THE BINARY GOLD FUNCTION AND ITS *c*-BOOMERANG CONNECTIVITY TABLE

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ABSTRACT. Here, we give a complete description of the *c*-Boomerang Connectivity Table for the Gold function over finite fields of even characteristic, by using double Weil sums. In the process we generalize a result of Boura and Canteaut (IACR Trans. Symmetric Cryptol. 2018(3) : 290-310, 2018) for the classical boomerang uniformity.

1. INTRODUCTION

Let \mathbb{F}_q be the finite field with $q = p^n$ elements, where p is a prime and n is a positive integer. The multiplicative cyclic group of nonzero elements of the finite field is denoted by $\mathbb{F}_q^* = \langle g \rangle$, where g is a primitive element of \mathbb{F}_q . A Weil sum is an important character sum defined as follows

$$\sum_{x \in \mathbb{F}_q} \chi(F(x)),$$

where χ is an additive character of \mathbb{F}_q and F(x) is a polynomial in $\mathbb{F}_q[x]$. It is well-known that a polynomial F(x) over finite field \mathbb{F}_q is a permutation polynomial (PP) if and only if its Weil sum $\sum_{x \in \mathbb{F}_q} \chi(F(x)) = 0$ for all nontrivial additive characters χ of \mathbb{F}_q . Permutation polynomials are a very important class of polynomials as they have applications in coding theory and cryptography, especially in the substitution boxes (S-boxes) of the block ciphers. The security of the S-boxes relies on certain properties of the function F(x), e.g., its differential uniformity, boomerang uniformity, nonlinearity etc.

Recently, Cid et al. [4] introduced a "new tool" for analyzing the boomerang style attack proposed by Wagner [17]. This new tool is usually referred to as Boomerang Connectivity Table (BCT). Boura and Canteaut [2] further studied BCT and coined the term boomerang uniformity, which is essentially the maximum value in the BCT. Li et al. [9] provided new insights in the study of BCT and presented an equivalent technique to compute BCT, which does not require the compositional inverse of the permutation polynomial F(x) at all. In fact, Li et al. [9] also gave a characterization of BCT in terms of Walsh transform and gave a class of permutation polynomial with boomerang uniformity 4.

Recently, Stănică [12] extended the notion of BCT and boomerang uniformity. In fact, he defined what he termed as c-BCT and c-boomerang uniformity for an arbitrary polynomial function F over \mathbb{F}_q and for any $c \neq 0 \in \mathbb{F}_q$. Let $a, b \in \mathbb{F}_q$, then the entry of the c-Boomerang Connectivity Table (c-BCT) at $(a, b) \in \mathbb{F}_{p^n} \times \mathbb{F}_{p^n}$, denoted as ${}_c\mathcal{B}_F(a, b)$, is the number of solutions in $\mathbb{F}_{p^n} \times \mathbb{F}_{p^n}$ of the following system

(1.1)
$$\begin{cases} F(x) - cF(y) = b \\ F(x+a) - c^{-1}F(y+a) = b. \end{cases}$$

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The c-boomerang uniformity of F is defined as

$$\beta_{F,c} = \max_{a,b \in \mathbb{F}_{p^n}^*} {}_{c} \mathcal{B}_F(a,b).$$

In yet other recent papers, Stănică [13, 14] further studied the *c*-BCT for the swapped inverse function and also gave an elegant description of the *c*-BCT entries of the power map in terms of double Weil sums. He further simplified his expressions for the Gold function x^{p^k+1} over \mathbb{F}_{p^n} , for all $1 \leq k < n$ and p odd. In this paper, we shall complement the work of [14] to the finite fields of even characteristic (p = 2).

The paper is structured as follows. Section 2 contains some preliminary results that will be used across the sections. Section 3 contains the characterization of c-BCT entries in terms of double Weil sums. For c = 1, we further simplify this expression in Section 4. In fact, Theorem 4.1 generalizes previously known results of Boura and Canteaut [2]. In Section 5, we consider the case when $c \in \mathbb{F}_{2^e} \setminus \{0, 1\}$, where $e = \gcd(k, n)$. In Section 6, we discuss the general case. Finally, in Section 7, we discuss the affine, extended affine and CCZ-equivalence as it relates to c-boomerang uniformity.

2. Preliminaries

We begin this section by first recalling the recent notion of *c*-differentials introduced in [7]. We shall assume that $q = 2^n$ for rest of the paper. For an (n, n)-function $F : \mathbb{F}_q \to \mathbb{F}_q$, and $c \in \mathbb{F}_q$, we define the (*multiplicative*) *c*-derivative of *F* with respect to $a \in \mathbb{F}_q$ to be the function

$$D_a F(x) = F(x+a) - cF(x)$$
, for all $x \in \mathbb{F}_q$.

Further, for $a, b \in \mathbb{F}_q$, we let the entries of the *c*-Difference Distribution Table (*c*-DDT) be defined by ${}_c\Delta_F(a,b) = \#\{x \in \mathbb{F}_q : F(x+a) - cF(x) = b\}$. We call the quantity

$$\delta_{F,c} = \max \left\{ {}_{c} \Delta_{F}(a,b) \, | \, a, b \in \mathbb{F}_{q}, \text{ and } a \neq 0 \text{ if } c = 1 \right\},\$$

the *c*-differential uniformity of F. Note that the case c = 1 corresponds to the usual notion of differential uniformity. The interested reader may refer to [1, 8, 10, 15, 16, 18] for some recent results concerning *c*-differential uniformity.

The following theorem is a "binary" analogue of [14, Theorem 1], which gives a nice connection between c-BCT and c-DDT entries of the power map x^d over \mathbb{F}_{2^n} .

Theorem 2.1. Let $F(x) = x^d$ be a power function on \mathbb{F}_q , $q = 2^n$ and $c \in \mathbb{F}_q^*$. Then, for fixed $b \in \mathbb{F}_q^*$, the c-Boomerang Connectivity Table entry ${}_c\mathcal{B}_F(1,b)$ at (1,b) is given by

$$\frac{1}{q} \left(\sum_{w \in \mathbb{F}_q} (c\Delta_F(w, b) + c^{-1}\Delta_F(w, b)) \right) - 1 + \frac{1}{q^2} \sum_{\alpha, \beta \in \mathbb{F}_q, \alpha \beta \neq 0} \chi_1(b(\alpha + \beta)) S_{\alpha, \beta} S_{\alpha c, \beta c^{-1}},$$

with

$$S_{\alpha,\beta} = \sum_{x \in \mathbb{F}_q} \chi_1\left(\alpha x^d\right) \chi_1\left(\beta (x+1)^d\right)$$

= $\frac{1}{(q-1)^2} \sum_{j,k=0}^{q-2} G(\bar{\psi}_j, \chi_1) G(\bar{\psi}_k, \chi_1) \sum_{x \in \mathbb{F}_q} \psi_1\left((\alpha x^d)^j (\beta (x+1)^d)^k\right),$

where χ_1 is the canonical additive character of the additive group of \mathbb{F}_q , ψ_k is the k-th multiplicative character of the multiplicative group of \mathbb{F}_q and $G(\psi, \chi)$ is the Gauss sum.

We shall now state some lemmas that will be used in the sequel. The following lemma is well-known and has been used in various contexts.

Lemma 2.2. Let $e = \operatorname{gcd}(k, n)$. Then

$$gcd(2^{k}+1,2^{n}-1) = \begin{cases} 1 & \text{if } n/e \text{ is odd,} \\ 2^{e}+1 & \text{if } n/e \text{ is even.} \end{cases}$$

We shall also use the following lemma, which appeared in [5], describing the number of roots in \mathbb{F}_{2^n} of a linearized polynomial $u^{2^k} x^{2^{2k}} + ux$, where $u \in \mathbb{F}_{2^n}^*$.

Lemma 2.3. [5, Theorem 3.1] Let g be a primitive element of \mathbb{F}_{2^n} and let e = gcd(n, k). For any $u \in \mathbb{F}_{2^n}^*$ consider the linearized polynomial $L_u(x) = u^{2^k} x^{2^{2k}} + ux$ over \mathbb{F}_{2^n} . Then for the equation $L_u(x) = 0$, the following are true:

- (1) If n/e is odd, then there are 2^e solutions to this equation for any choice of $u \in \mathbb{F}_{2^n}^*$;
- (2) If n/e is even and $u = g^{t(2^e+1)}$ for some t, then there are 2^{2e} solutions to the equation;
- (3) If n/e is even and $u \neq g^{t(2^e+1)}$ for any t, then x = 0 is the only solution.

The explicit expression for the Weil sum of the form $\sum_{x \in \mathbb{F}_{2^n}} \chi_1(ux^{2^k+1}+vx)$, where $u, v \in \mathbb{F}_{2^n}$, is obtained in [5]. In what follows, we shall denote the Weil sum $\sum_{x \in \mathbb{F}_q} \chi(ux^{2^k+1}+vx)$ by $\mathfrak{S}(u,v)$. The following lemma gives the explicit expression for $\mathfrak{S}(u,0)$.

Lemma 2.4. [5] Let χ be any nontrivial additive character of \mathbb{F}_q and g be the primitive element of the cyclic group \mathbb{F}_q^* . The following hold:

(1) If n/e is odd, then

$$\sum_{x \in \mathbb{F}_q} \chi(ux^{2^k+1}) = \begin{cases} q & \text{if } u = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let n/e be even so that n = 2m for some integer m. Then

$$\sum_{x \in \mathbb{F}_q} \chi(ux^{2^k+1}) = \begin{cases} (-1)^{m/e} 2^m & \text{if } u \neq g^{t(2^e+1)} \text{ for any integer } t, \\ (-1)^{\frac{m}{e}+1} 2^{m+e} & \text{if } u = g^{t(2^e+1)} \text{ for some integer } t. \end{cases}$$

From Lemma 2.2, it is easy to see that when n/e is odd, the power map $x^{2^{k+1}}$ permutes \mathbb{F}_{2^n} . Therefore if $u \neq 0$, there exists a unique element $\gamma \in \mathbb{F}_q^*$ such that $\gamma^{2^{k+1}} = u$ and hence

$$\begin{split} \mathfrak{S}(u,v) &= \sum_{x \in \mathbb{F}_q} \chi(ux^{2^k+1} + vx) \\ &= \sum_{x \in \mathbb{F}_q} \chi(x^{2^k+1} + v\gamma^{-1}x) \\ &= \mathfrak{S}(1,v\gamma^{-1}). \end{split}$$

The following lemma gives the expression for the Weil sum $\mathfrak{S}(1, v)$ for $v \neq 0$ and n/e odd. Lemma 2.5. [5, Theorem 4.2] Let $v \neq 0$ and n/e is odd. Then

$$\mathfrak{S}(1,v) = \begin{cases} 0 & \text{if } \operatorname{Tr}_e(v) \neq 1, \\ \left(\frac{2}{n/e}\right)^e 2^{\frac{n+e}{2}} & \text{if } \operatorname{Tr}_e(v) = 1, \end{cases}$$

where $\left(\frac{2}{n/e}\right)$ is the Jacobi symbol.

In the case when $u, v \neq 0$ and n/e is even, the Weil sum $\mathfrak{S}(u, v)$ depends whether or not the linearized polynomial $L_u(x) = u^{2^k} x^{2^{2k}} + ux$ is a permutation of \mathbb{F}_{2^n} . The following lemma gives the expression for Weil sum $\mathfrak{S}(u, v)$ for $u, v \neq 0$ and n/e even.

Lemma 2.6. [5, Theorem 5.3] Let $u, v \in \mathbb{F}_q^*$ and n/e is even so n = 2m for some integer m. Then

(1) If $u \neq g^{t(2^e+1)}$ for any integer t then L_u is a PP. Let $x_u \in \mathbb{F}_q$ be the unique solution of the equation $L_u(x) = v^{2^k}$. Then

$$\mathfrak{S}(u,v) = (-1)^{m/e} 2^m \chi_1(u x_u^{2^k+1}).$$

(2) If $u = g^{t(2^{e}+1)}$ for some integer t, then $\mathfrak{S}(u,v) = 0$ unless the equation $L_u(x) = v^{2^{k}}$ is solvable. If the equation $L_u(x) = v^{2^{k}}$ is solvable with some solution, say x_u , then

$$\mathfrak{S}(u,v) = \begin{cases} (-1)^{m/e} 2^m \chi_1(u x_u^{2^k+1}) & \text{if } \operatorname{Tr}_e(u) \neq 0, \\ (-1)^{\frac{m}{e}+1} 2^{m+e} \chi_1(u x_u^{2^k+1}) & \text{if } \operatorname{Tr}_e(u) = 0. \end{cases}$$

3. The binary Gold function

In this section, we shall give the explicit expression for the *c*-BCT entries of the Gold function x^{2^k+1} over \mathbb{F}_{2^n} , for all $c \neq 0$. Recall that the *c*-boomerang uniformity of a power function $F(x) = x^d$ over \mathbb{F}_{2^n} is given by $\max_{b \in \mathbb{F}_{2^n}^*} c\mathcal{B}_F(1,b)$, where $c\mathcal{B}_F(1,b)$ is the number of solutions in $\mathbb{F}_q \times \mathbb{F}_q$, $q = 2^n$ of the following system

(3.1)
$$\begin{cases} x^d + cy^d = b\\ (x+1)^d + c^{-1}(y+1)^d = b. \end{cases}$$

As done in [14], for $b \neq 0$ and fixed $c \neq 0$, the number of solutions $(x, y) \in \mathbb{F}_q^2$ of the system (3.1) is given by

$${}_{c}\mathcal{B}_{F}(1,b) = \frac{1}{q^{2}} \sum_{x,y \in \mathbb{F}_{q}} \sum_{\alpha \in \mathbb{F}_{q}} \chi_{1} \left(\alpha \left(x^{d} + cy^{d} + b \right) \right) \sum_{\beta \in \mathbb{F}_{q}} \chi_{1} \left(\beta \left((x+1)^{d} + c^{-1}(y+1)^{d} + b \right) \right)$$
$$= \frac{1}{q^{2}} \sum_{\alpha,\beta \in \mathbb{F}_{q}} \chi_{1} \left(b \left(\alpha + \beta \right) \right) \sum_{x \in \mathbb{F}_{q}} \chi_{1} \left(\alpha x^{d} + \beta (x+1)^{d} \right) \sum_{y \in \mathbb{F}_{q}} \chi_{1} \left(c\alpha y^{d} + c^{-1}\beta (y+1)^{d} \right)$$
$$= \frac{1}{q^{2}} \sum_{\alpha,\beta \in \mathbb{F}_{q}} \chi_{1} \left(b \left(\alpha + \beta \right) \right) S_{\alpha,\beta} S_{c\alpha,c^{-1}\beta},$$

where $S_{\alpha,\beta} = \sum_{x \in \mathbb{F}_q} \chi_1 \left(\alpha x^d + \beta (x+1)^d \right)$. Therefore, the problem of computing the *c*-BCT entry ${}_{c}\mathcal{B}_{F}(1,b)$ is reduced to the computation of the product of the Weil sums $S_{\alpha,\beta}$ and $S_{c\alpha,c^{-1}\beta}$. Now, in the particular case when $d = 2^k + 1$, i.e., for the Gold case, we shall further simplify the expression for $S_{\alpha,\beta}$ as follows:

$$S_{\alpha,\beta} = \sum_{x \in \mathbb{F}_q} \chi_1 \left(\alpha x^{2^k + 1} + \beta (x+1)^{2^k + 1} \right)$$

= $\chi_1(\beta) \sum_{x \in \mathbb{F}_q} \chi_1((\alpha + \beta) x^{2^k + 1}) \ \chi_1(\beta x^{2^k} + \beta x)$
= $\chi_1(\beta) \sum_{x \in \mathbb{F}_q} \chi_1((\alpha + \beta) x^{2^k + 1}) \ \chi_1((\beta^{2^{n-k}} x)^{2^k} + \beta x)$

$$= \chi_1(\beta) \sum_{x \in \mathbb{F}_q} \chi_1((\alpha + \beta)x^{2^{k+1}}) \ \chi_1((\beta^{2^{n-k}} + \beta)x)$$

$$= \chi_1(\beta) \sum_{x \in \mathbb{F}_q} \chi_1((\alpha + \beta)x^{2^{k+1}} + (\beta^{2^{n-k}} + \beta)x)$$

$$= \chi_1(\beta) \sum_{x \in \mathbb{F}_q} \chi_1(Ax^{2^{k+1}} + Bx),$$

where $A = \alpha + \beta$ and $B = \beta^{2^{n-k}} + \beta$. Here one may note that A = 0 if and only if $\alpha = \beta$. Also, B = 0 if and only if $\beta \in \mathbb{F}_{2^e}$, since

$$B = 0 \Leftrightarrow \beta^{2^{n-k}} = \beta$$

$$\Leftrightarrow \beta^{2^{n-k}-1} = 1$$

$$\Leftrightarrow \beta^{2^{\gcd(n-k,n)}-1} = 1$$

$$\Leftrightarrow \beta^{2^{e}-1} = 1, \text{ (as } \gcd(n-k,n) = e)$$

$$\Leftrightarrow \beta \in \mathbb{F}_{2^{e}}.$$

Now we shall calculate $S_{\alpha,\beta}$ in two cases, namely, n/e odd and n/e even, respectively. **Case 1**: n/e is odd.

In this case, if $\alpha = \beta$ and $\beta \in \mathbb{F}_{2^e}$, then $S_{\alpha,\beta} = q\chi_1(\beta)$. If $\alpha = \beta$ and $\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$ then $S_{\alpha,\beta} = 0$. In the event of $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^e}$, again we have $S_{\alpha,\beta} = 0$. Finally, if $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$, by Lemma 2.5 we have,

$$S_{\alpha,\beta} = \begin{cases} 0 & \text{if } \operatorname{Tr}_e(B\gamma^{-1}) \neq 1, \\ \left(\frac{2}{n/e}\right)^e 2^{\frac{n+e}{2}} \chi_1(\beta) & \text{if } \operatorname{Tr}_e(B\gamma^{-1}) = 1, \end{cases}$$

where $\gamma \in \mathbb{F}_q$ is the unique element such that $\gamma^{2^k+1} = A$. Case 2: n/e is even.

Let n = 2m, for some positive integer m and g be a primitive element of the finite field \mathbb{F}_q . When $\alpha = \beta$ and $\beta \in \mathbb{F}_{2^e}$ then $S_{\alpha,\beta} = q\chi_1(\beta)$. If $\alpha = \beta$ and $\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$ then again $S_{\alpha,\beta} = 0$. In the event of $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^e}$, by Lemma 2.4 we have

$$S_{\alpha,\beta} = \begin{cases} (-1)^{m/e} 2^m \chi_1(\beta) & \text{if } A \neq g^{t(2^e+1)} \text{ for any integer } t, \\ (-1)^{\frac{m}{e}+1} 2^{m+e} \chi_1(\beta) & \text{if } A = g^{t(2^e+1)} \text{ for some integer } t. \end{cases}$$

Finally, when $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$, we shall consider two cases depending on whether or not the linearized polynomial $L_A(x) = A^{2^k} x^{2^{2k}} + Ax$ is a permutation polynomial. From Lemma 2.3, L_A is a permutation polynomial if and only if n/e is even and $A \neq g^{t(2^e+1)}$ for any integer t. Therefore, when n/e is even and $A \neq g^{t(2^e+1)}$ for any integer t, the equation $L_A(x) = B^{2^k}$ will have a unique solution, say x_A . Therefore, by Lemma 2.6, we have

$$S_{\alpha,\beta} = (-1)^{m/e} 2^m \chi_1(\beta) \chi_1(A x_A^{2^k+1}).$$

Now if the linearized polynomial L_A is not permutation, i.e., n/e is even and $A = g^{t(2^e+1)}$ for some integer t, we again have two cases depending on whether or not the equation $L_A(x) = B^{2^k}$ is solvable. In the case when equation $L_A(x) = B^{2^k}$ is solvable, let x_A be one of its solution. Therefore, by Lemma 2.6 we have,

$$S_{\alpha,\beta} = \begin{cases} (-1)^{\frac{m}{e}+1} 2^{m+e} \chi_1(\beta) \chi_1 \left(A x_A^{2^k+1}\right) & \text{if } \operatorname{Tr}_e(A) = 0, \\ (-1)^{\frac{m}{e}} 2^m \chi_1(\beta) \chi_1 \left(A x_A^{2^k+1}\right) & \text{if } \operatorname{Tr}_e(A) \neq 0. \end{cases}$$

If $L_A(x) = B^{2^k}$ is not solvable, again, by Lemma 2.6, $S_{\alpha,\beta} = 0$. Thus we have computed $S_{\alpha,\beta}$ in all possible cases. Similarly, we can find $S_{c\alpha,c^{-1}\beta}$ by putting $c\alpha$ and $c^{-1}\beta$ in place of α and β , respectively. We shall now explicitly compute the *c*-BCT entry ${}_{c}B_{F}(1,b)$ for $c = 1, c \in \mathbb{F}_{2^e} \setminus \{0,1\}$ and $c \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$ in the forthcoming sections.

4. The case c = 1

When c = 1, $S_{\alpha,\beta}$ and $S_{c\alpha,c^{-1}\beta}$ coincide, therefore for any fixed $b \neq 0$, the c-BCT entry is given by,

$${}_{1}\mathcal{B}_{F}(1,b) = \frac{1}{q^{2}} \sum_{\alpha,\beta \in \mathbb{F}_{q}} \chi_{1} \left(b \left(\alpha + \beta \right) \right) S_{\alpha,\beta}^{2}.$$

Let us denote $T_b = S^2_{\alpha,\beta}$. Now we shall consider two cases, namely, n/e odd and n/e even, respectively.

Case 1: n/e is odd. We consider the following subcases.

(1) If $\alpha = \beta$ and $\beta \in \mathbb{F}_{2^e}$, then

$$T_b^{[1]} = q^2 \chi_1(\beta)^2 = q^2.$$

(2) If $\alpha = \beta$ and $\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$, then

$$\Gamma_b^{[2]} = 0.$$

(3) If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^e}$, then

$$T_b^{[3]} = 0.$$

(4) If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$ then

$$T_b^{[4]} = \begin{cases} 0 & \text{if } \operatorname{Tr}_e(B\gamma^{-1}) \neq 1, \\ 2^{n+e} & \text{if } \operatorname{Tr}_e(B\gamma^{-1}) = 1. \end{cases}$$

Nyberg [11, Proposition 3] showed that the differential uniformity of the Gold function $x \mapsto x^{2^{k+1}}$ over \mathbb{F}_{2^n} is 2^e , where $e = \gcd(k, n)$. Also, from [4], we know that the boomerang uniformity of the APN function equals 2. Boura and Canteaut [2, Proposition 8] proved that when n/e is odd and $n \equiv 2 \pmod{4}$, then the differential as well as the boomerang uniformity of the Gold function $x \mapsto x^{2^{k+1}}$ is 4. Our first theorem in this section generalizes the two previously mentioned results, and gives the boomerang uniformity of the Gold function for any parameters, when $\frac{n}{e}$ is odd. Note that we would require the notion of Walsh-Hadamard transform in the proof of this theorem, which is defined as follows.

For $f : \mathbb{F}_{2^n} \to \mathbb{F}_2$ we define the Walsh-Hadamard transform to be the integer-valued function

$$\mathcal{W}_f(u) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + \operatorname{Tr}(ux)}, u \in \mathbb{F}_{2^n}.$$

The Walsh transform $\mathcal{W}_F(a, b)$ of an (n, m)-function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^m}$ at $a \in \mathbb{F}_{2^n}, b \in \mathbb{F}_{2^m}$ is the Walsh-Hadamard transform of its component function $\operatorname{Tr}_1^m(bF(x))$ at a, that is,

$$\mathcal{W}_F(a,b) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\operatorname{Tr}_1^m(bF(x)) - \operatorname{Tr}_1^n(ax)}$$

Theorem 4.1. Let $F(x) = x^{2^k+1}$, $1 \le k < n$, be a function on \mathbb{F}_q , $q = 2^n$, $n \ge 2$. Let c = 1 and n/e be odd, where $e = \gcd(k, n)$. Then the c-BCT entry ${}_1\mathcal{B}_F(1, b)$ of F at (1, b) is

$$_{1}\mathcal{B}_{F}(1,b) = 0, \ or, \ 2^{e},$$

if
$$\operatorname{Tr}_{e}\left(b^{\frac{1}{2}}\right) = 0$$
, respectively, $\operatorname{Tr}_{e}\left(b^{\frac{1}{2}}\right) \neq 0$.

Proof. For every α, β , let $A = \alpha + \beta, B = \beta^{2^{-k}} + \beta$, and $\gamma \in \mathbb{F}_q$ be the unique element such that $\gamma^{2^k+1} = A$. Further, let

$$\mathcal{A} = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha = \beta \in \mathbb{F}_{2^e} \}, \\ \mathcal{B} = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha = \beta \in \mathbb{F}_q \setminus \mathbb{F}_{2^e} \}, \\ \mathcal{C} = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha \neq \beta \text{ and } \beta \in \mathbb{F}_{2^e} \}, \\ \mathcal{D} = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid \alpha \neq \beta \text{ and } \beta \in \mathbb{F}_q \setminus \mathbb{F}_{2^e} \}, \\ \mathcal{E} = \{ (\alpha, \beta) \in \mathcal{D} \mid \operatorname{Tr}_e(B\gamma^{-1}) \neq 1 \}, \\ \mathcal{F} = \{ (\alpha, \beta) \in \mathcal{D} \mid \operatorname{Tr}_e(B\gamma^{-1}) = 1 \}. \end{cases}$$

Then,

$${}_{1}\mathcal{B}_{F}(1,b) = \frac{1}{q^{2}} \left(\sum_{(\alpha,\beta)\in\mathcal{A}} \chi_{1}(b(\alpha+\beta))T_{b}^{[1]} + \sum_{(\alpha,\beta)\in\mathcal{B}} \chi_{1}(b(\alpha+\beta))T_{b}^{[2]} + \sum_{(\alpha,\beta)\in\mathcal{C}} \chi_{1}(b(\alpha+\beta))T_{b}^{[3]} + \sum_{(\alpha,\beta)\in\mathcal{F}} \chi_{1}(b(\alpha+\beta))T_{b}^{[4]} \right)$$
$$= \frac{1}{q^{2}} \left(\sum_{(\alpha,\beta)\in\mathcal{A}} q^{2} + \sum_{(\alpha,\beta)\in\mathcal{F}} \chi_{1}(b(\alpha+\beta))2^{n+e} \right)$$
$$= 2^{e} + \frac{2^{e}}{2^{n}} \sum_{(\alpha,\beta)\in\mathcal{F}} \chi_{1}(b(\alpha+\beta)).$$

As customary, $t^{-1} = t^{2^n-2}$, rendering $0^{-1} = 0$. For each $\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$, we let (if $\beta \in \mathbb{F}_{2^e}$, $Y_{\beta} = \mathbb{F}_{2^n}$)

$$Y_{\beta} = \left\{ \gamma^{-1} \in \mathbb{F}_{2^n} : \operatorname{Tr}_e\left((\beta^{2^{-k}} + \beta)\gamma^{-1} \right) = 1 \right\},$$

and

$$T_{\beta} = \left\{ d \in \mathbb{F}_{2^n} : \operatorname{Tr}_e((\beta^{2^{-k}} + \beta)d) = 0 \right\} = \langle \beta^{2^{-k}} + \beta \rangle^{\perp_e}.$$

We shall use below that when $\frac{n}{e}$ is odd, then $\operatorname{Tr}_e(1) = 1$. We label by $\langle S \rangle_e$ the \mathbb{F}_{2^e} -linear subspace in \mathbb{F}_{2^n} generate by S and we write S^{\perp_e} , for the trace orthogonal (via the relative trace Tr_e) of the subspace $\langle S \rangle_e$ (if e = 1, we drop the subscripts). Since $\operatorname{Tr}_e(1) = 1$, then, $(\beta^{2^{-k}} + \beta)^{-1} \in Y_\beta$. If $\gamma_1^{-1}, \gamma_2^{-1} \in Y_\beta$, then $\gamma_1^{-1} + \gamma_2^{-1} \in T_\beta$, of cardinality $|T_\beta| = 2^{n-1}$. Reciprocally, if $\gamma^{-1} \in Y_\beta$ and $d \in T_\beta$, it is easy to see that $\gamma^{-1} + d \in Y_\beta$. Therefore, Y_β is the affine subspace $Y_\beta = \gamma_\beta + T_\beta$, where $\gamma_\beta = (\beta^{2^{-k}} + \beta)^{-1}$.

Next, we observe that the kernel of $\phi : \beta \mapsto \beta^{2^{-k}} + \beta$, say ker (ϕ) , is an \mathbb{F}_2 -linear space of dimension e (in fact, it is exactly \mathbb{F}_{2^e}) and the image of ϕ , say Im (ϕ) , is an \mathbb{F}_2 -linear space of dimension n - e. Further, we show that Im $(\phi)^{\perp_e} = \ker(\phi)$. We use below the fact that $\operatorname{Tr}_e(x^{2^e}) = \operatorname{Tr}_e(x)$ and $e \mid k$. Let $u \in \operatorname{Im}(\phi)^{\perp_e}$, that is, for all $\beta \in \mathbb{F}_{2^n}$,

$$0 = \operatorname{Tr}_{e}(u(\beta^{2^{-k}} + \beta)) = \operatorname{Tr}_{e}(u\beta^{2^{-k}}) + \operatorname{Tr}_{e}(u\beta) = \operatorname{Tr}_{e}(u^{2^{k}}\beta) + \operatorname{Tr}_{e}(u\beta) = \operatorname{Tr}_{e}((u + u^{2^{k}})\beta),$$

and so, $u^{2^k} + u = 0$, which shows the claim. For easy referral, if we speak of the dimension of an F_{2^e} -linear space S, we shall be using the notation dim_e S (no subscript if e = 1).

We will be using below the Poisson summation formula (see [3, Corollary 8.9] and [6, Theorem 2.15]), which states that if $f : \mathbb{F}_{2^n} \to \mathbb{R}$ and S is a subspace of \mathbb{F}_{2^n} of dimension dim S, then

$$\sum_{\substack{\in \alpha+S}} \mathcal{W}_f(u)(-1)^{\operatorname{Tr}(\beta u)} = 2^{\dim S}(-1)^{\operatorname{Tr}(\alpha\beta)} \sum_{\substack{u\in\beta+S^{\perp}}} f(u)(-1)^{\operatorname{Tr}(\alpha u)},$$

and in particular,

u

$$\sum_{u \in S} \mathcal{W}_f(u) = 2^{\dim S} \sum_{u \in S^{\perp}} f(u).$$

Now, we are able to compute our sum (labelling $\alpha = \beta + \gamma^{2^k+1}$, and writing $\phi^{-1}(t) = \{\beta : \phi(\beta) = t\}$; we also note that when $\frac{n}{e}$ is odd, $\gcd(2^k + 1, 2^n - 1) = 1$, and so $\gamma \mapsto \gamma^{2^k+1}$ is a permutation)

$${}_{1}\mathcal{B}_{F}(1,b) = 2^{e} + \frac{2^{e}}{2^{n}} \sum_{\substack{\beta \in \mathbb{F}_{2^{n}} \setminus \mathbb{F}_{2^{e}}, \gamma \in \mathbb{F}_{2^{n}} \\ \operatorname{Tr}_{e}\left((\beta^{2^{-k}} + \beta)\gamma^{-1}\right) = 1}} \chi_{1}\left(b\gamma^{2^{k}+1}\right)}$$
$$= 2^{e} + \frac{2^{e}}{2^{n}} \sum_{\beta \in \mathbb{F}_{2^{n}} \setminus \mathbb{F}_{2^{e}}} \sum_{\gamma^{-1} \in Y_{\beta}} \chi_{1}\left(b\gamma^{2^{k}+1}\right)}$$
$$= 2^{e} + \frac{2^{e}}{2^{n}} \sum_{\beta \in \mathbb{F}_{2^{n}}} \sum_{x \in (\beta^{2^{-k}} + \beta)^{-1} + \langle\beta^{2^{-k}} + \beta\rangle^{\perp e}} \chi_{1}\left(bx^{-2^{k}-1}\right)$$

(we used here that $Y_{\beta} = (\beta^{2^{-k}} + \beta)^{-1} + T_{\beta}$; we also added

 $\beta \in \mathbb{F}_{2^e}$, as it contributes 0 to the inner sum)

$$= 2^{e} + \frac{2^{e}}{2^{n}} \sum_{\beta \in \mathbb{F}_{2^{n}}} 2^{-\dim S} \sum_{u \in \left(\langle \beta^{2^{-k}} + \beta \rangle^{\perp e} \right)^{\perp}} \mathcal{W}_{g_{\beta}}(u) (-1)^{\operatorname{Tr}\left(u(\beta^{2^{-k}} + \beta)^{-1} \right)}$$

(by Poisson summation with $S^{\perp} = \langle \beta^{2^{-k}} + \beta \rangle^{\perp_e}$, and $g_{\beta}(x) = \chi_1(bx^{-2^k-1})$). We now analyze the \mathbb{F}_2 -linear space

$$\left(\langle \beta^{2^{-k}} + \beta \rangle^{\perp_e}\right)^{\perp} = \{ x \in \mathbb{F}_{2^n} : \operatorname{Tr}(dx) = 0, \forall d \text{ with } \operatorname{Tr}_e(d(\beta^{2^{-k}} + \beta)) = 0 \}.$$

Further, \mathbb{F}_{2^n} has dimension n/e as an \mathbb{F}_{2^e} -linear space and so, $\dim_e \langle \beta^{2^{-k}} + \beta \rangle^{\perp_e} = \frac{n}{e} - 1$ as an \mathbb{F}_{2^e} -linear space, and since \mathbb{F}_{2^e} has dimension e as an \mathbb{F}_2 -linear space, then $\dim \langle \beta^{2^{-k}} + \beta \rangle^{\perp_e} = n - e$ as an \mathbb{F}_2 -linear space. Thus, $\dim \left(\langle \beta^{2^{-k}} + \beta \rangle^{\perp_e} \right)^{\perp} = e$. Moreover, $\operatorname{Tr}_e(\beta^{2^{-k}} + \beta) = 0$ and if $u \in \mathbb{F}_{2^e}$ then $\operatorname{Tr}_e(u(\beta^{2^{-k}} + \beta) = u\operatorname{Tr}_e(\beta^{2^{-k}} + \beta) = 0$, and consequently (since the dimensions match and $(\beta^{2^{-k}} + \beta)\mathbb{F}_{2^e} \subseteq S$)

$$S = \left(\langle \beta^{2^{-k}} + \beta \rangle^{\perp_e} \right)^{\perp} = (\beta^{2^{-k}} + \beta) \mathbb{F}_{2^e}$$

We are now ready to continue the computation, thus,

$${}_{1}\mathcal{B}_{F}(1,b) = 2^{e} + \frac{2^{e}}{2^{n}} 2^{-e} \sum_{\beta \in \mathbb{F}_{2^{n}}} \sum_{u \in (\beta^{2^{-k}} + \beta)\mathbb{F}_{2^{e}}} \mathcal{W}_{g_{\beta}}(u) (-1)^{\operatorname{Tr}\left(u(\beta^{2^{-k}} + \beta)^{-1}\right)}$$
$$= 2^{e} + \frac{2^{e}}{2^{n}} 2^{-e} \sum_{\beta \in \mathbb{F}_{2^{n}}} \sum_{d' \in \mathbb{F}_{2^{e}}} \mathcal{W}_{g_{\beta}}(d'(\beta^{2^{-k}} + \beta)) (-1)^{\operatorname{Tr}(d')}$$

where δ_0 is the Dirac symbol, defined by $\delta_0(c) = 1$, if c = 0, and 0, otherwise. Thus, ${}_{1}\mathcal{B}_{F}(1,b) \in \{0,2^{e}\},$ and the claim of our theorem is shown.

Case 2: n/e is even.

(1) If $\alpha = \beta$ and $\beta \in \mathbb{F}_{2^e}$, then

$$T_b^{[1]} = q^2 \chi_1(\beta)^2 = q^2.$$

(2) If $\alpha = \beta$ and $\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$, then

$$T_b^{[2]} = 0.$$

(3) If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^e}$, then

$$T_b^{[3]} = \begin{cases} 2^n & \text{if } A \neq g^{t(2^e+1)} \text{ for any integer } t, \\ 2^{n+2e} & \text{if } A = g^{t(2^e+1)} \text{ for some integer } t. \end{cases}$$

- (4) If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$, then (a) If $A \neq g^{t(2^e+1)}$ for any integer t, then

$$T_{b}^{[4(a)]} = 2^{n}$$

- (b) If $A = g^{t(2^e+1)}$ for some integer t, then (i) If the equation $L_A(x) = B^{2^k}$ is not solvable, where $L_A(x) = A^{2^k} x^{2^{2k}} + Ax$, then

$$T_b^{[4(b)(i)]} = 0$$

(ii) If the equation $L_A(x) = B^{2^k}$ is solvable, then

$$T_b^{[4(b)(ii)]} = \begin{cases} 2^n & \text{if } \operatorname{Tr}_e(A) \neq 0, \\ 2^{n+2e} & \text{if } \operatorname{Tr}_e(A) = 0. \end{cases}$$

Now we shall summarize the above discussion in the following theorem.

Theorem 4.2. Let $F(x) = x^{2^k+1}$, $1 \le k < n$ be a function on \mathbb{F}_{2^n} , $n \ge 2$. Let c = 1 and n/e be even, where $e = \gcd(k, n)$. Then the c-BCT entry ${}_{1}\mathcal{B}_{F}(1, b)$ of F at (1, b) is given by

$$2^{e} + \frac{1}{2^{n}} \sum_{(\alpha,\beta)\in\mathcal{G}\cup\mathcal{I}\cup\mathcal{K}} \chi_{1}(b(\alpha+\beta)) + \frac{2^{2e}}{2^{n}} \sum_{(\alpha,\beta)\in\mathcal{H}\cup\mathcal{L}} \chi_{1}(b(\alpha+\beta)),$$

with $A = \alpha + \beta$, $B = \beta^{2^{n-k}} + \beta$, $L_A(x) = A^{2^k} x^{2^{2k}} + Ax$, and

$$\begin{aligned} \mathcal{G} = &\{(\alpha, \beta) \in \mathcal{C} \mid A \neq g^{t(2^e+1)} \text{ for any integer } t\}, \\ \mathcal{H} = &\{(\alpha, \beta) \in \mathcal{C} \mid A = g^{t(2^e+1)} \text{ for some integer } t\}, \\ \mathcal{I} = &\{(\alpha, \beta) \in \mathcal{D} \mid A \neq g^{t(2^e+1)} \text{ for any integer } t\}, \\ \mathcal{K} = &\{(\alpha, \beta) \in \mathcal{D} \mid A = g^{t(2^e+1)} \text{ for some integer } t, \ \operatorname{Tr}_e(A) \neq 0, \\ L_A(x) = B^{2^k} \text{ is solvable}\}, \\ \mathcal{L} = &\{(\alpha, \beta) \in \mathcal{D} \mid A = g^{t(2^e+1)} \text{ for some integer } t, \ \operatorname{Tr}_e(A) = 0, \\ L_A(x) = B^{2^k} \text{ is solvable}\}. \end{aligned}$$

Proof. For the proof, we need to define

$$\mathcal{J} = \{ (\alpha, \beta) \in \mathcal{D} \mid A = g^{t(2^e+1)} \text{ for an integer } t, L_A(x) = B^{2^k} \text{ is not solvable} \}$$

Then

$$\begin{split} {}_{1}\mathcal{B}_{F}(1,b) &= \frac{1}{q^{2}} \left(\sum_{(\alpha,\beta)\in\mathcal{A}} \chi_{1}(b(\alpha+\beta))T_{b}^{[1]} + \sum_{(\alpha,\beta)\in\mathcal{B}} \chi_{1}(b(\alpha+\beta))T_{b}^{[2]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{G}} \chi_{1}(b(\alpha+\beta))T_{b}^{[3]} + \sum_{(\alpha,\beta)\in\mathcal{H}} \chi_{1}(b(\alpha+\beta))T_{b}^{[4(b)(i)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{I}} \chi_{1}(b(\alpha+\beta))T_{b}^{[4(b)(ii)]} + \sum_{(\alpha,\beta)\in\mathcal{J}} \chi_{1}(b(\alpha+\beta))T_{b}^{[4(b)(ii)]} \right) \\ &= \frac{1}{q^{2}} \left(\sum_{(\alpha,\beta)\in\mathcal{A}} q^{2} + 2^{n} \sum_{(\alpha,\beta)\in\mathcal{G}\cup\mathcal{I}\cup\mathcal{K}} \chi_{1}(b(\alpha+\beta)) + 2^{n+2e} \sum_{(\alpha,\beta)\in\mathcal{H}\cup\mathcal{L}} \chi_{1}(b(\alpha+\beta)) \right) \\ &= 2^{e} + \frac{1}{2^{n}} \sum_{(\alpha,\beta)\in\mathcal{G}\cup\mathcal{I}\cup\mathcal{K}} \chi_{1}(b(\alpha+\beta)) + \frac{2^{2e}}{2^{n}} \sum_{(\alpha,\beta)\in\mathcal{H}\cup\mathcal{L}} \chi_{1}(b(\alpha+\beta)). \end{split}$$

This completes the proof.

Corollary 4.3. Let $F(x) = x^{2^k+1}$, $1 \le k < n$, be a function on \mathbb{F}_q , $n \ge 2$. Let c = 1 and n/e be even, where $e = \gcd(k, n)$. With the notations of the previous theorem, the c-boomerang uniformity of F satisfies

$$\beta_{F,c} \leq 2^e + 2^{-n} |\mathcal{G} \cup \mathcal{I} \cup \mathcal{K}| + 2^{2e-n} |\mathcal{H} \cup \mathcal{L}|.$$

5. The case $c \in \mathbb{F}_{2^e} \setminus \{0, 1\}.$

Since the case c = 1 has already been considered in the previous section, throughout this section we assume that $c \neq 1$. Notice that when $c \in \mathbb{F}_{2^e}^*$, $\beta \in \mathbb{F}_{2^e} \Leftrightarrow \beta c^{-1} \in \mathbb{F}_{2^e}$. Recall that

for any fixed $b \neq 0$, the *c*-BCT entry is given by,

$${}_{c}\mathcal{B}_{F}(1,b) = \frac{1}{q^{2}} \sum_{\alpha,\beta \in \mathbb{F}_{q}} \chi_{1}\left(b\left(\alpha + \beta\right)\right) S_{\alpha,\beta}S_{c\alpha,c^{-1}\beta}.$$

Let us denote $T_b = S_{\alpha,\beta}S_{c\alpha,c^{-1}\beta}$ (we will use superscripts to point out the case we are in, for its value). Recall that $A = \alpha + \beta$ and $B = \beta^{2^{n-k}} + \beta$. Let us denote $\gamma = A^{\frac{1}{2^{k+1}}}, A' = c\alpha + c^{-1}\beta$ and $B' = (c^{-1}\beta)^{2^{n-k}} + c^{-1}\beta$. It is easy to observe that the conditions B = 0 and B' = 0 are equivalent. Now we shall consider two cases namely, $\frac{n}{e}$ odd and $\frac{n}{e}$ even, respectively. Case 1: $\frac{n}{e}$ is odd.

- (1) Let A = 0, B = 0. (a) If A' = 0, B' = 0, then $T_{\iota}^{[1(a)]} = q^2 \chi_1((1+c^{-1})\beta).$ (b) If $A' \neq 0, B' = 0$, then $S_{c\alpha,c^{-1}\beta} = 0$ and hence $T_{1}^{[1(b)]} = 0.$
- (2) Let $A = 0, B \neq 0$. In this case $S_{\alpha,\beta} = 0$ and hence

$$T_{h}^{[2]} = 0$$

(3) Let $A \neq 0, B = 0$. Again $S_{\alpha,\beta} = 0$ and hence

$$T_b^{[3]} = 0$$

- (4) Let $A \neq 0, B \neq 0$. (a) Assume $A' = 0, B' \neq 0$, then $S_{c\alpha,c^{-1}\beta} = 0$ and hence $T_{h}^{[4(a)]} = 0.$
 - (b) Assume $A' \neq 0, B' \neq 0$. In this case, recall that $\gamma^{2^k+1} = A$ and let $\gamma' \in \mathbb{F}_q$ such that $(\gamma')^{2^k+1} = A'$. (i) If $\operatorname{Tr}_e(B\gamma^{-1}) \neq 1$, then $S_{\alpha,\beta} = 0$ and hence

 - $T_{b}^{[4(b)(i)]} = 0.$ (ii) If $\operatorname{Tr}_e(B\gamma^{-1}) = 1$ and $\operatorname{Tr}_e(B'(\gamma')^{-1}) \neq 1$, then $S_{c\alpha,c^{-1}\beta} = 0$ and hence $T_{b}^{[4(b)(ii)]} = 0.$ (iii) If $\operatorname{Tr}_e(B\gamma^{-1}) = 1$ and $\operatorname{Tr}_e(B'(\gamma')^{-1}) = 1$, then $T_{\mathbf{h}}^{[4(b)(iii)]} = 2^{n+e} \chi_1((1+c^{-1})\beta).$

We now use the above discussion in the following theorem.

Theorem 5.1. Let $F(x) = x^{2^k+1}$, $1 \le k < n$ be a function on \mathbb{F}_{2^n} , $n \ge 2$. Let $c \in \mathbb{F}_{2^e} \setminus \{0, 1\}$ and n/e be odd, where $e = \gcd(k, n)$. Then the c-BCT entry ${}_{c}\mathcal{B}_{F}(1, b)$ of F at (1, b) is given by

$$1 + \frac{2^e}{2^n} \sum_{(\alpha,\beta)\in\mathcal{F}\cap\mathcal{F}'} \chi_1(b\alpha + (1+c^{-1}+b)\beta)),$$

where

$$\mathcal{F} = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid A, B \neq 0 \text{ and } \operatorname{Tr}_e(B\gamma^{-1}) = 1 \},$$

$$\mathcal{F}' = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid A', B' \neq 0 \text{ and } \operatorname{Tr}_e(B'(\gamma')^{-1}) = 1 \},$$

and $A = \alpha + \beta$, $B = \beta^{2^{n-k}} + \beta$, $A' = c\alpha + c^{-1}\beta$ and $B' = (c^{-1}\beta)^{2^{n-k}} + c^{-1}\beta$, $\gamma = A^{\frac{1}{2^{k+1}}}$, $\gamma' = A'^{\frac{1}{2^{k+1}}}$. *Proof.* Let

$$\mathcal{A}' = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid c\alpha = c^{-1}\beta \text{ and } c^{-1}\beta \in \mathbb{F}_{2^e} \},$$

$$\mathcal{B}' = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid c\alpha = c^{-1}\beta \text{ and } c^{-1}\beta \in \mathbb{F}_q \setminus \mathbb{F}_{2^e} \},$$

$$\mathcal{C}' = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid c\alpha \neq c^{-1}\beta \text{ and } c^{-1}\beta \in \mathbb{F}_{2^e} \},$$

$$\mathcal{D}' = \{ (\alpha, \beta) \in \mathbb{F}_q^2 \mid c\alpha \neq c^{-1}\beta \text{ and } c^{-1}\beta \in \mathbb{F}_q \setminus \mathbb{F}_{2^e} \},$$

$$\mathcal{E}' = \{ (\alpha, \beta) \in \mathcal{D}' \mid \operatorname{Tr}_e(B'(\gamma')^{-1}) \neq 1 \}.$$

Then,

$$\begin{split} {}_{c}\mathcal{B}_{F}(1,b) &= \frac{1}{q^{2}} \left(\sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{A}'} \chi_{1}(b(\alpha+\beta))T_{b}^{[1(\alpha)]} + \sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{C}'} \chi_{1}(b(\alpha+\beta))T_{b}^{[1(b)]} \right. \\ &+ \sum_{(\alpha,\beta)\in\mathcal{B}} \chi_{1}(b(\alpha+\beta))T_{b}^{[2]} + \sum_{(\alpha,\beta)\in\mathcal{C}} \chi_{1}(b(\alpha+\beta))T_{b}^{[3]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{D}\cap\mathcal{B}'} \chi_{1}(b(\alpha+\beta))T_{b}^{[4(\alpha)]} + \sum_{(\alpha,\beta)\in\mathcal{F}} \chi_{1}(b(\alpha+\beta))T_{b}^{[4(b)(i)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{F}\cap\mathcal{E}'} \chi_{1}(b(\alpha+\beta))T_{b}^{[4(b)(ii)]} + \sum_{(\alpha,\beta)\in\mathcal{F}\cap\mathcal{F}'} \chi_{1}(b(\alpha+\beta))T_{b}^{[4(b)(ii)]} \right) \\ &= \sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{A}'} \chi_{1}(b\alpha+(1+c^{-1}+b)\beta)) \\ &+ \frac{2^{e}}{2^{n}} \sum_{(\alpha,\beta)\in\mathcal{F}\cap\mathcal{F}'} \chi_{1}(b\alpha+(1+c^{-1}+b)\beta)) \\ &= 1 + \frac{2^{e}}{2^{n}} \sum_{(\alpha,\beta)\in\mathcal{F}\cap\mathcal{F}'} \chi_{1}(b\alpha+(1+c^{-1}+b)\beta)). \end{split}$$

This completes the proof.

Corollary 5.2. Let $F(x) = x^{2^k+1}$, $1 \le k < n$, be a function on \mathbb{F}_q , $n \ge 2$. Let $c \in \mathbb{F}_{2^e} \setminus \{0, 1\}$ and n/e be odd, where $e = \gcd(k, n)$. With the notations of the previous theorem, the *c*boomerang uniformity of *F* satisfies

$$\beta_{F,c} \le 1 + 2^{e-n} |\mathcal{F} \cap \mathcal{F}'|.$$

Case 2: n/e is even.

(1) Let
$$A = 0, B = 0.$$

(a) If $A' = 0, B' = 0$, then
 $T_b^{[1(a)]} = \chi_1((1 + c^{-1})\beta) q^2.$
(b) If $A' \neq 0, B' = 0$, let
 $\mathcal{G}' = \{(\alpha, \beta) \in \mathcal{C}' \mid A' \neq g^{t(2^e+1)} \text{ for any integer } t\},$
 $\mathcal{H}' = \{(\alpha, \beta) \in \mathcal{C}' \mid A' = g^{t(2^e+1)} \text{ for some integer } t\}.$

Then,

$$T_{b}^{[1(b)]} = \begin{cases} (-1)^{\frac{m}{e}} 2^{m+n} \chi_{1}((1+c^{-1})\beta) & \text{if } (\alpha,\beta) \in \mathcal{A} \cap \mathcal{G}', \\ (-1)^{\frac{m}{e}+1} 2^{m+n+e} \chi_{1}((1+c^{-1})\beta) & \text{if } (\alpha,\beta) \in \mathcal{A} \cap \mathcal{H}'. \end{cases}$$

(2) Let $A = 0, B \neq 0$. In this case $S_{\alpha,\beta} = 0$ and hence

$$T_b^{[2]} = 0.$$

(3) Let A ≠ 0, B = 0.
(a) If A' = 0, B' = 0, then T_b^[3(a)] is given by

$$\int (-1)^{\frac{m}{e}} 2^{m+n} \chi_1((1+c^{-1})\beta) \qquad \text{if } (\alpha, \beta) + if ($$

$$\begin{cases} (-1)^{\frac{m}{e}} 2^{m+n} \chi_1((1+c^{-1})\beta) & \text{if } (\alpha,\beta) \in \mathcal{A}' \cap \mathcal{G}, \\ (-1)^{\frac{m}{e}+1} 2^{m+n+e} \chi_1((1+c^{-1})\beta) & \text{if } (\alpha,\beta) \in \mathcal{A}' \cap \mathcal{H}. \end{cases}$$

(b) If
$$A' \neq 0, B' = 0$$
, then

$$T_b^{[3(b)]} = \begin{cases} 2^n \chi_1((1+c^{-1})\beta) & \text{if } (\alpha,\beta) \in \mathcal{G} \cap \mathcal{G}', \\ -2^{n+e} \chi_1((1+c^{-1})\beta) & \text{if } (\alpha,\beta) \in \mathcal{G} \cap \mathcal{H}', \\ -2^{n+e} \chi_1((1+c^{-1})\beta) & \text{if } (\alpha,\beta) \in \mathcal{H} \cap \mathcal{G}', \\ 2^{n+2e} \chi_1((1+c^{-1})\beta) & \text{if } (\alpha,\beta) \in \mathcal{H} \cap \mathcal{H}'. \end{cases}$$

(4) Let $A \neq 0, B \neq 0$.

(a) If
$$A' = 0, B' \neq 0$$
, then $S_{c\alpha,c^{-1}\beta} = 0$ and hence

$$T_{h}^{[4(a)]} = 0.$$

(b) If
$$A' \neq 0, B' \neq 0$$
, let
 $\mathcal{I}' = \{(\alpha, \beta) \in \mathcal{D}' \mid A' \neq g^{t(2^e+1)} \text{ for any integer } t\},$
 $\mathcal{J}' = \{(\alpha, \beta) \in \mathcal{D}' \mid A' = g^{t(2^e+1)} \text{ for some integer } t,$
 $L_{A'}(x) = (B')^{2^k} \text{ is not solvable}\},$
 $\mathcal{K}' = \{(\alpha, \beta) \in \mathcal{D}' \mid A' = g^{t(2^e+1)} \text{ for some integer } t,$
 $\operatorname{Tr}_e(A') \neq 0, L_{A'}(x) = (B')^{2^k} \text{ is solvable}\},$
 $\mathcal{L}' = \{(\alpha, \beta) \in \mathcal{D}' \mid A' = g^{t(2^e+1)} \text{ for some integer } t,$
 $\operatorname{Tr}_e(A') = 0, L_{A'}(x) = (B')^{2^k} \text{ is solvable}\}.$

Then,

$$T_b^{[4(b)]} = \begin{cases} 2^n \cdot M & if \ (\alpha, \beta) \in (\mathcal{I} \cup \mathcal{K}) \cap (\mathcal{I}' \cup \mathcal{K}'), \\ 0 & if \ (\alpha, \beta) \in (\mathcal{I} \cup \mathcal{K} \cup \mathcal{L}) \cap \mathcal{J}', \\ -2^{n+e} \cdot M & if \ (\alpha, \beta) \in (\mathcal{I} \cup \mathcal{K}) \cap \mathcal{L}', \\ 0 & if \ (\alpha, \beta) \in \mathcal{J} \cap (\mathcal{I}' \cup \mathcal{J}' \cup \mathcal{K}' \cup \mathcal{L}'), \\ -2^{n+e} \cdot M & if \ (\alpha, \beta) \in \mathcal{L} \cap (\mathcal{I}' \cup \mathcal{K}'), \\ 2^{n+2e} \cdot M & if \ (\alpha, \beta) \in \mathcal{L} \cap \mathcal{L}', \end{cases}$$

where $M = \chi_1((1+c^{-1})\beta)\chi_1\left(AA'x_A^{2^k+1}x_{A'}^{2^k+1}\right)$ and $x_A, x_{A'}$ are the solutions of the equations $L_A(x) = B^{2^k}$ and $L_{A'}(x) = (B')^{2^k}$, respectively.

We now summarize the above discussion in the following theorem.

Theorem 5.3. Let $F(x) = x^{2^k+1}$, $1 \le k < n$ be a function on \mathbb{F}_{2^n} , $n \ge 2$. Let $c \in \mathbb{F}_{2^e} \setminus \{0, 1\}$ and n/e be even, where $e = \gcd(k, n)$. With the previous notations, the c-BCT entry ${}_{c}\mathcal{B}_{F}(1, b)$ of F at (1, b) is given by

$$\begin{split} &\frac{1}{q^2} \left(\sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{A}'} \chi_1(b(\alpha+\beta)) T_b^{[1(\alpha)]} + \sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{G}'} \chi_1(b(\alpha+\beta)) T_b^{[1(b)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{H}'} \chi_1(b(\alpha+\beta)) T_b^{[1(b)]} + \sum_{(\alpha,\beta)\in\mathcal{A}'\cap\mathcal{G}} \chi_1(b(\alpha+\beta)) T_b^{[3(\alpha)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{H}} \chi_1(b(\alpha+\beta)) T_b^{[3(\alpha)]} + \sum_{(\alpha,\beta)\in\mathcal{G}\cap\mathcal{G}'} \chi_1(b(\alpha+\beta)) T_b^{[3(b)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{H}\cap\mathcal{H}'} \chi_1(b(\alpha+\beta)) T_b^{[3(b)]} + \sum_{(\alpha,\beta)\in\mathcal{H}\cap\mathcal{G}'} \chi_1(b(\alpha+\beta)) T_b^{[3(b)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{H}\cap\mathcal{H}'} \chi_1(b(\alpha+\beta)) T_b^{[3(b)]} + \sum_{(\alpha,\beta)\in\mathcal{L}\cap(\mathcal{I}'\cup\mathcal{K}')} \chi_1(b(\alpha+\beta)) T_b^{[4(b)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{L}\cap\mathcal{L}'} \chi_1(b(\alpha+\beta)) T_b^{[4(b)]} + \sum_{(\alpha,\beta)\in\mathcal{L}\cap(\mathcal{I}'\cup\mathcal{K}')} \chi_1(b(\alpha+\beta)) T_b^{[4(b)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{L}\cap\mathcal{L}'} \chi_1(b(\alpha+\beta)) T_b^{[4(b)]} \right). \end{split}$$

6. The general case

Since the case $c \in \mathbb{F}_{2^e}$ has already been considered in previous sections, throughout this section we assume that $c \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$. Recall that for any fixed $b \neq 0$, the *c*-BCT entry is given by,

$${}_{c}\mathcal{B}_{F}(1,b) = \frac{1}{q^{2}} \sum_{\alpha,\beta \in \mathbb{F}_{q}} \chi_{1}\left(b\left(\alpha + \beta\right)\right) S_{\alpha,\beta} S_{c\alpha,c^{-1}\beta}.$$

Let us denote $T_b = S_{\alpha,\beta}S_{c\alpha,c^{-1}\beta}$. Recall that $A = \alpha + \beta$, $B = \beta^{2^{n-k}} + \beta$, $A' = c\alpha + c^{-1}\beta$ and $B' = (c^{-1}\beta)^{2^{n-k}} + c^{-1}\beta$. Notice that, when $c \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$ then $\beta \in \mathbb{F}_{2^e}^*$, and so, $\beta c^{-1} \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$, otherwise $c \in \mathbb{F}_{2^e}$. Thus B = 0 = B' if and only if $\beta = 0$. Also, observe that the conditions A = 0 = A' if and only if $\alpha = 0 = \beta$. Now we shall consider two cases namely, $\frac{n}{e}$ is odd and $\frac{n}{e}$ is even, respectively.

Case 1: $\frac{n}{e}$ is odd.

(1) Let A = 0, B = 0.

Notice that the cases $A' = 0, B' \neq 0$, and $A' \neq 0, B' = 0$ would not arise, therefore, we shall calculate T_b in remaining two cases only.

(a) If A' = 0, B' = 0, then

$$T_b^{[1(a)]} = \chi_1((1+c^{-1})\beta) q^2.$$

(b) If $A' \neq 0, B' \neq 0$, then

$$T_b^{[1(b)]} = \begin{cases} 0 & \text{if } \operatorname{Tr}_e(B'(\gamma')^{-1}) \neq 1, \\ \left(\frac{2}{n/e}\right)^e 2^{\frac{3n+e}{2}} \chi_1((1+c^{-1})\beta) & \text{if } \operatorname{Tr}_e(B'(\gamma')^{-1}) = 1. \end{cases}$$

(2) Let $A = 0, B \neq 0$. In this case $S_{\alpha,\beta} = 0$ and hence

$$T_{h}^{[2]} = 0.$$

(3) Let $A \neq 0, B = 0$. Again, $S_{\alpha,\beta} = 0$ and hence

$$T_{h}^{[3]} = 0$$

(4) Let $A \neq 0, B \neq 0$. (a) If A' = 0, B' = 0, then

$$T_b^{[4(a)]} = \begin{cases} 0 & \text{if } \operatorname{Tr}_e(B\gamma^{-1}) \neq 1, \\ \left(\frac{2}{n/e}\right)^e 2^{\frac{3n+e}{2}} \chi_1((1+c^{-1})\beta) & \text{if } \operatorname{Tr}_e(B\gamma^{-1}) = 1. \end{cases}$$

(b) If $A'=0,B'\neq 0,$ then $S_{c\alpha,c^{-1}\beta}=0$ and hence

$$T_b^{[4(b)]} = 0$$

(c) If $A' \neq 0, B' = 0$, then again $S_{c\alpha,c^{-1}\beta} = 0$ and hence

$$T_b^{[4(c)]} = 0.$$

(d) If $A' \neq 0, B' \neq 0$, then the only relevant case is and

$$T_b^{[4(d)]} = \begin{cases} 2^{n+e}\chi_1((1+c^{-1})\beta) & \text{ if } (\alpha,\beta) \in \mathcal{F} \cap \mathcal{F}', \\ 0 & \text{ otherwise.} \end{cases}$$

We now summarize the above discussion in the following theorem.

Theorem 6.1. Let $F(x) = x^{2^{k+1}}$, $1 \le k < n$ be a function on \mathbb{F}_{2^n} , $n \ge 2$. Let $c \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$ and n/e be odd, where $e = \gcd(k, n)$. Then the c-BCT entry ${}_c\mathcal{B}_F(1, b)$ of F at (1, b) is given by

$$1 + \frac{2^{\frac{e}{2}}}{2^n} \sum_{\substack{(\alpha,\beta)\in(\mathcal{A}\cap\mathcal{F}')\cup(\mathcal{A}'\cap\mathcal{F})\\}} \chi_1(b\alpha + (1+c^{-1}+b)\beta)) + \frac{2^e}{2^n} \sum_{\substack{(\alpha,\beta)\in\mathcal{F}\cap\mathcal{F}'\\}} \chi_1(b\alpha + (1+c^{-1}+b)\beta)).$$

Proof.

$${}_{c}\mathcal{B}_{F}(1,b) = \frac{1}{q^{2}} \left(\sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{A}'} \chi_{1}(b(\alpha+\beta))T_{b}^{[1(a)]} + \sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{F}'} \chi_{1}(b(\alpha+\beta))T_{b}^{[1(b)]} \right)$$
$$+ \sum_{(\alpha,\beta)\in\mathcal{F}\cap\mathcal{A}'} \chi_{1}(b(\alpha+\beta))T_{b}^{[4(a)]} + \sum_{(\alpha,\beta)\in\mathcal{F}\cap\mathcal{F}'} \chi_{1}(b(\alpha+\beta))T_{b}^{[4(d)]} \right)$$
$$= 1 + \left(\frac{2}{n/e}\right)^{e} \cdot 2^{\frac{e-n}{2}} \sum_{(\alpha,\beta)\in(\mathcal{A}\cap\mathcal{F}')\cup(\mathcal{A}'\cap\mathcal{F})} \chi_{1}(b\alpha+(1+c^{-1}+b)\beta))$$
$$+ 2^{e-n} \sum_{(\alpha,\beta)\in\mathcal{F}\cap\mathcal{F}'} \chi_{1}(b\alpha+(1+c^{-1}+b)\beta)).$$

Corollary 6.2. Let $F(x) = x^{2^k+1}$, $1 \le k < n$, be a function on \mathbb{F}_q , $n \ge 2$. Let $c \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$ and n/e be odd, where $e = \gcd(k, n)$. With the notations of the previous theorem, the cboomerang uniformity of F satisfies

$$\beta_{F,c} \le 1 + \left(\frac{2}{n/e}\right)^e \cdot 2^{\frac{e-n}{2}} |(\mathcal{A} \cap \mathcal{F}') \cup (\mathcal{A}' \cap \mathcal{F})| + 2^{e-n} |\mathcal{F} \cap \mathcal{F}'|.$$

Case 2: n/e is even.

- (1) Let A = 0, B = 0. Notice that the cases $A' = 0, B' \neq 0$, and $A' \neq 0, B' = 0$ would not arise, therefore, we shall calculate T_b in remaining two cases only.
 - (a) If A' = 0, B' = 0, then

$$T_b^{[1(a)]} = \chi_1((1+c^{-1})\beta) q^2.$$

(b) If $A' \neq 0, B' \neq 0$, then

$$T_b^{[1(b)]} = \begin{cases} (-1)^{\frac{m}{e}} 2^{m+n} M' & \text{ if } (\alpha, \beta) \in \mathcal{A} \cap (\mathcal{I}' \cup \mathcal{K}'), \\ 0 & \text{ if } (\alpha, \beta) \in \mathcal{A} \cap \mathcal{J}', \\ (-1)^{\frac{m}{e}+1} 2^{m+n+e} M' & \text{ if } (\alpha, \beta) \in \mathcal{A} \cap \mathcal{L}', \end{cases}$$

where $M' = \chi_1((1+c^{-1})\beta)\chi_1(A'x_{A'}^{2^k+1})$. (2) Let $A = 0, B \neq 0$. In this case $S_{\alpha,\beta} = 0$ and hence

$$T_b^{[2]} = 0$$

- (3) Let $A \neq 0, B = 0$. Notice that the case A' = 0, B' = 0 would not arise. Now we shall calculate T_b in the remaining cases.
 - (a) If $A' = 0, B' \neq 0$, then $S_{c\alpha,c^{-1}\beta} = 0$ and hence

$$T_b^{[3(a)]} = 0.$$

(b) If $A' \neq 0, B' = 0$, then

$$T_b^{[3(b)]} = \begin{cases} 2^n \chi_1((1+c^{-1})\beta) & if \ (\alpha,\beta) \in \mathcal{G} \cap \mathcal{G}', \\ -2^{n+e} \chi_1((1+c^{-1})\beta) & if \ (\alpha,\beta) \in \mathcal{G} \cap \mathcal{H}', \\ -2^{n+e} \chi_1((1+c^{-1})\beta) & if \ (\alpha,\beta) \in \mathcal{H} \cap \mathcal{G}', \\ 2^{n+2e} \chi_1((1+c^{-1})\beta) & if \ (\alpha,\beta) \in \mathcal{H} \cap \mathcal{H}'. \end{cases}$$

(c) If $A' \neq 0, B' \neq 0$, then

$$T_{b}^{[3(c)]} = \begin{cases} 2^{n}M' & if \ (\alpha,\beta) \in \mathcal{G} \cap (\mathcal{I}' \cup \mathcal{K}'), \\ 0 & if \ (\alpha,\beta) \in (\mathcal{G} \cup \mathcal{H}) \cap \mathcal{J}', \\ -2^{n+e}M' & if \ (\alpha,\beta) \in \mathcal{G} \cap \mathcal{L}', \\ -2^{n+e}M' & if \ (\alpha,\beta) \in \mathcal{H} \cap (\mathcal{I}' \cup \mathcal{K}'), \\ 2^{n+2e}M' & if \ (\alpha,\beta) \in \mathcal{H} \cap \mathcal{L}'). \end{cases}$$

(4) Let $A \neq 0, B \neq 0$. (a) If A' = 0, B' = 0, then

$$T_b^{[4(a)]} = \begin{cases} (-1)^{\frac{m}{e}} 2^{m+n} M'' & \text{if } (\alpha, \beta) \in \mathcal{A}' \cap (\mathcal{I} \cup \mathcal{K}), \\ 0 & \text{if } (\alpha, \beta) \in \mathcal{A}' \cap \mathcal{J}, \\ (-1)^{\frac{m}{e}+1} 2^{m+n+e} M'' & \text{if } (\alpha, \beta) \in \mathcal{A}' \cap \mathcal{L}, \end{cases}$$

where $M'' = \chi_1((1+c^{-1})\beta)\chi_1(Ax_A^{2^k+1}).$

(b) If $A' = 0, B' \neq 0$, then $S_{c\alpha,c^{-1}\beta} = 0$ and hence

$$T_{b}^{[4(b)]} = 0.$$

(c) If $A' \neq 0, B' = 0$, then

$$T_b^{[4(c)]} = \begin{cases} 2^n M'' & \text{if } (\alpha, \beta) \in \mathcal{G}' \cap (\mathcal{I} \cup \mathcal{K}), \\ 0 & \text{if } (\alpha, \beta) \in (\mathcal{G}' \cup \mathcal{H}') \cap \mathcal{J}, \\ -2^{n+e} M'' & \text{if } (\alpha, \beta) \in \mathcal{G}' \cap \mathcal{L}, \\ -2^{n+e} M'' & \text{if } (\alpha, \beta) \in \mathcal{H}' \cap (\mathcal{I} \cup \mathcal{K}), \\ 2^{n+2e} M'' & \text{if } (\alpha, \beta) \in \mathcal{H}' \cap \mathcal{L}. \end{cases}$$

(d) If $A' \neq 0, B' \neq 0$, then

$$T_b^{[4(d)]} = \begin{cases} 2^n M''' & \text{if } (\alpha, \beta) \in (\mathcal{I} \cup \mathcal{K}) \cap (\mathcal{I}' \cup \mathcal{K}'), \\ 0 & \text{if } (\alpha, \beta) \in (\mathcal{I} \cup \mathcal{K} \cup \mathcal{L}) \cap \mathcal{J}', \\ -2^{n+e} M''' & \text{if } (\alpha, \beta) \in (\mathcal{I} \cup \mathcal{K}) \cap \mathcal{L}', \\ 0 & \text{if } (\alpha, \beta) \in \mathcal{J} \cap (\mathcal{I}' \cup \mathcal{J}' \cup \mathcal{K}' \cup \mathcal{L}'), \\ -2^{n+e} M''' & \text{if } (\alpha, \beta) \in \mathcal{L} \cap (\mathcal{I}' \cup \mathcal{K}'), \\ 2^{n+2e} M''' & \text{if } (\alpha, \beta) \in \mathcal{L} \cap \mathcal{L}', \end{cases}$$

where
$$M''' = \chi_1((1+c^{-1})\beta)\chi_1(Ax_A^{2^k+1} + A'x_{A'}^{2^k+1}).$$

We now summarize the above discussion in the form of following theorem.

Theorem 6.3. Let $F(x) = x^{2^{k+1}}$, $1 \le k < n$ be a function on \mathbb{F}_{2^n} , $n \ge 2$. Let $c \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^e}$ and n/e be even, where $e = \gcd(k, n)$. With the prior notations, the c-BCT entry ${}_{c}\mathcal{B}_{F}(1, b)$ of F at (1, b) is given by

$$\begin{split} &\frac{1}{q^2} \left(\sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{A}'} \chi_1(b(\alpha+\beta)) T_b^{[1(\alpha)]} + \sum_{(\alpha,\beta)\in\mathcal{A}\cap(\mathcal{I}'\cup\mathcal{K}')} \chi_1(b(\alpha+\beta)) T_b^{[1(b)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{A}\cap\mathcal{L}'} \chi_1(b(\alpha+\beta)) T_b^{[1(b)]} + \sum_{(\alpha,\beta)\in\mathcal{G}\cap\mathcal{G}'} \chi_1(b(\alpha+\beta)) T_b^{[3(b)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{G}\cap\mathcal{H}'} \chi_1(b(\alpha+\beta)) T_b^{[3(b)]} + \sum_{(\alpha,\beta)\in\mathcal{H}\cap\mathcal{G}'} \chi_1(b(\alpha+\beta)) T_b^{[3(c)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{H}\cap\mathcal{H}'} \chi_1(b(\alpha+\beta)) T_b^{[3(c)]} + \sum_{(\alpha,\beta)\in\mathcal{H}\cap(\mathcal{I}'\cup\mathcal{K}')} \chi_1(b(\alpha+\beta)) T_b^{[3(c)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{H}\cap\mathcal{L}'} \chi_1(b(\alpha+\beta)) T_b^{[3(c)]} + \sum_{(\alpha,\beta)\in\mathcal{H}\cap(\mathcal{I}'\cup\mathcal{K})} \chi_1(b(\alpha+\beta)) T_b^{[3(c)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{H}\cap\mathcal{L}'} \chi_1(b(\alpha+\beta)) T_b^{[4(a)]} + \sum_{(\alpha,\beta)\in\mathcal{G}'\cap(\mathcal{I}\cup\mathcal{K})} \chi_1(b(\alpha+\beta)) T_b^{[4(a)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{G}'\cap\mathcal{L}} \chi_1(b(\alpha+\beta)) T_b^{[4(c)]} + \sum_{(\alpha,\beta)\in\mathcal{H}'\cap(\mathcal{I}\cup\mathcal{K})} \chi_1(b(\alpha+\beta)) T_b^{[4(c)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{H}'\cap\mathcal{L}} \chi_1(b(\alpha+\beta)) T_b^{[4(c)]} + \sum_{(\alpha,\beta)\in\mathcal{H}'\cap(\mathcal{I}\cup\mathcal{K})} \chi_1(b(\alpha+\beta)) T_b^{[4(c)]} \\ &+ \sum_{(\alpha,\beta)\in\mathcal{H}'\cap\mathcal{L}} \chi_1(b(\alpha+\beta)) T_b^{[4(c)]} + \sum_{(\alpha,\beta)\in\mathcal{H}'\cap(\mathcal{I}\cup\mathcal{K})} \chi_1(b(\alpha+\beta)) T_b^{[4(d)]} \end{split}$$

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$$+ \sum_{(\alpha,\beta)\in(\mathcal{I}\cup\mathcal{K})\cap\mathcal{L}'} \chi_1(b(\alpha+\beta))T_b^{[4(d)]} + \sum_{(\alpha,\beta)\in(\mathcal{I}'\cup\mathcal{K}')\cap\mathcal{L}} \chi_1(b(\alpha+\beta))T_b^{[4(d)]}$$
$$+ \sum_{(\alpha,\beta)\in\mathcal{L}'\cap\mathcal{L}} \chi_1(b(\alpha+\beta))T_b^{[4(d)]} \Bigg).$$

7. DISCUSSION ON EQUIVALENCE

Boura and Canteaut [2] showed that the BCT table is preserved under the affine equivalence but not under the extended affine equivalence (and consequently under the CCZ-equivalence). It is quite natural to ask a similar question in the context of *c*-BCT. It is straightforward to see that in the case of even characteristic, *c*-BCT and c^{-1} -BCT entries of an (n, n)-function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ are the same under the transformations $x \mapsto x + a$ and $y \mapsto y + a$, since the *c*-boomerang system

$$\begin{cases} F(x) + cF(y) = b\\ F(x+a) + c^{-1}F(y+a) = b \end{cases}$$

becomes

$$\begin{cases} F(x) + c^{-1}F(y) = b\\ F(x+a) + cF(y+a) = b \end{cases}$$

We consider the binomial $G(x) = x^{2^k+1} + ux^{2^{n-k}+1} \in \mathbb{F}_{2^n}[x]$, which is a PP if and only if $\frac{n}{e}$ is odd and $u \neq g^{t(2^e-1)}$, where $e = \gcd(n,k) = \gcd(n-k,k)$ and g is the primitive element of \mathbb{F}_{2^n} . Notice that $G(x) = (L \circ F)(x)$ where $L(x) = x^{2^k} + ux$ and $F(x) = x^{2^{n-k}+1}$. When n = 6, k = 2 and u = g, where g is a root of the primitive polynomial $y^6 + y^4 + y^3 + y + 1$ over \mathbb{F}_2 , then L(x) and G(x) are PP. It is easy to see from the Table 1 in the Appendix A that the *c*-BCT is not preserved under the (output applied) affine equivalence. However, if the affine transformation is applied to the input, that is, $G(x) = (F \circ L)(x)$, then the *c*-BCT spectrum is preserved, as was the case for the *c*-differential uniformity.

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APPENDIX A.

Let $G(x) = x^5 + gx^{17} = (x^4 + gx) \circ x^{17} \in \mathbb{F}_{2^6}[x]$, where g is a root of the primitive polynomial $y^6 + y^4 + y^3 + y + 1$ over \mathbb{F}_2 . The following Table 1 gives the set of the c-BCT entries for x^{17} as well as $G(x) = x^5 + gx^{17}$ for all $c \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$. In view of the discussion in Section 7, it is sufficient to compute the set of c-BCT entries for either of c or c^{-1} as they are going to be exactly the same.

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С	Set of c-BCT entries of x^{17}	Set of c-BCT entries of $G(x)$
g	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5\}$
a^2	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5\}$
g^3	$\{0,1,2,3,5\}$	$\{0,1,2,3,4,5\}$
g^4	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5\}$
g^5	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5,6\}$
g^6	$\{0,1,2,3,5\}$	$\{0,1,2,3,4,5,6\}$
g^7	$\{0,1,2,3,4,5\}$	$\{0,1,2,3,4,5,6\}$
g^8	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5\}$
g^9	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5,6\}$
g^{10}	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5\}$
g^{11}	$\{0,1,2,3\}$	$\{0,1,2,3,4,5\}$
q^{12}	$\{0,1,2,3,5\}$	$\{0,1,2,3,4,5\}$
g^{13}	$\{0,1,2,3\}$	$\{0,1,2,3,4,5\}$
g^{14}	$\{0,1,2,3,4,5\}$	$\{0,1,2,3,4,5,6\}$
g^{15}	$\{0,1,2,3,5\}$	$\{0,1,2,3,4,5\}$
g^{16}	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5\}$
g^{17}	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5,6\}$
g^{18}	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5,6\}$
g^{19}	$\{0,1,2,3\}$	$\{0,1,2,3,4,5\}$
g^{20}	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5,6\}$
g^{21}	$\{0,1,4\}$	{0,1,4}
g^{22}	$\{0,1,2,3\}$	$\{0,1,2,3,4,5\}$
g^{23}	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5\}$
g^{24}	$\{0,1,2,3,5\}$	$\{0,1,2,3,4,5,6\}$
g^{25}_{26}	$\{0,1,2,3\}$	$\{0,1,2,3,4,5\}$
g^{26}	$\{0,1,2,3\}$	$\{0,1,2,3,4,5\}$
g^{27}	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5,6\}$
g^{28}_{-99}	$\{0,1,2,3,4,5\}$	$\{0,1,2,3,4,5,6\}$
g^{29}	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5\}$
g^{30}	$\{0,1,2,3,5\}$	$\{0,1,2,3,4,5,6\}$
g^{31}	$\{0,1,2,3,4\}$	$\{0,1,2,3,4,5\}$

Table 1. *c*-BCT entries of x^{17} and $x^5 + gx^{17}$