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The Berezin Transform and Invariant Differential Operators

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Abstract: The Berezin calculus is important to quantum mechanics (creation-annihilation operators) and operator theory (Toeplitz operators). We study the basic Berezin transform (linking the contravariant and covariant symbol) for all bounded symmetric domains, and express it in terms of invariant differential operators.

0. Introduction

There are two equivalent ways to define the Wick calculus of operators on $\mathbb{R}^n$: the first one is based on creation and annihilation operators, a generalization of which constitutes a basic tool in quantum field theory. It is the alternative definition, based on reproducing kernel function theory, that leads to the generalization first defined and studied by Berezin [B1].

The Berezin calculus is of interest to theoretical physicists in that it constitutes a canonical quantization procedure associated with a fairly general class of phase spaces. Also, recent physics literature has shown much interest for coherent states, and Perelomov's book [P1] devotes a chapter to Berezin's theory. The operators obtained (also known as Toeplitz operators) generate some of the most interesting geometrically-structured C^*-algebras [BBCZ, U1]. Beyond applications to operator theory and partial differential equations [U8], Berezin operators are used in C^*-algebraic index theory and in deformation quantization of symmetric spaces [CGR, BLU].

Let M be a measure space, and let H be a closed subspace of $L^2(M)$, with orthogonal projection $E: L^2(M) \to H$. Then, given any bounded function f on M, the Berezin operator with contravariant symbol f is the linear operator $\sigma^*(f)$ on H given by

$$\sigma^*(f)h := E(fh). \tag{0.1}$$

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Assuming that E has a reproducing kernel $E(z,w)$ on $M \times M$ and denoting as e_w the coherent state at $w \in M$ characterized by $e_w(z) = E(z,w)/E(w,w)^{1/2}$, one defines the covariant symbol $\sigma(A)$ of any bounded operator A on H by

$$\sigma(A)(w) := (e_w \mid Ae_w) \tag{0.2}$$

so that the maps σ and σ^* are, in a suitable sense, adjoint to each other. The Berezin transform, whose computation for a certain class of M's constitutes the main purpose of the present paper, is the map $\sigma\sigma^*$ from symbols to symbols.

The Wick calculus is the specialization that corresponds to $M = \mathbb{C}^n = \{x + i\xi;\ x \in \mathbb{R}^n, \xi \in \mathbb{R}^n\}$ with the measure $\varrho^n \cdot \exp -\pi\varrho(|x|^2 + |\xi|^2)\,dx\,d\xi$, and to the (Segal-Bargmann) realization of H as the image of $L^2(\mathbb{R}^n, dx)$ under some isometry. Then as is well-known, one has in this case

$$\sigma\sigma^* = \exp\left(-\frac{1}{4\pi\varrho}\Delta\right), \tag{0.3}$$

where Δ is the standard (non-negative) Laplacian on $M = \mathbb{R}^{2n}$.

We shall be interested here in the case when M is a complex symmetric domain of the non-compact type, and $H = H_\lambda$ is the so-called λ-Bergman space of holomorphic functions on M. In view of the existence of a semi-simple transitive group G of isometries of M, and of the associated (projective) representation of G in H_λ, a complete characterization of the Berezin map $\sigma\sigma^*$ is possible: the problem is to express it, in the spectral-theoretic sense, as an explicit function of the "generalized Laplacians" $\Delta_1, \ldots, \Delta_r$ that generate the (commutative) algebra of G-invariant differential operators on M. At least formally, some answer was given by Berezin himself [B2], for the case of classical domains only, and in terms depending on the classification: however, no proof was given. We give a complete statement and proof, valid for the exceptional domains as well as for the classical ones, and independent of the classification. Our basic tool, the theory of Jordan algebras, permits a unified description of the domains themselves and turns out to be exactly what is needed for a complete description of N-invariant eigenfunctions. Then, the proof can be completed with the help of Harish-Chandra's spherical functions and of Fourier transformation theory [H1]: however, going beyond formal arguments in the spectral-theoretic treatment (not attempted by Berezin) depends on a number of inequalities of a geometric nature which are not quite trivial. The end of the paper is concerned with topics involving the possible limits of the theory as λ goes to infinity (cf. also [BLU]). The authors wish to thank the Mittag-Leffler Institute in Stockholm, where this work was initiated, for its hospitality in the Fall of 1990.

1. The Berezin Quantization

The Berezin-Wick quantization [B1] is based on reproducing kernel functions. Let M be a smooth manifold, endowed with a smooth positive measure μ. Consider the Lebesgue space

$$L^2(M) := L^2(M, d\mu) \tag{1.1}$$

with the inner product (conjugate-linear in the first variable)

$$(h \mid k) := \int_M \overline{h(z)}\,k(z)\,d\mu(z)\,. \tag{1.2}$$

Let H be a closed subspace of $L^2(M)$, consisting of continuous functions on M that do not all vanish at any single point of M. Consider the orthogonal projection $E: L^2(M) \to H$. Now suppose that H has a *reproducing kernel* with respect to $d\mu$. This means that there exists a (necessarily unique) continuous function

$$(z, w) \mapsto E(z, w) \tag{1.3}$$

on $M \times M$ such that for every $w \in M$, the function E_w defined by

$$E_w(z) := E(z, w) \tag{1.4}$$

belongs to H, and the identity

$$(E_w \mid h) = h(w) \tag{1.5}$$

holds for all $w \in M$ and $h \in H$. In particular, $E(z, w) = \overline{E(w, z)}$ and we have

$$E_w(w) = \|E_w\|^2 \tag{1.6}$$

for the norm $\|\cdot\|$ on the Hilbert space $L^2(M)$. Since (1.6) is strictly positive by an assumption on H made above, we may define the "coherent state" e_w as

$$e_w(z) := \frac{E_w(z)}{\|E_w\|} = \frac{E(z, w)}{E(w, w)^{1/2}} \tag{1.7}$$

and consider the measure

$$d\mu_0(w) := E(w, w)\, d\mu(w) \tag{1.8}$$

on M. For $h_1, h_2 \in L^2(M)$, let

$$(h_1 h_2^*)h := h_1(h_2 \mid h) \qquad (h \in L^2(M)) \tag{1.9}$$

be the associated rank 1 operator. The importance of reproducing kernel functions lies in the consequences of the following:

1.10 Lemma. *There exists a weak integral decomposition*

$$E = \int_M E_w E_w^* \, d\mu(w) = \int_M e_w e_w^* \, d\mu_0(w) . \tag{1.11}$$

Proof. By definition, we have to show that

$$(k \mid Eh) = \int_M (k \mid E_w)(E_w \mid h)\, d\mu(w) = \int_M (k \mid e_w)(e_w \mid h)\, d\mu_0(w) \tag{1.12}$$

for all $h, k \in L^2(M)$. This follows from (1.5) and (1.7) if $h, k \in H$. If h or k is orthogonal to H, then both sides of (1.12) vanish. Q.E.D.

1.13 Definition. *For every $f \in L^\infty(M)$, the bounded operator $\sigma^*(f)$ on H defined by*

$$\sigma^*(f)h := E(fh) \qquad (h \in H) \tag{1.14}$$

is called the Toeplitz operator with symbol f, or the Berezin operator with contravariant symbol f. For any bounded operator A on H, the bounded function $\sigma(A)$ on M defined by

$$\sigma(A)(z) = \frac{\operatorname{trace} A(E_z E_z^*)}{\operatorname{trace} E_z E_z^*} = \operatorname{trace} A(e_z e_z^*) = (e_z \mid Ae_z) \tag{1.15}$$

is called the Berezin covariant symbol of A. The Berezin transform is the map $\sigma\sigma^*$ *from* $L^\infty(M)$ *to* $L^\infty(M)$*, i.e.*

$$(\sigma\sigma^* f)(z) = (e_z \mid (\sigma^*(f) e_z) = (e_z \mid f e_z). \tag{1.16}$$

1.17 Lemma. *For* $f \in L^\infty(M)$*, there is a weak integral decomposition*

$$\sigma^*(f) = \int_M f(w) E_w E_w^* d\mu(w) = \int_M f(w) e_w e_w^* d\mu_0(w). \tag{1.18}$$

Proof. For $h, k \in H$, one may write

$$(k \mid \sigma^*(f) h) = (k \mid E(fh)) = (k \mid fh) = \int_M \overline{k(w)} f(w) h(w) d\mu(w)$$

$$= \int_M (k \mid E_w) f(w) (E_w \mid h) d\mu(w). \qquad \text{Q.E.D.}$$

1.19 Proposition. *Let* $\mathscr{L}^2(H)$ *be the Hilbert space of all Hilbert-Schmidt operators on* H*. Then the transformations* σ *and* σ^* *map* $\mathscr{L}^2(H)$ *into* $L^2(M, d\mu_0)$ *(resp.,* $L^2(M, d\mu_0)$ *into* $\mathscr{L}^2(H)$*) and are adjoint to each other. Moreover, the Berezin transform extends as an integral operator on* $L^2(M, d\mu_0)$ *with operator norm* ≤ 1 *and kernel*

$$K(z, w) = \frac{|E(z, w)|^2}{E(z, z) E(w, w)} = |(e_z \mid e_w)|^2. \tag{1.20}$$

Proof. If $f \in L^\infty(M)$ and $z \in M$, one has

$$\sigma\sigma^*(f)(z) = (e_z \mid \sigma^*(f) e_z) = \int_M f(w) |(e_w \mid e_z)|^2 d\mu_0(w)$$

according to Lemma 1.17. Since Lemma 1.10 implies

$$\int |(e_w \mid e_z)|^2 d\mu_0(w) = \|e_z\|^2 = 1 \tag{1.21}$$

for all z, the operator $\sigma\sigma^*$, initially defined on $L^\infty(M)$, extends as a bounded operator on $L^2(M, d\mu_0)$, with norm ≤ 1. If $f \in L^\infty(M) \cap L^1(M, d\mu_0)$, Lemma 1.17 shows that the operator $\sigma^*(f)$ is trace-class with a trace-norm $\leq \|f\|_{L^1(M, d\mu_0)}$. Also, for any bounded operator A, one may write

$$\begin{aligned} \text{trace}(A^* \cdot \sigma^*(f)) &= \text{trace}\left(A^* \int_M f(w) e_w e_w^* d\mu_0(w)\right) \\ &= \int_M f(w) \, \text{trace}(A^* e_w e_w^*) \, d\mu_0(w) = \int_M f(w) (e_w \mid A^* e_w) d\mu_0(w) \\ &= \int_M \overline{(e_w \mid A e_w)} f(w) d\mu_0(w) = \int_M \overline{\sigma(A)(w)} f(w) d\mu_0(w). \end{aligned} \tag{1.22}$$

If $A = \sigma^*(f)$, this identity, together with the Cauchy-Schwarz inequality and the part already proven of the present proposition, implies that

$$\|A\|^2_{\mathscr{L}^2(H)} = \mathrm{Tr}(A^*A) = \int_M f(w)\overline{\sigma\sigma^*(f)(w)}\,d\mu_0(w) \leq \|f\|^2_{L^2(M,d\mu_0)}.$$

Thus the map σ^* extends as a bounded map (with norm ≤ 1) from $L^2(M, d\mu_0)$ into $\mathscr{L}^2(H)$. That σ and σ^* are formally adjoint to each other follows from (1.22) again, which proves also that σ is bounded from $\mathscr{L}^2(H)$ to $L^2(M, d\mu_0)$. Q.E.D.

The basic *examples* of Berezin quantization come from connected *complex* manifolds M, endowed with a strictly positive smooth measure μ on M. Then

$$H^2(M) = H^2(M, d\mu) := \{h \in L^2(M, d\mu) : h \text{ holomorphic}\} \tag{1.23}$$

is a closed subspace of $L^2(M)$ which has a reproducing kernel $E(z, w)$ if one assumes that, for each $z \in M$, there exists $h \in H^2(M)$ with $h(z) \neq 0$. It is well-known [U2, P2] that under a slightly stronger non-degeneracy assumption M is naturally endowed with a *Kähler* structure with associated symplectic form

$$\omega = \frac{i}{2\pi}\,\partial\bar{\partial}\log E(z, z), \tag{1.24}$$

where ∂ and $\bar{\partial}$ are the Dolbeault differentiation operators.

Let $\gamma: M \to N$ be a biholomorphic mapping between two complex connected manifolds M and N; assume that $d\mu$ (resp. $d\nu$) is a given smooth strictly positive measure on M (resp. N) and that there exists a non-vanishing holomorphic function φ on M such that $|\varphi|^2\,d\mu = \gamma^* d\nu$. Then, if any of the two spaces $H^2(M, d\mu)$ and $H^2(N, d\nu)$ satisfies the non-degeneracy conditions that permit to construct a Kähler metric, so does the other. Moreover, the map $h \mapsto \varphi \cdot (h \circ \gamma)$ is an isometry from $H^2(N, d\nu)$ onto $H^2(M, d\mu)$, and the reproducing kernels E_N and E_M of these two spaces, with respect to $d\nu$ and $d\mu$ respectively, are linked by

$$E_M(z, w) = \varphi(z)\,E_N(\gamma z, \gamma w)\,\overline{\varphi(w)}. \tag{1.25}$$

Also, for every $z \in M$,

$$\begin{aligned}(\gamma^* d\nu_0)(z) &= E_N(\gamma z, \gamma z)\,|\varphi(z)|^2\,d\mu(z)\\ &= |\varphi(z)|^2\,\frac{E_N(\gamma z, \gamma z)}{E_M(z, z)}\,d\mu_0(z) = d\mu_0(z).\end{aligned}$$

It follows that the measure $d\mu_0$ on M, as well as the Kähler metric on M built from $d\mu$, is invariant under the group G of all biholomorphic transformations γ of M that satisfy the property that the (Radon-Nikodym) derivative $\dfrac{\gamma^* d\mu}{d\mu}$ is the square of the modulus of some holomorphic function. In the case when G acts on M transitively, $d\mu_0$ is a constant times the measure on M associated with its Kähler structure. Besides acting on M, G has a natural action on $H^2(M, d\mu)$, namely the one that associates to $\gamma^{-1} \in G$ the transformation

$$U(\gamma^{-1})h := (h \circ \gamma) \cdot \varphi \tag{1.26}$$

with the notations above and $N = M$. However, this is only a projective representation in general, i.e. a representation up to constant factors of modulus 1 since only $|\varphi|^2$, not φ, depends solely on γ. The Berezin quantization is covariant under these two actions of G: as a consequence, the Berezin map $\sigma\sigma^*: L^2(M, d\mu_0) \to L^2(M, d\mu_0)$ commutes with the action of G on M.

1.27 Example. Let $M = \mathbb{C}^n$ and fix a constant $\varrho > 0$. Consider the probability measure

$$d\mu_\varrho(z) := \varrho^n e^{-\pi\varrho|z|^2} dV(z), \tag{1.28}$$

where dV denotes the standard Lebesgue measure on $\mathbb{C}^n$. Then the *Segal-Bargmann space*

$$H^2(\mathbb{C}^n) := H^2(\mathbb{C}^n, d\mu_\varrho) \tag{1.29}$$

is a closed subspace of $L^2(\mathbb{C}^n, d\mu_\varrho)$, with inner product

$$(h \mid k)_\varrho := \int_{\mathbb{C}^n} \overline{h(z)} k(z) \, d\mu_\varrho(z). \tag{1.30}$$

For any fixed $w \in \mathbb{C}^n$, we have

$$\varrho^n \int_{\mathbb{C}^n} e^{-\pi\varrho|z-w|^2} dV(z) = 1,$$

since $dV(z)$ is translation invariant. This implies

$$e^{\pi\varrho|w|^2} = \int_{\mathbb{C}^n} |e^{\pi\varrho z \cdot \bar{w}}|^2 \, d\mu_\varrho(z) = (e^{\pi\varrho z \cdot \bar{w}} \mid e^{\pi\varrho z \cdot \bar{w}})_\varrho .$$

By polarization it follows that the reproducing kernel of (1.29) is given by

$$E_\varrho(z, w) = E_w(z) = e^{\pi\varrho z \cdot \bar{w}} \tag{1.31}$$

and

$$e_w(z) = e^{\pi\varrho z \cdot \bar{w}} e^{-\frac{\pi\varrho}{2} w \cdot \bar{w}} \tag{1.32}$$

is the coherent state at w. Comparing (1.28) and (1.31), we can see that the measure $d\mu_0$ has the form

$$d\mu_0 = E_\varrho(z, z) \, d\mu_\varrho(z) = \varrho^n dV(z). \tag{1.33}$$

By (1.31), the symplectic form associated with the Kähler structure on $\mathbb{C}^n$ under consideration is

$$\omega = \frac{i}{2\pi} \partial\bar{\partial}(\pi\varrho|z|^2) = \frac{i\varrho}{2} \partial\Big(\sum_j z_j \, d\bar{z}_j\Big) = \frac{i\varrho}{2} \sum_j dz_j \wedge d\bar{z}_j = \varrho \sum dx_j \wedge d\xi_j$$

if $z = x + i\xi$. Note that $d\mu_0$ is exactly the measure associated with the symplectic form ω and that the standard interpretation of ϱ^{-1} is that of Planck's constant. The associated Berezin quantization is just the Wick calculus: more precisely the covariant (resp. contravariant, when it exists) symbol of an operator, evaluated at z, is just its Wick (resp. anti-Wick) symbol, evaluated at $\bar{z}$. By Proposition 1.19 and (1.32), the "Wick transform" $\sigma_\varrho \sigma_\varrho^*$ has the integral kernel

$$K_\varrho(z, w) = \frac{e^{\pi\varrho z \cdot \bar{w}} e^{\pi\varrho w \cdot \bar{z}}}{e^{\pi\varrho z \cdot \bar{z}} e^{\pi\varrho w \cdot \bar{w}}} = e^{-\pi\varrho|z-w|^2}$$

with respect to (1.33). This implies (0.3). Since $e^{-\pi\varrho|z|^2}$ is invariant under the group $U(n)$ (acting on $\mathbb{C}^n$ in a linear way) and since, for all $w \in \mathbb{C}^n$, the function $z \mapsto e^{\pi\varrho|z|^2} e^{-\pi\varrho|z-w|^2}$ is the square of the modulus of some entire function, the measure $d\mu_0$ as well as the Kähler structure of $\mathbb{C}^n$ is invariant under the group generated by $U(n)$ and by translations; also, the Berezin quantization is covariant under the natural actions of this group on $\mathbb{C}^n$ and $H^2_\varrho(\mathbb{C}^n)$. As is well-known, to get a true (not projective) representation, one has to substitute the Heisenberg group for the group of translations.

1.33 Example. Let M be a domain in $\mathbb{C}^n$, endowed with the Lebesgue measure dV. Assume that M is equivalent to a bounded domain under a biholomorphic transformation. Then the *Bergman space*

$$H^2(M) := H^2(M, dV) \tag{1.34}$$

is a closed subspace of $L^2(M, dV)$ with a reproducing "Bergman" kernel $E_M(z, w)$. The associated quantization (1.18) is called the *Bergman-Toeplitz quantization.* It is covariant under the natural actions of the group G of all biholomorphic transformations of M; this time one has a true representation. If $\gamma: M \to N$ is a biholomorphic mapping between domains M and N in $\mathbb{C}^n$, the invariance property (1.25) becomes

$$\operatorname{Det} \partial\gamma(z)\, E_N(\gamma z, \gamma w) \overline{\operatorname{Det} \partial\gamma(w)} = E_M(z, w) \tag{1.35}$$

for all $z, w \in M$. Here $\partial\gamma$ denotes the complex derivative.

2. Invariant Differential Operators on Symmetric Domains

The bounded symmetric domains (Cartain domains) are the most studied class of Kähler manifolds (of non-compact type). In one complex dimension, every Riemann surface of genus ≥ 2 is the quotient of the unit disk (the only bounded symmetric domain in $\mathbb{C}$) by a discrete group of Moebius transformations. In higher dimensions there is no such "uniformization theorem" but the characterization of Kähler manifolds covered by symmetric domains is one of the most active areas in modern differential geometry [Y1].

As explained in the introduction, our main result (Theorem 3.43) is an explicit formula expressing the Berezin transform (1.16), for any symmetric domain, in terms of invariant differential operators (generalized Laplacians). For the proof we need detailed information about the joint spectral behavior of those (commuting) operators.

In general let $M = G/K$ be a non-compact symmetric space of rank r and consider the algebra $\mathrm{Diff}(M)^G$ of all (scalar) differential operators D on M which are invariant under the natural action of G:

$$D(f \circ g) = (Df) \circ g$$

for all $f \in \mathscr{C}_0^\infty(M)$ and $g \in G$. It is known [H2] that $\mathrm{Diff}(M)^G$ is a polynomial algebra

$$\mathrm{Diff}(M)^G = \mathbb{R}[\Delta_1, \ldots, \Delta_r]$$

in r algebraically independent commuting operators $\Delta_1, \ldots, \Delta_r$ (called "generalized Laplacians", since one of the generators can be chosen as the Laplace-Beltrami operator Δ_M). Let $\mathfrak{g} = \mathscr{k} \oplus \mathscr{p}$ denote the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, with $\mathscr{k}$ the Lie algebra of K. Let $\mathfrak{a} \subset \mathscr{p}$ be a maximal abelian subspace. The dimension of $\mathfrak{a}$ coincides with the *rank* of M. Let $\mathfrak{a}^{\#}$ denote the real dual space of $\mathfrak{a}$. For $\alpha \in \mathfrak{a}^{\#}$, put

$$\mathfrak{g}_{\alpha} := \{B \in \mathfrak{g} : [A, B] = \alpha(A) \cdot B \; \forall A \in \mathfrak{a}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_{\alpha} \neq \{0\}$, then α is called a *real root* of $\mathfrak{g}$ relative to $\mathfrak{a}$ and $\mathfrak{g}_{\alpha}$ is the corresponding root space. We have the root decompositon

$$\mathfrak{g} = \mathfrak{a} \oplus \mathscr{m} \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_{\alpha} \quad \text{(direct sum)}, \tag{2.1}$$

where $\Sigma \subset \mathfrak{a}^{\#}$ is the set of real roots and $\mathscr{m} = \{B \in \mathscr{k} : [A, B] = 0 \;\; \forall A \in \mathfrak{a}\}$. For each $\alpha \in \Sigma$, the number $m_{\alpha} := \dim \mathfrak{g}_{\alpha}$ is called the multiplicity of α (note the difference with the complex root decomposition of $\mathfrak{g}^{\mathbb{C}}$, where every root has multiplicity one). Let Σ_{+} denote the positive roots, with respect to some ordering. Then we have the *Iwasawa decomposition* $\mathfrak{g} = \mathscr{n} \oplus \mathfrak{a} \oplus \mathscr{k}$ (cf. [H2; p. 26]) where

$$\mathscr{n} := \sum_{\alpha \in \Sigma_{+}} \mathfrak{g}_{\alpha}. \tag{2.2}$$

The convex open subset $\mathfrak{a}_{+} := \{A \in \mathfrak{a} : \alpha(A) > 0 \;\; \forall \alpha \in \Sigma_{+}\}$ is called the Weyl chamber. We put

$$\varrho := \frac{1}{2} \sum_{\alpha \in \Sigma_{+}} m_{\alpha} \alpha. \tag{2.3}$$

Now consider the associated Iwasawa decomposition $G = NAK$ of G [H2]. Denoting the base point of M by o, define $\tilde{M} := A \cdot o \subset M$. Then every $z \in M$ can be written uniquely as $z = n \cdot a$, $n \in N$, $a \in \tilde{M}$. Since this decomposition has certain transversality properties [H2] one can define the N-*radial part* (also called *orispherical radial part*) of $D \in \mathrm{Diff}(M)^{G}$ to be the differential operator $\tilde{D}$ on $\tilde{M}$ determined by

$$\tilde{D}\tilde{f} := (Df)^{\tilde{}} \tag{2.4}$$

for all N-*invariant* functions $f \in \mathscr{C}_{0}^{\infty}(M)$. Here $\tilde{f}$ denotes the restriction of f to $\tilde{M}$.

In contrast to the (similarly defined) K-radial parts which are rather complicated the N-radial parts have a simple description: Fix a basis $A_1, \ldots, A_r$ of $\mathfrak{a}$. Then every $a \in \tilde{M}$ has the form

$$a = \exp\left(\sum_{j} t_j A_j\right) \cdot o, \tag{2.5}$$

where the "coordinates" $t_1, \ldots, t_r \in \mathbb{R}$ are uniquely determined. By [K1; Theorem 1], every N-radial part $\tilde{D}$, for $D \in \mathrm{Diff}(M)^{G}$, has *constant coefficients* when expressed in terms of $t_1, \ldots, t_r$. More precisely [K1; Theorem 2], we have

$$\tilde{D} = q\left(\frac{\partial}{\partial t_1} - \varrho_1, \ldots, \frac{\partial}{\partial t_r} - \varrho_r\right), \tag{2.6}$$

where q is any polynomial invariant under the Weyl group W and $\varrho_j := \varrho(A_j)$ for $1 \leq j \leq r$. The operator $D \in \mathrm{Diff}(M)^{G}$ satisfying (2.6) is uniquely determined by q and will be denoted by $D = D_q$.

Using the Iwasawa decomposition of G, we define a smooth function $l : G \to \mathfrak{a}$ by putting $g = n \exp(l(g)) k$ for every $g \in G$. Now let $\alpha \in \mathfrak{a}^{\#} \otimes \mathbb{C}$ be a complex-valued linear form on $\mathfrak{a}$. Then

$$\Delta^{\alpha}(g \cdot o) := e^{\alpha(l(g))} \tag{2.7}$$

defines an N-invariant smooth function $\Delta^\alpha : M \to \mathbb{C}$ since $l(ngk) = l(g)$ for all $n \in N$ and $k \in K$. On $\tilde{D}$, Δ^α has the values

$$\Delta^\alpha\left(\exp\left(\sum_j t_j A_j\right)\cdot o\right) = \prod_j e^{t_j \alpha(A_j)} = \prod_j e^{t_j \alpha_j} \tag{2.8}$$

since $l(\exp A) = A$ for all $A \in \alpha$. Here $\alpha_j := \alpha(A_j)$. By [H3; Lemma 5.15 and (35)], we have

$$\tilde{D}\Delta^\alpha = q(\alpha_1 - \varrho_1, \ldots, \alpha_r - \varrho_r)\Delta^\alpha \tag{2.9}$$

for all $D \in \mathrm{Diff}(M)^G$.

We will now give an explicit construction for the N-invariant eigenfunctions Δ^α in case M has a *classical root system* (type A, C, D, B, BC). This includes all hermitian symmetric spaces (even the two exceptional ones). It is well-known [L1] that, besides the Lie-theoretic approach, these spaces M can also be described in *Jordan algebraic* terms. Let X be a real Jordan algebra [BK] with product $x \circ y$ and unit element e, satisfying the "formally-real" condition $x^2 + y^2 = 0 \Rightarrow x = y = 0$. Basic examples are the space $X = \mathscr{H}_r(\mathbb{K})$ of all self-adjoint $(r \times r)$-matrices x over $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (quaternions), with the anti-commutator product $x \circ y := (xy + yx)/2$, and the "spin factors" $X = \mathbb{R}^{1+n}$ of all vectors $x = (x_0, x_1, \ldots, x_n) = (x_0, x')$, with product $x \circ y = (x_0 y_0 + x' \cdot y', x_0 y' + y_0 x')$. The open subset

$$\Lambda := \{x^2 : x \in X \text{ invertible}\} \tag{2.10}$$

is a convex cone. For the matrix algebras, Λ is the cone of positive definite matrices, whereas for the spin factor Λ is the forward light cone. The linear automorphism group

$$GL(\Lambda) := \{P \in GL(X) : P(\Lambda) = \Lambda\} \tag{2.11}$$

is transitive on Λ. $GL(\Lambda)$ is a reductive Lie group, whose Lie algebra

$$\mathfrak{gl}(\Lambda) = \{M \in \mathfrak{gl}(X) : \exp(tM) \in GL(\Lambda)\ \forall t \in \mathbb{R}\} \tag{2.12}$$

has the commutator bracket $[M, N] := MN - NM$. By [U3, U4] the Lie algebra $\mathfrak{g}_\Lambda := \mathfrak{gl}(\Lambda)$ has a Cartan decomposition $\mathfrak{g}_\Lambda = \mathscr{k}_\Lambda \oplus \mathscr{p}_\Lambda$, $\mathscr{k}_\Lambda$ is the Lie algebra of $O(\Lambda) := \{P \in GL(\Lambda) : Pe = e\}$ and $\mathscr{p}_\Lambda = \{M_x : x \in X\}$ consists of all *multiplication operators*

$$M_x y = x \circ y \tag{2.13}$$

on X. For $x, y \in X$ define an element of $\mathfrak{g}_\Lambda$ by putting

$$x \,\Box\, y := M_{x\circ y} + [M_x, M_y]. \tag{2.14}$$

Assuming from now on that Λ is an irreducible symmetric cone, we have the *Peirce decomposition*

$$X = \sum_{1 \le i \le j \le r}^{\oplus} X_{ij} \tag{2.15}$$

induced by a maximal orthogonal system $e_1, \ldots, e_r$ of idempotents in X [BK]. Here

$$X_{ij} = X_{ji} = \left\{x \in X : e_k \circ x = \frac{\delta_{ik} + \delta_{jk}}{2} x \text{ for } 1 \le k \le r\right\}. \tag{2.16}$$

2.17 Proposition. *The Lie algebra* $\mathfrak{g}_\Lambda$ *has a decomposition*

$$\mathfrak{g}_\Lambda = \sum_{1\le i,j\le r}^{\oplus} \mathfrak{g}_{ij} \oplus m\,, \tag{2.18}$$

where

$$\mathfrak{g}_{ij} := \{x \square e_i : x \in X_{ij}\} \tag{2.19}$$

and

$$m = \{M \in \mathfrak{g}_\Lambda : M e_k = 0 \ \forall 1 \le k \le r\}\,. \tag{2.20}$$

Proof. By (2.14), we have $\mathfrak{g}_{ij} \subset \mathfrak{g}_\Lambda$. According to the next Proposition 2.23, the spaces $\mathfrak{g}_{ij}$ form a direct sum. Now let $M \in \mathfrak{g}_\Lambda$ be arbitrary. Applying the Peirce composition rules [L2] to (the derivation and multiplication part of) M, it follows that $Me_j = \sum_i x_{ij}$ with $x_{ij} \in X_{ij}$. Then

$$A := \sum_i x_{ii} \square e_i + 2\sum_{i\ne k} x_{ik} \square e_i \in \mathfrak{g}_\Lambda$$

satisfies

$$Ae_j = x_{jj} + 2\sum_{j\ne k} x_{jk}/2 = Me_j$$

for all j, showing that $A - M \in m$. Q.E.D.

The subspace

$$\mathfrak{a} := \sum_i \mathfrak{g}_{ii} \tag{2.21}$$

of p_Λ is maximal abelian, with basis $M_{e_1}, \ldots, M_{e_r}$. Let $M^{\#}_{e_i}$ denote the dual basis of $\mathfrak{a}^{\#}$, i.e.

$$M^{\#}_{e_i} M_{e_j} = \delta_{ij} \quad \text{(Kronecker symbol)}\,. \tag{2.22}$$

2.23 Proposition. *The splitting* (2.18) *is the real root decomposition of* $\mathfrak{g}_\Lambda$ *relative to* $\mathfrak{a}$. *More precisely,* $\mathfrak{g}_{ij}$ *is the root space for*

$$\frac{1}{2}(M^{\#}_{e_j} - M^{\#}_{e_i}) \qquad (i \ne j)\,. \tag{2.24}$$

Proof. Putting $\{xyz\} := (x \square y)z$, the Jordan triple identity [L1; (2.8)] gives

$$[x \square y, u \square v] = x \square \{yuv\} - \{uvx\} \square y \tag{2.25}$$

for all $x, y, u, v \in X$. For every $x \in X_{ij}$ we have

$$[e_i \square e, x \square e_i] = \{e_i e x\} \square e_i - x \square \{e_i e_i e\} = -\frac{1}{2} x \square e_i$$

and

$$[e_j \square e, x \square e_i] = \{e_j e x\} \square e_i - x \square \{e_i e_j e\} = \frac{1}{2} x \square e_i\,.$$

Since $[e_k \square e, x \square e_i] = 0$ for $k \notin \{i, j\}$, the assertion follows. Q.E.D.

Choose an ordering of the roots such that

$$\mathfrak{n}_\Lambda := \sum_{i<j}^{\oplus} \mathfrak{g}_{ij} \tag{2.26}$$

is the nilpotent part of the Iwasawa decomposition $\mathfrak{g}_\Lambda = \mathfrak{n}_\Lambda \oplus \mathfrak{a} \oplus \mathfrak{k}_\Lambda$. Then the Weyl chamber is given by

$$\mathfrak{a}_+ = \left\{ \sum_{j=1}^{r} t_j M_{e_j} : t_1 < \ldots < t_r \right\}.$$

Put $G_\Lambda := GL(\Lambda)^0$, $K_\Lambda := O(\Lambda)^0$ (identity components) and consider the Iwasawa decompositon

$$G_\Lambda = N_\Lambda A K_\Lambda \tag{2.27}$$

associated with (2.26), where $A := \exp(\mathfrak{a})$. There exists a unique K_Λ-invariant inner product $(\mid)$ on the tangent space $T_e(\Lambda) = \mathfrak{p}_\Lambda$ such that the vectors $M_{e_1}, \ldots, M_{e_r}$ are orthonormal. Let $t_1, \ldots, t_r$ denote the corresponding coordinates on $\mathfrak{a}$, and endow Λ with the induced G_Λ-invariant Riemannian metric. One can show that $\Lambda = G_\Lambda/K_\Lambda$ is a Riemannian symmetric space (of type $A_{r-1} \times A_1$), with symmetry at $e \in \Lambda$ given by $x \mapsto x^{-1}$ (Jordan algebra inverse). With this structure, Λ is called a *symmetric cone*. One can show [L2] that

$$a := \dim X_{ij} \qquad (i < j) \tag{2.28}$$

is independent of $i < j$ and of the choice of $e_1, \ldots, e_r$. It is called the *characteristic multiplicity* of Λ. By Proposition 2.23 we have

$$\varrho = \frac{a}{2} \sum_{i<j} \frac{1}{2} (M_{e_j}^{\#} - M_{e_i}^{\#}) = \frac{a}{4} \sum_j (2j - r - 1) M_{e_j}^{\#}$$

and the Weyl group W_Λ consists of all permutations of $t_1, \ldots, t_r$. Hence (2.6) implies that the N_Λ-radial parts of $D \in \mathrm{Diff}(\Lambda)^{G_\Lambda}$ are given by

$$\tilde{D} = q\left(\frac{\partial}{\partial t_1} + \frac{a}{4}(r-1), \frac{\partial}{\partial t_2} + \frac{a}{4}(r-3), \ldots, \frac{\partial}{\partial t_r} + \frac{a}{4}(1-r) \right), \tag{2.29}$$

where q is any *symmetric* polynomial.

The explicit description of the N_Λ-invariant eigenfunctions Δ^α, $\alpha \in \mathfrak{a}^\# \otimes \mathbb{C}$, defined in (2.7), is based on the existence of the *Jordan algebra determinant*

$$\Delta : X \to \mathbb{R} \tag{2.30}$$

which is a polynomial of degree r uniquely determined by "Cramer's rule"

$$x^{-1} = \frac{\mathrm{grad}_x \Delta}{\Delta(x)} \qquad (x \in X \text{ invertible})$$

and the normalization $\Delta(e) = 1$. Using the decomposition (2.15) consider the subalgebra

$$X_l := \sum_{1 \le i \le j \le l} X_{ij} \qquad (1 \le l \le r) \tag{2.31}$$

of X. This algebra has rank l and unit element $e_1 + \ldots + e_l$. Define polynomials Δ_l on X by putting

$$\Delta_l(x) := \Delta_{X_l}(P_l x),$$

where $P_l : X \to X_l$ is the orthogonal projection and Δ_{X_l} is the determinant of X_l. Given any complex vector $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$ put

$$\Delta_\alpha(x) := \Delta_1(x)^{\alpha_1 - \alpha_2} \cdot \Delta_2(x)^{\alpha_2 - \alpha_3} \ldots \Delta_r(x)^{\alpha_r} \tag{2.32}$$

for all $x \in \Lambda$. (Note that $\Delta_1, \ldots, \Delta_r$ are positive on Λ.)

2.33 Proposition. *For any $\alpha \in \mathbb{C}^r$, the function* (2.32) *is an N_Λ-invariant joint eigenfunction for all $D \in \mathrm{Diff}(\Lambda)^{G_\Lambda}$. More precisely, we have*

$$D\Delta^\alpha = q\left(\alpha_1 + \frac{a}{4}(r-1), \alpha_2 + \frac{a}{4}(r-3), \ldots, \alpha_r + \frac{a}{4}(1-r)\right) \cdot \Delta^\alpha, \tag{2.34}$$

where q is the symmetric polynomial associated with D via (2.29), *and a is the characteristic multiplity* (2.28).

Proof. It was shown in [U5, KS2] that the polynomials $\Delta_1, \ldots, \Delta_r$ and hence the functions Δ^α are invariant under the group N_Λ. Since $e = e_1 + \ldots + e_r$, we have

$$\exp\left(\sum_j t_j M_{e_j}\right) \cdot e = \sum_j \exp(t_j M_{e_j}) \cdot e_j = \sum_j e^{t_j} e_j,$$

and obtain

$$\begin{aligned}\Delta^\alpha\left(\exp\left(\sum_j t_j M_{e_j}\right) \cdot e\right) &= \Delta^\alpha\left(\sum_j e^{t_j} e_j\right) \\ &= (e^{t_1})^{\alpha_1 - \alpha_2} (e^{t_1 + t_2})^{\alpha_2 - \alpha_3} \ldots (e^{t_1 + \ldots + t_r})^{\alpha_r} \\ &= e^{t_1 \alpha_1 + t_2 \alpha_2 + \ldots + t_r \alpha_r}.\end{aligned}$$

Hence Δ^α coincides with the function (2.7) corresponding to the linear form

$$\sum_j \alpha_j M_{e_j}^\# . \tag{2.35}$$

Now the assertion follows from (2.9). Q.E.D.

We will now consider the *hermitian* symmetric spaces of non-compact type. These spaces have both a bounded (Harish-Chandra) realization [H2] and an unbounded realization as a "Siegel domain" over a symmetric cone Λ [L1, KU]. It is more convenient to work in the unbounded realization, and we first consider the special case of "tube domains" (Siegel domains of the first kind).

The complexification $Z = X \oplus iX$ of a real Jordan algebra X has a *Jordan triple product*

$$\{z_1 z^* z_2\} := (z_1 \circ z^*) \circ z_2 - (z_1 \circ z_2) \circ z^* + (z_2 \circ z^*) \circ z_1 \tag{2.36}$$

obtained by combining the (complexified) Jordan algebra product $z_1 \circ z_2$ and the involution $(x + i\xi)^* := x - i\xi$ of Z. Let Λ be the symmetric cone of X. The half-space

$$\Pi := \Lambda \oplus iX = \{z \in Z : z + z^* \in \Lambda\} \tag{2.37}$$

is called a *tube domain* with real base Λ. The group Aut(Π) of all biholomorphic automorphisms of Π is transitive since it contains $GL(\Lambda)$ and the translations

$$z \mapsto z + i\xi \qquad (\xi \in X). \tag{2.38}$$

Aut(Π) is a semi-simple Lie group, whose Lie algebra $\mathfrak{g} = \operatorname{aut}(\Pi)$ consists of all completely integrable holomorphic vector fields $h(z)\frac{\partial}{\partial z}$ on Π, with commutator bracket

$$\left[h(z)\frac{\partial}{\partial z}, k(z)\frac{\partial}{\partial z}\right] = (\partial h(z)\cdot k(z) - \partial k(z)\cdot h(z))\frac{\partial}{\partial z}. \tag{2.39}$$

Here ∂ denotes the complex derivative. Identifying $M \in \mathfrak{gl}(Z)$ with the vector field $Mz\frac{\partial}{\partial z}$, we see that $\mathfrak{g}_\Lambda$, endowed with the commutator (2.39), becomes a subalgebra of $\mathfrak{g}$. The vector field $z\frac{\partial}{\partial z} \in \mathfrak{g}_\Lambda$ induces a gradation $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1}$, with $\mathfrak{g}^\lambda = \left\{A \in \mathfrak{g}: \left[z\frac{\partial}{\partial z}, A\right] = \lambda A\right\}$. We have

$$\mathfrak{g}^1 = \left\{i\xi\frac{\partial}{\partial z}: \xi \in X\right\}, \qquad \mathfrak{g}^0 = \mathfrak{g}_\Lambda, \qquad \mathfrak{g}^{-1} = \left\{i\{z\xi z\}\frac{\partial}{\partial z}: \xi \in X\right\}, \tag{2.40}$$

where $\{\,\}$ denotes the Jordan triple product (2.36). One can show [KU] that $\mathfrak{g}^{-1}$ is the conjugate of $\mathfrak{g}^1$ under the automorphism $z \mapsto z^{-1}$. By [KU, L1], the Lie algebra $\mathfrak{g}$ has a Cartan decompositon $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \mathfrak{k}_\Lambda \oplus \left\{i(\xi - \{z\xi z\})\frac{\partial}{\partial z}: \xi \in X\right\}$$

is the Lie algebra of $\operatorname{Aut}_e(\Pi) := \{g \in \operatorname{Aut}(\Pi): g\cdot e = e\}$, and

$$\mathfrak{p} = \mathfrak{p}_\Lambda \oplus \left\{i(\xi + \{z\xi z\})\frac{\partial}{\partial z}: \xi \in X\right\}.$$

The subspace $\mathfrak{a} \subset \mathfrak{p}_\Lambda$ defined in (2.21) is still maximal abelian in $\mathfrak{p}$. Consider the Peirce decompositon (2.15).

2.41 Proposition. *The Lie algebra* $\mathfrak{g}$ *has a decomposition*

$$\mathfrak{g} = \sum_{i\le j}\mathfrak{g}^1_{ij} \oplus \sum_{i,j}\mathfrak{g}^0_{ij} \oplus \sum_{i\le j}\mathfrak{g}^{-1}_{ij} \oplus \mathfrak{m}, \tag{2.42}$$

where

$$\begin{aligned}\mathfrak{g}^1_{ij} &= \left\{i\xi\frac{\partial}{\partial z}: \xi \in X_{ij}\right\},\\ \mathfrak{g}^0_{ij} &= \left\{x \,\square\, e_i\frac{\partial}{\partial z}: \xi \in X_{ij}\right\},\\ \mathfrak{g}^{-1}_{ij} &= \left\{i\{z\xi z\}\frac{\partial}{\partial z}: \xi \in X_{ij}\right\}.\end{aligned} \tag{2.43}$$

Proof. By (2.40) and Proposition 2.17, the subspaces (2.43) belong to $\mathfrak{g}$. In view of the Peirce decomposition (2.15), it is clear that $\mathfrak{g}^{\pm1}$ are spanned by the respective subspaces. For $\mathfrak{g}^0$, apply (2.40) and Proposition 2.17. Q.E.D.

2.44 Propositon. *The splitting* (2.42) *is the real root decomposition of* $\mathfrak{g}$ *relative to* $\mathfrak{a}$. *More precisely, we have*

(i) $\mathfrak{g}_{ij}^{\pm 1}$ *is the root space for* $\pm\frac{1}{2}(M_{e_i}^{\#} + M_{e_j}^{\#})$ $(i \leq j)$,

(ii) $\mathfrak{g}_{ij}^{0}$ *is the root space for* $\frac{1}{2}(M_{e_j}^{\#} - M_{e_i}^{\#})$ $(i \neq j)$.

Proof. Let $i \leq j$ and $\xi \in X_{ij}$. Then (2.39) implies

$$\left[M_{e_k}, \xi \frac{\partial}{\partial z}\right] = \{\xi e e_k\} \frac{\partial}{\partial z} = \frac{\delta_{ik} + \delta_{jk}}{2} \xi \frac{\partial}{\partial z}$$

and, similarly, by the Jordan triple identity (2.25),

$$\begin{aligned}\left[\{z\xi z\} \frac{\partial}{\partial z}, M_{e_k}\right] &= (\{\{zee_k\}\xi z\} + \{z\xi\{zee_k\}\} - \{\{z\xi z\}ee_k\}) \frac{\partial}{\partial z} \\ &= \{z\{e_k e\xi\}z\} \frac{\partial}{\partial z} = \frac{\delta_{ik} + \delta_{jk}}{2} \{z\xi z\} \frac{\partial}{\partial z}.\end{aligned}$$

This proves (i). For (ii), apply Proposition 2.23. Q.E.D.

Choose an ordering of the roots such that

$$\mathfrak{n} = \sum_{i \leq j} \mathfrak{g}_{ij}^{1} \oplus \sum_{i<j} \mathfrak{g}_{ij}^{0} = \mathfrak{g}^{1} \oplus \mathfrak{n}_{\Lambda} \tag{2.45}$$

is the nilpotent part of the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Then the Weyl chamber is given by

$$\mathfrak{a}_{+} = \left\{ \sum_{j=1}^{r} t_j M_{e_j} : 0 < t_1 < \ldots < t_r \right\}. \tag{2.46}$$

Put $G := \mathrm{Aut}(\Pi)^0$, $K := \mathrm{Aut}_e(\Pi)^0$ (identity components) and consider the Iwasawa decomposition

$$G = NAK \tag{2.47}$$

associated with (2.45), where $A := \exp(\mathfrak{a})$. There exists a unique K-invariant inner product $(\,|\,)$ on the tangent space $T_e(\Pi) \approx \mathfrak{p}$ such that the vectors $M_{e_1}, \ldots, M_{e_r}$ are orthonormal. Let $t_1, \ldots, t_r$ denote the corresponding coordinates on $\mathfrak{a}$, and endow Π with the induced G-invariant hermitian metric. One can show that $\Pi = G/K$ is a hermitian symmetric space (of type C or D), with symmetry at $e \in \Lambda \subset \Pi$ given by $z \mapsto z^{-1}$ (Jordan algebra inverse). With this structure, Π is called a *symmetric tube domain.* By Proposition 2.44, we have

$$\begin{aligned}\varrho &= \frac{a}{2}\left(\sum_{i<j} \frac{1}{2}(M_{e_j}^{\#} - M_{e_i}^{\#}) + \frac{1}{2}(M_{e_j}^{\#} + M_{e_i}^{\#})\right) + \frac{1}{2}\sum_{j} M_{e_j}^{\#} \\ &= \sum_{j} \left(\frac{1}{2} + \frac{a}{2}(j-1)\right) M_{e_j}^{\#},\end{aligned} \tag{2.48}$$

and the Weyl group W consists of all *signed permutations*

$$(t_1, \ldots, t_r) \mapsto (\varepsilon_1 t_{\sigma(1)}, \ldots, \varepsilon_r t_{\sigma(r)})$$

with $\varepsilon_i \in \{\pm 1\}$ and $\sigma \in \mathscr{S}_r$ (symmetric group). Hence (2.29) implies that the N-radial parts of $D \in \mathrm{Diff}(\Pi)^G$ are given by

$$\tilde{D} = q\left(\frac{\partial}{\partial t_1} - \frac{1}{2}, \frac{\partial}{\partial t_2} - \frac{1}{2} - \frac{a}{2}, \dots, \frac{\partial}{\partial t_r} - \frac{1}{2} - \frac{a}{2}(r-1)\right), \tag{2.49}$$

where q is any *even symmetric* polynomial (i.e., a symmetric polynomial in $t_1^2, \dots, t_r^2$).

2.50 Example. Let Δ_Π be the Laplace-Beltrami operator on the Riemannian manifold Π. By [H2; Proposition II. 3.8], its N-radial part is given by

$$\tilde{\Delta}_\Pi = \sum_j \left(\frac{\partial}{\partial t_j} - \varrho_j\right)^2 - \sum_j \varrho_j^2 = \sum_j \left(\frac{\partial^2}{\partial t_j^2} - 2\varrho_j \frac{\partial}{\partial t_j}\right),$$

where

$$\varrho_j = \varrho(M_{e_j}) = \frac{1}{2} + \frac{a}{2}(j-1).$$

It follows that Δ_Π corresponds to the even symmetric polynomial

$$q = \sum_j t_j^2 - \sum_j \varrho_j^2.$$

Thus $\Delta_\Pi = Q_1 - \sum_j \varrho_j^2$, where $Q_1 \in \mathrm{Diff}(\Pi)^G$ corresponds to the polynomial $q_1 := \sum_j t_j^2$. For $r = 1$ we have $\varrho_1 = \frac{1}{2}$, and hence $\Delta_\Pi = Q_1 - \frac{1}{4}$.

By (2.45), N is generated by N_A and the translations (2.38). It follows that for every $\alpha \in \mathbb{C}^r$, the functions

$$\Delta^\alpha(z) := \Delta^\alpha(\mathrm{Re}\, z) \tag{2.51}$$

on Π are N-invariant and correspond to the linear form (2.35) in $\mathfrak{a}^\# \otimes \mathbb{C}$. Therefore (2.9) implies

2.52 Proposition. *For every $\alpha \in \mathbb{C}^r$, the function* (2.51) *is a joint eigenfunction for all $D \in \mathrm{Diff}(\Pi)^G$. More precisely, we have*

$$D\Delta^\alpha = q\left(\alpha_1 - \frac{1}{2}, \alpha_2 - \frac{1}{2} - \frac{a}{2}, \dots, \alpha_r - \frac{1}{2} - \frac{a}{2}(r-1)\right)\Delta^\alpha, \tag{2.53}$$

where q is the even symmetric polynomial associated with D via (2.49).

2.54 Example. Consider the forward light cone

$$\Lambda_{1,n} := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0 > \sqrt{x_1^2 + \dots + x_n^2}\}$$

of dimension $n + 1 \geq 3$, which is a symmetric cone of rank 2. We have $a = n - 1$ in this case. The tube domain Π is best realized in $\mathbb{C}^{n+1}$ by mapping X onto $\mathbb{R} \times i\mathbb{R}^n$. Consider the orthogonal idempotents

$$e_1 := \left(\frac{1}{2}, \frac{i}{2}, 0, \dots, 0\right), \quad e_2 = \bar{e}_1 = \left(\frac{1}{2}, -\frac{i}{2}, 0, \dots, 0\right).$$

Then the unit element is $e = e_1 + e_2 = (1, 0, \dots, 0)$. The normalized inner product on $\mathbb{C}^{n+1}$ is $(z \mid w) = 2z \cdot \bar{w} = 2\sum_j z_j \bar{w}_j$. With these conventions, we have for the complexified minors

$$\Delta_1(z) = (z \mid e_1) = 2z \cdot e_2 = z_0 - iz_1, \qquad \Delta_2(z) = z \cdot z = \sum_j z_j^2 .$$

For vectors $x = (x_0, ix_1, \dots, ix_n) \in X$ we obtain

$$\Delta_1(x) = x_0 - i(ix_1) = x_0 + x_1 \tag{2.55}$$

and

$$\Delta_2(x) = x_0^2 + (ix_1)^2 + \dots + (ix_n)^2 = x_0^2 - x_1^2 - \dots - x_n^2 . \tag{2.56}$$

Now let $\alpha \in \mathbb{C}^2$ and write $\alpha = (\alpha_1, \alpha_2)$. By (2.55) and (2.56),

$$\Delta^\alpha(x) = \Delta_1(x)^{\alpha_1 - \alpha_2} \Delta_2(x)^{\alpha_2} = (x_0 + x_1)^{\alpha_1 - \alpha_2}(x_0^2 - x_1^2 - \dots - x_n^2)^{\alpha_2} .$$

By (2.53), every $D \in \mathrm{Diff}(\Pi)^G$ satisfies

$$D\Delta^\alpha = q\left(\alpha_1 - \frac{1}{2}, \alpha_2 - \frac{n}{2}\right)\Delta^\alpha ,$$

where q is the associated even symmetric polynomial. In particular, for $q(t) = t_1^2 + t_2^2 - \dfrac{n^2+1}{4}$ we obtain $D\Delta^\alpha = (\alpha_1(\alpha_1 - 1) + \alpha_2(\alpha_2 - n))\Delta^\alpha$, whereas $q(t) = \left(t_1^2 - \dfrac{n^2}{4}\right)\left(t_2^2 - \dfrac{n^2}{4}\right)$ yields $D\Delta^\alpha = \alpha_2(\alpha_2 - n)\left(\alpha_1 + \dfrac{n-1}{2}\right)\left(\alpha_1 - \dfrac{n+1}{2}\right)\Delta^\alpha$.

We finally discuss the general case of Siegel domains of the second kind. Consider a complex vector space $Z = U \oplus V$ such that $U = X \oplus iX$ is the complexification of a formally-real Jordan algebra X with unit element e and symmetric cone Λ, and let $\Phi : V \times V \to U$ be a sesqui-linear mapping (linear in the firsts variable, antilinear in the second) such that $\Phi(v_1, v_2)^* = \Phi(v_2, v_1)$ and $\Phi(v, v)$ belongs to the closure of Λ for all $v, v_1, v_2 \in V$. Assume also that $\Phi(v, v) \neq 0$ whenever $v \neq 0$. Then

$$\Pi = \Pi(\Lambda, \Phi) := \{(u, v) \in U \oplus V : u + u^* - \Phi(v, v) \in \Lambda\} \tag{2.57}$$

is called the *Siegel domain* associated with Λ and Φ. For the special case $V = \{0\}$, we have $Z = U$ and Π becomes the tube domain with real base Λ. It can be shown [L1] that any symmetric domain equivalent with a bounded domain of $\mathbb{C}^n$ admits such a realization Π as a Siegel domain (of tube type if $V = \{0\}$, of type B or BC if $V \neq \{0\}$). Then Z carries a *Jordan triple product* $\{z_1 z^* z_2\}$ generalizing (2.36) which is anti-linear in z and symmetric bilinear in (z_1, z_2) such that $z_1 \circ z_2 := \{z_1 e^* z_2\}$ defines a Jordan algebra on Z (which is non-unital if $V \neq \{0\}$). In terms of this product, the geodesic symmetry around the base point $e \in \Lambda \subset \Pi$ has the form [L1; Proposition 10.12] $S(u, v) = (u^{-1}, -u^{-1} \circ v)$ for all $(u, v) \in \Pi \subset U \oplus V$. The symmetries at the other points of Π can be computed from this via a transitive group of affine transformations of Π, generated by $GL(\Lambda)$ and the "quasi-translations"

$$(u, v) \mapsto \left(u + ia + \Phi(b, v) + \frac{\Phi(b, b)}{2}, v + b\right), \tag{2.58}$$

where $a \in iX$ and $b \in V$.

The holomorphic automorphism group $\mathrm{Aut}(\Pi)$ of Π is a semi-simple Lie group, whose Lie algebra $\mathfrak{g} = \mathrm{aut}(\Pi)$ endowed with the commutator bracket (2.39), has a canonical gradation $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^{1/2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1/2} \oplus \mathfrak{g}^{-1}$ induced by the vector field

$$I := \{ee^*z\}\,\frac{\partial}{\partial z} = u\,\frac{\partial}{\partial u} + \frac{v}{2}\,\frac{\partial}{\partial v} \in \mathfrak{g}$$

via $\mathfrak{g}^\lambda := \{A \in \mathfrak{g} : [I, A] = \lambda A\}$. We have

$$\mathfrak{g}^1 = \left\{i\xi\,\frac{\partial}{\partial z} : \xi \in X\right\}, \qquad \mathfrak{g}^{1/2} = \left\{(\beta + 2\{e\beta^*z\})\,\frac{\partial}{\partial z} : \beta \in V\right\}, \tag{2.59}$$

$$\mathfrak{g}^0 = \mathit{l} \oplus \left\{\{ze^*x\}\,\frac{\partial}{\partial z} : x \in X\right\} \tag{2.60}$$

and

$$\begin{aligned}\mathfrak{g}^{-1/2} &= \left\{(2\{\beta e^*z\} + \{z\beta^*z\})\,\frac{\partial}{\partial z} : \beta \in V\right\},\\ \mathfrak{g}^{-1} &= \left\{i\{z\xi z\}\,\frac{\partial}{\partial z} : \xi \in X\right\}.\end{aligned} \tag{2.61}$$

Here $\mathit{l} \approx \mathscr{k}_\Lambda$ consists of all Jordan triple derivations M of Z vanishing at e.

In the non-tube-type setting, we have the complex Peirce decompositions

$$U = \sum_{1\le i\le j\le r} X_{ij}^{\mathbb{C}}, \qquad V = \sum_{1\le j\le r} V_j, \tag{2.62}$$

where

$$V_j := \left\{v \in V : \{e_k e_k^* v\} = \frac{\delta_{jk}}{2}\,v \;\forall 1 \le k \le r\right\}.$$

One can show [L1] that

$$b = \dim V_j \tag{2.63}$$

is independent of j. The numbers a (defined in (2.28)) and b are called the *characteristic multiplicities* of Π.

2.64 Proposition. *The Lie algebra* $\mathfrak{g}$ *has a decomposition*

$$\mathfrak{g} = \sum_{i\le j} \mathfrak{g}_{ij}^1 \oplus \sum_i \mathfrak{g}_i^{1/2} \oplus \sum_{i,j} \mathfrak{g}_{ij}^0 \oplus \sum_i \mathfrak{g}_i^{-1/2} \oplus \sum_{i\le j} \mathfrak{g}_{ij}^{-1} \oplus \mathscr{m}, \tag{2.65}$$

where

(i) $$\mathfrak{g}_{ij}^1 = \left\{i\xi\,\frac{\partial}{\partial z} : \xi \in X_{ij}\right\}, \qquad \mathfrak{g}_j^{1/2} = \left\{(\beta + 2\{z\beta^*e\})\,\frac{\partial}{\partial z} : \beta \in V_j\right\},$$

(ii) $$\mathfrak{g}_{ij}^0 = \left\{x \,\Box\, e_i\,\frac{\partial}{\partial z} : x \in X_{ij}\right\},$$

(iii) $$\mathfrak{g}_j^{-1/2} = \left\{(\{z\beta^*z\} + 2\{ze^*\beta\})\,\frac{\partial}{\partial z} : \beta \in V_j\right\},$$

$$\mathfrak{g}_{ij}^{-1} = \left\{i\{z\xi z\}\,\frac{\partial}{\partial z} : \xi \in X_{ij}\right\}.$$

Proof. By (2.59), (2.60), (2.61) and Proposition 2.17, the subspaces (i), (ii), (iii) belong to $\mathfrak{g}$. In view of the Peirce decomposition (2.15) and (2.62), it is clear that $\mathfrak{g}^{\pm 1}$ and $\mathfrak{g}^{\pm 1/2}$ are spanned by the respective subspaces. For $\mathfrak{g}^0$, apply (2.60) and Proposition 2.17. Q.E.D.

2.66 Proposition. *The splitting* (2.65) *is the root decomposition of* $\mathfrak{g}$ *relative to* $\mathfrak{a}$. *More precisely, we have*

(i) $\mathfrak{g}_{ij}^{\pm 1}$ *is the root space for* $\pm\frac{1}{2}(M_{e_i}^{\#} + M_{e_j}^{\#})$ $(i \le j)$,

(ii) $\mathfrak{g}_j^{\pm 1/2}$ *is the root space for* $\pm\frac{1}{2} M_{e_j}^{\#}$ $(1 \le j \le r)$,

and

(iii) $\mathfrak{g}_{ij}^0$ *is the root space for* $\frac{1}{2}(M_{e_j}^{\#} - M_{e_i}^{\#})$ $(i \ne j)$.

Proof. Let $1 \le j \le r$ and $b \in V_j$. Then

$$\begin{aligned}&\left[\{e_k e^* z\}\frac{\partial}{\partial z}, (b + 2\{eb^* z\})\frac{\partial}{\partial z}\right]\\ &= (\{be^* e_k\} + 2(\{\{zb^* e\} e^* e_k\} - \{\{ze^* e_k\} b^* e\}))\frac{\partial}{\partial z}\\ &= (\{be^* e_k\} + 2\{z\{be^* e_k\}^* e\})\frac{\partial}{\partial z} = \frac{\delta_{jk}}{2}(b + 2\{zb^* e\})\frac{\partial}{\partial z}\end{aligned}$$

as follows from the Jordan triple identity (2.25) and the fact that $\{eb^* e_k\} = 0$. Similarly,

$$\begin{aligned}&\left[(\{zb^* z\} + 2\{be^* z\})\frac{\partial}{\partial z}, \{e_k e^* z\}\frac{\partial}{\partial z}\right]\\ &= (2\{\{ze^* e_k\} b^* z\} + 2\{\{ze^* e_k\} e^* b\} - \{\{zb^* z\} e^* e_k\} - 2\{\{ze^* b\} e^* e_k\})\frac{\partial}{\partial z}\\ &= (\{z\{be^* e_k\}^* z\} + 2\{ze^*\{be^* e_k\}\})\frac{\partial}{\partial z} = \frac{\delta_{jk}}{2}(\{zb^* z\} + 2\{ze^* b\})\frac{\partial}{\partial z}.\end{aligned}$$

This proves (ii). For (i) and (iii), apply Proposition 2.41 and 2.17, respectively. Q.E.D.

Choose an ordering of the roots such that

$$\mathfrak{n} = \sum_{i \le j} \mathfrak{g}_{ij}^1 \oplus \sum_j \mathfrak{g}_j^{1/2} \oplus \sum_{i<j} \mathfrak{g}_{ij}^0 = \mathfrak{g}^1 \oplus \mathfrak{g}^{1/2} \oplus \mathfrak{n}_\Lambda \tag{2.67}$$

is the nilpotent part of the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Then the Weyl chamber is still given by (2.47). Consider the associated Iwasawa decomposition

$$G = NAK \tag{2.68}$$

of $G := \mathrm{Aut}(\Pi)^0$. By Proposition 2.66 we have

$$\begin{aligned}\varrho &= \frac{a}{2}\left(\sum_{i<j}\frac{1}{2}(M_{e_j}^{\#} - M_{e_i}^{\#}) + \frac{1}{2}(M_{e_j}^{\#} + M_{e_i}^{\#})\right) + \frac{2b}{2}\sum_j \frac{1}{2} M_{e_j}^{\#} + \frac{1}{2}\sum_j M_{e_j}^{\#}\\ &= \sum_j \left(\frac{1}{2} + \frac{a}{2}(j-1) + \frac{b}{2}\right) M_{e_j}^{\#}.\end{aligned} \tag{2.69}$$

and the Weyl group W consists of all signed permutations of $t_1, \ldots, t_r$. Hence (2.6) implies that the N-radial parts of $D \in \mathrm{Diff}(\Pi)^G$ are given by

$$\tilde{D} = q\left(\frac{\partial}{\partial t_1} - \frac{b+1}{2}, \frac{\partial}{\partial t_2} - \frac{b+1}{2} - \frac{a}{2}, \ldots, \frac{\partial}{\partial t_r} - \frac{b+1}{2} - \frac{a}{2}(r-1)\right), \quad (2.70)$$

where q is an even symmetric polynomial. For $b = 0$, this formula specializes to (2.49). By (2.67), N is generated by N_Λ and the quasi-translations (2.58). It follows that the functions

$$\Delta^\alpha(u, v) := \Delta^\alpha\left(\mathrm{Re}(u) - \frac{\Phi(v, v)}{2}\right) \quad (2.71)$$

on Π, for $\alpha \in \mathbb{C}^r$, are N-invariant and correspond to the linear form (2.35) in $\mathfrak{a}^\# \otimes \mathbb{C}$. Therefore (2.9) implies

2.72 Proposition. *For every $\alpha \in \mathbb{C}^r$, the function* (2.71) *is a joint eigenfunction for all $D \in \mathrm{Diff}(\Pi)^G$. More precisely,*

$$D\Delta^\alpha = q\left(\alpha_1 - \frac{b+1}{2}, \alpha_2 - \frac{b+1}{2} - \frac{a}{2}, \ldots, \alpha_r - \frac{b+1}{2} - \frac{a}{2}(r-1)\right)\Delta^\alpha,$$

where q is the even symmetric polynomial associated with D via (2.70).

2.73 Lemma. *There exists a unique homomorphism*

$$N_\Lambda A \ni P \mapsto \tilde{P} \in GL(V)$$

satisfying $\Phi(\tilde{P}v_1, \tilde{P}v_2) = P\Phi(v_1, v_2)$ for all $v_1, v_2 \in V$, and

$$\mathrm{Det}\,\tilde{P} = (\mathrm{Det}\,P)^{n_2/2n_1} = \Delta(Pe)^{n_2/r}. \quad (2.74)$$

Proof. Since $N_\Lambda A$ is a simply-connected Lie group [H2; Theorem 5.1], one can exponentiate the corresponding facts relative to the Lie subalgebra $\mathfrak{n}_\Lambda \oplus \mathfrak{a}$ of $\mathfrak{g}_\Lambda$. By [KU, p. 195], the Lie algebra $\mathfrak{g}$ satisfies

$$\mathfrak{g}^0 = \left\{Mu\frac{\partial}{\partial u} + \tilde{M}v\frac{\partial}{\partial v} : M \in \mathfrak{g}_\Lambda, \tilde{M} \in \mathfrak{gl}(V), M\Phi(v, v) \right.$$
$$\left. = \Phi(\tilde{M}v, v) + \Phi(v, \tilde{M}v)\ \forall v \in V\right\}.$$

In order to show that $\tilde{M}$ is uniquely determined by M, assume $\tilde{M}v\dfrac{\partial}{\partial v} \in \mathfrak{g}^0$ for some $\tilde{M} \in \mathfrak{gl}(V)$. According to [KU; Corollary 5.3 (iii)], we have $\{(\tilde{M}v_1)\alpha^* v_2\} + \{v_1\alpha^*(\tilde{M}v_2)\} = 0$ for all $\alpha \in X$ and $v_1, v_2 \in V$. Therefore $\tilde{M} = 0$, and $M \mapsto \tilde{M}$ is a well-defined homomorphism from $\mathfrak{g}_\Lambda$ to $\mathfrak{gl}(V)$, which proves the first part of the lemma. According to [KU; Theorem 2.5], $\mathfrak{g}^0$ contains all commutators

$$\frac{1}{2}\left[\{z\alpha^* z\}\frac{\partial}{\partial z}, \beta\frac{\partial}{\partial z}\right] = \{\beta\alpha^* z\}\frac{\partial}{\partial z} = \{\beta\alpha^* u\}\frac{\partial}{\partial u} + \{\beta\alpha^* v\}\frac{\partial}{\partial v}$$

for all $\alpha, \beta \in X$. Thus $Mu = \{\beta\alpha^* u\}$ corresponds to $\tilde{M}v = \{\beta\alpha^* v\}$. Now let $M \in \mathfrak{n}_\Lambda \oplus \mathfrak{a}$. In case $M \in \mathfrak{n}_\Lambda$, we have $\tilde{M} \in \mathfrak{n}$ by (2.67) and hence

$\operatorname{trace} \tilde{M} = \operatorname{trace} M = 0$. In case $M \in \mathfrak{a}$, we may assume $M = M_{e_k}$ for some $1 \leq k \leq r$ and obtain

$$Mx_{ij} = \{e_k e^* x_{ij}\} = \{e_k e_k^* x_{ij}\} = \frac{\delta_{ik} + \delta_{jk}}{2} x_{ij}$$

and

$$\tilde{M} v_j = \{e_k e^* v_j\} = \{e_k e_k^* v_j\} = \frac{\delta_{jk}}{2} v_j$$

for all $x_{ij} \in X_{ij}$ and $v_j \in V_j$. Therefore

$$\operatorname{trace} M = \sum_{1 \leq i < j \leq r} a \cdot \frac{\delta_{ik} + \delta_{jk}}{2} + \sum_{j=1}^{r} \delta_{jk} = 1 + \frac{a}{2}(r-1)$$

and

$$\operatorname{trace} \tilde{M} = \sum_{j=1}^{r} b \cdot \frac{\delta_{jk}}{2} = \frac{b}{2}.$$

Since $\frac{n_2}{2n_1} = \frac{b}{2 + a(r-1)}$, we obtain $\operatorname{trace} \tilde{M} = \frac{n_2}{2n_1} \operatorname{trace} M$ for all $M \in \mathscr{n}_\Lambda \oplus \mathfrak{a}$.

Q.E.D.

3. The Berezin Transform and Invariant Differential Operators

In this section we study the Berezin transform for an arbitrary complex symmetric domain and express it in terms of canonically chosen invariant differential operators. We may assume that the domain is irreducible and is given in its realization Π as a Siegel domain (2.57), including the tube domain case ($V = \{0\}$) as a special case. Then Λ is an irreducible symmetric cone (but is not irreducible as a symmetric space) in a euclidean Jordan algebra X, U is the complexification of X and $\Phi: V \times V \to U$ is a sesqui-linear mapping as described in Sect. 2. We put $Z = U \oplus V$, and write $Z_1 := U$, $Z_2 := V$. We put $n_i := \dim_{\mathbb{C}} Z_i$ and obtain the genus p of Π as

$$p = \frac{2n_1 + n_2}{r}, \tag{3.1}$$

where r is the rank of Π. Then $n := n_1 + n_2 = \dim_{\mathbb{C}} Z$. Let $dV(z_2)$ be the Lebesgue measure on Z_2 associated with the inner product $(z_2 \mid w_2) := (\Phi(z_2, w_2) \mid e)$. Given any $t \in \Lambda$, and any function h holomorphic on Z_2 such that

$$\int_{Z_2} |h(z_2)|^2 e^{-2\pi(\Phi(z_2,z_2) \mid t)} dV(z_2) < \infty,$$

one has

$$\int_{Z_2} e^{-2\pi(\Phi(z_2,z_2) - \Phi(w_2,z_2) \mid t)} h(z_2) dV(z_2) = 2^{-n_2} \Delta(t)^{-\frac{n_2}{r}} h(w_2) \tag{3.2}$$

for every $w_2 \in Z_2$. This follows from the reproducing property of the Segal-Bargmann kernel (1.31) in the case when $t = e$, and the general case follows from an application of Lemma 2.73. In particular, we obtain for all $z_2, w_2 \in Z_2$,

$$\int_{Z_2} e^{-2\pi(\Phi(z_2,z_2)-\Phi(z_2,w_2)-\Phi(w_2,z_2)\,|\,t)} dV(z_2) = 2^{-n_2} \Delta(t)^{-n_2/r} e^{2\pi(\Phi(w_2,w_2)\,|\,t)}. \tag{3.3}$$

Now let dx and $dV(z_1)$ denote the Lebesgue measures on X and Z_1, respectively associated with the euclidean structure of X. For all $\lambda > \frac{n_1}{r} - 1$ and $t \in \Lambda$, we have [F1]

$$\int_{\Lambda} e^{-2\pi(x\,|\,t)} \Delta(x)^{\lambda - \frac{n_1}{r}} dx = \Gamma_{\Lambda}(\lambda)\, \Delta(2\pi t)^{-\lambda}, \tag{3.4}$$

where

$$\Gamma_{\Lambda}(\lambda) := (2\pi)^{\frac{n_1 - r}{2}} \prod_{j=1}^{r} \Gamma\left(\lambda - \frac{a}{2}(j-1)\right) \tag{3.5}$$

is the *Γ-function* of the cone Λ [F1]. Assuming $\lambda > p - 1$, consider the measure

$$d\mu_{\lambda}(z_1, z_2) = \pi^{-n} \frac{\Gamma_{\Lambda}(\lambda)}{\Gamma_{\Lambda}\left(\lambda - \frac{n}{r}\right)} \Delta(z_1 + z_1^* - \Phi(z_2, z_2))^{\lambda - p} dV(z_1) dV(z_2) \tag{3.6}$$

on Π and the associated *λ-Bargmann space*

$$H^2_{\lambda}(\Pi) := H^2(\Pi, d\mu_{\lambda}) \tag{3.7}$$

of holomorphic functions on Π, with inner product denoted as $(h \mid k)_{\lambda}$. Let us consider also, on $\Lambda \times Z_2$, the measure

$$dm(t, z_2) := e^{-2\pi(\Phi(z_2,z_2)\,|\,t)} \Delta(t)^{-\frac{n_1}{r}} dt\, dV(z_2) \tag{3.8}$$

and the Hilbert space $L^2_{\text{hol}}(\Lambda \times Z_2)$ consisting of all functions $u(t, z_2)$ which are square-integrable with respect to dm and, for almost all t, are holomorphic with respect to z_2. As a consequence of (3.4) with λ replaced by $\lambda - \frac{n_1}{r}$, and of Plancherel's formula, one can see that the map $u \mapsto u^{\#}$, with

$$u^{\#}(z_1, z_2) = 2^{-\frac{n_2}{2}} (2\pi)^{\frac{r\lambda}{2}} \Gamma_{\Lambda}(\lambda)^{-\frac{1}{2}} \int_{\Lambda} e^{-2\pi(z_1\,|\,t)} u(t, z_2) \Delta(t)^{\frac{\lambda - p}{2}} dt \tag{3.9}$$

is an isometry from $L^2_{\text{hol}}(\Lambda \times Z_2)$ into $H^2_{\lambda}(\Pi)$: it is classical but harder to prove [KS1] that it is onto. Setting, for any $w = (w_1, w_2) \in \Pi$,

$$\begin{aligned} \varepsilon^{\lambda}_{w}(t, z_2) = 2^{\frac{n_2}{2}} \frac{(2\pi)^{\frac{\lambda r}{2}}}{\Gamma_{\Lambda}(\lambda)^{1/2}} \Delta(w_1 + w_1^* - \Phi(w_2, w_2))^{\lambda/2} \\ \Delta(t)^{\frac{\lambda}{2} + \frac{n_2}{2r}} e^{-2\pi(w_1^* - \Phi(z_2, w_2)\,|\,t)}, \end{aligned} \tag{3.10}$$

one can see, as a consequence of (3.3) and (3.9), that $\varepsilon^{\lambda}_{w}$ has norm 1 in $L^2_{\text{hol}}(\Lambda \times Z_2)$: also,

$$\Delta(w_1 + w_1^* - \Phi(w_2, w_2))^{\lambda/2} u^{\#}(w) = (\varepsilon^{\lambda}_{w} \mid u), \tag{3.11}$$

where the inner product is taken in the same Hilbert space; finally, one may write

$$e_w^\lambda(z) := (\varepsilon_w^\lambda)^\#(z) = \frac{E_\lambda(z,w)}{E_\lambda(w,w)^{1/2}}, \tag{3.12}$$

where

$$E_\lambda(z,w) = \Delta(z_1 + w_1^* - \Phi(z_2,w_2))^{-\lambda}. \tag{3.13}$$

From this it follows that $E_\lambda(z,w)$ is the reproducing kernel of $H^2_\lambda(\Pi)$ and that e_w^λ is the normalized coherent state at w. Also, as a consequence of (3.13), one has

$$|(e_z^\lambda \mid e_w^\lambda)_\lambda|^2 = \delta(z,w)^{-2\lambda} \tag{3.14}$$

with

$$\delta(z,w) := \frac{|\Delta(z_1 + w_1^* - \Phi(z_2,w_2))|}{\Delta(z_1 + z_1^* - \Phi(z_2,z_2))^{1/2}\,\Delta(w_1 + w_1^* - \Phi(w_2,w_2))^{1/2}} \tag{3.15}$$

for $z = (z_1, z_2)$, $w = (w_1, w_2)$ in Π. According to (1.8), we put

$$d\mu_0(z) = \pi^{-n}\frac{\Gamma_\Lambda(\lambda)}{\Gamma_\Lambda\left(\lambda - \frac{n}{r}\right)}\,\Delta(z_1 + z_1^* - \Phi(z_2,z_2))^{-p}\,dV(z_1)dV(z_2). \tag{3.16}$$

Note that (3.16) agrees with the measure (3.6), for $\lambda = 0$, up to a constant factor depending on λ. For $\lambda = p$, we obtain the "standard" Bergman space, with reproducing kernel

$$E_\Pi(z,w) = \Delta(z_1 + w_1^* - \Phi(z_2,w_2))^{-p}. \tag{3.17}$$

3.18 Lemma. *The function δ defined in* (3.15) *satisfies the "triangle inequality"*

$$\delta(z,w) \le 2^r \delta(z,y)\delta(y,w)$$

for all $z, w, y \in \Pi$.

Proof. By Aut(Π)-invariance, we may assume $y = e$. According to (3.15), we have to show $|\Delta(z_1+w_1^* - \Phi(z_2,w_2))| \le |\Delta(z_1+e)|\cdot|\Delta(e+w_1^*)|$ for all points $z = (z_1, z_2)$, $w = (w_1, w_2)$ in Π. Taking the $(-p)$-th power and using (3.17), we have to show

$$|E_\Pi(e,w)|\,|E_\Pi(z,e)| \le |E_\Pi(z,w)|. \tag{3.19}$$

Now consider the Cayley transformation $\gamma: \Omega \to \Pi$ associated with the bounded realization Ω of Π [L1]. Then $e = \gamma(0)$ and $z = \gamma(x)$, $w = \gamma(y)$ for some $x, y \in \Omega$. Multiplying both sides of (3.19) by $|\mathrm{Det}\,\partial\gamma(0)|^2 \cdot |\mathrm{Det}\,\partial\gamma(x)|\,|\mathrm{Det}\,\partial\gamma(y)|$ and using the transformation formula (1.35) for the (standard) Bergman kernels, it follows from the formula $\mathrm{Det}\,\partial\gamma(0) = 2^{\mathrm{pr}/2}$ that we have to show

$$|E_\Omega(0,y)|\cdot|E_\Omega(x,0)| \le 2^{\mathrm{pr}}|E_\Omega(x,y)|.$$

Now the Bergman kernel of Ω is given by $E_\Omega(z,w) = \Delta(z,w)^{-p}$, where $\Delta: Z\times Z \to \mathbb{C}$ is the so-called Jordan triple determinant [L1]. Since $E_\Omega(0,y) = 1 = E_\Omega(x,0)$ and $|\Delta(x,y)| \le 2^r$ for all $x, y \in \Omega$, the assertion follows. Q.E.D.

Consider the Berezin calculus $f \mapsto \sigma_\lambda^*(f)$ associated with $H^2_\lambda(\Pi)$ (for $\lambda > p-1$). Since (3.13) is a power of (3.17) and Π is simply-connected, it follows from (1.25) and (3.6), that, for any $g \in \mathrm{Aut}(\Pi)$, $g^* d\mu_\lambda/d\mu_\lambda$ is the square of the modulus of some holomorphic function on Π. According to Sect. 1, the Berezin calculus is covariant under some projective representation π_λ of $G := \mathrm{Aut}(\Pi)^0$ on $H^2_\lambda(\Pi)$ (a continuation of the holomorphic discrete series). In particular, the Berezin transform $\sigma_\lambda \sigma_\lambda^*$ commutes with the action $f \mapsto f \circ g$ of G on symbols. We now need to define $\sigma_\lambda^*(f)$ for certain unbounded symbols: we first define the possible domain of such an operator.

3.20 Definition. *Given $\lambda > p-1$ and $N \geq 0$, define W_λ^N as the subspace consisting of all functions $h \in H_\lambda^2(\Pi)$ such that*

$$\|h\|_{N,\lambda}^2 := \int_\Pi \delta(e,z)^{2N} |(e_z^\lambda \mid h)_\lambda|^2 \, d\mu_0(z) < \infty .$$

When $N = 0$, W_λ^N coincides as a Hilbert space with $H_\lambda^2(\Pi)$, since Lemma 1.10 implies for all $h \in H_\lambda^2(\Pi)$,

$$(h \mid h)_\lambda = \int_\Pi |(e_z^\lambda \mid h)_\lambda|^2 \, d\mu_0(z) . \tag{3.21}$$

The projective representation π_λ of G on $H_\lambda^2(\Pi)$ acts on the coherent states in the following way: Given $g \in G$ and $z \in \Pi$, one has

$$\pi_\lambda(g) e_z^\lambda = \theta \cdot e_{gz}^\lambda \tag{3.22}$$

for some $\theta \in \mathbb{C}$ (with $|\theta| = 1$) depending on (g, z). Since

$$\delta(e, g \cdot z)^{2N} \leq 2^{2rN} \delta(e, g \cdot e)^{2N} \delta(e,z)^{2N}$$

it is clear that $\pi_\lambda(g)$ defines a continuous linear automorphism of W_λ^N for all g. Only when $g \in K$ is $\pi_\lambda(g)$ an isometry.

3.23 Proposition. *Given $N \geq 0$, assume that $\lambda > p - 1 + N$. Then* (i) *the coherent states e_w^λ belong to W_λ^N;* (ii) *the map $f \mapsto \sigma_\lambda^*(f)$, initially defined from $L^\infty(\Pi)$ to the space of bounded operators on $H_\lambda^2(\Pi)$, can be (uniquely) extended as a continuous map from the Banach space of all symbols f such that*

$$\sup_{z \in \Pi} |\delta(e,z)^{-N} f(z)| < \infty \tag{3.24}$$

to the space $\mathscr{L}(W_\lambda^N, H_\lambda^2(\Pi))$ consisting of all bounded linear operators from W_λ^N to $H_\lambda^2(\Pi)$; (iii) *for every f satisfying* (3.24), *the function*

$$(\sigma_\lambda \sigma_\lambda^* f) := (e_z^\lambda \mid \sigma_\lambda^*(f) e_z^\lambda)_\lambda \tag{3.25}$$

(well-defined in view of (i) *and* (ii)*) is linked to f by the integral equation (independent of N)*

$$(\sigma_\lambda \sigma_\lambda^* f)(z) = \int_\Pi \delta(z,w)^{-2\lambda} f(w) \, d\mu_0(w) . \tag{3.26}$$

Proof. From Lemma 1.10, one has

$$\int_\Pi |(e_z^\lambda \mid e_w^\lambda)_\lambda|^2 \, d\mu_0(z) = 1$$

for all $w \in \Pi$ if $\lambda > p-1$. For the coherent state e_e^λ (at $z = e$), we obtain

$$\|e_e^\lambda\|_{N,\lambda}^2 = \int_\Pi \delta(e,z)^{2N} |(e_z^\lambda \mid e_e^\lambda)_\lambda|^2 \, d\mu_0(z) = \int_\Pi \delta(e,z)^{2N-2\lambda} \, d\mu_0(z)$$

which is finite if $\lambda - N > p-1$. Thus $e_e^\lambda \in W_\lambda^N$ under this assumption. Since W_λ^N is invariant under any $\pi_\lambda(g)$, the assertion (i) follows from (3.22). If $f \in L^\infty(\Pi)$, Lemma 1.17 shows that

$$(k \mid \sigma_\lambda^*(f)h)_\lambda = \int_\Pi f(z)\,(k \mid e_z^\lambda)_\lambda \, d\mu_0(z)$$

for all $h, k \in H_\lambda^2(\Pi)$. Using (3.21), we obtain

$$|(k \mid \sigma_\lambda^*(f)h)_\lambda| \le \|k\|_{H_\lambda^2(\Pi)} \cdot \left[\int_\Pi |f(z)|^2 \, |(e_z^\lambda \mid h)_\lambda|^2 \, d\mu_0(z) \right]^{1/2} . \tag{3.27}$$

By Definition 3.20, it follows that the left-hand side of (3.27) extends as a sesquilinear form of $(k,h) \in H_\lambda^2(\Pi) \times W_\lambda^N$ if f satisfies (3.24). This proves (ii). In view of (3.27), the integral link from f to $\sigma_\lambda \sigma_\lambda^*(f)$ is a consequence of Proposition 1.19 in case f is a bounded symbol. If $\lambda > p-1+N$ and f satisfies (3.24), both sides of (3.26) are, for any fixed $z \in \Pi$, continuous linear forms in f. Thus (3.26) holds for all f in the Banach space determined by (3.24) and all $z \in \Pi$. Q.E.D.

3.28 Lemma. *For all points $z = (z_1, z_2)$ and $w = (w_1, w_2)$ in Π, we have*

$$\delta(z,w) \ge \delta\big(\tfrac{1}{2}(z_1 + z_1^* - \Phi(z_2,z_2)), \tfrac{1}{2}(w_1 + w_1^* - \Phi(w_2,w_2))\big) .$$

Proof. The mapping $\varphi(z_1,z_2) := \frac{1}{2}(z_1 + z_1^* - \Phi(z_2,z_2))$ maps Π onto $\Lambda \subset \Pi$ and satisfies $\varphi|_\Lambda = \mathrm{id}$. Now put $ia := (z_1^* - z_1)/2$, $b := -z_2$ and consider the associated "quasi-translation" g defined by (2.58). Then $g(z) = (\varphi(z), 0)$ and $\varphi(g(w)) = \varphi(w)$. Since δ is G-invariant, it follows that $\delta(z,w) = \delta(g(z), g(w)) = \delta(\varphi(z), g(w))$. Thus we have to show $\delta(x,w) \ge \delta(x, \varphi(w))$ for all $x \in \Lambda$ and $w \in \Pi$. Using (3.15), this is equivalent to

$$|\Delta(x + w_1^*)| \ge \Delta\big(x + \tfrac{1}{2}(w_1 + w_1^* - \Phi(w_2,w_2))\big) . \tag{3.29}$$

Write $w_1 = y + i\eta$ with $y \in \Lambda$ and $\eta \in \mathbb{R}^{n_1}$. Then

$$\begin{aligned} \Delta(x + w_1^*) &= \Delta(x + y - i\eta) \\ &= \Delta(P_{x+y}^{1/2}(e - iP_{x+y}^{-1/2}\eta)) \\ &= \Delta(x+y)\,\Delta(e - iP_{x+y}^{-1/2}\eta) . \end{aligned}$$

Here $P_x := 2M_x^2 - M_{x^2}$ is the so-called quadratic representation operator on $\mathbb{C}^{n_1}$. It satisfies the properties that $P_x^{1/2}$ preserves Λ and sends e to x. Since

$|\Delta(e - iP_{x+y}^{-1/2}\eta)| \geq 1$ by spectral theory [L1], it follows that $|\Delta(x+w_1^*)| \geq \Delta(x+y)$. On the other hand

$$\begin{aligned}\Delta\big(x + \tfrac{1}{2}(w_1 + w_1^* - \Phi(w_2, w_2))\big) &= \Delta\big(x + y - \tfrac{1}{2}\Phi(w_2, w_2)\big)\\ &= \Delta\big(P_{x+y}^{1/2}\big(e - \tfrac{1}{2}P_{x+y}^{-1/2}\Phi(w_2, w_2)\big)\big)\\ &= \Delta(x+y)\,\Delta\big(e - \tfrac{1}{2}P_{x+y}^{-1/2}\Phi(w_2, w_2)\big)\\ &\leq \Delta(x+y),\end{aligned}$$

since $\Phi(w_2, w_2)$ lies in the closure of Λ for all $w_2 \in Z_2$. Combining both inequalities, we obtain (3.29). Q.E.D.

3.30 Proposition. *Let a be a measurable function on Λ, satisfying*

$$|a(x)| \leq C\delta\left(e, \frac{x}{2}\right)^N \qquad (x \in \Lambda) \tag{3.31}$$

for some $C > 0$ and $N \geq 0$; assume $\lambda > p - 1 + N$. Then the function

$$f(z) = a(z_1 + z_1^* - \Phi(z_2, z_2))$$

on Π satisfies (3.24), and one has

$$(\sigma_\lambda \sigma_\lambda^* f)(z) = b(z_1 + z_1^* - \Phi(z_2, z_2)),$$

where b is linked to a through the equations

$$g(s) = \frac{\Delta(2\pi s)^{\lambda - \frac{n}{r}}}{\Gamma_\Lambda\left(\lambda - \dfrac{n}{r}\right)} \int_\Lambda e^{-2\pi(s\,|\,y)} a(y)\,\Delta(y)^{\lambda - p}\,dy$$

and

$$b(x) = \frac{\Delta(2\pi x)^\lambda}{\Gamma_\Lambda(\lambda)} \int_\Lambda e^{-2\pi(s\,|\,x)} g(s)\,\Delta(s)^{\lambda - \frac{n_1}{r}}\,ds.$$

Proof. By Lemma 3.28, f satisfies (3.24). Using Proposition 3.23 and (3.26), we obtain

$$\begin{aligned}(\sigma_\lambda \sigma_\lambda^* f)(z) = \pi^{-n} &\frac{\Gamma_\Lambda(\lambda)}{\Gamma_\Lambda\left(\lambda - \dfrac{n}{r}\right)}\,\Delta(z_1 + z_1^* - \Phi(z_2, z_2))^\lambda\\ &\times \int a(2x - \Phi(w_2, w_2))\,\Delta(2x - \Phi(w_2, w_2))^{\lambda - p}\\ &\times |\Delta(z_1 + x - i\xi - \Phi(z_2, w_2))|^{-2\lambda}\,dx\,d\xi\,dV(w_2).\end{aligned}$$

Here we put $w_1 = x + i\xi$, with $\xi \in X \approx \mathbb{R}^{n_1}$, $w_2 \in Z_2$ and $y = \frac{1}{2}\Phi(w_2, w_2)$ running through Λ. Applying (3.4) and Plancherel's formula, we obtain

$$\begin{aligned}&\int |\Delta(z_1 + x - i\xi - \Phi(z_2, w_2))|^{-2\lambda}\,d\xi\\ &= \frac{(2\pi)^{2\lambda r}}{\Gamma_\Lambda(\lambda)^2} \int_\Lambda e^{-4\pi\,\mathrm{Re}(z_1 + x - \Phi(z_2, w_2)\,|\,s)}\,\Delta(s)^{2\lambda - \frac{2n_1}{r}}\,ds.\end{aligned}$$

Thus

$$(\sigma_\lambda \sigma_\lambda^* f)(z) = \frac{(2\pi)^{2\lambda r}\pi^{-n}}{\Gamma_\Lambda(\lambda)\Gamma_\Lambda\left(\lambda - \frac{n}{r}\right)} \Delta(z_1 + z_1^* - \Phi(z_2, z_2))^\lambda \times \int_{Z_2 \times \Lambda \times \Lambda} e^{-2\pi(z_1 + z_1^* + 2y + \Phi(w_2, w_2) - \Phi(z_2, w_2) - \Phi(w_2, z_2) \mid s)} \times a(2y)\,\Delta(2y)^{\lambda - p}\,\Delta(s)^{2\lambda - \frac{2n_1}{r}}\, dV(w_2)\, ds\, dy\,.$$

Using (3.3) to compute the integral with respect to w_2, and changing y to $\frac{y}{2}$, we obtain

$$(\sigma_\lambda \sigma_\lambda^* f)(z) = \frac{(2\pi)^{2\lambda r - n}}{\Gamma_\Lambda(\lambda)\Gamma_\Lambda\left(\lambda - \frac{n}{r}\right)} \Delta(z_1 + z_1^* - \Phi(z_2, z_2))^\lambda \times \int_{\Lambda \times \Lambda} e^{-2\pi(s \mid z_1 + z_1^* - \Phi(z_2, z_2))} \Delta(s)^{2\lambda - \frac{n}{r} - \frac{n_1}{r}} \times e^{-2\pi(s \mid y)} a(y)\,\Delta(y)^{\lambda - p}\, ds\, dy\,.$$

Q.E.D

The function g that acts as an intermediary from a to b has the following interpretation: By combining the Laplace transform (associated with the cone Λ) and the Segal-Bargmann realization of $L^2(\mathbb{R}^{n_2})$, one can construct a Hilbert space isometry from $L^2(\Lambda \times \mathbb{R}^{n_2}, \Delta(t)^{-n_1/r}\, dt\, dx_2)$ onto $H^2_\lambda(\Pi)$. Then, on functions of (t, x_2), $\sigma_\lambda^*(f)$ is just the operator of multiplication by the function $g(t)$.

3.32 Lemma. *For any $1 \leq k \leq r$, the k-th minor Δ_k on Λ satisfies the inequalities*

$$\Delta_k(x) \leq C_k \delta(e, x)^{2k} \tag{3.33}$$

and

$$\Delta_k(x)^{-1} \leq C_k \delta(e, x)^{2(r-k+1)} \tag{3.34}$$

for all $x \in \Lambda$, where $C_k > 0$ is independent of x.

Proof. The norm $|x| := (x \mid x)^{1/2}$ on X is K_Λ-invariant and thus satisfies $|x| \leq \Delta(e + x)$ for all $x \in \Lambda$ (use a spectral decomposition of x). Since $\Delta(x) \leq \Delta(e + x)$ (by spectral theory) and Δ_k is k-homogeneous, we have

$$\Delta(x)^k \cdot \Delta_k(x) \leq C_k \cdot \Delta(e + x)^{2k}\,.$$

Since

$$\delta(e, x) = 2^{-r} \frac{\Delta(e + x)}{\Delta(x)^{1/2}}\,,$$

the first assertion (3.33) follows. Now consider the idempotent $c := e_1 + \ldots + e_k$ and the associated Peirce decomposition [L1]

$$X = X_1(c) \oplus X_{1/2}(c) \oplus X_0(c)\,.$$

Then Δ_k is the determinant function of $X_1(c)$. Let Δ^*_{r-k} denote the determinant function of $X_0(c)$, so that $\Delta(x+z) = \Delta_k(x)\,\Delta^*_{r-k}(z)$ for all $x \in X_1(c)$ and $z \in X_0(c)$. For every $b \in X_{1/2}(c)$, the endomorphism $2b \,\Box\, e_1 \in \mathfrak{g}_\Lambda$ is nilpotent and satisfies

$$\exp(2b \,\Box\, e_1)(e_1 + w) = e_1 + w + b + P_b e_1$$

for all $w \in X_0(c)$, as follows from the Peirce multiplication rules [L1]. Putting $w := e_2 - P_b e_1$, we obtain

$$\begin{aligned}\Delta(e+b) = \Delta(e_1 + e_2 + b) &= \Delta(\exp(2b \,\Box\, e_1)(e_1 + w)) \\ &= \Delta(e_1 + w) = \Delta(e - P_b e_1) \leq 1\,,\end{aligned}$$

since $P_b e_1 \in \bar{\Lambda}$. Now let $x = x_1 + x_{1/2} + x_0 \in \Lambda$. Then $x_1 + x_0 \in \Lambda$ and

$$\begin{aligned}\Delta(x) &= \Delta(P^{1/2}_{x_1+x_0}(e + P^{-1/2}_{x_1+x_0} x_{1/2})) \\ &= \Delta(x_1 + x_0)\,\Delta(e + P^{-1/2}_{x_1+x_0} x_{1/2}) \leq \Delta(x_1 + x_0) = \Delta_k(x)\,\Delta^*_{r-k}(x)\,,\end{aligned}$$

since $P^{-1/2}_{x_1+x_0} x_{1/2} \in X_{1/2}(c)$. Since there exists $\theta \in K_\Lambda$ such that $\Delta^*_{r-k}(x) = \Delta_{r-k}(\theta x)$ for all $x \in \Lambda$ [L1], we obtain from (3.33) and the K_Λ-invariance of δ:

$$\begin{aligned}\Delta_k(x)^{-1} &\leq \Delta^*_{r-k}(x)\,\Delta(x)^{-1} = \frac{\Delta_{r-k}(\theta x)}{\Delta(\theta x)} \\ &\leq C \cdot \delta(e, \theta x)^{2(r-k)}\,\delta(e, \theta x)^2 = C \cdot \delta(e, x)^{2(r-k+1)}\,. \qquad \text{Q.E.D.}\end{aligned}$$

As a consequence of Lemma 3.32, there exists for each $\alpha \in \mathbb{C}^r$ a constant $N \geq 0$ such that

$$\sup_{x \in \Lambda} \Delta^\alpha(x)\,\delta\left(e, \frac{x}{2}\right)^{-N} < \infty\,. \tag{3.35}$$

For $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{C}^r$ with $\operatorname{Re}\beta_j > (j-1)\,\frac{a}{2}$, define

$$\Gamma_\Lambda(\beta) := (2\pi)^{\frac{n_1 - r}{2}} \prod_{j=1}^{r} \Gamma\left(\beta_j - \frac{a}{2}(j-1)\right). \tag{3.36}$$

Identifying λ with $(\lambda, \ldots, \lambda) \in \mathbb{C}^r$, (3.36) specializes to (3.5).

3.37 Lemma. *For $\alpha \in \mathbb{C}^r$, choose N satisfying* (3.35) *and let $\lambda > p - 1 + N$. Then the function $f(z) := \Delta^\alpha(z_1 + z_1^* - \Phi(z_2, z_2))$ on Π satisfies*

$$\sigma_\lambda \sigma^*_\lambda(f) = \frac{\Gamma_\Lambda\left(\alpha + \lambda - \frac{n}{r}\right)\Gamma_\Lambda(-\hat{\alpha} + \lambda)}{\Gamma_\Lambda\left(\lambda - \frac{n}{r}\right)\Gamma_\Lambda(\lambda)}\, f\,,$$

where $-\hat{\alpha} = (-\alpha_r, \ldots, -\alpha_1)$.

Proof. Generalizing (3.4), we have the identity [F1]

$$\int_\Lambda e^{-2\pi(s \mid y)}\,\Delta^\beta(y)\,\Delta(y)^{-\frac{n_1}{r}}\,dy = \Gamma_\Lambda(\beta)\,\Delta^\beta((2\pi s)^{-1}) \tag{3.38}$$

for all $\beta \in \mathbb{C}^r$ with $\mathrm{Re}\,\beta_j > (j-1)\frac{a}{2}$. With $\beta := \alpha + \lambda - \frac{n}{r}$ and for λ large enough, the function $g(s)$ associated with $a(x) = \Delta^\alpha(x)$ as in Proposition 3.30 becomes

$$g(s) = \frac{\Gamma_\Lambda\left(\alpha + \lambda - \frac{n}{r}\right)}{\Gamma_\Lambda\left(\lambda - \frac{n}{r}\right)} \Delta^\alpha((2\pi s)^{-1}).$$

Choosing $\theta \in K_\Lambda$ with $\theta^2 = \mathrm{id}$ and $\theta e_j = e_{r-j+1}$ for all $1 \le j \le r$ [F1], we have

$$\Delta^\beta(s^{-1}) = \Delta^{-\hat{\beta}}(\theta s)$$

for all $\beta \in \mathbb{C}^r$ and $s \in \Lambda$ and obtain (with the notations of Proposition 3.30)

$$b(x) = \frac{\Gamma_\Lambda\left(\alpha + \lambda - \frac{n}{r}\right)}{\Gamma_\Lambda(\lambda)\Gamma_\Lambda\left(\lambda - \frac{n}{r}\right)} \Delta(x)^\lambda \int\limits_\Lambda e^{-(s\,|\,x)} \Delta^{\alpha-\lambda}(s^{-1}) \Delta(s)^{-\frac{n_1}{r}}\, ds$$

and, since $(\theta x)^{-1} = \theta(x^{-1})$,

$$\begin{aligned}\int\limits_\Lambda e^{-(s\,|\,x)} \Delta^{\alpha-\lambda}(s^{-1}) \Delta(s)^{-\frac{n_1}{r}}\, ds &= \int\limits_\Lambda e^{-(s\,|\,x)} \Delta^{\hat{\alpha}+\lambda}(\theta s) \Delta(s)^{-\frac{n_1}{r}}\, ds \\ &= \Gamma_\Lambda(-\hat{\alpha} + \lambda) \Delta^{-\hat{\alpha}+\lambda}((\theta x)^{-1}) \\ &= \Gamma_\Lambda(-\hat{\alpha} + \lambda) \Delta^{\alpha-\lambda}(x).\end{aligned}$$

This completes the proof when λ is large enough, but $\sigma_\lambda \sigma_\lambda^*(f)$ is well-defined under the sole assumption that $\lambda > p - 1 + N$: to prove Lemma 3.37 it thus suffices to show that $\sigma_\lambda \sigma_\lambda^*(f)$, as given by (3.26), extends as a holomorphic function of λ for λ complex with $\mathrm{Re}\,\lambda > p-1+N$. Now, as a consequence of (3.35) and of Lemma 3.28, one has $|f(w)| \le c\delta(e,w)^N$ for some constant c, and using Lemma 3.18 it remains to be shown that $\delta(e,w)^{-2\lambda+N}$ is summable with respect to $d\mu_0(w)$ under the assumption made on λ, this follows from (3.14) and (1.21). Q.E.D.

Now consider the Iwasawa decomposition (2.68) of G. For each $\nu \in \mathfrak{a}^\# \approx \mathbb{R}^r$, define the *spherical function*

$$\varphi^\nu(g) := \int\limits_K e^{(\varrho + i\nu)\,l(kg)} dk$$

on G; φ^ν depends only on the orbit of ν under the Weyl group W. Using (2.71), we may identify φ^ν with the K-invariant function

$$\varphi^\nu(z) = \int\limits_K \Delta^{\varrho + i\nu}(k \cdot z)\, dk$$

on Π.

3.39 Proposition. *Put* $N = 1 + \left(\frac{n_1}{r} - 1\right)(r+4) + \frac{n_2}{r}$ *and let* $\lambda > p - 1 + N$. *Then one has for all* $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$,

$$\sigma_\lambda \sigma_\lambda^*(\varphi^\nu) = \prod_{j=1}^{r} \frac{\Gamma\left(i\nu_j + \lambda - \frac{p-1}{2}\right)\Gamma\left(-i\nu_j + \lambda - \frac{p-1}{2}\right)}{\Gamma\left(-\varrho_j + \lambda - \frac{p-1}{2}\right)\Gamma\left(\varrho_j + \lambda - \frac{p-1}{2}\right)} \varphi^\nu .$$

Proof. By (2.69), we have $\varrho_j = \frac{1}{2} + \frac{a}{2}(j-1) + \frac{b}{2}$ with $\frac{a}{2}(r-1) = \frac{n_1}{r} - 1$ and $b = \frac{n_2}{r}$. By Lemma 3.32,

$$\Delta^\varrho = \Delta_1^{\varrho_1 - \varrho_2} \cdots \Delta_{r-1}^{\varrho_{r-1} - \varrho_r} \Delta_r^{\varrho_r}$$

satisfies $|\Delta^\varrho(x)| \leq C \cdot \delta(e, x)^N$ for all $x \in \Lambda$, if $N = a(r + (r-1) + \ldots + 2) + 1 + a(r-1) + b = 1 + \frac{a}{2}(r-1)(r+4) + b$. Since $\lambda > p - 1 + N$, we may apply Lemma 3.37 with $\alpha = \varrho + i\nu$ and average over K, since $\sigma_\lambda \sigma_\lambda^*$ commutes with the action of G on symbols. By (3.36), we have

$$(2\pi)^{\frac{r-n_1}{2}} \Gamma_\Lambda\left(\alpha + \lambda - \frac{n}{r}\right) = \prod_j \Gamma\left(\varrho_j + i\nu_j + \lambda - \frac{n}{r} - \frac{a}{2}(j-1)\right)$$
$$= \prod_j \Gamma\left(\lambda + i\nu_j - \frac{p-1}{2}\right)$$

and the other Γ-factors can be expressed in a similar way. Q.E.D.

Recall [H2] that the space spanned by the G-translates of φ^ν has a natural inner product: denoting as $\mathscr{H}_\nu$ its (Hilbert-space) completion, one then has the direct integral decomposition

$$L^2(\Pi) = \int_{\mathfrak{a}^\#/W} \mathscr{H}_\nu |\mathbf{c}(\nu)|^{-2} \, d\nu\,, \qquad T_\Pi = \int_{\mathfrak{a}^\#/W} T_\nu |\mathbf{c}(\nu)|^{-2} \, d\nu\,, \tag{3.40}$$

where $\mathbf{c}$ is Harish-Chandra's $\mathbf{c}$-function. T_Π is the natural action of G in $L^2(\Pi)$ and T_ν is the (irreducible) spherical representation of G in $\mathscr{H}_\nu$. Given any W-invariant function F on $\mathfrak{a}^\# \approx \mathbb{R}^r$, continuous and real-valued, one can define the G-invariant self-adjoint operator $\hat{F}$ on $L^2(\Pi)$ by the formula

$$\hat{F} f = \int F(\nu) f_\nu |\mathbf{c}(\nu)|^{-2} \, d\nu\,, \qquad f = \int f_\nu |\mathbf{c}(\nu)|^{-2} \, d\nu\,. \tag{3.41}$$

The domain of $\hat{F}$ is defined as the space of functions f such that

$$\int |F(\nu)|^2 \, \|f_\nu\|_\nu^2 \, |\mathbf{c}(\nu)|^{-2} \, d\nu < \infty$$

and $\hat{F}$ is bounded if F is a bounded function. For each $k = 1, \ldots, r$, let $Q_k \in \mathrm{Diff}(\Pi)^G$ be the differential operator associated with the polynomial

$$q_k(t_1, \ldots, t_r) = -\sum_{j=1}^{r} t_j^{2k} \tag{3.42}$$

via (2.70). The direct integral decomposition above diagonalizes simultaneously the operators $Q_1, \ldots, Q_r : Q_k = \hat{F}_k$ if $F_k(\nu) = q_k(i\nu)$. Moreover, since any W-invariant function of ν can be expressed as a function of $(q_1(i\nu), \ldots, q_r(i\nu))$, it is clear that, for any bounded W-invariant continuous function F on $\mathfrak{a}^\#$, the bounded operator $\hat{F}$ is, in the spectral-theoretic sense, a function of the (commuting) operators Q_k.

3.43 Theorem. *For $\lambda > p-1$, let $G_\lambda = G_\lambda(q_1, \ldots, q_r)$ be the function, defined and analytic in a neighborhood of $([0,\infty[)^r$ in $\mathbb{C}^r$, characterized by the identity*

$$G_\lambda(q_1(i\nu), \ldots, q_r(i\nu)) = \prod_{j=1}^{r} \frac{\Gamma\left(i\nu_j + \lambda - \frac{p-1}{2}\right)\Gamma\left(-i\nu_j + \lambda - \frac{p-1}{2}\right)}{\Gamma\left(\varrho_j + \lambda - \frac{p-1}{2}\right)\Gamma\left(-\varrho_j + \lambda - \frac{p-1}{2}\right)} \tag{3.44}$$

for all $\nu \in \mathbb{R}^r$, with $q_k(t)$ defined in (3.42). *Then the Berezin transform $\sigma_\lambda\sigma_\lambda^*$ is the bounded operator on $L^2(\Pi)$ defined in the spectral-theoretic sense as*

$$\sigma_\lambda\sigma_\lambda^* = G_\lambda(Q_1, \ldots, Q_r), \tag{3.45}$$

with Q_k associated with q_k via (2.70).

Proof. If λ is large enough, the assertion follows from Proposition 3.39, from the fact that $\sigma_\lambda\sigma_\lambda^*$ commutes with T_Π and from the direct integral decomposition (3.40). Since $\frac{a}{2} \le \varrho_j \le \frac{1}{2} + \frac{a}{2}(r-1) + \frac{b}{2} = \frac{p-1}{2}$ for all j, both sides of (3.44) are still defined as bounded operators whenever $\lambda > p-1$. For such λ, $\sigma_\lambda\sigma_\lambda^*$ has the kernel $\delta(z,w)^{-2\lambda}$ with respect to the measure

$$d\mu_0(w) = \pi^{-n} \prod_{j=1}^{r} \frac{\Gamma\left(\lambda - \varrho_j + \frac{n}{r} - \frac{p-1}{2}\right)}{\Gamma\left(\lambda - \varrho_j - \frac{p-1}{2}\right)} \times \Delta(w_1 + w_1^* - \Phi(w_2, w_2))^{-p}\, dV(w_1)\, dV(w_2)$$

(cf. (3.16) and (3.36)). Let P be the polynomial in r variables such that

$$P(q_1(i\nu), \ldots, q_r(i\nu)) = \prod_{j=1}^{r} \left[\left(\lambda - \frac{p-1}{2}\right)^2 + \nu_j^2\right]$$

and consider the differential operator $D = P(Q_1, \ldots, Q_r)$ on Π. As a consequence of the functional equation of the Γ-function we obtain

$$D_z(\delta(z,w)^{-2\lambda}) = \delta(z,w)^{-2(\lambda+1)} \prod_{j=1}^{r} \left(\varrho_j + \lambda - \frac{p-1}{2}\right)\left(\lambda - \varrho_j + \frac{n}{r} - \frac{p-1}{2}\right) \tag{3.46}$$

whenever λ is large enough. Here D_z is D acting on the z-variable. Since both sides of (3.46) are analytic in λ, for λ complex with $\mathrm{Re}(\lambda) > p-1$, (3.46) remains valid for real $\lambda > p-1$. Using the functional equation of the Γ-function in the reverse direction, one may easily extend (3.44) to all $\lambda > p-1$. Q.E.D.

4. Limit and Expansions as $\lambda \to \infty$

In the case of the Wick calculus, the Berezin transform is, as recalled in (0.3) the operator $\exp\left(-\frac{1}{4\pi\varrho}\Delta\right)$, with $\Delta = -\sum\left(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial \xi_j^2}\right)$: then, if $h \in L^2(\mathbb{C}^n, dV)$ lies in the domain of $\exp\frac{1}{4\pi\varrho}\Delta$, one has the strongly convergent expansion (as $\varrho^{-1}\to 0$)

$$\exp\left(-\frac{1}{4\pi\varrho}\Delta\right)h = \sum_{k\geq 0}\frac{1}{k!}\left(-\frac{1}{4\pi\varrho}\right)^k \Delta^k h\,.$$

Nothing of the kind, with ϱ replaced by λ, can be valid, even formally, for the Berezin transform characterized by Theorem 3.43, for, as a function of λ, the function G_λ has an essential singularity at infinity. This forbids to base on the formal inversion of such a series (as is sometimes done in the physics literature) any claim for the validity of a "composition formula" for Berezin contravariant or covariant symbols: indeed, if it is trivial to compute the covariant symbol of the composition of two operators of the form $\sigma_\lambda^*(f)$, there is no going back from the covariant to some contravariant symbol of the product.

What remains of these formal considerations is the result of an application of Stirling's formula for $\Gamma(x+1)$ to the various factors involved in G_λ. Indeed, the unbounded factors essentially cancel out and, using again the Weyl group symmetry of G_λ as a function of ν, one gets an asymptotic expansion (as remarked by Berezin [B3] who did not, however, make the meaning of this expansion specific)

$$G_\lambda(Q_1, \ldots, Q_r) \sim \sum_{k\geq 0}\lambda^{-k}P_k(Q_1, \ldots, Q_r)\,,$$

where the P_k's are polynomials: the meaning of it is that if a symbol $h \in L^2(\Pi)$ is such that $(\Pi Q_j^{\beta-j})h$ belongs to $L^2(\Pi)$ whenever $\sum\beta_j$ is less than some number depending on k, then the difference

$$\lambda^k\left[G_\lambda(Q_1, \ldots, Q_r) - \sum_{j=0}^{k-1}\lambda^{-j}P_j(Q_1, \ldots, Q_r)\right]h$$

lies in a bounded subset of $L^2(\Pi)$ as $\lambda \to \infty$. From this one can get an asymptotic expansion, in powers of λ^{-1}, of the covariant symbol of a product $\sigma_\lambda^*(h_1)\sigma_\lambda^*(h_2)$ if h_1 and h_2 have sufficiently many derivatives in $L^2(\Pi)$ in the sense explained above: only, this is meaningless for fixed λ; also, one has to make assumptions about the *contravariant* symbols of the two factors whereas the conclusion concerns the much softer *covariant* symbol of the product. (In contrast, for the Weyl calculus associated with the Poincaré half-plane, there is for fixed λ an exact (integral) composition formula [U7] involving only one type of symbol: such a formula does probably admit generalizations to the higher rank case, depending on generalizations of special function theory.)

One may also wish to have a look at possible *limits* as $\lambda \to \infty$ of the Berezin quantization: several are possible, among which we shall briefly describe one that is best associated with the bounded realization Ω. It is common knowledge [O1] that a suitably renormalized version of the measure $d\mu_\lambda$ on Ω tends, as $\lambda \to \infty$, to the measure associated with the Segal-Bargmann space $H^2(\mathbb{C}^n)$ recalled in Sect. 1: indeed, considering the "Jordan triple determinant" Δ [U4] and the probability measure

$$d\mu_\lambda(z) = \pi^{-n} \frac{\Gamma_\Lambda(\lambda)}{\Gamma_\Lambda\left(\lambda - \frac{p}{2}\right)} \Delta(z,z)^{\lambda-p} dV(z)$$

on Ω, the expansion $\Delta(z,z) \sim 1 - |z|^2 + o(|z|^2)$ valid for z near 0 implies that

$$\lambda^{n-\frac{p}{2}} d\mu_\lambda\left(\left(\frac{\pi}{\lambda}\right)^{1/2} z\right) \to e^{-\pi|z|^2}$$

as $\lambda \to \infty$. This makes it plausible that, under some pair of transformations (one, the obvious dilatation, acting between the phase spaces, the second one acting between the Hilbert spaces involved), the Berezin calculus will contract to the Wick calculus: this is, indeed, the case; let us describe the somewhat subtler Hilbert space transformation involved.

For any $\lambda > p - 1$, let $H^2_\lambda(\Omega)$ be the λ-Bergman space defined in (3.7). It contains the polynomial algebra $\mathscr{P}(Z)$ as a dense subspace. Under the natural action of $K := GL(\Omega)$, $\mathscr{P}(Z)$ has a decomposition

$$\mathscr{P}(Z) = \sum_{\alpha \in \vec{\mathbb{N}}^r}^{\oplus} \mathscr{P}^\alpha(Z),$$

where $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ satisfies $\alpha_1 \geq \ldots \geq \alpha_r$ [S1, U5, KS2]. The K-invariant inner products on $\mathscr{P}^\alpha(Z)$, regarded as a subspace of $H^2_\lambda(\Omega)$ or $H^2(Z)$, respectively, are proportional for each $\alpha \in \vec{\mathbb{N}}^r$, and one can show [FK]

$$(h \mid k)_Z = ((\lambda))_\alpha (h \mid k)_\lambda \tag{4.1}$$

for all $h, k \in \mathscr{P}^\alpha(Z)$, where

$$((\lambda))_\alpha = \frac{\Gamma_\Lambda(\lambda + \alpha)}{\Gamma_\Lambda(\lambda)} = \prod_{j=1}^{r} \left(\lambda - \frac{a}{2}(j-1)\right)_{\alpha_j} \tag{4.2}$$

is the "multi-Pochhammer symbol." This implies

4.3 Proposition. *There exists a Hilbert space isomorphism*

$$H^2_\lambda(\Omega) \xrightarrow[\approx]{\mathscr{F}_\lambda} H^2(Z)$$

such that

$$\mathscr{F}_\lambda h = \frac{1}{\sqrt{((\lambda))_\alpha}} h$$

for every $h \in \mathscr{P}^\alpha(Z)$.

As a consequence of Proposition 4.3, we may, for any polynomial $f \in \mathscr{P}(Z)$, consider the commuting diagram

$$\begin{array}{ccc} H^2_\lambda(\Omega) & \xrightarrow{\mathscr{F}_\lambda} & H^2(Z) \\ {\scriptstyle \sigma^*_\lambda(f)} \downarrow & & \downarrow {\scriptstyle \mathrm{Ad}(\mathscr{F}_\lambda)\sigma^*_\lambda(f)} \\ H^2_\lambda(\Omega) & \xrightarrow[\mathscr{F}_\lambda]{} & H^2(Z). \end{array}$$

Here $\sigma_\lambda^*(f)$ is the Berezin operator (1.14) with symbol f, and we put $\mathrm{Ad}(\mathscr{F}_\lambda)T := \mathscr{F}_\lambda T \mathscr{F}_\lambda^*$. On the other hand, we may deform the underlying phase space Ω via the biholomorphic mapping

$$\begin{aligned} \Omega &\xrightarrow{\varphi_\lambda} \sqrt{\lambda}\Omega \\ z &\longmapsto \sqrt{\lambda} z \,. \end{aligned}$$

The mapping

$$\begin{aligned} H_\lambda^2(\Omega) &\underset{\approx}{\longrightarrow} H^2\big(\sqrt{\lambda}\Omega\big) \\ h &\longmapsto h \circ \varphi_\lambda^{-1} \end{aligned}$$

defines a Hilbert space $H^2(\sqrt{\lambda}\Omega)$ of holomorphic functions on $\sqrt{\lambda}\Omega$. For any real-analytic polynomial $f: Z \to \mathbb{C}$ there is a commuting diagram

$$\begin{array}{ccc} H_\lambda^2(\Omega) & \longrightarrow & H^2(\sqrt{\lambda}\Omega) \\ {\scriptstyle \sigma_\lambda^*(f\circ\varphi_\lambda)} \big\downarrow & & \big\downarrow {\scriptstyle \sigma^*(f)} \\ H_\lambda^2(\Omega) & \longrightarrow & H^2(\sqrt{\lambda}\Omega) \end{array}$$

for the respective Toeplitz operators. In this sense the correct active symbol, as $\lambda \to \infty$, is $f \circ \varphi_\lambda$.

4.4 Theorem. *For every real-analytic polynomial $f: Z \to \mathbb{C}$,*

$$\lim_{\lambda\to\infty} \mathrm{Ad}(\mathscr{F}_\lambda)\sigma_\lambda^*(f \circ \varphi_\lambda) = \sigma^*(f)$$

is the flat Berezin operator with contravariant symbol f (cf. Example 1.27 with $\varrho = 1$).

Proof. Write

$$f = \sum_i \bar{f}_i g_i \quad \text{(finite sum)},$$

where $f_i, g_i \in \mathscr{P}(Z)$. Then the flat Berezin quantized operator satisfies

$$\sigma^*(f) = \sum_i \sigma^*(f_i)^* \quad \sigma^*(g_i)$$

and a similar relation holds for $\sigma_\lambda^*(f)$. We may therefore assume that $f \in \mathscr{P}(Z)$. We may even assume that $f(z) = (z \mid b)$ is linear. Then

$$\sigma_\lambda^*(f \circ \varphi_\lambda)h(z) = \sqrt{\lambda} h(z)(z \mid b) \,.$$

Now suppose $h \in \mathscr{P}^\alpha(Z)$ for some $\alpha \in \vec{\mathbb{N}}^r$. Then [U6] implies

$$h(z)(z \mid b) \in \sum_{j=1}^r \mathscr{P}^{\alpha+\varepsilon_j}(Z) \,,$$

where we put

$$\alpha + \varepsilon_j := (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \ldots, \alpha_r) \,,$$

provided this multi-index belongs to $\vec{\mathbb{N}}^r$. Using (4.1) and denoting the $(\alpha+\varepsilon_j)$-component by a subscript, we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{\lambda}}\,\mathrm{Ad}(\mathscr{F}_\lambda)\sigma_\lambda^*(f\circ\varphi_\lambda)h(z)\\
&=\frac{1}{\sqrt{\lambda}}\mathscr{F}_\lambda\sigma_\lambda^*(f\circ\varphi_\lambda)\mathscr{F}_\lambda^{*}h(z)=\sqrt{((\lambda))_\alpha}\,\mathscr{F}_\lambda h(z)\,(z\mid b)\\
&=\sqrt{((\lambda))_\alpha}\sum_j \mathscr{F}_\lambda(h(z)\,(z\mid b))_{\alpha+\varepsilon_j}\\
&=\sqrt{((\lambda))_\alpha}\sum_j \frac{1}{\sqrt{((\lambda))_{\alpha+\varepsilon_j}}}(h(z)\cdot(z\mid b))_{\alpha+\varepsilon_j}\\
&=\sum_j\sqrt{\frac{((\lambda))_\alpha}{((\lambda))_{\alpha+\varepsilon_j}}}\,(h(z)\cdot(z\mid b))_{\alpha+\varepsilon_j}\\
&=\sum_j\frac{1}{\sqrt{\lambda-\frac{a}{2}(j-1)+\alpha_j}}(h(z)\cdot(z\mid b))_{\alpha+\varepsilon_j}\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{\lambda\to\infty}\mathrm{Ad}(\mathscr{F}_\lambda)\sigma_\lambda^*(f\circ\varphi_\lambda)h(z)\\
&=\lim_{\lambda\to\infty}\sum_j\frac{\sqrt{\lambda}}{\sqrt{\lambda-\frac{a}{2}(j-1)+\alpha_j}}(h(z)\cdot(z\mid b))_{\alpha+\varepsilon_j}\\
&=\sum_j(h(z)\cdot(z\mid b))_{\alpha+\varepsilon_j}=h(z)\cdot(z\mid b)=\sigma^*(f)h(z)\,. \qquad \text{Q.E.D.}
\end{aligned}$$

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