



The Beta Exponential Power Series Distribution

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Abstract

In this paper, we investigate to propose a new statistical distribution based on power series. We introduce a new family of distributions which are constructed based on a latent complementary risk problem and are obtained by compounding Beta Exponential (BE) and Power Series distributions. The new distribution contains, as special sub-models, several important distributions which are discussed in the literature, such as Beta Exponential Poisson (BEP) distribution, Beta Exponential Geometric (BEG) distribution, Beta Exponential Logarithmic (BEL) distribution, Beta Exponential Binomial (BEB) distribution as special cases. The hazard function of the BEPS distributions can be increasing, decreasing or bathtub shaped among others. The comprehensive mathematical properties of the new distribution is provided such as closed-form expressions for the density, cumulative distribution, survival function, failure rate function, the r -th raw moment, maximum likelihood estimation and also the moments of order statistics. The proposed type of distributions is used to modeling simulated and real datasets.

Keywords Beta exponential · Power series distributions · Hazard function · Maximum likelihood estimation

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1 Introduction

In the analyzing of lifetime data, it is common that researchers use the Exponential and Generalized Exponential distributions. It is well-known that Exponential can have only constant hazard function and Generalized Exponential distribution can have only monotone hazard functions. Since these distributions are very well-known distributions for modeling lifetime data in reliability and medical studies, one of the interesting points in the data analysis can be searching for distributions that have some properties which enable them to describe the lifetime of some devices.

In the concept of lifetime data analysis, new classes of distributions were proposed based on the Beta Exponential distribution. Interesting reviews of some of these models is presented by [1]. Also, they defined the cumulative distribution function (CDF) of the Generalized Exponential (GE) distribution. The distribution introduced by them is also named the Exponentiated Exponential distribution. Clearly, the Exponential distribution is a particular case of the GE distribution, [1] also investigated some of their properties. The Exponentiated Weibull (EW) distribution, introduced by [2], extends the GE distribution. The Exponentiated Weibull also was studied by [3, 4] and [5]. Nadarajah and Kotz [6] introduced the exponentiated type of distributions such as Exponentiated Gumbel and Exponentiated Frechet distributions by generalizing the Gumbel, Frechet and other distributions in the same way that the GE distribution extends the Exponential distribution. They also provide some mathematical properties for each Exponentiated distribution. Eugene et al. [7] defined the Beta-Bornal (BN) distribution. General expressions for the moments of the BN distribution were derived by [8]. Nadarajah and Kotz [9] considered the Beta Gumbel (BG) distribution and provided closed-form expressions for the moments, the asymptotic distribution of the extreme order statistics and discussed the maximum likelihood estimation procedure. Moreover, [10] worked with the Beta Exponential (BE) distribution and obtained the moment generating function, the first four cumulants, the asymptotic distribution of the extreme order statistics and presented the maximum likelihood estimation. Barreto-Souza et al. [11] proposed the generalization of BE distribution and Raffiq et al. [12] proposed The Marshall–Olkin inverted Nadarajah–Haghighi distribution to analyze lifetime data. In addition, [13] introduced a new generalization of Gumbel–Weibull distribution with application in analysis COVID-19 data.

Recently, by pioneering statistical learning and its related concepts, data science played main role in the analyzing of lifetime data. The effect of data science was not restricted in only computational aspects. Also, it helped theoretical studies. Olsen et al. [14] and Shi et al. [15] discussed and introduced the role and usage of data mining in modern data analysis. Moreover, [16] proposed a new type of Lomax distribution using statistical learning literature. For more information about usage of statistical learning concepts see [17].

In this paper we propose a new lifetime model, called Beta Exponential Power Series distributions. We obtain some of its mathematical properties. Some structural properties of the new distribution are studied. The method of maximum likelihood is used for estimating the model parameters. We illustrate the usefulness of the proposed model by applications to real data. Accordingly in Sect. 2, the new class of BEPS distribution is introduced. In Sect. 3, some properties of this distribution is calculated

such as quantiles and order statistics. In Sect. 3, the parameters are estimated by MLE method. In Sect. 4, the numerical experiments is provided for analysis of proposed distribution based on simulated datasets.

2 The Class of BEPS Distribution

The three-parameter distribution known as Beta Exponential (BE) distribution, was introduced by [1]. The cumulative distribution function (cdf) of the BE distribution with four parameters $a > 0$, $b > 0$, $\lambda > 0$ is given by

$$\begin{aligned} I_y(a, b) &= \frac{B_y(a, b)}{B(a, b)} \\ B_y(a, b) &= \int_0^y w^{a-1}(1-w)^{b-1}dw \\ G(y) &= I_{1-\exp(-\lambda x)}(a, b) = \frac{B_{1-\exp(-\lambda x)}(a, b)}{B(a, b)}, \end{aligned} \quad (1)$$

and the density function of the BE distribution is given by

$$g(y) = \frac{\lambda \exp(-b\lambda x)(1 - \exp(-\lambda x))^{a-1}}{B(a, b)}. \quad (2)$$

The beauty and importance of this distribution lies in its ability to model monotone as well as non-monotone failure rates, which are quite common in lifetime problems and reliability. Let N be a random variable denoting the number of failure causes, $N = 1, 2, \dots$ and considering N following a power series distribution (truncated at zero) with probability function given by

$$p(N = n) = \frac{a_n \theta^n}{C(\theta)}, \quad n = 1, 2, \dots \text{ and } \theta \in (0, S). \quad (3)$$

where a_1, a_2, \dots is a sequence of non-negative real numbers, where at least one of them is strictly positive, S is a positive number no greater than the ratio of convergence of the power series $\sum_{n=1}^{\infty} a_n \theta^n$ and $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$. The Beta Exponential Power Series distribution, denoted by BEPS $(\alpha, a, b, \lambda, \theta)$, is defined by the marginal cdf of $X = \min\{Y_1, Y_2, \dots, Y_N\}$, i.e.,

$$F(x) = 1 - \frac{C(\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha))}{C(\theta)}. \quad (4)$$

where $\alpha, a, b > 0, \lambda > 0, \theta > 0$. The pdf of the BEPS $(\alpha, a, b, \lambda, \theta)$ is given by

$$f(x) = \frac{\alpha \theta \lambda}{C(\theta) B(a, b)} A. \quad (5)$$

Table 1 Useful quantities of some power series distributions

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C(\theta)^{-1}$	Θ
Poisson	$n!^{-1}$	$e^\theta - 1$	e^θ	e^θ	$\log(\theta + 1)$	$\theta \in (0, \infty)$
Logarithmic	n^{-1}	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - e^{-\theta}$	$\theta \in (0, 1)$
Geometric	1	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(\theta + 1)^{-1}$	$\theta \in (0, 1)$
Binomial	$\binom{m}{n}$	$(\theta + 1)^m - 1$	$m(\theta + 1)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta - 1)^{\frac{1}{m}} - 1$	$\theta \in (0, 1)$

The survival function and hazard rate function of the BEPS distribution are given, respectively, by

$$s(x) = \frac{C(\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha))}{C(\theta)},$$

and

$$h(x) = \frac{\alpha\theta\lambda}{C(\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha))B(a, b)}A. \tag{6}$$

where

$$A = C'(\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha))(I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1} \exp(-\lambda x)(1 - \exp(-\lambda x))^{a-1} \exp(-\lambda x)^{b-1},$$

2.1 Beta Exponential Poisson Distribution

The Beta Exponential Poisson (BEP) distribution is a special case of BEPS distributions with $a_n = \frac{1}{n!}$ and $C(\theta) = e^\theta - 1$. Using cdf (4), the CDF of Beta Exponential Poisson (BEP) distribution is given by

$$F_{BEP}(x) = \left\{ 1 - \frac{(\exp(\theta - \theta[I_{1-\exp(-\lambda x)}(a, b)]^\alpha) - 1)}{\exp(\theta) - 1} \right\} .. \tag{7}$$

Where $\alpha, a, b > 0, \lambda, \theta > 0$. The associated pdf and hazard rate function of this distribution are given, respectively, by

$$f_{BEP}(x) = \frac{\alpha\lambda\theta}{(\exp(\theta) - 1)B(a, b)}B. \tag{8}$$

and

$$h_{BEP}(x) = \frac{\alpha\lambda\theta}{(\exp(\theta - \theta[I_{1-\exp(-\lambda x)}(a, b)]^\alpha) - 1)B(a, b)}B. \tag{9}$$

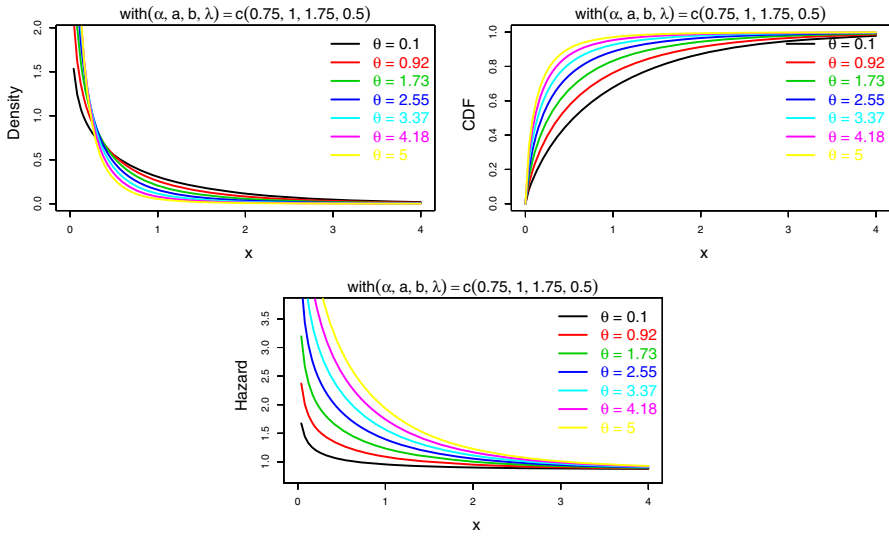


Fig. 1 Plots of density function, cumulative distribution function and hazard function for BEP

where

$$B = \exp(\theta(1 - [I_{1-\exp(-\lambda x)}(a, b)]^\alpha)) \exp(-\lambda x) (1 - \exp(-\lambda x))^{a-1} \exp(-\lambda x)^{b-1} (I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1}$$

The plots of pdf, cdf and hazard rate function of BEP distribution for some values of $\alpha, a, b, \lambda, \theta$ are given in Fig. 1.

Models that present bathtub-shaped failure rate are very useful in survival analysis. The modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields. The new distribution due to its flexibility in accommodating all the forms of the risk function seems to be an important distribution that can be used in a variety of problems in modeling survival data.

2.2 Beta Exponential Geometric Distribution

The Beta Exponential Geometric (BEG) distribution is a special case of BEPS distributions with $a_n = 1$ and $C(\theta) = \theta(1 - \theta)^{-1}$. Using cdf (4), the cdf of Beta Exponential Geometric (BEG) distribution is given by

$$F_{BEG}(x) = 1 - \frac{(1 - \theta)[1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha]}{1 - \theta[1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha]}$$

where $\alpha, a, b, \lambda, \theta > 0$. The pdf is

$$f_{BEG}(x) = \frac{\alpha\lambda(1 - \theta)}{(\exp(\lambda) - 1)(1 - \theta[1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha])^2 B(a, b)} C.$$

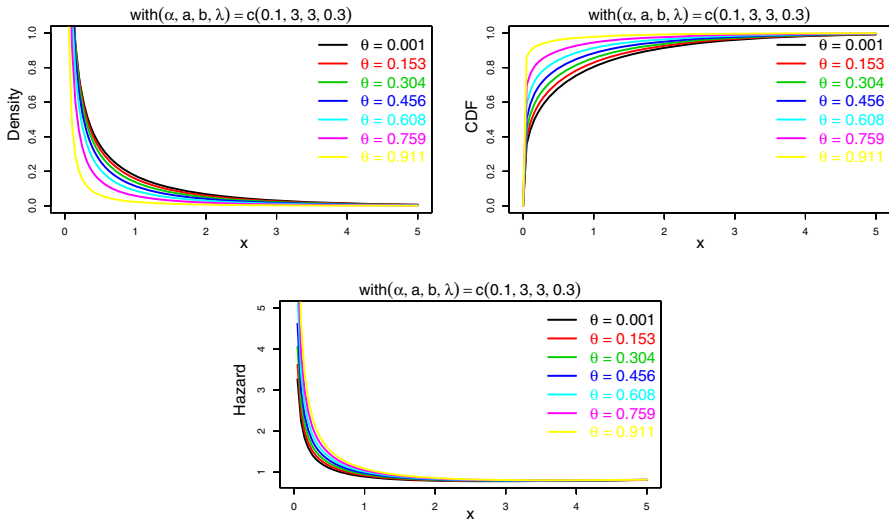


Fig. 2 Plots of density function, cumulative distribution function and hazard function for BEG

The hazard rate function of BEG distribution is given by

$$h_{BEG}(x) = \frac{\alpha\lambda}{[1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha](1 - \theta[1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha])B(a, b)} C.$$

where

$$C = \exp(-\lambda x)(1 - \exp(-\lambda x))^{a-1} \exp(-\lambda x)^{b-1} (I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1}$$

The plots of pdf, cdf and hazard rate function of BEG distribution for some values of $\alpha, a, b, \lambda, \theta$ are given in Fig. 2.

2.3 Beta Exponential Logarithmic Distribution

The Beta Exponential Logarithmic (BEL) distribution is a special case of BEPS distributions with $a_n = \frac{1}{n}$ and $C(\theta) = -\log(1 - \theta)$. Using cdf (4), the cdf of Beta Exponential Logarithmic (BEL) distribution is given by

$$F_{BEL}(x) = 1 - \frac{\log[\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]}{\log(1 - \theta)}, \tag{10}$$

where $\alpha, a, b, \lambda, \theta > 0$. The associated pdf and hazard rate functions are given, respectively, by

$$f_{BEL}(x) = \frac{\lambda\alpha}{\log(1 - \theta)[1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha]B(a, b)} D.$$

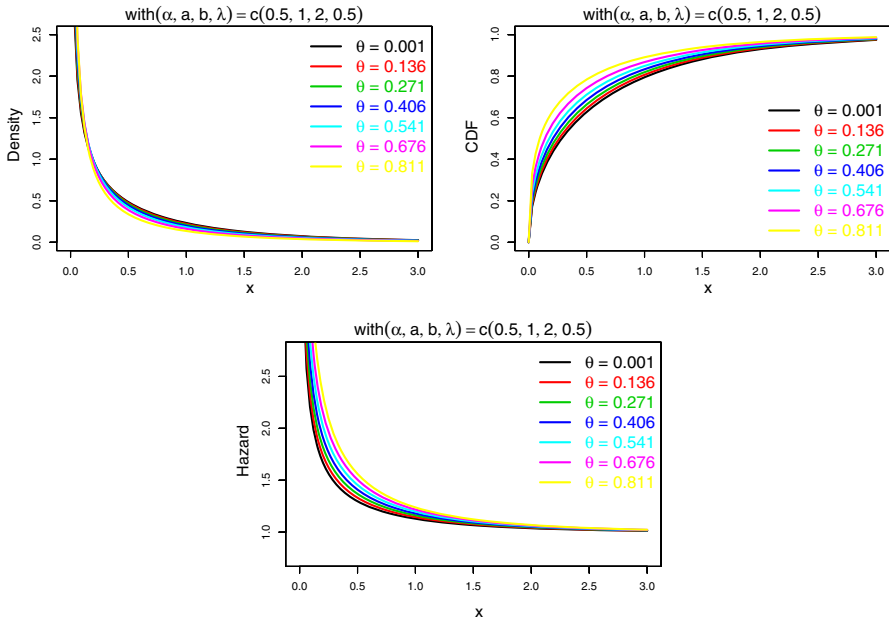


Fig. 3 Plots of density function, cumulative distribution function and hazard function for BEL

and

$$h_{BEL}(x) = \frac{\lambda\alpha}{\log(1 - \theta)[1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha] \log[\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)] B(a, b)} D. \tag{11}$$

where

$$D = \exp(-\lambda x)(1 - \exp(-\lambda x))^{a-1} \exp(-\lambda x)^{b-1} (I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1}$$

The plots of pdf, cdf and hazard rate function of BEL distribution for some values of $\alpha, a, b, \lambda, \theta$ are given in Fig. 3.

2.4 Beta Exponential Binomial Distribution

The Beta Exponential Binomial (BEB) distribution is a special case of the BEPS distribution with $a_n = \binom{m}{n}$ and $C(\theta) = (\theta + 1)^m - 1$ where m ($n \leq m$) is the number of replicates. The pdf and cdf of the BEB distribution is given, by

$$F_{BEB}(x) = \frac{1 - [\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]^m - 1}{(\theta + 1)^m - 1}.$$

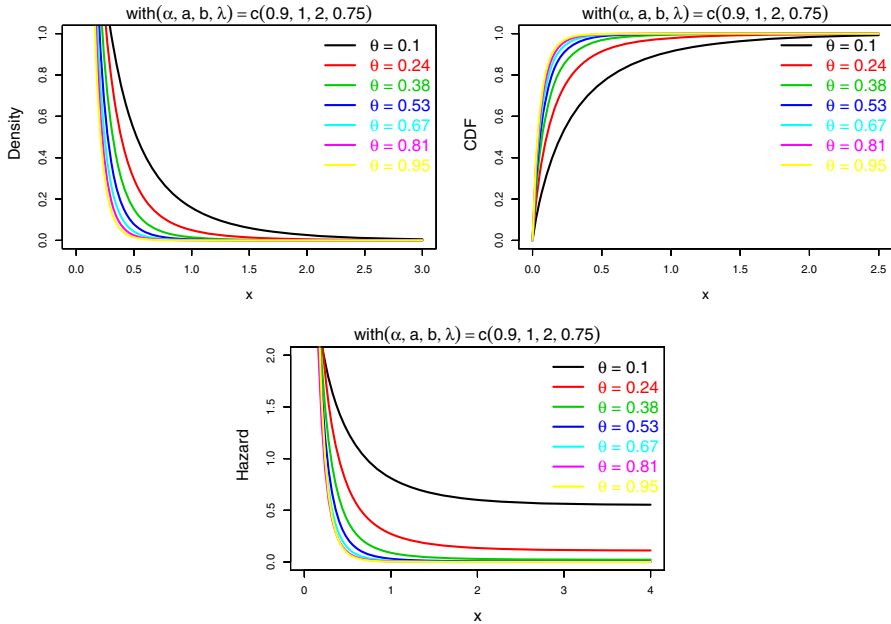


Fig. 4 Plots of density function, cumulative distribution function and hazard function for BEB

where $\alpha, a, b, \lambda, \theta$ The associated pdf and hazard rate functions are given, respectively, by

$$f_{BEB}(x) = \frac{\alpha\lambda\theta m}{(\theta + 1)^{m-1} B(a, b)} E.$$

and

$$h_{BEB}(x) = \frac{\alpha\lambda\theta m}{([\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]^m - 1) B(a, b)} E.$$

where

$$E = \{[\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]^{m-1} \exp(-\lambda x)(1 - \exp(-\lambda x))^{a-1} \exp(-\lambda x)^{b-1} (I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1}\}$$

If $m = 1$, then the density function in (12) changes to the density of BE distribution.

3 Statistical Properties

In this section, we proposed some of the basic statistical properties of the BEPS. For examples, we provide Quantiles and Order statistic

Quantiles, Moments and Order Statistics

The quantiles of a distribution can be used in data generation from a distribution. The quantile x_q of the BEPS($\alpha, a, b, \lambda, \theta$) is the real solution of the following equation:

$$\int_{-\infty}^{x_q} f(x)dx = q,$$

$$\frac{C(\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha))}{C(\theta)} = q,$$

$$C(\theta) = \theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha) \quad (12)$$

The above equation has no closed form solution in x_q , so we have to use a numerical technique such as a Newton-Raphson method to get the quantile. The pdf $f_{i:n}$ of the i th order statistic for a random sample X_1, X_2, \dots, X_n from the BEPS distribution is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x)F(x)^{i-1}[1-F(x)]^{n-i}$$

$$= \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left[\frac{C(\theta(1 - [I_{1-\exp(-\lambda x)}(a, b)]^\alpha))}{C(\theta)} \right]^{j+i-1} \quad (13)$$

and the cdf is

$$F_{i:n}(x) = \sum_{k=i}^n \binom{n}{k} F(x)^k [1-F(x)]^{n-k}$$

$$= \sum_{k=i}^n \sum_{j=0}^{n-k} \binom{n-k}{j} \binom{n}{k} (-1)^j \left[\frac{C(\theta(1 - [I_{1-\exp(-\lambda x)}(a, b)]^\alpha))}{C(\theta)} \right]^{j+k} \quad (14)$$

Carrasco et al. [18] obtained an infinite representation for the r th moment of the BE(a, b, λ) distribution. If Y has the BE(a, b, λ), the r th moment of Y say μ'_r , is given as follows

$$\mu'_r = \sum_{j=0}^{\infty} \sum_{i_1, \dots, i_r=1}^{\infty} w_j \frac{A_{i_1, \dots, i_r} \Gamma(S_r + 1)}{(\alpha(b+j))^{S_r}} \quad (15)$$

where $A_{i_1, \dots, i_r} = a_{i_1} \dots a_{i_r}$, $S_r = i_1 + \dots + i_r$, $a_i = \frac{(-1)^{i+1} i^{-2} \lambda^{i-1}}{(i-1)!}$ and

$$w_j = w_j(a, b) = \frac{(-1)^j \Gamma(a)}{B(a, b) \Gamma(a-j)(b+j)!}$$

Let that $Y_{i:n}$ be the i th order statistic of a random sample from the BE distribution. The r th moment of BEPS $(\alpha, a, b, \lambda, \theta)$, is given as follows

$$\begin{aligned} \mu_r &= E(X^r) = \sum_{i=1}^{\infty} P(N = n) E(Y_{(n)}^r) \\ &= \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} n \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a)}{B(a, b) \Gamma(a - j)(b + j) j!} \sum_{i_1, \dots, i_r=1}^{\infty} \frac{A_{i_1, \dots, i_r} \Gamma(S_r + 1)}{(\alpha(b + j))^{\frac{S_r}{\gamma}}} \end{aligned} \quad (16)$$

According to the Eq. (16), we can conclude that:

$$\begin{aligned} E_{BEPS}(X) &= \mu_1 \\ Var_{BEPS}(X) &= \mu_2 - \mu_1^2. \end{aligned}$$

Hence, expectation and variance of BEPS can be immediately implied from (16). Also, based on the results given in (15), the measures of skewness and kurtosis of the BEPS $(\alpha, a, b, \lambda, \theta)$ can be obtained according to the following relations, respectively,

$$Skewness_{BEPS} = \frac{\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3}{(\mu_2 - \mu_1)^{\frac{3}{2}}},$$

and

$$Kurtosis_{BEPS} = \frac{\mu_4 - 4\mu_3\mu_1 + 6\mu_1^2\mu_2 - 3\mu_1^4}{(\mu_2 - \mu_1)^2}.$$

In Fig. 5, we show the behavior of the Galton' skewness [19] and Moors' kurtosis [20] as a functions of θ for the values of α, a, b and λ . The behavior of different types of BEPS is different against the variation of θ . But, skewness and kurtosis are generally decreasing meaningfully by increasing the values of α, a, b and λ .

4 Estimation and Inference

Standard statistical techniques such as method of maximum likelihood can always be used for parametric estimation. The likelihood equations, given the complete or censored failure data set, can be derived and solved. Parameter estimation is usually a difficult problem even for a five parameter BEPS distribution. Methods like the maximum likelihood estimation will not yield a closed form solution. Different methods can be used to estimate the model parameters. Among these methods, the Maximum Likelihood Estimation method is the most commonly used method for model estimation. In this subsection, we use the maximum likelihood procedure to derive the point and interval estimates of the parameters. Calculating the first partial derivatives of L with respect to $\alpha, a, b, \sigma, \theta$ and equating each to zero, we get the likelihood equations

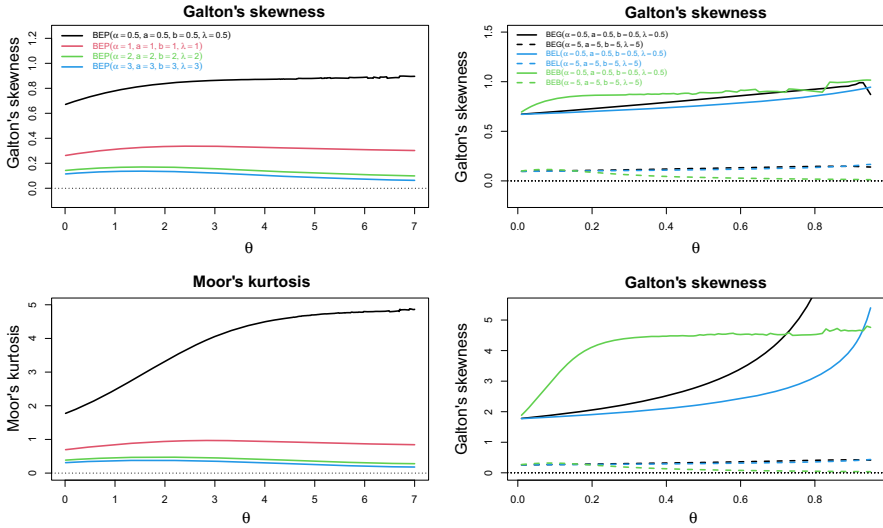


Fig. 5 The effect of θ on Galton's skewness and Moors' kurtosis for different values of α, a, b and λ

in the following system of nonlinear equations of α, a, b, σ and θ .

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln(I_{U(x_i)}(a, b)) - \theta (I_{U(x)}(a, b))^\alpha \ln(I_{U(x)}(a, b)) T_{i1}. \tag{17}$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + \left(1 - (I_{U(x)}(a, b))^\alpha\right) - n \frac{C'(\theta)}{C(\theta)} \tag{18}$$

$$\begin{aligned} \frac{\partial L}{\partial a} &= \frac{\frac{\partial(B(a,b))^{-1}}{\partial a}}{(B(a, b))^{-1}} + (\alpha - 1) \sum_{i=1}^n \frac{\frac{\partial(B(a,b))^{-1}}{\partial a} T_{i2} + (B(a, b))^{-1} T_{i3}}{I_{U(x)}(a, b)} \\ &\quad - \sum_{i=1}^n \alpha \theta (I_{U(x)}(a, b))^{\alpha-1} \left(\frac{\partial(B(a, b))^{-1}}{\partial a} T_{i2} + (B(a, b))^{-1} T_{i3}\right) T_{i1}. \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\partial L}{\partial b} &= \frac{\frac{\partial(B(a,b))^{-1}}{\partial b}}{(B(a, b))^{-1}} (\alpha - 1) \sum_{i=1}^n \frac{\frac{\partial(B(a,b))^{-1}}{\partial b} T_{i2} + (B(a, b))^{-1} T_{i4}}{I_{U(x)}(a, b)} \\ &\quad - \sum_{i=1}^n \alpha \theta (I_{U(x)}(a, b))^{\alpha-1} \left(\frac{\partial(B(a, b))^{-1}}{\partial b} T_{i2} + (B(a, b))^{-1} T_{i4}\right) T_{i1} \end{aligned} \tag{20}$$

where

$$\begin{aligned} T_1 &= \frac{C''(\theta(1 - (I_{U(x)}(a, b))^\alpha))}{C'(\theta(1 - (I_{U(x)}(a, b))^\alpha))} \\ T_2 &= \int_0^{1 - \exp(-\lambda x)} t^{a-1} (1 - t)^{b-1} dt \end{aligned}$$

$$T_3 = \int_0^{1-\exp(-\lambda x)} t^{a-1} \ln t (1-t)^{b-1} dt$$

and

$$T_4 = \int_0^{1-\exp(-\lambda x)} t^{a-1} \ln(1-t)(1-t)^{b-1} dt$$

To find out the maximum likelihood estimators of α , a , b , λ and θ , we have to solve the above system of nonlinear equations with respect to α , a , b , λ and θ . As it seems, this system has no closed form solution in α , a , b , λ and θ . Then we have to use a numerical technique method, such as Newton-Raphson method, to obtain the solution.

5 Numerical Studies

5.1 Simulation Study

In the simulation study, we examine the precision of maximum likelihood estimators for $BEPs(\alpha, a, b, \lambda, \theta)$. The mean of square error (MSE) for each parameter as comparison tool. We follow the following steps:

1. Consider 4 sample size $n = 50, 100, 200, 400$ for $(\alpha, a, b, \lambda, \theta) = (1.25, 1, 0.75, 1.75, 0.5)$
2. Repeat the above step $N = 5000$ times
3. Calculate the MSE of each parameter over the 100 iterations.

The results of generating random sample and calculating the MSE values are prepared in Table 2. As we can see, by increasing n , MSE decreases. Actually, when $n = 400$, MSE have the minimum amount except α in BEL . Generally, based on MSE , we can clearly conclude that we can obtain more precise estimations when we have more sample data.

Now, we check the behavior of ML estimation of the model parameters when n tends to infinity. We simulate 1000 repetition of simulated data for $(\alpha, a, b, \lambda, \theta) = (1.25, 1, 0.75, 1.75, 1)$ with size of $n = 5000$. Then, we consider the average of their MSE over increasing of sample size. Figure 6 shows the effect of increasing sample size n on the amount of MSE for ML estimation of the model parameters. As it can be seen, the MSE decreases significantly by increasing n . In the most of the cases, it converges to 0 around $n = 5000$. Also, there is an exceptions, and the MSE increases for MLE of λ in BEP .

5.2 Real Data Applications

From the current study, it is hoped that BEPS distribution can be used more widely in both a theoretical and an applicable aspect. In this section, we analyze a real data sets to demonstrate the performance of the BEPS distribution in practice. This is a sample of 50 components taken from [21]. We fit the BEPS distributions family over

Table 2 MSE of ML estimation of parameters for different sample sizes

Distribution	n	$MSE(\alpha)$	$MSE(a)$	$MSE(b)$	$MSE(\lambda)$	$MSE(\theta)$
BEP	50	1.600	4.342	2.801	3.832	2.119
	100	1.403	3.050	2.699	3.694	2.060
	200	1.054	2.318	3.172	3.579	1.802
	400	0.996	1.544	3.132	3.480	1.378
BEG	50	1.527	2.311	1.881	2.237	0.155
	100	1.324	2.329	2.668	1.516	0.139
	200	1.078	1.887	2.901	1.004	0.126
	400	0.857	1.044	2.229	0.754	0.127
BEL	50	1.263	2.425	1.493	3.245	0.198
	100	1.699	2.079	1.222	2.592	0.183
	200	2.035	2.426	1.336	1.957	0.168
	400	2.148	2.345	0.822	1.552	0.166
BEB	50	3.347	2.206	1.001	4.356	0.208
	100	3.643	1.128	0.578	3.949	0.197
	200	1.571	0.713	0.435	4.124	0.184
	400	0.933	0.428	0.401	4.103	0.179

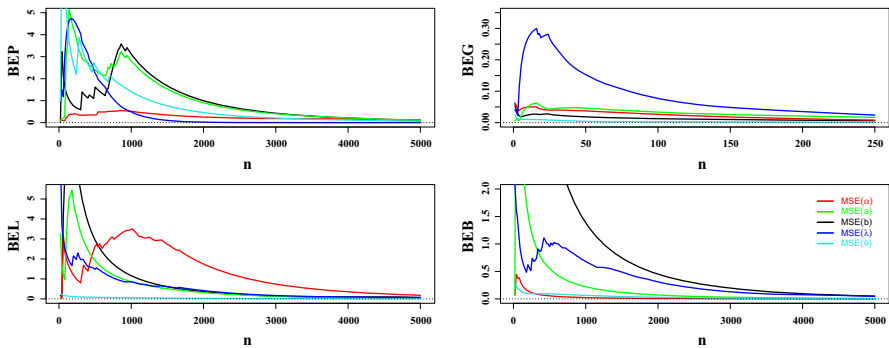


Fig. 6 The values of MSE for each parameter against increasing sample sizes

the widely used Aarset data and analyze the obtained results. It is worth to note that the Aarset data was also reported and analyzed in [2, 4, 22, 23] studies and it contains lifetimes of 50 components, which possess a bathtub-shaped failure rate property. The data contains the times to failure of 50 devices put on life test at time 0, from [21].

The submodels of *BEPS* contained the exponential power series (proposed by [24]), beta exponential and exponential distributions are considered for comparison with the performance of the BEPS distributions family and also for applying likelihood ratio tests. Tables 3 and 4 shows values and Descriptive statistics of the Aarset data. In additions Fig. 7 show histogram and the approximation of density curve of the data.

Table 5 shows the *MLEs* of parameters of the models (distributions) fitted to Aarset data. Also, we can find the values of *AIC*, *BIC* and $-2 \log -likelihood (-2\log(L))$

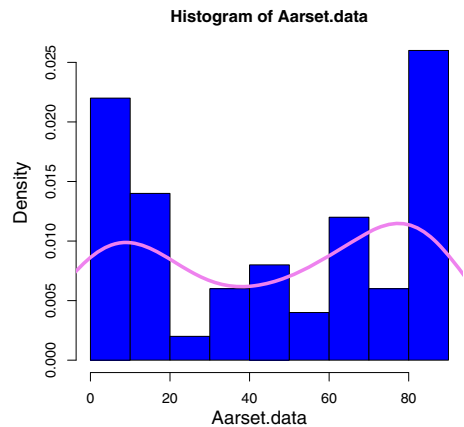
Table 3 Lifetimes of 50 devices

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86									

Table 4 Descriptive values of Aarset data

Min	1st.Qu	Median	Mean	3rd.Qu	Max	Skewness	Kurtosis
0.10	13.50	48.50	45.69	81.25	86.00	−0.14	1.41

Fig. 7 Plots of density function, cumulative distribution function and hazard function



) for the fitted models which are considered for model comparison. According to AIC and $-2 \log(L)$, BEL is the best model. But, based on BIC , BE with a slight difference with BEL has a better performance. In continue, we analyze the performance of fitted models by other approaches to reach the best conclusion about the best fitted model on Aarset data.

For more comparison and reaching better decision about the best model fitted over the Aarset data, we apply likelihood ratio (LR) test. We should calculate the maximum amounts of log-likelihoods of null and alternative hypothesis to obtain the LR statistics for testing some sub-models of the $BEP S$ distribution. In LR test, if Θ be the parameter space of problem, we divide the parameter space as $\Theta = (\Theta_0, \Theta_1)$. Θ_0 is the parameter space of model under null hypothesis (H_0) and Θ_1 is the parameter space of model under alternative hypothesis (H_1). For example, if we want to asses the BE model against BEL , Θ_0 is the parameter space under the assumption of BE distribution and Θ_1 is the parameter space when we consider BEL for data. $\hat{\Theta}_0$ and $\hat{\Theta}_1$ be MLE under H_0 and H_1 , the LR statistic for this problem is calculated as:

$$LR = -2 \log \left[\frac{L(\hat{\Theta}_0)}{L(\hat{\Theta}_1)} \right] = 476.24 - 468.78 = 7.46.$$

Table 5 MLE values of the model parameters, and the values of AIC, BIC and $-2\log(L)$ for Aarset data

Distribution	α	a	b	λ	θ	AIC	BIC	$-2\log(L)$
BEP	1.002324913	0.601624576	0.113833203	0.175731116	4.2e-08	486.44	496.00	476.44
BEG	1.183133957	0.376450289	0.081115057	0.259416258	0	485.11	494.67	475.11
BEL	132.019453122	0.003302091	0.211916428	0.135870949	1.8e-4	478.78	488.34	468.78
EP	0.996530443			0.021755384	2.5e-08	488.12	493.85	482.12
EG	0.964617636			0.021404086	0	489.58	497.22	481.58
EL	0.776080731			0.018641036	1e-09	485.99	491.73	479.99
BE		0.523653541	0.084671895	0.235204516		482.24	487.98	476.24
E				0.021888543		486.18	490.00	482.18

Table 6 Likelihood ratio test results for Aarset data

Model	LR statistic	<i>p</i> value
BEP vs. EP	5.68098	0.058
BEP vs. BE	−0.19645	1.000
BEP vs. E	5.74278	0.125
BEG vs. EG	6.46815	0.011
BEG vs. BE	1.13127	0.568
BEG vs. E	7.07050	0.070
BEL vs. EL	11.21232	0.004
BEL vs. BE	7.46121	0.024
BEL vs. E	13.40045	0.004

Table 7 Goodness-of-fit tests for Aarset data

Distribution	Statistic			<i>p</i> value
	<i>AD</i> *	<i>W</i> *	<i>K</i> − <i>S</i>	
BEP	2.84282	0.50019	0.19070	0.05269
BEG	2.58339	0.45259	0.18094	0.07572
BEL	2.04316	0.32219	0.15827	0.16327
EP	3.67365	0.51814	0.18972	0.05469
EG	3.58586	0.52633	0.18395	0.06783
EL	3.33391	0.57274	0.20064	0.03570
BE	2.78103	0.49463	0.19017	0.05376
E	3.70812	0.52404	0.19107	0.05194

Table 6 shows the information about *LR* test for some submodels against *BEPS* family. For the aforementioned example about testing *BE* versus *BEL*, *p* − value = 0.024 and we have another evidence to obtain *BEL* as the best model fitted over the Aarset data.

Now, we examine the goodness of fit for the fitted models over the Aarset data. we consider and compute the Kolmogorov–Smirnov (*K* − *S*), the Cramer–von Mises (*W**) and Anderson–Darling (*AD**) statistics for testing the goodness of fit. The smaller values of *W**) and *AD**) indicate the better fitting of the model. [25] provided detailed information about Anderson–Darling and Cramer–von Mises tests.

Table 7 presents the values of *AD**, *W**) and *K* − *S* statistics, also it gives *p* − values of *K* − *S* test. According to the Table 7, *BEL* has the minimum values of *AD**) and *W**) statistics. In addition, except *EL*, other distributions are fit for the data based on *K* − *S* test.

Figures 8 and 9, show the estimated survival function plot, TTT plot and Kaplan–Meier curve of the proposed models for the Aarset data. These figures show that the *BEL* distribution has better fit than the other involved distributions.

Fig. 8 Estimated survival function and the empirical survival for Aarset data

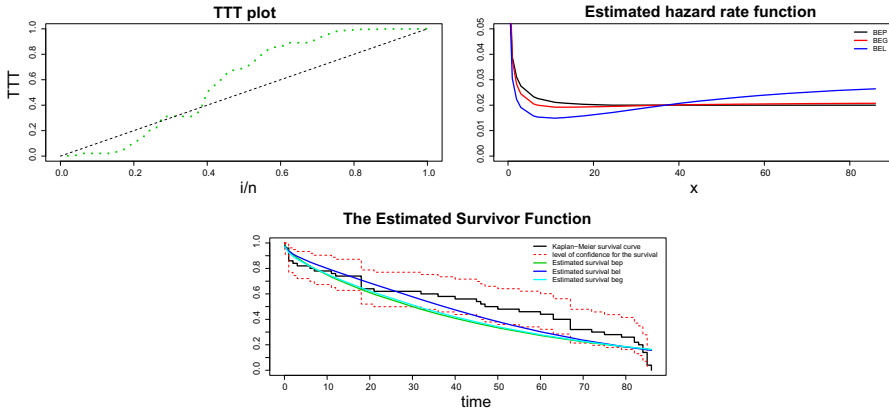
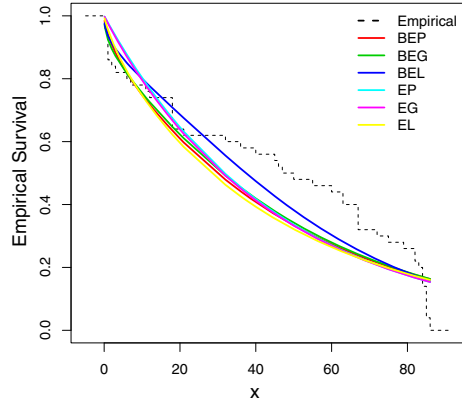


Fig. 9 Empirical TTT-plot (top left), estimated hazard rate function (top right), estimated survival functions (bottom) of three fitted models for Aarset data

Conclusion

We introduce a five parameter lifetime family of distributions that is called Beta Exponential Power Series (BEPS). This type of distributions is a new mixed distribution of the Beta Exponential and power series distribution. Beta Exponential Poisson (BEP), beta Exponential Geometric (BEG), Beta Exponential Logarithmic (BEL), and Beta Exponential Binomial (BEB) are the distributions of this family of statistical distributions. Exponential power series (EP) and Beta Exponential (BE) distributions are the special cases of this type of distributions. Furthermore, we provide a mathematical treatment of this distribution including the order statistics. Also, we provide explicit expressions for the density function of the order statistics and their moments. Several properties of the *BEPS* distribution such as quantiles and moments are provided. *BEPS* has better fitting over Aarset data in comparison of submodels such as *EP* and *BE*.

For the future studies, the *BEPS* family can be extended by alpha power distributions as a new way to model lifetime data. Similarly, *BE* and *EP* can be considered for the extension by alpha power distributions.

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Data Availability The dataset is a sample of 50 components taken from Aarset (1987). The data can be found in the Table 1 in Aarset (1987) which is cited in the references of the paper.

Code Availability Not Applicable.

Conflict of interest Authors state no conflict of interest.

Ethical approval The conducted research is not related to either human or animals use.

Ethical statements All of the followed procedures were in accordance with the ethical and scientific standards. Authors declare that this manuscript is the result of their independent creation under the reviewers' comment and this work does not cause any harm to human or society.

Appendix

Proof of the formulas of Beta Exponential Power Series distribution, Beta Exponential Poisson distribution, Beta Exponential Geometric distribution, Beta Exponential Logarithmic distribution, Beta Exponential Binomial distribution, respectively,

Proof (BEPS):

$$\begin{aligned}
 G(x|N) &= P(X \leq x|N) \\
 &= 1 - P(X > x|N) = 1 - P(\min(Y_1, \dots, Y_N > x|N) \\
 &= 1 - P(Y_1 > x, \dots, Y_N > x|N) \\
 &= 1 - (P(Y_1 > x|N))^N \\
 &= 1 - (1 - P(Y_1 \leq x|N))^N \\
 &= 1 - (1 - P(Z_{i1} \leq x, \dots, Z_{i\alpha} \leq x))^N \\
 &= 1 - (1 - (P(Z_{i1} \leq x))^\alpha)^N \\
 &= 1 - (1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)^N \\
 G(x) &= \sum_{n=1}^{\infty} G(x|N)P(N = n) \\
 &= \sum_{n=1}^{\infty} \{1 - [1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha]^n\} \times \frac{a(n)\theta^n}{C(\theta)}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} \frac{\theta^n a(n)}{C(\theta)} - \sum_{n=1}^{\infty} \frac{\theta^n a(n)[1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha]^n}{C(\theta)} \\
 &= 1 - \frac{1}{C(\theta)} \sum_{n=1}^{\infty} a(n)[\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]^n \\
 &= 1 - \frac{C(\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha))}{C(\theta)} \\
 g(x) &= \frac{\partial G(x)}{\partial x} \\
 g(x) &= \frac{1}{\partial x} \left\{ 1 - \frac{1}{C(\theta)} C(\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)) \right\} \\
 g(x) &= \frac{1}{\partial x} \left\{ 1 - \frac{1}{C(\theta)} C(\theta(1 - (\frac{1}{B(a, b)} \int_0^{1-\exp(-\lambda x)} w^{a-1}(1-w)^{b-1} dw)^\alpha)) \right\} \\
 g(x) &= \frac{\alpha \theta \lambda}{C(\theta) B(a, b)} C'(\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)) (I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1} \\
 &\quad \exp(-\lambda x)(1 - \exp(-\lambda x))^{a-1} \exp(-\lambda x)^{b-1}
 \end{aligned}$$

Proof (BEP):

$$\begin{aligned}
 P(N = n) &= \frac{\lambda}{(\exp(\lambda) - 1)n!} \\
 a(n) &= \frac{1}{n!} \\
 c(\lambda) &= \exp(\lambda) - 1 \\
 G(x) &= \sum_{n=1}^{\infty} G(x|N)P(N = n) \\
 &= \sum_{n=1}^{\infty} \{1 - [1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha]^n\} \times \left\{ \frac{\lambda}{(\exp(\lambda) - 1)n!} \right\} \\
 &= \frac{1}{\exp(\lambda) - 1} \left\{ \sum_{i=1}^{\infty} \frac{\lambda^i}{i!} - \sum_{i=1}^{\infty} \frac{[\lambda - \lambda(I_{1-\exp(-\lambda x)}(a, b))^\alpha]^i}{i!} \right\} \\
 &= \frac{1}{\exp(\lambda) - 1} \{(\exp(\lambda) - 1) - (\exp(\lambda - \lambda[I_{1-\exp(-\lambda x)}(a, b))^\alpha] - 1)\} \\
 &= \left\{ 1 - \frac{(\exp(\theta - \theta[I_{1-\exp(-\lambda x)}(a, b)]^\alpha) - 1)}{\exp(\lambda) - 1} \right\} \\
 g(x) &= \frac{\partial G(x)}{\partial x} \\
 g(x) &= \frac{1}{\partial x} \left\{ 1 - \frac{(\exp(\lambda - \lambda[I_{1-\exp(-\lambda x)}(a, b)]^\alpha) - 1)}{\exp(\lambda) - 1} \right\} \\
 &= \frac{\alpha \lambda^\theta}{(\exp(\lambda) - 1)B(a, b)} \exp(\lambda(1 - [I_{1-\exp(-\lambda x)}]^\alpha)) \exp(-\lambda x)
 \end{aligned}$$

$$(1 - \exp(-\lambda x))^{a-1} \exp(-\lambda x)^{b-1} I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1}$$

Proof (BEG):

$$\begin{aligned} a(n) &= 1 \\ C(\theta) &= \theta(1 - \theta) \\ P(N = n) &= \frac{\theta^n}{\theta(1 - \theta)^{-1}} = \theta^{n-1}(1 - \theta) \\ G(x) &= \sum_{n=1}^{\infty} G(x|N)P(N = n) \\ &= \sum_{n=1}^{\infty} \{1 - [1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}]^n\} \times \{\theta^{n-1}(1 - \theta)\} \\ &= \sum_{n=1}^{\infty} \theta^{n-1}(1 - \theta) - \sum_{n=1}^{\infty} [1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}]^n [\theta^{n-1}(1 - \theta)] \\ &= (1 - \theta) \sum_{n=1}^{\infty} \theta^{n-1} - \theta^{-1}(1 - \theta) \sum_{n=1}^{\infty} \theta^n [1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}]^n \\ &= (1 - \theta) \frac{1}{1 - \theta} - \theta^{-1}(1 - \theta) \sum_{n=1}^{\infty} [\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha})]^n \\ &= \frac{1 - (1 - \theta)1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}}{1 - \theta[1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}]} \\ &= 1 - \frac{(1 - \theta)[1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}]}{1 - \theta[1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}]} \\ g(x) &= \frac{\partial G(x)}{\partial x} \\ g(x) &= \frac{1}{\partial x} \left\{ 1 - \frac{(1 - \theta)[1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}]}{1 - \theta[1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}]} \right\} \\ g(x) &= \frac{\alpha \lambda (1 - \theta) \exp(-\lambda x) (1 - \exp(-\lambda x))^{a-1} \exp(-\lambda x)^{b-1} (I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1}}{B(a, b) (1 - \theta[1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}])^2} \end{aligned}$$

Proof (BEL):

$$\begin{aligned} P(N = n) &= \frac{\theta^n}{-n \log(1 - \theta)} \\ a(n) &= \frac{1}{n} \\ C(\theta) &= -\log(1 - \theta) \\ G(x) &= \sum_{n=1}^{\infty} G(x|N)P(N = n) \\ &= \sum_{n=1}^{\infty} \{1 - [1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha}]^n\} \times \left\{ -\frac{\theta^n}{n \log(1 - \theta)} \right\} \\ &= -\frac{1}{\log(1 - \theta)} \sum_{n=1}^{\infty} \frac{\theta^n}{n} + \frac{1}{\log(1 - \theta)} \sum_{n=1}^{\infty} \frac{[\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^{\alpha})]^n}{n} \end{aligned}$$

$$\begin{aligned}
 \log(1-x) &= -\sum_{n=1}^{\infty} \frac{x^n}{n} \\
 G(x) &= \frac{1}{\log(1-\theta)} \left\{ -\sum_{n=1}^{\infty} \frac{\theta^n}{n} + \sum_{n=1}^{\infty} \frac{[\theta(1 - (I_{1-\exp(-\lambda x)})^\alpha)]^n}{n} \right\} \\
 &= \frac{1}{\log(1-\theta)} \left\{ \log(1-\theta) - \sum_{n=1}^{\infty} \frac{-[\theta(1 - (I_{1-\exp(-\lambda x)})^\alpha)]^n}{n} \right\} \\
 &= 1 - \frac{\log[\theta(1 - (I_{1-\exp(-\lambda x)})^\alpha)]}{\log(1-\theta)} \\
 g(x) &= \frac{\partial G(x)}{\partial x} \\
 g(x) &= \frac{1}{\partial x} \left\{ 1 - \frac{\log[\theta(1 - (I_{1-\exp(-\lambda x)})^\alpha)]}{\log(1-\theta)} \right\} \\
 g(x) &= \frac{\lambda \alpha}{\log(1-\theta) B(a, b)} \frac{\exp(-\lambda x)(1 - \exp(-\lambda x))^{a-1} \exp(-\lambda x)^{b-1} (I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1}}{[1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha]}
 \end{aligned}$$

Proof(BEB):

$$\begin{aligned}
 a(n) &= \binom{m}{n} \\
 C(\theta) &= (\theta + 1)^m - 1 \\
 P(N = n) &= \frac{\binom{m}{n} \theta}{(\theta + 1)^m - 1} \\
 G(x) &= \sum_{n=1}^m G(x|N) P(N = n) \\
 &= \sum_{n=1}^m \left\{ 1 - [1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha]^n \right\} \times \left\{ \frac{\binom{m}{n} \theta}{(\theta + 1)^m - 1} \right\} \\
 &= \frac{1}{(\theta + 1)^m - 1} \left\{ \sum_{n=1}^m \binom{m}{n} \theta^n - \sum_{n=1}^m \binom{m}{n} [\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]^n 1^{m-n} \right\} \\
 &= \frac{1}{(\theta + 1)^m - 1} \left\{ \sum_{n=1}^m \binom{m}{n} \theta^n 1^{m-n} - \binom{m}{0} \theta^0 1^{m-0} \right. \\
 &\quad \left. - \sum_{n=1}^m \binom{m}{n} [(\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]^n - 1 \right\} \\
 &= 1 - \frac{[\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]^m - 1}{(\theta + 1)^m - 1} \\
 g(x) &= \frac{\partial G(x)}{\partial x} \\
 g(x) &= \frac{1}{\partial x} \left\{ 1 - \frac{[\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]^m - 1}{(\theta + 1)^m - 1} \right\} \\
 g(x) &= \frac{\alpha \lambda \theta m}{((\theta + 1)^m - 1) B(a, b)} \\
 &\quad \left\{ [\theta(1 - (I_{1-\exp(-\lambda x)}(a, b))^\alpha)]^{m-1} \exp(-\lambda x)(1 - \exp(-\lambda x))^{\alpha-1} \right. \\
 &\quad \left. \exp(-\lambda x)^{b-1} (I_{1-\exp(-\lambda x)}(a, b))^{\alpha-1} \right\}
 \end{aligned}$$

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