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THE BEST BOUNDS IN WALLIS' INEQUALITY

CHAO-PING CHEN AND FENG QI

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Abstract. For all natural numbers n, let $n!!$ denote a double factorial. Then

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}.$$

The constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible. From this, the well-known Wallis' inequality is improved.

1. Introduction

A double factorial $n!!$ can be defined by

$$(2m)!! = \prod_{i=1}^{m}(2i) \quad \text{and} \quad (2m-1)!! = \prod_{i=1}^{m}(2i-1)$$

for any given positive integer m. Let

$$P_n = \frac{(2n-1)!!}{(2n)!!}. \tag{1}$$

Then we have

$$\frac{1}{2\sqrt{n}} < \frac{\sqrt{2}}{\sqrt{(2n+1)\pi}} < P_n < \frac{2}{\sqrt{(4n+1)\pi}} < \frac{1}{\sqrt{3n+1}} < \frac{1}{\sqrt{2n+1}} < \frac{1}{\sqrt{2n}} \tag{2}$$

for $n > 1$. The inequality (2) is called Wallis' inequality in [18, p. 103].

The lower and upper bounds of P_n in (2) are frequently cited and applied by mathematicians. The smallest upper bound $\frac{2}{\sqrt{(4n+1)\pi}}$ and the largest lower bound $\frac{\sqrt{2}}{\sqrt{(2n+1)\pi}}$ in (2), that is, the inequalities

$$\frac{\sqrt{2}}{\sqrt{(2n+1)\pi}} < P_n < \frac{2}{\sqrt{(4n+1)\pi}} \tag{3}$$

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are obtained by N. D. Kazarinoff. See [16, pp. 47–48 and pp. 65–67]. We can rewrite inequality (3) as

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < P_n < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}} \tag{4}$$

for $n \in \mathbb{N}$.

The important use of formula (4) is to give a particular case of Wallis' formula (see [4, p. 259]) by taking $x = \frac{\pi}{2}$ in (6):

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{[(2n)!!]^2}{[(2n-1)!!]^2(2n+1)} = \prod_{n=1}^{\infty}\left[\frac{(2n)^2}{(2n-1)(2n+1)}\right]. \tag{5}$$

The Wallis formula follows originally from the infinite product representation of the sine (see [12, 23]):

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right). \tag{6}$$

The Wallis formula can also be expressed as

$$\frac{\pi}{2} = \left[4^{\zeta(0)}e^{-\zeta'(0)}\right]^2; \tag{7}$$

see [12], where ζ is the Riemann zeta function [11].

A derivation of the Wallis formula from $\zeta'(0)$ using the Hadamard product [10] for the Riemann zeta function $\zeta(s)$ due to Y. L. Yung can be found in [12]. The Wallis formula can also be reversed to derive $\zeta'(0)$ from the Wallis formula without using the Hadamard product [22].

It is noted that Wallis's sine (cosine) formula [13, 14] is as follows:

$$\int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} = \begin{cases} \frac{\pi}{2}\cdot\frac{(n-1)!!}{n!!} & \text{for } n \text{ even},\\ \frac{(n-1)!!}{n!!} & \text{for } n \text{ odd}, \end{cases} \tag{8}$$

where Γ is a gamma function.

An inequality involving the term P_n is obtained by the second author and coworkers in [21] by using Tchebysheff's integral inequality.

It is well known that factorials and their "continuous" extension play an eminent role, for instance, in combinatorics, graph theory, and special functions.

For more information on the Wallis formula, please refer to [1, p. 258], [5, 6, 7], [8, pp. 17–28], [15, p. 468], [17, pp. 63–64], and the references therein.

In this article, we will refine inequality (4). More precisely, we will ask for two best possible constants A and B such that the double inequality

$$\frac{1}{\sqrt{\pi(n+A)}} \le P_n \le \frac{1}{\sqrt{\pi(n+B)}} \tag{9}$$

holds for all natural number n. In other words, the constants $A = \frac{4}{\pi} - 1$ and $B = \frac{1}{4}$ cannot be replaced by smaller and larger numbers, respectively, in (9).

2. Lemmas

Lemma 1. *For $x > 0$, we have*

$$\frac{2x+1}{x(4x+1)} < \frac{\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)} - \frac{\Gamma'(x)}{\Gamma(x)}, \tag{10}$$

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right), \quad x \to \infty. \tag{11}$$

The proof of inequality (10) is given in [2, 3, 19], and the proof of the asymptotic expansion (11) can be found in [9] and [20, p. 378]. See also [1, p. 257].

Remark 1. Replacing x by $x + \frac{1}{2}$ in (10) yields

$$\frac{4x+4}{(2x+1)(4x+3)} < \frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)}. \tag{12}$$

Lemma 2. *For $x > 0$, we have*

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \frac{2x+1}{\sqrt{4x+3}}. \tag{13}$$

Proof. Define for a positive real number x,

$$f(x) = \ln(2x+1) - \frac{1}{2}\ln(4x+3) - \ln\Gamma(x+1) + \ln\Gamma\left(x+\frac{1}{2}\right).$$

Differentiating $f(x)$ gives us

$$f'(x) = \frac{2}{2x+1} - \frac{2}{4x+3} - \left[\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)}\right].$$

Utilizing (12), we obtain

$$f'(x) > \frac{2}{2x+1} - \frac{2}{4x+3} - \frac{4x+4}{(2x+1)(4x+3)} = 0.$$

Therefore, $f(x)$ is strictly increasing in $(0,\infty)$, and

$$f(x) > f(0) = \frac{1}{2}\ln\frac{\pi}{3} > 0,$$

which leads to inequality (13). □

Corollary 1. *For all natural numbers n, we have*

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \frac{2n+1}{\sqrt{4n+3}}. \tag{14}$$

Corollary 2. *The sequence*

$$\{Q_n\}_{n=1}^{\infty} \triangleq \left\{\left[\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 - n\right\}_{n=1}^{\infty} \tag{15}$$

is strictly decreasing.

Proof. The inequality $Q_{n+1} < Q_n$ is equivalent to

$$\left[\frac{\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)}\right]^2 - (n+1) < \left[\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 - n,$$

which can be rewritten by using $\Gamma(x+1) = x\Gamma(x)$ as

$$\left[\frac{n+1}{n+\frac{1}{2}} \cdot \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 - 1 < \left[\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2,$$

$$\left[\frac{(n+1)^2}{\left(n+\frac{1}{2}\right)^2} - 1\right]\left[\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 < 1,$$

$$\left[\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 < \frac{\left(n+\frac{1}{2}\right)^2}{(n+1)^2-\left(n+\frac{1}{2}\right)^2},$$
$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} < \frac{2n+1}{\sqrt{4n+3}},$$

which is a special case of inequality (14) by taking $x = n$. Therefore, the proof of monotonicity follows. □

3. Main results

Now we give the main results of this paper.

Theorem 1. *For all natural numbers n, we have*

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{16}$$

The constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible.

Proof. Since

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad 2^n n! = (2n)!!,$$

the double inequality (16) is equivalent to

$$\frac{1}{4} < Q_n = \left[\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\right]^2 - n \le \frac{4}{\pi} - 1. \tag{17}$$

From the monotonicity of the sequence Q_n provided in Corollary 2, it follows that

$$\lim_{n\to\infty} Q_n < Q_n \le Q_1 = \frac{4}{\pi} - 1.$$

Using the asymptotic formula (11), we conclude from

$$Q_n = n\left[n^{-\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} - 1\right]\left[n^{-\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} + 1\right]$$

that

$$\lim_{n\to\infty} Q_n = \frac{1}{4}.$$

Thus, inequality (17) follows. The proof is complete. □

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Department of Applied Mathematics and Informatics, Research Institute of Applied Mathematics, Henan Polytechnic University, Jiaozuo City, Henan 454000, People's Republic of China

E-mail address: `chenchaoping@hpu.edu.cn`

Department of Applied Mathematics and Informatics, Research Institute of Applied Mathematics, Henan Polytechnic University, Jiaozuo City, Henan 454000, People's Republic of China

E-mail address: `qifeng@hpu.edu.cn`

E-mail address: `fengqi618@member.ams.org`

URL: `http://rgmia.vu.edu.au/qi.html`