if we define

$$t_\lambda = t_\lambda(|u_0|, |R_\lambda(u_0)|, \nu, L, q)$$

by the condition

$$2c_s\lambda^{s/2}L^{-3/2}\int_0^{t_\lambda} \exp\left(\frac{4\pi^2}{L^2}\nu\lambda\tau\right)|u(\tau)|\,\|u(\tau)\|\,d\tau \leqq \frac{q-1}{q}|R_\lambda u_0|, \tag{3.13}$$

then

$$\exp\left(\frac{4\pi^2}{L^2}\nu\lambda t\right)|R_\lambda(u(t))| \geqq \frac{1}{q}|R_\lambda u_0| \tag{3.14}$$

for $t \leqq t_\lambda$. Using (3.14) and (3.13) in (3.10) we deduce that

$$|\sqrt{1-(\beta_\lambda(u(t)))^2} - \sqrt{1-(\beta_\lambda(u_0))^2}| \leqq q-1 \tag{3.15}$$

for all $t \leqq t_\lambda$. Finally, the familiar two consequences of the energy estimate

$$|u(t)| \leqq |u_0|\exp\left(-\nu\frac{4\pi^2}{L^2}t\right), \quad \text{for all} \quad t \geqq 0, \tag{3.16}$$

$$\nu\int_0^\infty \|u(t)\|^2\,dt \leqq \frac{|u_0|^2}{2}, \quad \text{for all} \quad t \geqq 0 \tag{3.17}$$

are easily used in order to produce a lower bound for t_λ expressed entirely in terms of the initial data; therefore, we have $t_\lambda \geqq T_\lambda$ with T_λ given by

$$\sqrt{\frac{\nu T_\lambda}{L^2}}\exp\left(4\pi^2(\lambda-1)\frac{\nu T_\lambda}{L^2}\right) = \left(\sqrt{2}\,c_s\lambda^{s/2}\frac{|u_0|}{\nu L^{1/2}}\right)^{-1}\frac{|R_\lambda u_0|}{|u_0|}\cdot\frac{q-1}{q}. \tag{3.18}$$

We remark that the groups $\frac{\nu T}{L^2}, \frac{|u_0|}{\nu L^{1/2}}$ are nondimensional.

We see that as long as there is no complete depletion of the λ energy level there is no significant change in the β_λ Beltrami number. The proof of Theorem 3.1 is complete.

Now we are going to describe possible situations in which

$$|\sqrt{1-(\beta_\lambda(u(t)))^2} - \sqrt{1-(\beta_\lambda(u_0))^2}| \leqq q-1$$

for all $t \geqq 0$. It is well known ([9]) that for any initial datum $u_0 \in H^1$ leading to a globally defined strong solution of the 3-D periodic Navier-Stokes equation there exists $\lambda(u_0) \geqq 1$, an integer which can be represented as a sum of three squares of integers and which satisfies

$$\lim_{t\to\infty}\left(\frac{\log|u(t)|}{t}\right) = \lim_{t\to\infty}\left(\frac{\log\|u(t)\|}{t}\right) = -\frac{4\pi^2}{L^2}\nu\lambda(u_0). \tag{3.19}$$

Moreover,

$$|u(t)| \leqq c|u_0| \exp\left(-\frac{4\pi^2}{L^2}\nu t\lambda_0(u_0)\right), \tag{3.20}$$

$$\|u(t)\| \leqq c\|u_0\| \exp\left(-\frac{4\pi^2}{L^2}\nu t\lambda_0(u_0)\right) \tag{3.21}$$

for all $t \geqq 0$. The constant c is nondimensional and can be chosen uniformly for $\lambda(u_0) + |u_0|$ bounded. Using (3.20) and (3.21) in (3.13) we see that (3.13) will be true with $t_\lambda = \infty$ provided $\lambda < 2\lambda(u_0)$ and

$$2c_s c^2 |u_0|\, \|u_0\|\, \lambda^{s/2} L^{-3/2} \leqq \frac{4\pi^2}{L^2}\nu(2\lambda(u_0)-\lambda)\frac{q-1}{q}|R_\lambda u_0|.$$

Introducing the nondimensional number

$$a = \frac{\lambda^{s/2}}{2\lambda(u_0)-\lambda}\left(\frac{|u_0|}{|R_\lambda u_0|}\right)\left(\frac{L^{1/2}\|u_0\|}{\nu}\right), \tag{3.22}$$

we see that (3.13) is valid, with $t_\lambda = \infty$, if

$$a \leqq \frac{q-1}{q}(2c_s c^2)^{-1}. \tag{3.23}$$

Thus we have proved

Corollary 3.1. *Let $u_0 \in H^1$. Assume $u(t)$ is a smooth solution of the 3-D Navier-Stokes equation. Assume that $q > 1$, $s > 3/2$ are fixed. Let $\lambda(u_0)$ be the positive integer defined by*

$$\lim_{t\to\infty}\left(\frac{\log|u(t)|}{t}\right) = -\frac{4\pi^2}{L^2}\nu\lambda(u_0).$$

Let λ be a positive integer satisfying

$$\lambda(u_0) \leqq \lambda < 2\lambda(u_0).$$

Assume that the nondimensional number a defined in (3.22) *satisfies the condition* (3.23). *Then, for all $t \geqq 0$ simultaneously we have*

$$\exp\left(\frac{4\pi^2}{L^2}\nu\lambda t\right)|R_\lambda(u(t))| \geqq \frac{1}{q}|R_\lambda u_0|$$

and

$$|\sqrt{1-(\beta_\lambda(u(t)))^2} - \sqrt{1-(\beta_\lambda(u_0))^2}| \leqq q-1.$$

We note that, since $\lambda(u_0)$ is not known explicitly and since it may vary in discontinuous ways, the "smallness condition" (3.23) can very well be vacuous. For instance this is surely the case if we have $\lambda < \lambda(u_0)$ because, obviously in that case $\exp\left(\frac{4\pi^2}{L^2}\nu\lambda t\right)R_\lambda(u(t))\to 0$ in view of (3.20). If $\lambda=\lambda(u_0)$ from [9] we know that $\lim\limits_{t\to\infty}\exp\left(\frac{4\pi^2}{L^2}\lambda t\right)R_\lambda(u(t))=V_\lambda\neq 0$. Therefore, in that case

$$\int_{t_0}^{\infty}\frac{|u(t)|\,\|u(t)\|}{|R_\lambda u(t)|}dt$$

is finite. Foias and Saut proved that the sets $M_k=\{u_0\in H^1\,|\,\lambda(u_0)\geqq\lambda_k\}$ are smooth (analytic) finite codimension, unbounded manifolds in H^1. Here $\{\lambda_k\}$ is the increasing sequence of positive integers which are sums of three nonnegative integers.

From this result it follows that condition (3.23) can be fulfilled by many initial data, including many for which the quantity $\frac{L^{1/2}\|u_0\|}{\nu}$ is not small.

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References

1. Chorin, A.J.: Fluid Mech. **57**, 785–796 (1973)
2. Corcos, G.M., Lin, S.J.: J. Fluid Mech. **139**, 67–84 (1984)
3. Pelz, R., Yakhot, V., Orszag, S.A., Shtilman, L., Levich, E.: Velocity-vorticity patterns in turbulent flow. Phys. Rev. Lett. **54**, 2505–2509 (1985)
4. Yakhot, V., Orszag, S.A., Yakhot, A., Panda, R., Frisch, U., Kraichnan, R.: Weak interactions and local order in strong turbulence. (Preprint Nov. 3, 1986) submitted to Phys. Rev. Lett.
5. Brachet, M.E., Meneguzzi, M., Sulem, P.L.: Phys. Rev. Lett. **57**, 683–686 (1986)
6. Dombre, T., Frisch, U., Greene, J.M., Henon, M., Mehr, A., Soward, A.M.: J. Fluid Mech. **167**, 353–367 (1986)
7. Bayly, B.: Ph. D. Thesis, Princeton University, June 1986
8. Majda, A.: Commun. Pure Appl. Math. **39**, 187–220 (1986)
9. Foias, C., Saut, J.C.: Indiana Univ. Math. J. Vol. **33**, No. 3, May-June 1984, pp. 459–477
10. Ladyzhenskaia, O.A.: The mathematical theory of viscous incompressible flow. New York: Gordon & Breach, Second ed. 1969
11. Constantin, P., Foias, C.: Navier-Stokes equations. Chicago, IL: The University of Chicago Press (to appear)
12. Temam, R.: Navier-Stokes equations: theory and numerical analysis. Amsterdam, New York: North-Holland 1977
13. Kraichnan, R.H.: Helical turbulence and absolute equilibrium. J. Fluid Mech. **59**, 745–752 (1973)
14. André, J.C., Lesieur, M.: Influence of helicity on the evolution of isotropic turbulence at high Reynolds numbers. J. Fluid Mech. **81**, 187–207 (1977)

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