# THE BINARY GOLD FUNCTION AND ITS $c$-BOOMERANG CONNECTIVITY TABLE 

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#### Abstract

Here, we give a complete description of the $c$-Boomerang Connectivity Table for the Gold function over finite fields of even characteristic, by using double Weil sums. In the process we generalize a result of Boura and Canteaut (IACR Trans. Symmetric Cryptol. $2018(3): 290-310,2018)$ for the classical boomerang uniformity.


## 1. Introduction

Let $\mathbb{F}_{q}$ be the finite field with $q=p^{n}$ elements, where $p$ is a prime and $n$ is a positive integer. The multiplicative cyclic group of nonzero elements of the finite field is denoted by $\mathbb{F}_{q}^{*}=\langle g\rangle$, where $g$ is a primitive element of $\mathbb{F}_{q}$. A Weil sum is an important character sum defined as follows

$$
\sum_{x \in \mathbb{F}_{q}} \chi(F(x)),
$$

where $\chi$ is an additive character of $\mathbb{F}_{q}$ and $F(x)$ is a polynomial in $\mathbb{F}_{q}[x]$. It is well-known that a polynomial $F(x)$ over finite field $\mathbb{F}_{q}$ is a permutation polynomial (PP) if and only if its Weil sum $\sum_{x \in \mathbb{F}_{q}} \chi(F(x))=0$ for all nontrivial additive characters $\chi$ of $\mathbb{F}_{q}$. Permutation polynomials are a very important class of polynomials as they have applications in coding theory and cryptography, especially in the substitution boxes (S-boxes) of the block ciphers. The security of the S-boxes relies on certain properties of the function $F(x)$, e.g., its differential uniformity, boomerang uniformity, nonlinearity etc.

Recently, Cid et al. [4] introduced a "new tool" for analyzing the boomerang style attack proposed by Wagner [17]. This new tool is usually referred to as Boomerang Connectivity Table (BCT). Boura and Canteaut [2] further studied BCT and coined the term boomerang uniformity, which is essentially the maximum value in the BCT. Li et al. 9 provided new insights in the study of BCT and presented an equivalent technique to compute BCT , which does not require the compositional inverse of the permutation polynomial $F(x)$ at all. In fact, Li et al. [9] also gave a characterization of BCT in terms of Walsh transform and gave a class of permutation polynomial with boomerang uniformity 4.

Recently, Stănică [12 extended the notion of BCT and boomerang uniformity. In fact, he defined what he termed as $c$-BCT and $c$-boomerang uniformity for an arbitrary polynomial function $F$ over $\mathbb{F}_{q}$ and for any $c \neq 0 \in \mathbb{F}_{q}$. Let $a, b \in \mathbb{F}_{q}$, then the entry of the $c$-Boomerang Connectivity Table $\left(c\right.$-BCT) at $(a, b) \in \mathbb{F}_{p^{n}} \times \mathbb{F}_{p^{n}}$, denoted as ${ }_{c} \mathcal{B}_{F}(a, b)$, is the number of solutions in $\mathbb{F}_{p^{n}} \times \mathbb{F}_{p^{n}}$ of the following system

$$
\left\{\begin{array}{l}
F(x)-c F(y)=b  \tag{1.1}\\
F(x+a)-c^{-1} F(y+a)=b
\end{array}\right.
$$

[^0]The $c$-boomerang uniformity of $F$ is defined as

$$
\beta_{F, c}=\max _{a, b \in \mathbb{F}_{p^{n}}^{*}} \mathcal{B}_{F}(a, b) .
$$

In yet other recent papers, Stănică [13, 14] further studied the $c$-BCT for the swapped inverse function and also gave an elegant description of the $c$-BCT entries of the power map in terms of double Weil sums. He further simplified his expressions for the Gold function $x^{p^{k}+1}$ over $\mathbb{F}_{p^{n}}$, for all $1 \leq k<n$ and $p$ odd. In this paper, we shall complement the work of [14] to the finite fields of even characteristic $(p=2)$.

The paper is structured as follows. Section 2 contains some preliminary results that will be used across the sections. Section [3 contains the characterization of $c$-BCT entries in terms of double Weil sums. For $c=1$, we further simplify this expression in Section 4. In fact, Theorem 4.1 generalizes previously known results of Boura and Canteaut [2]. In Section 5 , we consider the case when $c \in \mathbb{F}_{2^{e}} \backslash\{0,1\}$, where $e=\operatorname{gcd}(k, n)$. In Section 6, we discuss the general case. Finally, in Section 7, we discuss the affine, extended affine and CCZ-equivalence as it relates to c-boomerang uniformity.

## 2. Preliminaries

We begin this section by first recalling the recent notion of $c$-differentials introduced in 77. We shall assume that $q=2^{n}$ for rest of the paper. For an $(n, n)$-function $F: \mathbb{F}_{q} \rightarrow \mathbb{F}_{q}$, and $c \in \mathbb{F}_{q}$, we define the (multiplicative) $c$-derivative of $F$ with respect to $a \in \mathbb{F}_{q}$ to be the function

$$
{ }_{c} D_{a} F(x)=F(x+a)-c F(x), \text { for all } x \in \mathbb{F}_{q} .
$$

Further, for $a, b \in \mathbb{F}_{q}$, we let the entries of the $c$-Difference Distribution Table ( $c$-DDT) be defined by ${ }_{c} \Delta_{F}(a, b)=\#\left\{x \in \mathbb{F}_{q}: F(x+a)-c F(x)=b\right\}$. We call the quantity

$$
\delta_{F, c}=\max \left\{{ }_{c} \Delta_{F}(a, b) \mid a, b \in \mathbb{F}_{q}, \text { and } a \neq 0 \text { if } c=1\right\},
$$

the $c$-differential uniformity of $F$. Note that the case $c=1$ corresponds to the usual notion of differential uniformity. The interested reader may refer to [1, 8, 10, 15, 16, 18] for some recent results concerning $c$-differential uniformity.

The following theorem is a "binary" analogue of [14, Theorem 1], which gives a nice connection between $c$-BCT and $c$-DDT entries of the power map $x^{d}$ over $\mathbb{F}_{2^{n}}$.
Theorem 2.1. Let $F(x)=x^{d}$ be a power function on $\mathbb{F}_{q}, q=2^{n}$ and $c \in \mathbb{F}_{q}^{*}$. Then, for fixed $b \in \mathbb{F}_{q}^{*}$, the $c$-Boomerang Connectivity Table entry ${ }_{c} \mathcal{B}_{F}(1, b)$ at $(1, b)$ is given by

$$
\frac{1}{q}\left(\sum_{w \in \mathbb{F}_{q}}\left({ }_{c} \Delta_{F}(w, b)+{ }_{c^{-1}} \Delta_{F}(w, b)\right)\right)-1+\frac{1}{q^{2}} \sum_{\alpha, \beta \in \mathbb{F}_{q}, \alpha \beta \neq 0} \chi_{1}(b(\alpha+\beta)) S_{\alpha, \beta} S_{\alpha c, \beta c^{-1}},
$$

with

$$
\begin{aligned}
S_{\alpha, \beta} & =\sum_{x \in \mathbb{F}_{q}} \chi_{1}\left(\alpha x^{d}\right) \chi_{1}\left(\beta(x+1)^{d}\right) \\
& =\frac{1}{(q-1)^{2}} \sum_{j, k=0}^{q-2} G\left(\bar{\psi}_{j}, \chi_{1}\right) G\left(\bar{\psi}_{k}, \chi_{1}\right) \sum_{x \in \mathbb{F}_{q}} \psi_{1}\left(\left(\alpha x^{d}\right)^{j}\left(\beta(x+1)^{d}\right)^{k}\right)
\end{aligned}
$$

where $\chi_{1}$ is the canonical additive character of the additive group of $\mathbb{F}_{q}, \psi_{k}$ is the $k$-th multiplicative character of the multiplicative group of $\mathbb{F}_{q}$ and $G(\psi, \chi)$ is the Gauss sum.

We shall now state some lemmas that will be used in the sequel. The following lemma is well-known and has been used in various contexts.

Lemma 2.2. Let $e=\operatorname{gcd}(k, n)$. Then

$$
\operatorname{gcd}\left(2^{k}+1,2^{n}-1\right)= \begin{cases}1 & \text { if } n / e \text { is odd }, \\ 2^{e}+1 & \text { if } n / e \text { is even } .\end{cases}
$$

We shall also use the following lemma, which appeared in [5], describing the number of roots in $\mathbb{F}_{2^{n}}$ of a linearized polynomial $u^{2^{k}} x^{2^{2 k}}+u x$, where $u \in \mathbb{F}_{2^{n}}^{*}$.
Lemma 2.3. 5. Theorem 3.1] Let $g$ be a primitive element of $\mathbb{F}_{2^{n}}$ and let $e=\operatorname{gcd}(n, k)$. For any $u \in \mathbb{F}_{2^{n}}^{*}$ consider the linearized polynomial $L_{u}(x)=u^{2^{k}} x^{2^{2 k}}+u x$ over $\mathbb{F}_{2^{n}}$. Then for the equation $L_{u}(x)=0$, the following are true:
(1) If $n / e$ is odd, then there are $2^{e}$ solutions to this equation for any choice of $u \in \mathbb{F}_{2^{n}}^{*}$;
(2) If $n / e$ is even and $u=g^{t\left(2^{e}+1\right)}$ for some $t$, then there are $2^{2 e}$ solutions to the equation;
(3) If $n / e$ is even and $u \neq g^{t\left(2^{e}+1\right)}$ for any $t$, then $x=0$ is the only solution.

The explicit expression for the Weil sum of the form $\sum_{x \in \mathbb{F}_{2^{n}}} \chi_{1}\left(u x^{2^{k}+1}+v x\right)$, where $u, v \in$ $\mathbb{F}_{2^{n}}$, is obtained in [5]. In what follows, we shall denote the Weil sum $\sum_{x \in \mathbb{F}_{q}} \chi\left(u x^{2^{k}+1}+v x\right)$ by $\mathfrak{S}(u, v)$. The following lemma gives the explicit expression for $\mathfrak{S}(u, 0)$.

Lemma 2.4. [5] Let $\chi$ be any nontrivial additive character of $\mathbb{F}_{q}$ and $g$ be the primitive element of the cyclic group $\mathbb{F}_{q}^{*}$. The following hold:
(1) If $n / e$ is odd, then

$$
\sum_{x \in \mathbb{F}_{q}} \chi\left(u x^{2^{k}+1}\right)= \begin{cases}q & \text { if } u=0 \\ 0 & \text { otherwise } .\end{cases}
$$

(2) Let $n / e$ be even so that $n=2 m$ for some integer $m$. Then

$$
\sum_{x \in \mathbb{F}_{q}} \chi\left(u x^{2^{k}+1}\right)= \begin{cases}(-1)^{m / e} 2^{m} & \text { if } u \neq g^{t\left(2^{e}+1\right)} \text { for any integer } t \\ (-1)^{\frac{m}{e}+1} 2^{m+e} & \text { if } u=g^{t\left(2^{e}+1\right)} \text { for some integer } t .\end{cases}
$$

From Lemma 2.2, it is easy to see that when $n / e$ is odd, the power map $x^{2^{k}+1}$ permutes $\mathbb{F}_{2^{n}}$. Therefore if $u \neq 0$, there exists a unique element $\gamma \in \mathbb{F}_{q}^{*}$ such that $\gamma^{2^{k}+1}=u$ and hence

$$
\begin{aligned}
\mathfrak{S}(u, v) & =\sum_{x \in \mathbb{F}_{q}} \chi\left(u x^{2^{k}+1}+v x\right) \\
& =\sum_{x \in \mathbb{F}_{q}} \chi\left(x^{2^{k}+1}+v \gamma^{-1} x\right) \\
& =\mathfrak{S}\left(1, v \gamma^{-1}\right) .
\end{aligned}
$$

The following lemma gives the expression for the Weil sum $\mathfrak{S}(1, v)$ for $v \neq 0$ and $n / e$ odd.
Lemma 2.5. [5, Theorem 4.2] Let $v \neq 0$ and $n / e$ is odd. Then

$$
\mathfrak{S}(1, v)= \begin{cases}0 & \text { if } \operatorname{Tr}_{e}(v) \neq 1 \\ \left(\frac{2}{n / e}\right)^{e} 2^{\frac{n+e}{2}} & \text { if } \operatorname{Tr}_{e}(v)=1\end{cases}
$$

where $\left(\frac{2}{n / e}\right)$ is the Jacobi symbol.

In the case when $u, v \neq 0$ and $n / e$ is even, the Weil sum $\mathfrak{S}(u, v)$ depends whether or not the linearized polynomial $L_{u}(x)=u^{2^{k}} x^{2^{2 k}}+u x$ is a permutation of $\mathbb{F}_{2^{n}}$. The following lemma gives the expression for Weil sum $\mathfrak{S}(u, v)$ for $u, v \neq 0$ and $n / e$ even.
Lemma 2.6. [5, Theorem 5.3] Let $u, v \in \mathbb{F}_{q}^{*}$ and $n / e$ is even so $n=2 m$ for some integer $m$. Then
(1) If $u \neq g^{t\left(2^{e}+1\right)}$ for any integer $t$ then $L_{u}$ is a PP. Let $x_{u} \in \mathbb{F}_{q}$ be the unique solution of the equation $L_{u}(x)=v^{2^{k}}$. Then

$$
\mathfrak{S}(u, v)=(-1)^{m / e} 2^{m} \chi_{1}\left(u x_{u} 2^{2^{k}+1}\right)
$$

(2) If $u=g^{t\left(2^{e}+1\right)}$ for some integer $t$, then $\mathfrak{S}(u, v)=0$ unless the equation $L_{u}(x)=v^{2^{k}}$ is solvable. If the equation $L_{u}(x)=v^{2^{k}}$ is solvable with some solution, say $x_{u}$, then

$$
\mathfrak{S}(u, v)= \begin{cases}(-1)^{m / e} 2^{m} \chi_{1}\left(u x_{u}^{2^{k}+1}\right) & \text { if } \operatorname{Tr}_{e}(u) \neq 0 \\ (-1)^{\frac{m}{e}+1} 2^{m+e} \chi_{1}\left(u x_{u}^{2^{k}+1}\right) & \text { if } \operatorname{Tr}_{e}(u)=0\end{cases}
$$

## 3. The binary Gold function

In this section, we shall give the explicit expression for the $c$-BCT entries of the Gold function $x^{2^{k}+1}$ over $\mathbb{F}_{2^{n}}$, for all $c \neq 0$. Recall that the $c$-boomerang uniformity of a power function $F(x)=x^{d}$ over $\mathbb{F}_{2^{n}}$ is given by $\max _{b \in \mathbb{F}_{2^{n}}}{ }_{c} \mathcal{B}_{F}(1, b)$, where ${ }_{c} \mathcal{B}_{F}(1, b)$ is the number of solutions in $\mathbb{F}_{q} \times \mathbb{F}_{q}, q=2^{n}$ of the following system

$$
\left\{\begin{array}{l}
x^{d}+c y^{d}=b  \tag{3.1}\\
(x+1)^{d}+c^{-1}(y+1)^{d}=b
\end{array}\right.
$$

As done in [14], for $b \neq 0$ and fixed $c \neq 0$, the number of solutions $(x, y) \in \mathbb{F}_{q}^{2}$ of the system (3.1) is given by

$$
\begin{aligned}
{ }_{c} \mathcal{B}_{F}(1, b) & =\frac{1}{q^{2}} \sum_{x, y \in \mathbb{F}_{q}} \sum_{\alpha \in \mathbb{F}_{q}} \chi_{1}\left(\alpha\left(x^{d}+c y^{d}+b\right)\right) \sum_{\beta \in \mathbb{F}_{q}} \chi_{1}\left(\beta\left((x+1)^{d}+c^{-1}(y+1)^{d}+b\right)\right) \\
& =\frac{1}{q^{2}} \sum_{\alpha, \beta \in \mathbb{F}_{q}} \chi_{1}(b(\alpha+\beta)) \sum_{x \in \mathbb{F}_{q}} \chi_{1}\left(\alpha x^{d}+\beta(x+1)^{d}\right) \sum_{y \in \mathbb{F}_{q}} \chi_{1}\left(c \alpha y^{d}+c^{-1} \beta(y+1)^{d}\right) \\
& =\frac{1}{q^{2}} \sum_{\alpha, \beta \in \mathbb{F}_{q}} \chi_{1}(b(\alpha+\beta)) S_{\alpha, \beta} S_{c \alpha, c^{-1} \beta},
\end{aligned}
$$

where $S_{\alpha, \beta}=\sum_{x \in \mathbb{F}_{q}} \chi_{1}\left(\alpha x^{d}+\beta(x+1)^{d}\right)$. Therefore, the problem of computing the $c$-BCT entry ${ }_{c} \mathcal{B}_{F}(1, b)$ is reduced to the computation of the product of the Weil sums $S_{\alpha, \beta}$ and $S_{c \alpha, c^{-1} \beta}$. Now, in the particular case when $d=2^{k}+1$, i.e., for the Gold case, we shall further simplify the expression for $S_{\alpha, \beta}$ as follows:

$$
\begin{aligned}
S_{\alpha, \beta} & =\sum_{x \in \mathbb{F}_{q}} \chi_{1}\left(\alpha x^{2^{k}+1}+\beta(x+1)^{2^{k}+1}\right) \\
& =\chi_{1}(\beta) \sum_{x \in \mathbb{F}_{q}} \chi_{1}\left((\alpha+\beta) x^{2^{k}+1}\right) \chi_{1}\left(\beta x^{2^{k}}+\beta x\right) \\
& =\chi_{1}(\beta) \sum_{x \in \mathbb{F}_{q}} \chi_{1}\left((\alpha+\beta) x^{2^{k}+1}\right) \chi_{1}\left(\left(\beta^{2^{n-k}} x\right)^{2^{k}}+\beta x\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\chi_{1}(\beta) \sum_{x \in \mathbb{F}_{q}} \chi_{1}\left((\alpha+\beta) x^{2^{k}+1}\right) \chi_{1}\left(\left(\beta^{2^{n-k}}+\beta\right) x\right) \\
& =\chi_{1}(\beta) \sum_{x \in \mathbb{F}_{q}} \chi_{1}\left((\alpha+\beta) x^{2^{k}+1}+\left(\beta^{2^{n-k}}+\beta\right) x\right) \\
& =\chi_{1}(\beta) \sum_{x \in \mathbb{F}_{q}} \chi_{1}\left(A x^{2^{k}+1}+B x\right)
\end{aligned}
$$

where $A=\alpha+\beta$ and $B=\beta^{2^{n-k}}+\beta$. Here one may note that $A=0$ if and only if $\alpha=\beta$. Also, $B=0$ if and only if $\beta \in \mathbb{F}_{2^{e}}$, since

$$
\begin{aligned}
B=0 & \Leftrightarrow \beta^{2^{n-k}}=\beta \\
& \Leftrightarrow \beta^{2^{n-k}-1}=1 \\
& \Leftrightarrow \beta^{2^{\operatorname{scd}(n-k, n)}-1}=1 \\
& \Leftrightarrow \beta^{2^{e}-1}=1, \quad(\operatorname{as} \operatorname{gcd}(n-k, n)=e) \\
& \Leftrightarrow \beta \in \mathbb{F}_{2^{e}} .
\end{aligned}
$$

Now we shall calculate $S_{\alpha, \beta}$ in two cases, namely, $n / e$ odd and $n / e$ even, respectively.
Case 1: $n / e$ is odd.
In this case, if $\alpha=\beta$ and $\beta \in \mathbb{F}_{2^{e}}$, then $S_{\alpha, \beta}=q \chi_{1}(\beta)$. If $\alpha=\beta$ and $\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$ then $S_{\alpha, \beta}=0$. In the event of $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^{e}}$, again we have $S_{\alpha, \beta}=0$. Finally, if $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$, by Lemma 2.5 we have,

$$
S_{\alpha, \beta}= \begin{cases}0 & \text { if } \operatorname{Tr}_{e}\left(B \gamma^{-1}\right) \neq 1 \\ \left(\frac{2}{n / e}\right)^{e} 2^{\frac{n+e}{2}} \chi_{1}(\beta) & \text { if } \operatorname{Tr}_{e}\left(B \gamma^{-1}\right)=1\end{cases}
$$

where $\gamma \in \mathbb{F}_{q}$ is the unique element such that $\gamma^{2^{k}+1}=A$.
Case 2: $n / e$ is even.
Let $n=2 m$, for some positive integer $m$ and $g$ be a primitive element of the finite field $\mathbb{F}_{q}$. When $\alpha=\beta$ and $\beta \in \mathbb{F}_{2^{e}}$ then $S_{\alpha, \beta}=q \chi_{1}(\beta)$. If $\alpha=\beta$ and $\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$ then again $S_{\alpha, \beta}=0$. In the event of $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^{e}}$, by Lemma 2.4 we have

$$
S_{\alpha, \beta}= \begin{cases}(-1)^{m / e} 2^{m} \chi_{1}(\beta) & \text { if } A \neq g^{t\left(2^{e}+1\right)} \text { for any integer } t \\ (-1)^{\frac{m}{e}+1} 2^{m+e} \chi_{1}(\beta) & \text { if } A=g^{t\left(2^{e}+1\right)} \text { for some integer } t .\end{cases}
$$

Finally, when $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$, we shall consider two cases depending on whether or not the linearized polynomial $L_{A}(x)=A^{2^{k}} x^{2^{2 k}}+A x$ is a permutation polynomial. From Lemma 2.3, $L_{A}$ is a permutation polynomial if and only if $n / e$ is even and $A \neq g^{t\left(2^{e}+1\right)}$ for any integer $t$. Therefore, when $n / e$ is even and $A \neq g^{t\left(2^{e}+1\right)}$ for any integer $t$, the equation $L_{A}(x)=B^{2^{k}}$ will have a unique solution, say $x_{A}$. Therefore, by Lemma [2.6, we have

$$
S_{\alpha, \beta}=(-1)^{m / e} 2^{m} \chi_{1}(\beta) \chi_{1}\left(A x_{A}^{2^{k}+1}\right) .
$$

Now if the linearized polynomial $L_{A}$ is not permutation, i.e, $n / e$ is even and $A=g^{t\left(2^{e}+1\right)}$ for some integer $t$, we again have two cases depending on whether or not the equation $L_{A}(x)=$ $B^{2^{k}}$ is solvable. In the case when equation $L_{A}(x)=B^{2^{k}}$ is solvable, let $x_{A}$ be one of its solution. Therefore, by Lemma 2.6 we have,

$$
S_{\alpha, \beta}= \begin{cases}(-1)^{\frac{m}{e}+1} 2^{m+e} \chi_{1}(\beta) \chi_{1}\left(A x_{A}^{2^{k}+1}\right) & \text { if } \operatorname{Tr}_{e}(A)=0, \\ (-1)^{\frac{m}{e}} 2^{m} \chi_{1}(\beta) \chi_{1}\left(A x_{A}^{2^{k}+1}\right) & \text { if } \operatorname{Tr}_{e}(A) \neq 0 .\end{cases}
$$

If $L_{A}(x)=B^{2^{k}}$ is not solvable, again, by Lemma 2.6, $S_{\alpha, \beta}=0$.
Thus we have computed $S_{\alpha, \beta}$ in all possible cases. Similarly, we can find $S_{c \alpha, c^{-1} \beta}$ by putting $c \alpha$ and $c^{-1} \beta$ in place of $\alpha$ and $\beta$, respectively. We shall now explicitly compute the $c$-BCT entry ${ }_{c} B_{F}(1, b)$ for $c=1, c \in \mathbb{F}_{2^{e}} \backslash\{0,1\}$ and $c \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$ in the forthcoming sections.

## 4. The case $c=1$

When $c=1, S_{\alpha, \beta}$ and $S_{c \alpha, c^{-1} \beta}$ coincide, therefore for any fixed $b \neq 0$, the $c$-BCT entry is given by,

$$
{ }_{1} \mathcal{B}_{F}(1, b)=\frac{1}{q^{2}} \sum_{\alpha, \beta \in \mathbb{F}_{q}} \chi_{1}(b(\alpha+\beta)) S_{\alpha, \beta}^{2} .
$$

Let us denote $T_{b}=S_{\alpha, \beta}^{2}$. Now we shall consider two cases, namely, $n / e$ odd and $n / e$ even, respectively.

Case 1: $n / e$ is odd. We consider the following subcases.
(1) If $\alpha=\beta$ and $\beta \in \mathbb{F}_{2^{e}}$, then

$$
T_{b}^{[1]}=q^{2} \chi_{1}(\beta)^{2}=q^{2} .
$$

(2) If $\alpha=\beta$ and $\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$, then

$$
T_{b}^{[2]}=0 .
$$

(3) If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^{e}}$, then

$$
T_{b}^{[3]}=0 .
$$

(4) If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$ then

$$
T_{b}^{[4]}= \begin{cases}0 & \text { if } \operatorname{Tr}_{e}\left(B \gamma^{-1}\right) \neq 1 \\ 2^{n+e} & \text { if } \operatorname{Tr}_{e}\left(B \gamma^{-1}\right)=1\end{cases}
$$

Nyberg [11, Proposition 3] showed that the differential uniformity of the Gold function $x \mapsto x^{2^{k}+1}$ over $\mathbb{F}_{2^{n}}$ is $2^{e}$, where $e=\operatorname{gcd}(k, n)$. Also, from 4], we know that the boomerang uniformity of the APN function equals 2. Boura and Canteaut [2, Proposition 8] proved that when $n / e$ is odd and $n \equiv 2(\bmod 4)$, then the differential as well as the boomerang uniformity of the Gold function $x \mapsto x^{2^{k}+1}$ is 4. Our first theorem in this section generalizes the two previously mentioned results, and gives the boomerang uniformity of the Gold function for any parameters, when $\frac{n}{e}$ is odd. Note that we would require the notion of Walsh-Hadamard transform in the proof of this theorem, which is defined as follows.

For $f: \mathbb{F}_{2^{n}} \rightarrow \mathbb{F}_{2}$ we define the Walsh-Hadamard transform to be the integer-valued function

$$
\mathcal{W}_{f}(u)=\sum_{x \in \mathbb{F}_{2^{n}}}(-1)^{f(x)+\operatorname{Tr}(u x)}, u \in \mathbb{F}_{2^{n}}
$$

The Walsh transform $\mathcal{W}_{F}(a, b)$ of an $(n, m)$-function $F: \mathbb{F}_{2^{n}} \rightarrow \mathbb{F}_{2^{m}}$ at $a \in \mathbb{F}_{2^{n}}, b \in \mathbb{F}_{2^{m}}$ is the Walsh-Hadamard transform of its component function $\operatorname{Tr}_{1}^{m}(b F(x))$ at $a$, that is,

$$
\mathcal{W}_{F}(a, b)=\sum_{x \in \mathbb{F}_{2^{n}}}(-1)^{\operatorname{Tr}_{1}^{m}(b F(x))-\operatorname{Tr}_{1}^{n}(a x)}
$$

Theorem 4.1. Let $F(x)=x^{2^{k}+1}, 1 \leq k<n$, be a function on $\mathbb{F}_{q}, q=2^{n}$, $n \geq 2$. Let $c=1$ and $n / e$ be odd, where $e=\operatorname{gcd}(k, n)$. Then the $c-B C T$ entry ${ }_{1} \mathcal{B}_{F}(1, b)$ of $F$ at $(1, b)$ is

$$
{ }_{1} \mathcal{B}_{F}(1, b)=0, \text { or, } 2^{e},
$$

if $\operatorname{Tr}_{e}\left(b^{\frac{1}{2}}\right)=0$, respectively, $\operatorname{Tr}_{e}\left(b^{\frac{1}{2}}\right) \neq 0$.
Proof. For every $\alpha, \beta$, let $A=\alpha+\beta, B=\beta^{2^{-k}}+\beta$, and $\gamma \in \mathbb{F}_{q}$ be the unique element such that $\gamma^{2^{k}+1}=A$. Further, let

$$
\begin{aligned}
\mathcal{A} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid \alpha=\beta \in \mathbb{F}_{2^{e}}\right\}, \\
\mathcal{B} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid \alpha=\beta \in \mathbb{F}_{q} \backslash \mathbb{F}_{2^{e}}\right\}, \\
\mathcal{C} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid \alpha \neq \beta \text { and } \beta \in \mathbb{F}_{2^{e}}\right\}, \\
\mathcal{D} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid \alpha \neq \beta \text { and } \beta \in \mathbb{F}_{q} \backslash \mathbb{F}_{2^{e}}\right\}, \\
\mathcal{E} & =\left\{(\alpha, \beta) \in \mathcal{D} \mid \operatorname{Tr}_{e}\left(B \gamma^{-1}\right) \neq 1\right\}, \\
\mathcal{F} & =\left\{(\alpha, \beta) \in \mathcal{D} \mid \operatorname{Tr}_{e}\left(B \gamma^{-1}\right)=1\right\} .
\end{aligned}
$$

Then,

$$
\begin{aligned}
{ }_{1} \mathcal{B}_{F}(1, b)= & \frac{1}{q^{2}}\left(\sum_{(\alpha, \beta) \in \mathcal{A}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1]}+\sum_{(\alpha, \beta) \in \mathcal{B}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[2]}+\sum_{(\alpha, \beta) \in \mathcal{C}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3]}+\right. \\
& \left.\sum_{(\alpha, \beta) \in \mathcal{E}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4]}+\sum_{\alpha, \beta \in \mathcal{F}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4]}\right) \\
= & \frac{1}{q^{2}}\left(\sum_{(\alpha, \beta) \in \mathcal{A}} q^{2}+\sum_{(\alpha, \beta) \in \mathcal{F}} \chi_{1}(b(\alpha+\beta)) 2^{n+e}\right) \\
= & 2^{e}+\frac{2^{e}}{2^{n}} \sum_{(\alpha, \beta) \in \mathcal{F}} \chi_{1}(b(\alpha+\beta)) .
\end{aligned}
$$

As customary, $t^{-1}=t^{2^{n}-2}$, rendering $0^{-1}=0$. For each $\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$, we let (if $\beta \in \mathbb{F}_{2^{e}}$, $\left.Y_{\beta}=\mathbb{F}_{2^{n}}\right)$

$$
Y_{\beta}=\left\{\gamma^{-1} \in \mathbb{F}_{2^{n}}: \operatorname{Tr}_{e}\left(\left(\beta^{2^{-k}}+\beta\right) \gamma^{-1}\right)=1\right\}
$$

and

$$
T_{\beta}=\left\{d \in \mathbb{F}_{2^{n}}: \operatorname{Tr}_{e}\left(\left(\beta^{2^{-k}}+\beta\right) d\right)=0\right\}=\left\langle\beta^{2^{-k}}+\beta\right\rangle^{\perp_{e}}
$$

We shall use below that when $\frac{n}{e}$ is odd, then $\operatorname{Tr}_{e}(1)=1$. We label by $\langle S\rangle_{e}$ the $\mathbb{F}_{2^{e}}$-linear subspace in $\mathbb{F}_{2^{n}}$ generate by $S$ and we write $S^{\perp_{e}}$, for the trace orthogonal (via the relative trace $\operatorname{Tr}_{e}$ ) of the subspace $\langle S\rangle_{e}$ (if $e=1$, we drop the subscripts). Since $\operatorname{Tr}_{e}(1)=1$, then, $\left(\beta^{2^{-k}}+\beta\right)^{-1} \in Y_{\beta}$. If $\gamma_{1}^{-1}, \gamma_{2}^{-1} \in Y_{\beta}$, then $\gamma_{1}^{-1}+\gamma_{2}^{-1} \in T_{\beta}$, of cardinality $\left|T_{\beta}\right|=2^{n-1}$. Reciprocally, if $\gamma^{-1} \in Y_{\beta}$ and $d \in T_{\beta}$, it is easy to see that $\gamma^{-1}+d \in Y_{\beta}$. Therefore, $Y_{\beta}$ is the affine subspace $Y_{\beta}=\gamma_{\beta}+T_{\beta}$, where $\gamma_{\beta}=\left(\beta^{2^{-k}}+\beta\right)^{-1}$.

Next, we observe that the kernel of $\phi: \beta \mapsto \beta^{2^{-k}}+\beta$, say $\operatorname{ker}(\phi)$, is an $\mathbb{F}_{2}$-linear space of dimension $e$ (in fact, it is exactly $\mathbb{F}_{2^{e}}$ ) and the image of $\phi$, say $\operatorname{Im}(\phi)$, is an $\mathbb{F}_{2}$-linear space of dimension $n-e$. Further, we show that $\operatorname{Im}(\phi)^{\perp_{e}}=\operatorname{ker}(\phi)$. We use below the fact that $\operatorname{Tr}_{e}\left(x^{2^{e}}\right)=\operatorname{Tr}_{e}(x)$ and $e \mid k$. Let $u \in \operatorname{Im}(\phi)^{\perp_{e}}$, that is, for all $\beta \in \mathbb{F}_{2^{n}}$,

$$
0=\operatorname{Tr}_{e}\left(u\left(\beta^{2^{-k}}+\beta\right)\right)=\operatorname{Tr}_{e}\left(u \beta^{2^{-k}}\right)+\operatorname{Tr}_{e}(u \beta)=\operatorname{Tr}_{e}\left(u^{2^{k}} \beta\right)+\operatorname{Tr}_{e}(u \beta)=\operatorname{Tr}_{e}\left(\left(u+u^{2^{k}}\right) \beta\right)
$$

and so, $u^{2^{k}}+u=0$, which shows the claim. For easy referral, if we speak of the dimension of an $F_{2^{e}}$-linear space $S$, we shall be using the notation $\operatorname{dim}_{e} S$ (no subscript if $e=1$ ).

We will be using below the Poisson summation formula (see [3, Corollary 8.9] and [6, Theorem 2.15]), which states that if $f: \mathbb{F}_{2^{n}} \rightarrow \mathbb{R}$ and $S$ is a subspace of $\mathbb{F}_{2^{n}}$ of dimension $\operatorname{dim} S$, then

$$
\sum_{u \in \alpha+S} \mathcal{W}_{f}(u)(-1)^{\operatorname{Tr}(\beta u)}=2^{\operatorname{dim} S}(-1)^{\operatorname{Tr}(\alpha \beta)} \sum_{u \in \beta+S^{\perp}} f(u)(-1)^{\operatorname{Tr}(\alpha u)}
$$

and in particular,

$$
\sum_{u \in S} \mathcal{W}_{f}(u)=2^{\operatorname{dim} S} \sum_{u \in S^{\perp}} f(u) .
$$

Now, we are able to compute our sum (labelling $\alpha=\beta+\gamma^{2^{k}+1}$, and writing $\phi^{-1}(t)=\{\beta$ : $\phi(\beta)=t\}$; we also note that when $\frac{n}{e}$ is odd, $\operatorname{gcd}\left(2^{k}+1,2^{n}-1\right)=1$, and so $\gamma \mapsto \gamma^{2^{k}+1}$ is a permutation)

$$
\begin{aligned}
& { }_{1} \mathcal{B}_{F}(1, b)=2^{e}+\frac{2^{e}}{2^{n}} \sum_{\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}, \gamma \in \mathbb{F}_{2 n}}} \chi_{1}\left(b \gamma^{2^{k}+1}\right) \\
& \operatorname{Tr}_{e}\left(\left(\beta^{2^{-k}}+\beta\right) \gamma^{-1}\right)=1 \\
& =2^{e}+\frac{2^{e}}{2^{n}} \sum_{\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}} \sum_{\gamma^{-1} \in Y_{\beta}} \chi_{1}\left(b 2^{2^{k}+1}\right) \\
& =2^{e}+\frac{2^{e}}{2^{n}} \sum_{\beta \in \mathbb{F}_{2^{n}}} \sum_{x \in\left(\beta^{2-k}+\beta\right)^{-1}+\left\langle\beta^{2-k}+\beta\right\rangle^{\perp}} \chi_{1}\left(b x^{-2^{k}-1}\right) \\
& \text { (we used here that } Y_{\beta}=\left(\beta^{2^{-k}}+\beta\right)^{-1}+T_{\beta} \text {; we also added } \\
& \beta \in \mathbb{F}_{2^{e}} \text {, as it contributes } 0 \text { to the inner sum) } \\
& =2^{e}+\frac{2^{e}}{2^{n}} \sum_{\beta \in \mathbb{F}_{2^{n}}} 2^{-\operatorname{dim} S} \sum_{u \in\left(\left\langle\beta^{2-k}+\beta\right\rangle^{\perp}\right)^{\perp}} \mathcal{W}_{g_{\beta}}(u)(-1)^{\operatorname{Tr}\left(u\left(\beta^{2^{-k}}+\beta\right)^{-1}\right)}
\end{aligned}
$$

(by Poisson summation with $S^{\perp}=\left\langle\beta^{2^{-k}}+\beta\right\rangle^{\perp_{e}}$, and $g_{\beta}(x)=\chi_{1}\left(b x^{-2^{k}-1}\right)$ ).
We now analyze the $\mathbb{F}_{2}$-linear space

$$
\left(\left\langle\beta^{2^{-k}}+\beta\right\rangle^{\perp}\right)^{\perp}=\left\{x \in \mathbb{F}_{2^{n}}: \operatorname{Tr}(d x)=0, \forall d \text { with } \operatorname{Tr}_{e}\left(d\left(\beta^{2^{-k}}+\beta\right)\right)=0\right\}
$$

Further, $\mathbb{F}_{2^{n}}$ has dimension $n / e$ as an $\mathbb{F}_{2^{e-}}$-linear space and so, $\operatorname{dim}_{e}\left\langle\beta^{2^{-k}}+\beta\right\rangle^{\perp_{e}}=\frac{n}{e}-1$ as an $\mathbb{F}_{2^{e}}$-linear space, and since $\mathbb{F}_{2^{e}}$ has dimension $e$ as an $\mathbb{F}_{2^{-}}$-linear space, then $\operatorname{dim}\left\langle\beta^{2^{-k}}+\right.$ $\beta\rangle^{\perp_{e}}=n-e$ as an $\mathbb{F}_{2}$-linear space. Thus, $\operatorname{dim}\left(\left\langle\beta^{2^{-k}}+\beta\right\rangle^{\perp_{e}}\right)^{\perp}=e$. Moreover, $\operatorname{Tr}_{e}\left(\beta^{2^{-k}}+\right.$ $\beta)=0$ and if $u \in \mathbb{F}_{2^{e}}$ then $\operatorname{Tr}_{e}\left(u\left(\beta^{2^{-k}}+\beta\right)=u \operatorname{Tr}_{e}\left(\beta^{2^{-k}}+\beta\right)=0\right.$, and consequently (since the dimensions match and $\left.\left(\beta^{2^{-k}}+\beta\right) \mathbb{F}_{2^{e}} \subseteq S\right)$

$$
S=\left(\left\langle\beta^{2^{-k}}+\beta\right\rangle^{\perp_{e}}\right)^{\perp}=\left(\beta^{2^{-k}}+\beta\right) \mathbb{F}_{2^{e}}
$$

We are now ready to continue the computation, thus,

$$
\begin{aligned}
& { }_{1} \mathcal{B}_{F}(1, b)=2^{e}+\frac{2^{e}}{2^{n}} 2^{-e} \sum_{\beta \in \mathbb{F}_{2^{n}}} \sum_{u \in\left(\beta^{2-k}+\beta\right) \mathbb{F}_{2^{e}}} \mathcal{W}_{g_{\beta}}(u)(-1)^{\operatorname{Tr}\left(u\left(\beta^{2^{-k}}+\beta\right)^{-1}\right)} \\
& =2^{e}+\frac{2^{e}}{2^{n}} 2^{-e} \sum_{\beta \in \mathbb{F}_{2^{n}}} \sum_{d^{\prime} \in \mathbb{F}_{2^{e}}} \mathcal{W}_{g_{\beta}}\left(d^{\prime}\left(\beta^{2^{-k}}+\beta\right)\right)(-1)^{\operatorname{Tr}\left(d^{\prime}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= 2^{e}+\frac{2^{e}}{2^{n}} 2^{-e} \sum_{\beta \in \mathbb{F}_{2^{n}}} \sum_{d^{\prime} \in \mathbb{F}_{2} e} \sum_{x \in \mathbb{F}_{2} n} \chi_{1}\left(b x^{-2^{k}-1}+d^{\prime} x\left(\beta^{2^{-k}}+\beta\right)+d^{\prime}\right) \\
&= 2^{e}+\frac{2^{e}}{2^{n}} 2^{-e} \sum_{d^{\prime} \in \mathbb{F}_{2^{e}}} \sum_{x \in \mathbb{F}_{2^{n}}} \chi_{1}\left(b x^{-2^{k}-1}+d^{\prime}\right) \sum_{\beta \in \mathbb{F}_{2^{n}}} \chi_{1}\left(d^{\prime} x\left(\beta^{2^{-k}}+\beta\right)\right) \\
&= 2^{e}+\frac{2^{e}}{2^{n}} 2^{-e} \sum_{d^{\prime} \in \mathbb{F}_{2^{e}}} \sum_{x \in \mathbb{F}_{2^{n}}} \chi_{1}\left(b x^{-2^{k}-1}+d^{\prime}\right) \sum_{\beta \in \mathbb{F}_{2^{n}}} \chi_{1}\left(\left(\left(d^{\prime} x\right)^{2^{k}}+d^{\prime} x\right) \beta\right) \\
&\left(\text { since } \operatorname{Tr}\left(d^{\prime} x\left(\beta^{2^{-k}}+\beta\right)\right)=\operatorname{Tr}\left(\left(\left(d^{\prime} x\right)^{2^{k}}+d^{\prime} x\right) \beta\right)=\operatorname{Tr}\left(d^{\prime}\left(x^{2^{k}}+x\right) \beta\right)\right) \\
&= 2^{e}+\frac{2^{e}}{2^{n}} 2^{n-e} \sum_{d^{\prime} \in \mathbb{F}_{2} e, x \in \mathbb{F}_{2^{n}}} \chi_{1}\left(b x^{-2^{k}-1}+d^{\prime}\right) \\
&= 2^{e}+\frac{2^{e}}{2^{n}} 2^{n-e} \sum_{d^{\prime}\left(x^{2^{k}}+x\right)=0} \chi_{1}\left(b x^{-2}+d^{\prime}\right)+\sum_{x \in \mathbb{F}_{2^{n}}} \chi_{1}\left(b x^{-2^{k}-1}\right) \\
&=2^{e}+\frac{2^{e}}{2^{n}} 2^{n-e} \sum_{d^{\prime} \in \mathbb{F}_{2^{e}, x \in \mathbb{F}_{2^{e}}}} \chi_{1}\left(b x^{-2}+d^{\prime}\right) \\
&= 2^{e}-2^{e} \delta_{0}\left(\operatorname{Tr}_{e}\left(b^{\frac{1}{2}, x \in \mathbb{F}_{2^{e}}}\right)\right),
\end{aligned}
$$

where $\delta_{0}$ is the Dirac symbol, defined by $\delta_{0}(c)=1$, if $c=0$, and 0 , otherwise. Thus, ${ }_{1} \mathcal{B}_{F}(1, b) \in\left\{0,2^{e}\right\}$, and the claim of our theorem is shown.

Case 2: $n / e$ is even.
(1) If $\alpha=\beta$ and $\beta \in \mathbb{F}_{2^{e}}$, then

$$
T_{b}^{[1]}=q^{2} \chi_{1}(\beta)^{2}=q^{2} .
$$

(2) If $\alpha=\beta$ and $\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$, then

$$
T_{b}^{[2]}=0 .
$$

(3) If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^{e}}$, then

$$
T_{b}^{[3]}= \begin{cases}2^{n} & \text { if } A \neq g^{t\left(2^{e}+1\right)} \text { for any integer } t \\ 2^{n+2 e} & \text { if } A=g^{t\left(2^{e}+1\right)} \text { for some integer } t .\end{cases}
$$

(4) If $\alpha \neq \beta$ and $\beta \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$, then
(a) If $A \neq g^{t\left(2^{e}+1\right)}$ for any integer $t$, then

$$
T_{b}^{[4(a)]}=2^{n}
$$

(b) If $A=g^{t\left(2^{e}+1\right)}$ for some integer $t$, then
(i) If the equation $L_{A}(x)=B^{2^{k}}$ is not solvable, where $L_{A}(x)=A^{2^{k}} x^{2^{2 k}}+A x$, then

$$
T_{b}^{[4(b)(i)]}=0 .
$$

(ii) If the equation $L_{A}(x)=B^{2^{k}}$ is solvable, then

$$
T_{b}^{[4(b)(i i)]}= \begin{cases}2^{n} & \text { if } \operatorname{Tr}_{e}(A) \neq 0 \\ 2^{n+2 e} & \text { if } \operatorname{Tr}_{e}(A)=0\end{cases}
$$

Now we shall summarize the above discussion in the following theorem.

Theorem 4.2. Let $F(x)=x^{2^{k}+1}, 1 \leq k<n$ be a function on $\mathbb{F}_{2^{n}}, n \geq 2$. Let $c=1$ and $n / e$ be even, where $e=\operatorname{gcd}(k, n)$. Then the $c-B C T$ entry ${ }_{1} \mathcal{B}_{F}(1, b)$ of $F$ at $(1, b)$ is given by

$$
2^{e}+\frac{1}{2^{n}} \sum_{(\alpha, \beta) \in \mathcal{G} \cup \mathcal{I} \cup \mathcal{K}} \chi_{1}(b(\alpha+\beta))+\frac{2^{2 e}}{2^{n}} \sum_{(\alpha, \beta) \in \mathcal{H} \cup \mathcal{L}} \chi_{1}(b(\alpha+\beta)),
$$

with $A=\alpha+\beta, B=\beta^{2^{n-k}}+\beta, L_{A}(x)=A^{2^{k}} x^{2^{2 k}}+A x$, and

$$
\begin{aligned}
\mathcal{G} & =\left\{(\alpha, \beta) \in \mathcal{C} \mid A \neq g^{t\left(2^{e}+1\right)} \text { for any integer } t\right\}, \\
\mathcal{H} & =\left\{(\alpha, \beta) \in \mathcal{C} \mid A=g^{t\left(2^{e}+1\right)} \text { for some integer } t\right\}, \\
\mathcal{I} & =\left\{(\alpha, \beta) \in \mathcal{D} \mid A \neq g^{t\left(2^{e}+1\right)} \text { for any integer } t\right\}, \\
\mathcal{K} & =\left\{(\alpha, \beta) \in \mathcal{D} \mid A=g^{t\left(2^{e}+1\right)} \text { for some integer } t, \operatorname{Tr}_{e}(A) \neq 0, L_{A}(x)=B^{2^{k}} \text { is solvable }\right\}, \\
\mathcal{L} & =\left\{(\alpha, \beta) \in \mathcal{D} \mid A=g^{t\left(2^{e}+1\right)} \text { for some integer } t, \operatorname{Tr}_{e}(A)=0, L_{A}(x)=B^{2^{k}} \text { is solvable }\right\} .
\end{aligned}
$$

Proof. For the proof, we need to define

$$
\mathcal{J}=\left\{(\alpha, \beta) \in \mathcal{D} \mid A=g^{t\left(2^{e}+1\right)} \text { for an integer } t, L_{A}(x)=B^{2^{k}} \text { is not solvable }\right\} .
$$

Then

$$
\begin{aligned}
{ }_{1} \mathcal{B}_{F}(1, b)= & \frac{1}{q^{2}}\left(\sum_{(\alpha, \beta) \in \mathcal{A}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1]}+\sum_{(\alpha, \beta) \in \mathcal{B}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[2]}\right. \\
& +\sum_{(\alpha, \beta) \in \mathcal{G}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3]}+\sum_{(\alpha, \beta) \in \mathcal{H}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3]} \\
& +\sum_{(\alpha, \beta) \in \mathcal{I}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(a)}+\sum_{(\alpha, \beta) \in \mathcal{J}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)(i)]} \\
& \left.+\sum_{(\alpha, \beta) \in \mathcal{K}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)(i i)]}+\sum_{(\alpha, \beta) \in \mathcal{L}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)(i i)]}\right) \\
= & \frac{1}{q^{2}}\left(\sum_{(\alpha, \beta) \in \mathcal{A}} q^{2}+2^{n} \sum_{(\alpha, \beta) \in \mathcal{G} \cup \mathcal{I} \cup \mathcal{K}} \chi_{1}(b(\alpha+\beta))+2^{n+2 e} \sum_{(\alpha, \beta) \in \mathcal{H} \cup \mathcal{L}} \chi_{1}(b(\alpha+\beta))\right) \\
= & 2^{e}+\frac{1}{2^{n}} \sum_{(\alpha, \beta) \in \mathcal{G} \cup \mathcal{I} \cup \mathcal{K}} \chi_{1}(b(\alpha+\beta))+\frac{2^{2 e}}{2^{n}} \sum_{(\alpha, \beta) \in \mathcal{H} \cup \mathcal{L}} \chi_{1}(b(\alpha+\beta)) .
\end{aligned}
$$

This completes the proof.
Corollary 4.3. Let $F(x)=x^{2^{k}+1}, 1 \leq k<n$, be a function on $\mathbb{F}_{q}, n \geq 2$. Let $c=1$ and $n / e$ be even, where $e=\operatorname{gcd}(k, n)$. With the notations of the previous theorem, the $c$-boomerang uniformity of $F$ satisfies

$$
\beta_{F, c} \leq 2^{e}+2^{-n}|\mathcal{G} \cup \mathcal{I} \cup \mathcal{K}|+2^{2 e-n}|\mathcal{H} \cup \mathcal{L}| .
$$

5. The case $c \in \mathbb{F}_{2^{e}} \backslash\{0,1\}$.

Since the case $c=1$ has already been considered in the previous section, throughout this section we assume that $c \neq 1$. Notice that when $c \in \mathbb{F}_{2^{e}}^{*}, \beta \in \mathbb{F}_{2^{e}} \Leftrightarrow \beta c^{-1} \in \mathbb{F}_{2^{e}}$. Recall that
for any fixed $b \neq 0$, the $c$-BCT entry is given by,

$$
{ }_{c} \mathcal{B}_{F}(1, b)=\frac{1}{q^{2}} \sum_{\alpha, \beta \in \mathbb{F}_{q}} \chi_{1}(b(\alpha+\beta)) S_{\alpha, \beta} S_{c \alpha, c^{-1} \beta}
$$

Let us denote $T_{b}=S_{\alpha, \beta} S_{c \alpha, c^{-1} \beta}$ (we will use superscripts to point out the case we are in, for its value). Recall that $A=\alpha+\beta$ and $B=\beta^{2^{n-k}}+\beta$. Let us denote $\gamma=A^{\frac{1}{2^{k}+1}}, A^{\prime}=c \alpha+c^{-1} \beta$ and $B^{\prime}=\left(c^{-1} \beta\right)^{2^{n-k}}+c^{-1} \beta$. It is easy to observe that the conditions $B=0$ and $B^{\prime}=0$ are equivalent. Now we shall consider two cases namely, $\frac{n}{e}$ odd and $\frac{n}{e}$ even, respectively.
Case 1: $\frac{n}{e}$ is odd.
(1) Let $A=0, B=0$.
(a) If $A^{\prime}=0, B^{\prime}=0$, then

$$
T_{b}^{[1(a)]}=q^{2} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) .
$$

(b) If $A^{\prime} \neq 0, B^{\prime}=0$, then $S_{c \alpha, c^{-1} \beta}=0$ and hence

$$
T_{b}^{[1(b)]}=0 .
$$

(2) Let $A=0, B \neq 0$. In this case $S_{\alpha, \beta}=0$ and hence

$$
T_{b}^{[2]}=0 .
$$

(3) Let $A \neq 0, B=0$. Again $S_{\alpha, \beta}=0$ and hence

$$
T_{b}^{[3]}=0
$$

(4) Let $A \neq 0, B \neq 0$.
(a) Assume $A^{\prime}=0, B^{\prime} \neq 0$, then $S_{c \alpha, c^{-1} \beta}=0$ and hence

$$
T_{b}^{[4(a)]}=0 .
$$

(b) Assume $A^{\prime} \neq 0, B^{\prime} \neq 0$. In this case, recall that $\gamma^{2^{k}+1}=A$ and let $\gamma^{\prime} \in \mathbb{F}_{q}$ such that $\left(\gamma^{\prime}\right)^{2^{k}+1}=A^{\prime}$.
(i) If $\operatorname{Tr}_{e}\left(B \gamma^{-1}\right) \neq 1$, then $S_{\alpha, \beta}=0$ and hence

$$
T_{b}^{[4(b)(i)]}=0 .
$$

(ii) If $\operatorname{Tr}_{e}\left(B \gamma^{-1}\right)=1$ and $\operatorname{Tr}_{\mathrm{e}}\left(\mathrm{B}^{\prime}\left(\gamma^{\prime}\right)^{-1}\right) \neq 1$, then $S_{c \alpha, c^{-1} \beta}=0$ and hence

$$
T_{b}^{[4(b)(i i)]}=0 .
$$

(iii) If $\operatorname{Tr}_{e}\left(B \gamma^{-1}\right)=1$ and $\operatorname{Tr}_{e}\left(B^{\prime}\left(\gamma^{\prime}\right)^{-1}\right)=1$, then

$$
T_{b}^{[4(b)(i i i)]}=2^{n+e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) .
$$

We now use the above discussion in the following theorem.
Theorem 5.1. Let $F(x)=x^{2^{k}+1}, 1 \leq k<n$ be a function on $\mathbb{F}_{2^{n}}, n \geq 2$. Let $c \in \mathbb{F}_{2^{e}} \backslash\{0,1\}$ and $n / e$ be odd, where $e=\operatorname{gcd}(k, n)$. Then the $c-B C T$ entry ${ }_{c} \mathcal{B}_{F}(1, b)$ of $F$ at $(1, b)$ is given by

$$
\left.1+\frac{2^{e}}{2^{n}} \sum_{(\alpha, \beta) \in \mathcal{F} \cap \mathcal{F}^{\prime}} \chi_{1}\left(b \alpha+\left(1+c^{-1}+b\right) \beta\right)\right),
$$

where

$$
\begin{aligned}
\mathcal{F} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid A, B \neq 0 \text { and } \operatorname{Tr}_{e}\left(B \gamma^{-1}\right)=1\right\}, \\
\mathcal{F}^{\prime} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid A^{\prime}, B^{\prime} \neq 0 \text { and } \operatorname{Tr}_{e}\left(B^{\prime}\left(\gamma^{\prime}\right)^{-1}\right)=1\right\},
\end{aligned}
$$

and $A=\alpha+\beta, B=\beta^{2^{n-k}}+\beta, A^{\prime}=c \alpha+c^{-1} \beta$ and $B^{\prime}=\left(c^{-1} \beta\right)^{2^{n-k}}+c^{-1} \beta, \gamma=A^{\frac{1}{2^{k}+1}}$, $\gamma^{\prime}=A^{\frac{1}{2^{k}+1}}$.

## Proof. Let

$$
\begin{aligned}
\mathcal{A}^{\prime} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid c \alpha=c^{-1} \beta \text { and } c^{-1} \beta \in \mathbb{F}_{2^{e}}\right\}, \\
\mathcal{B}^{\prime} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid c \alpha=c^{-1} \beta \text { and } c^{-1} \beta \in \mathbb{F}_{q} \backslash \mathbb{F}_{2^{e}}\right\}, \\
\mathcal{C}^{\prime} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid c \alpha \neq c^{-1} \beta \text { and } c^{-1} \beta \in \mathbb{F}_{2^{e}}\right\}, \\
\mathcal{D}^{\prime} & =\left\{(\alpha, \beta) \in \mathbb{F}_{q}^{2} \mid c \alpha \neq c^{-1} \beta \text { and } c^{-1} \beta \in \mathbb{F}_{q} \backslash \mathbb{F}_{2^{e}}\right\}, \\
\mathcal{E}^{\prime} & =\left\{(\alpha, \beta) \in \mathcal{D}^{\prime} \mid \operatorname{Tr}_{e}\left(B^{\prime}\left(\gamma^{\prime}\right)^{-1}\right) \neq 1\right\} .
\end{aligned}
$$

Then,

$$
\begin{aligned}
{ }_{c} \mathcal{B}_{F}(1, b) & =\frac{1}{q^{2}}\left(\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{A}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(a)]}+\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{C}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(b)]}\right. \\
& +\sum_{(\alpha, \beta) \in \mathcal{B}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[2]}+\sum_{(\alpha, \beta) \in \mathcal{C}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3]} \\
& +\sum_{(\alpha, \beta) \in \mathcal{D} \cap \mathcal{B}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(a)]}+\sum_{(\alpha, \beta) \in \mathcal{E}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)(i)]} \\
& \left.+\sum_{(\alpha, \beta) \in \mathcal{F} \cap \mathcal{E}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)(i i)]}+\sum_{(\alpha, \beta) \in \mathcal{F} \cap \mathcal{F}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)(i i i)])}\right) \\
& \left.=\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{A}^{\prime}} \chi_{1}\left(b \alpha+\left(1+c^{-1}+b\right) \beta\right)\right) \\
& \left.+\frac{2^{e}}{2^{n}} \sum_{(\alpha, \beta) \in \mathcal{F} \cap \mathcal{F}^{\prime}} \chi_{1}\left(b \alpha+\left(1+c^{-1}+b\right) \beta\right)\right) \\
& \left.=1+\frac{2^{e}}{2^{n}} \sum_{(\alpha, \beta) \in \mathcal{F} \cap \mathcal{F}^{\prime}} \chi_{1}\left(b \alpha+\left(1+c^{-1}+b\right) \beta\right)\right) .
\end{aligned}
$$

This completes the proof.
Corollary 5.2. Let $F(x)=x^{2^{k}+1}, 1 \leq k<n$, be a function on $\mathbb{F}_{q}, n \geq 2$. Let $c \in \mathbb{F}_{2^{e}} \backslash\{0,1\}$ and $n / e$ be odd, where $e=\operatorname{gcd}(k, n)$. With the notations of the previous theorem, the $c$ boomerang uniformity of $F$ satisfies

$$
\beta_{F, c} \leq 1+2^{e-n}\left|\mathcal{F} \cap \mathcal{F}^{\prime}\right| .
$$

Case 2: $n / e$ is even.
(1) Let $A=0, B=0$.
(a) If $A^{\prime}=0, B^{\prime}=0$, then

$$
T_{b}^{[1(a)]}=\chi_{1}\left(\left(1+c^{-1}\right) \beta\right) q^{2} .
$$

(b) If $A^{\prime} \neq 0, B^{\prime}=0$, let

$$
\begin{aligned}
\mathcal{G}^{\prime} & =\left\{(\alpha, \beta) \in \mathcal{C}^{\prime} \mid A^{\prime} \neq g^{t\left(2^{e}+1\right)} \text { for any integer } t\right\}, \\
\mathcal{H}^{\prime} & =\left\{(\alpha, \beta) \in \mathcal{C}^{\prime} \mid A^{\prime}=g^{t\left(2^{e}+1\right)} \text { for some integer } t\right\} .
\end{aligned}
$$

Then,

$$
T_{b}^{[1(b)]}= \begin{cases}(-1)^{\frac{m}{e}} 2^{m+n} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{A} \cap \mathcal{G}^{\prime}, \\ (-1)^{\frac{m}{e}+1} 2^{m+n+e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{A} \cap \mathcal{H}^{\prime} .\end{cases}
$$

(2) Let $A=0, B \neq 0$.

In this case $S_{\alpha, \beta}=0$ and hence

$$
T_{b}^{[2]}=0 .
$$

(3) Let $A \neq 0, B=0$.
(a) If $A^{\prime}=0, B^{\prime}=0$, then $T_{b}^{[3(a)]}$ is given by

$$
\begin{cases}(-1)^{\frac{m}{e}} 2^{m+n} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{A}^{\prime} \cap \mathcal{G} \\ (-1)^{\frac{m}{e}+1} 2^{m+n+e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{A}^{\prime} \cap \mathcal{H}\end{cases}
$$

(b) If $A^{\prime} \neq 0, B^{\prime}=0$, then

$$
T_{b}^{[3(b)]}= \begin{cases}2^{n} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{G} \cap \mathcal{G}^{\prime}, \\ -2^{n+e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{G} \cap \mathcal{H}^{\prime}, \\ -2^{n+e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{H} \cap \mathcal{G}^{\prime} \\ 2^{n+2 e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{H} \cap \mathcal{H}^{\prime} .\end{cases}
$$

(4) Let $A \neq 0, B \neq 0$.
(a) If $A^{\prime}=0, B^{\prime} \neq 0$, then $S_{c \alpha, c^{-1} \beta}=0$ and hence

$$
T_{b}^{[4(a)]}=0
$$

(b) If $A^{\prime} \neq 0, B^{\prime} \neq 0$, let

$$
\begin{aligned}
\mathcal{I}^{\prime}= & \left\{(\alpha, \beta) \in \mathcal{D}^{\prime} \mid A^{\prime} \neq g^{t\left(2^{e}+1\right)} \text { for any integer } t\right\}, \\
\mathcal{J}^{\prime}= & \left\{(\alpha, \beta) \in \mathcal{D}^{\prime} \mid A^{\prime}=g^{t\left(2^{e}+1\right)} \text { for some integer } t,\right. \\
& \left.L_{A^{\prime}}(x)=\left(B^{\prime}\right)^{2^{k}} \text { is not solvable }\right\}, \\
\mathcal{K}^{\prime}= & \left\{(\alpha, \beta) \in \mathcal{D}^{\prime} \mid A^{\prime}=g^{t\left(2^{e}+1\right)} \text { for some integer } t,\right. \\
& \operatorname{Tr}_{e}\left(A^{\prime}\right) \neq 0, L_{A^{\prime}}(x)=\left(B^{\prime} 2^{2^{k}} \text { is solvable }\right\}, \\
\mathcal{L}^{\prime}= & \left\{(\alpha, \beta) \in \mathcal{D}^{\prime} \mid A^{\prime}=g^{t\left(2^{e}+1\right)} \text { for some integer } t,\right. \\
& \left.\operatorname{Tr}_{e}\left(A^{\prime}\right)=0, L_{A^{\prime}}(x)=\left(B^{\prime}\right)^{2^{k}} \text { is solvable }\right\} .
\end{aligned}
$$

Then,

$$
T_{b}^{[4(b)]}= \begin{cases}2^{n} \cdot M & \text { if }(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K}) \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right), \\ 0 & \text { if }(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K} \cup \mathcal{L}) \cap \mathcal{J}^{\prime}, \\ -2^{n+e} \cdot M & \text { if }(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K}) \cap \mathcal{L}^{\prime}, \\ 0 & \text { if }(\alpha, \beta) \in \mathcal{J} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{J}^{\prime} \cup \mathcal{K}^{\prime} \cup \mathcal{L}^{\prime}\right), \\ -2^{n+e} \cdot M & \text { if }(\alpha, \beta) \in \mathcal{L} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right), \\ 2^{n+2 e} \cdot M & \text { if }(\alpha, \beta) \in \mathcal{L} \cap \mathcal{L}^{\prime},\end{cases}
$$

where $M=\chi_{1}\left(\left(1+c^{-1}\right) \beta\right) \chi_{1}\left(A A^{\prime} x_{A}^{2^{k}+1} x_{A^{\prime}}^{2^{k}+1}\right)$ and $x_{A}, x_{A^{\prime}}$ are the solutions of the equations $L_{A}(x)=B^{2^{k}}$ and $L_{A^{\prime}}(x)=\left(B^{\prime}\right)^{2^{k}}$, respectively.

We now summarize the above discussion in the following theorem.
Theorem 5.3. Let $F(x)=x^{2^{k}+1}, 1 \leq k<n$ be a function on $\mathbb{F}_{2^{n}}, n \geq 2$. Let $c \in \mathbb{F}_{2^{e}} \backslash\{0,1\}$ and $n / e$ be even, where $e=\operatorname{gcd}(k, n)$. With the previous notations, the $c-B C T$ entry ${ }_{c} \mathcal{B}_{F}(1, b)$ of $F$ at $(1, b)$ is given by

$$
\begin{aligned}
& \frac{1}{q^{2}}\left(\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{A}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(a)]}+\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{G}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(b)]}\right. \\
& \quad+\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{H}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(b)]}+\sum_{(\alpha, \beta) \in \mathcal{A}^{\prime} \cap \mathcal{G}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(a)]} \\
& \quad+\sum_{(\alpha, \beta) \in \mathcal{A}^{\prime} \cap \mathcal{H}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(a)]}+\sum_{(\alpha, \beta) \in \mathcal{G} \cap \mathcal{G}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(b)]} \\
& \quad+\sum_{(\alpha, \beta) \in \mathcal{G} \cap \mathcal{H}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(b)]}+\sum_{(\alpha, \beta) \in \mathcal{H} \cap \mathcal{G}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(b)]} \\
& \quad+\sum_{(\alpha, \beta) \in \mathcal{H} \cap \mathcal{H}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(b)]}++\sum_{(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K}) \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right)} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)]} \\
& \quad+\sum_{(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K}) \cap \mathcal{L}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)]}+\sum_{(\alpha, \beta) \in \mathcal{L} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right)} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)]} \\
& \quad+\sum_{(\alpha, \beta) \in \mathcal{L} \cap \mathcal{L}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(b)]) .}
\end{aligned}
$$

## 6. The general case

Since the case $c \in \mathbb{F}_{2^{e}}$ has already been considered in previous sections, throughout this section we assume that $c \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$. Recall that for any fixed $b \neq 0$, the $c$-BCT entry is given by,

$$
{ }_{c} \mathcal{B}_{F}(1, b)=\frac{1}{q^{2}} \sum_{\alpha, \beta \in \mathbb{F}_{q}} \chi_{1}(b(\alpha+\beta)) S_{\alpha, \beta} S_{c \alpha, c^{-1} \beta}
$$

Let us denote $T_{b}=S_{\alpha, \beta} S_{c \alpha, c^{-1} \beta}$. Recall that $A=\alpha+\beta, B=\beta^{2^{n-k}}+\beta, A^{\prime}=c \alpha+c^{-1} \beta$ and $B^{\prime}=\left(c^{-1} \beta\right)^{2^{n-k}}+c^{-1} \beta$. Notice that, when $c \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$ then $\beta \in \mathbb{F}_{2^{e}}^{*}$, and so, $\beta c^{-1} \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$, otherwise $c \in \mathbb{F}_{2^{e}}$. Thus $B=0=B^{\prime}$ if and only if $\beta=0$. Also, observe that the conditions $A=0=A^{\prime}$ if and only if $\alpha=0=\beta$. Now we shall consider two cases namely, $\frac{n}{e}$ is odd and $\frac{n}{e}$ is even, respectively.
Case 1: $\frac{n}{e}$ is odd.
(1) Let $A=0, B=0$.

Notice that the cases $A^{\prime}=0, B^{\prime} \neq 0$, and $A^{\prime} \neq 0, B^{\prime}=0$ would not arise, therefore, we shall calculate $T_{b}$ in remaining two cases only.
(a) If $A^{\prime}=0, B^{\prime}=0$, then

$$
T_{b}^{[1(a)]}=\chi_{1}\left(\left(1+c^{-1}\right) \beta\right) q^{2} .
$$

(b) If $A^{\prime} \neq 0, B^{\prime} \neq 0$, then

$$
T_{b}^{[1(b)]}= \begin{cases}0 & \text { if } \operatorname{Tr}_{e}\left(B^{\prime}\left(\gamma^{\prime}\right)^{-1}\right) \neq 1, \\ \left(\frac{2}{n / e}\right)^{e} 2^{\frac{3 n+e}{2}} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if } \operatorname{Tr}_{e}\left(B^{\prime}\left(\gamma^{\prime}\right)^{-1}\right)=1\end{cases}
$$

(2) Let $A=0, B \neq 0$. In this case $S_{\alpha, \beta}=0$ and hence

$$
T_{b}^{[2]}=0 .
$$

(3) Let $A \neq 0, B=0$. Again, $S_{\alpha, \beta}=0$ and hence

$$
T_{b}^{[3]}=0 .
$$

(4) Let $A \neq 0, B \neq 0$.
(a) If $A^{\prime}=0, B^{\prime}=0$, then

$$
T_{b}^{[4(a)]}= \begin{cases}0 & \text { if } \operatorname{Tr}_{e}\left(B \gamma^{-1}\right) \neq 1 \\ \left(\frac{2}{n / e}\right)^{e} 2^{\frac{3 n+e}{2}} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if } \operatorname{Tr}_{e}\left(B \gamma^{-1}\right)=1\end{cases}
$$

(b) If $A^{\prime}=0, B^{\prime} \neq 0$, then $S_{c \alpha, c^{-1} \beta}=0$ and hence

$$
T_{b}^{[4(b)]}=0 .
$$

(c) If $A^{\prime} \neq 0, B^{\prime}=0$, then again $S_{c \alpha, c^{-1} \beta}=0$ and hence

$$
T_{b}^{[4(c)]}=0 .
$$

(d) If $A^{\prime} \neq 0, B^{\prime} \neq 0$, then the only relevant case is and

$$
T_{b}^{[4(d)]}= \begin{cases}2^{n+e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{F} \cap \mathcal{F}^{\prime} \\ 0 & \text { otherwise }\end{cases}
$$

We now summarize the above discussion in the following theorem.
Theorem 6.1. Let $F(x)=x^{2^{k}+1}, 1 \leq k<n$ be a function on $\mathbb{F}_{2^{n}}, n \geq 2$. Let $c \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$ and $n / e$ be odd, where $e=\operatorname{gcd}(k, n)$. Then the $c-B C T$ entry ${ }_{c} \mathcal{B}_{F}(1, b)$ of $F$ at $(1, b)$ is given by

$$
\begin{aligned}
1+\frac{2^{\frac{e}{2}}}{2^{n}} & \left.\sum_{(\alpha, \beta) \in\left(\mathcal{A} \cap \mathcal{F}^{\prime}\right) \cup\left(\mathcal{A}^{\prime} \cap \mathcal{F}\right)} \chi_{1}\left(b \alpha+\left(1+c^{-1}+b\right) \beta\right)\right) \\
& \left.+\frac{2^{e}}{2^{n}} \sum_{(\alpha, \beta) \in \mathcal{F} \cap \mathcal{F}^{\prime}} \chi_{1}\left(b \alpha+\left(1+c^{-1}+b\right) \beta\right)\right) .
\end{aligned}
$$

Proof.

$$
\begin{aligned}
{ }_{c} \mathcal{B}_{F}(1, b)=\frac{1}{q^{2}} & \left(\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{A}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(a)]}+\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{F}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(b)]}\right. \\
& \left.+\sum_{(\alpha, \beta) \in \mathcal{F} \cap \mathcal{A}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(a)]}+\sum_{(\alpha, \beta) \in \mathcal{F} \cap \mathcal{F}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(d)]}\right) \\
=1+ & \left.\left(\frac{2}{n / e}\right)^{e} \cdot 2^{\frac{e-n}{2}} \sum_{(\alpha, \beta) \in\left(\mathcal{A} \cap \mathcal{F}^{\prime}\right) \cup\left(\mathcal{A}^{\prime} \cap \mathcal{F}\right)} \chi_{1}\left(b \alpha+\left(1+c^{-1}+b\right) \beta\right)\right) \\
& \left.+2^{e-n} \sum_{(\alpha, \beta) \in \mathcal{F} \cap \mathcal{F}^{\prime}} \chi_{1}\left(b \alpha+\left(1+c^{-1}+b\right) \beta\right)\right) .
\end{aligned}
$$

Corollary 6.2. Let $F(x)=x^{2^{k}+1}, 1 \leq k<n$, be a function on $\mathbb{F}_{q}, n \geq 2$. Let $c \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$ and $n / e$ be odd, where $e=\operatorname{gcd}(k, n)$. With the notations of the previous theorem, the $c$ boomerang uniformity of $F$ satisfies

$$
\beta_{F, c} \leq 1+\left(\frac{2}{n / e}\right)^{e} \cdot 2^{\frac{e-n}{2}}\left|\left(\mathcal{A} \cap \mathcal{F}^{\prime}\right) \cup\left(\mathcal{A}^{\prime} \cap \mathcal{F}\right)\right|+2^{e-n}\left|\mathcal{F} \cap \mathcal{F}^{\prime}\right| .
$$

Case 2: $n / e$ is even.
(1) Let $A=0, B=0$. Notice that the cases $A^{\prime}=0, B^{\prime} \neq 0$, and $A^{\prime} \neq 0, B^{\prime}=0$ would not arise, therefore, we shall calculate $T_{b}$ in remaining two cases only.
(a) If $A^{\prime}=0, B^{\prime}=0$, then

$$
T_{b}^{[1(a)]}=\chi_{1}\left(\left(1+c^{-1}\right) \beta\right) q^{2} .
$$

(b) If $A^{\prime} \neq 0, B^{\prime} \neq 0$, then

$$
T_{b}^{[1(b)]}= \begin{cases}(-1)^{\frac{m}{e}} 2^{m+n} M^{\prime} & \text { if }(\alpha, \beta) \in \mathcal{A} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right), \\ 0 & \text { if }(\alpha, \beta) \in \mathcal{A} \cap \mathcal{J}^{\prime}, \\ (-1)^{\frac{m}{e}+1} 2^{m+n+e} M^{\prime} & \text { if }(\alpha, \beta) \in \mathcal{A} \cap \mathcal{L}^{\prime},\end{cases}
$$

where $M^{\prime}=\chi_{1}\left(\left(1+c^{-1}\right) \beta\right) \chi_{1}\left(A^{\prime} x_{A^{\prime}}^{2^{k}+1}\right)$.
(2) Let $A=0, B \neq 0$. In this case $S_{\alpha, \beta}=0$ and hence

$$
T_{b}^{[2]}=0
$$

(3) Let $A \neq 0, B=0$. Notice that the case $A^{\prime}=0, B^{\prime}=0$ would not arise. Now we shall calculate $T_{b}$ in the remaining cases.
(a) If $A^{\prime}=0, B^{\prime} \neq 0$, then $S_{c \alpha, c^{-1} \beta}=0$ and hence

$$
T_{b}^{[3(a)]}=0 .
$$

(b) If $A^{\prime} \neq 0, B^{\prime}=0$, then

$$
T_{b}^{[3(b)]}= \begin{cases}2^{n} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{G} \cap \mathcal{G}^{\prime}, \\ -2^{n+e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{G} \cap \mathcal{H}^{\prime}, \\ -2^{n+e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{H} \cap \mathcal{G}^{\prime}, \\ 2^{n+2 e} \chi_{1}\left(\left(1+c^{-1}\right) \beta\right) & \text { if }(\alpha, \beta) \in \mathcal{H} \cap \mathcal{H}^{\prime} .\end{cases}
$$

(c) If $A^{\prime} \neq 0, B^{\prime} \neq 0$, then

$$
T_{b}^{[3(c)]}= \begin{cases}2^{n} M^{\prime} & \text { if }(\alpha, \beta) \in \mathcal{G} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right), \\ 0 & \text { if }(\alpha, \beta) \in(\mathcal{G} \cup \mathcal{H}) \cap \mathcal{J}^{\prime}, \\ -2^{n+e} M^{\prime} & \text { if }(\alpha, \beta) \in \mathcal{G} \cap \mathcal{L}^{\prime}, \\ -2^{n+e} M^{\prime} & \text { if }(\alpha, \beta) \in \mathcal{H} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right), \\ 2^{n+2 e} M^{\prime} & \text { if } \left.(\alpha, \beta) \in \mathcal{H} \cap \mathcal{L}^{\prime}\right) .\end{cases}
$$

(4) Let $A \neq 0, B \neq 0$.
(a) If $A^{\prime}=0, B^{\prime}=0$, then

$$
T_{b}^{[4(a)]}= \begin{cases}(-1)^{\frac{m}{e}} 2^{m+n} M^{\prime \prime} & \text { if }(\alpha, \beta) \in \mathcal{A}^{\prime} \cap(\mathcal{I} \cup \mathcal{K}), \\ 0 & \text { if }(\alpha, \beta) \in \mathcal{A}^{\prime} \cap \mathcal{J}, \\ (-1)^{\frac{m}{e}+1} 2^{m+n+e} M^{\prime \prime} & \text { if }(\alpha, \beta) \in \mathcal{A}^{\prime} \cap \mathcal{L},\end{cases}
$$

where $M^{\prime \prime}=\chi_{1}\left(\left(1+c^{-1}\right) \beta\right) \chi_{1}\left(A x_{A}^{2^{k}+1}\right)$.
(b) If $A^{\prime}=0, B^{\prime} \neq 0$, then $S_{c \alpha, c^{-1} \beta}=0$ and hence

$$
T_{b}^{[4(b)]}=0
$$

(c) If $A^{\prime} \neq 0, B^{\prime}=0$, then

$$
T_{b}^{[4(c)]}= \begin{cases}2^{n} M^{\prime \prime} & \text { if }(\alpha, \beta) \in \mathcal{G}^{\prime} \cap(\mathcal{I} \cup \mathcal{K}), \\ 0 & \text { if }(\alpha, \beta) \in\left(\mathcal{G}^{\prime} \cup \mathcal{H}\right) \cap \mathcal{J}, \\ -2^{n+e} M^{\prime \prime} & \text { if }(\alpha, \beta) \in \mathcal{G}^{\prime} \cap \mathcal{L}, \\ -2^{n+e} M^{\prime \prime} & \text { if }(\alpha, \beta) \in \mathcal{H}^{\prime} \cap(\mathcal{I} \cup \mathcal{K}), \\ 2^{n+2 e} M^{\prime \prime} & \text { if }(\alpha, \beta) \in \mathcal{H}^{\prime} \cap \mathcal{L} .\end{cases}
$$

(d) If $A^{\prime} \neq 0, B^{\prime} \neq 0$, then

$$
T_{b}^{[4(d)]}= \begin{cases}2^{n} M^{\prime \prime \prime} & \text { if }(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K}) \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right), \\ 0 & \text { if }(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K} \cup \mathcal{L}) \cap \mathcal{J}^{\prime}, \\ -2^{n+e} M^{\prime \prime \prime} & \text { if }(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K}) \cap \mathcal{L}^{\prime}, \\ 0 & \text { if }(\alpha, \beta) \in \mathcal{J} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{J}^{\prime} \cup \mathcal{K}^{\prime} \cup \mathcal{L}^{\prime}\right), \\ -2^{n+e} M^{\prime \prime \prime} & \text { if }(\alpha, \beta) \in \mathcal{L} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right), \\ 2^{n+2 e} M^{\prime \prime \prime} & \text { if }(\alpha, \beta) \in \mathcal{L} \cap \mathcal{L}^{\prime},\end{cases}
$$

where $M^{\prime \prime \prime}=\chi_{1}\left(\left(1+c^{-1}\right) \beta\right) \chi_{1}\left(A x_{A}^{2^{k}+1}+A^{\prime} x_{A^{\prime}}^{2^{k}+1}\right)$.
We now summarize the above discussion in the form of following theorem.
Theorem 6.3. Let $F(x)=x^{2^{k}+1}, 1 \leq k<n$ be a function on $\mathbb{F}_{2^{n}}, n \geq 2$. Let $c \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2^{e}}$ and $n / e$ be even, where $e=\operatorname{gcd}(k, n)$. With the prior notations, the $c-B C T$ entry ${ }_{c} \mathcal{B}_{F}(1, b)$ of $F$ at $(1, b)$ is given by

$$
\begin{aligned}
& \frac{1}{q^{2}}\left(\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{A}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(a)]}+\sum_{(\alpha, \beta) \in \mathcal{A} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right)} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(b)]}\right. \\
&+\sum_{(\alpha, \beta) \in \mathcal{A} \cap \mathcal{L}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[1(b)]}+\sum_{(\alpha, \beta) \in \mathcal{G} \cap \mathcal{G}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(b)]} \\
&+\sum_{(\alpha, \beta) \in \mathcal{G} \cap \mathcal{H}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(b)]}+\sum_{(\alpha, \beta) \in \mathcal{H} \cap \mathcal{G}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(b)]} \\
&+\sum_{(\alpha, \beta) \in \mathcal{H} \cap \mathcal{H}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(b)]}+\sum_{(\alpha, \beta) \in \mathcal{G} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right)} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(c)]} \\
&+\sum_{(\alpha, \beta) \in \mathcal{G} \cap \mathcal{L}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(c)]}+\sum_{(\alpha, \beta) \in \mathcal{H} \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right)} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(c)]} \\
& \quad+\sum_{(\alpha, \beta) \in \mathcal{H} \cap \mathcal{L}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[3(c)]}+\sum_{(\alpha, \beta) \in \mathcal{A}^{\prime} \cap(\mathcal{I} \cup \mathcal{K})} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(a)]} \\
& \quad+\sum_{(\alpha, \beta) \in \mathcal{A}^{\prime} \cap \mathcal{L}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(a)]}+\sum_{(\alpha, \beta) \in \mathcal{G}^{\prime} \cap(\mathcal{I} \cup \mathcal{K})} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(c)]} \\
& \quad+\sum_{(\alpha, \beta) \in \mathcal{G}^{\prime} \cap \mathcal{L}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(c)}+\sum_{(\alpha, \beta) \in \mathcal{H}^{\prime} \cap(\mathcal{I} \cup \mathcal{K})} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(c)]} \\
& \quad+\sum_{(\alpha, \beta) \in \mathcal{H} \cap \cap \mathcal{L}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(c)]}+\sum_{(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K}) \cap\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right)} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(d)]}
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{(\alpha, \beta) \in(\mathcal{I} \cup \mathcal{K}) \cap \mathcal{L}^{\prime}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(d)]}+\sum_{(\alpha, \beta) \in\left(\mathcal{I}^{\prime} \cup \mathcal{K}^{\prime}\right) \cap \mathcal{L}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(d)]} \\
& \left.+\sum_{(\alpha, \beta) \in \mathcal{L}^{\prime} \cap \mathcal{L}} \chi_{1}(b(\alpha+\beta)) T_{b}^{[4(d)]}\right) .
\end{aligned}
$$

## 7. DISCUSSION ON EQUIVALENCE

Boura and Canteaut [2] showed that the BCT table is preserved under the affine equivalence but not under the extended affine equivalence (and consequently under the CCZ-equivalence). It is quite natural to ask a similar question in the context of $c$-BCT. It is straightforward to see that in the case of even characteristic, $c$-BCT and $c^{-1}$-BCT entries of an $(n, n)$-function $F: \mathbb{F}_{2^{n}} \rightarrow \mathbb{F}_{2^{n}}$ are the same under the transformations $x \mapsto x+a$ and $y \mapsto y+a$, since the $c$-boomerang system

$$
\left\{\begin{array}{l}
F(x)+c F(y)=b \\
F(x+a)+c^{-1} F(y+a)=b
\end{array}\right.
$$

becomes

$$
\left\{\begin{array}{l}
F(x)+c^{-1} F(y)=b \\
F(x+a)+c F(y+a)=b
\end{array}\right.
$$

We consider the binomial $G(x)=x^{2^{k}+1}+u x^{2^{n-k}+1} \in \mathbb{F}_{2^{n}}[x]$, which is a PP if and only if $\frac{n}{e}$ is odd and $u \neq g^{t\left(2^{e}-1\right)}$, where $e=\operatorname{gcd}(n, k)=\operatorname{gcd}(n-k, k)$ and $g$ is the primitive element of $\mathbb{F}_{2^{n}}$. Notice that $G(x)=(L \circ F)(x)$ where $L(x)=x^{2^{k}}+u x$ and $F(x)=x^{2^{n-k}+1}$. When $n=6, k=2$ and $u=g$, where $g$ is a root of the primitive polynomial $y^{6}+y^{4}+y^{3}+y+1$ over $\mathbb{F}_{2}$, then $L(x)$ and $G(x)$ are PP. It is easy to see from the Table $\square$ in the Appendix A that the $c$-BCT is not preserved under the (output applied) affine equivalence. However, if the affine transformation is applied to the input, that is, $G(x)=(F \circ L)(x)$, then the $c$-BCT spectrum is preserved, as was the case for the $c$-differential uniformity.

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## References

[1] D. Bartoli, M. Calderini, On construction and (non)existence of c-(almost) perfect nonlinear functions, Finite Fields Appl., 72 (2021), 101835.
[2] C. Boura, A. Canteaut, On the boomerang uniformity of cryptographic Sboxes, IACR Trans. Symmetric Cryptol., vol. 2018, no. 3, 290-310, 2018.
[3] C. Carlet, Boolean functions for cryptography and error correcting codes, In: Y. Crama, P. Hammer (eds.), Boolean Methods and Models, Cambridge Univ. Press, Cambridge, pp. 257-397, 2010.
[4] C. Cid, T. Huang, T. Peyrin, Y. Sasaki, and L. Song, Boomerang connectivity table: a new cryptanalysis tool. In: Nielsen J., Rijmen V. (eds.), Advances in Cryptology-EUROCRYPT 2018, LNCS 10821, Springer, Cham, pp. 683-714, 2018.
[5] R. S. Coulter, On the evaluation of a class of Weil sums in characteristic 2, New Zealand J. of Math., vol. 28, pp. 171-184, 1999.
[6] T. W. Cusick, P. Stănică, Cryptographic Boolean Functions and Applications (Ed. 2), Academic Press, San Diego, CA, 2017.
[7] P. Ellingsen, P. Felke, C. Riera, P. Stănică, A. Tkachenko, C-differentials, multiplicative uniformity and (almost) perfect c-nonlinearity, IEEE Trans. Inform. Theory 66(9) (2020), 5781-5789.
[8] S. U. Hasan, M. Pal, C. Riera, P. Stănică, On the c-differential uniformity of certain maps over finite fields, Des. Codes Cryptogr. 89(2) (2021), 221-239.
[9] K. Li, L. Qu, B. Sun, C. Li, New results about the boomerang uniformity of permutation polynomials, IEEE Trans. Inform. Theory 65(11) (2019), 7542-7553.
[10] S. Mesnager, C. Riera, P. Stănică, H. Yan and Z. Zhou, Investigations on c-(almost) perfect nonlinear functions, IEEE Trans. Inform. Theory (2021), https://doi.org/10.1109/TIT.2021.3081348
[11] K. Nyberg, Differentially uniform mappings for cryptography. In: Helleseth T. (eds.), Advances in Cryptology-EUROCRYPT 1993, LNCS 765, Springer, Berlin, Heidelberg, pp. 55-64, 1994.
[12] P. Stănică, Investigations on c-boomerang uniformity and perfect nonlinearity, Discrete Appl. Math., Vol. 304, 297-314, 2021.
[13] P. Stănică, Low c-differential and c-boomerang uniformity of the swapped inverse function, Discrete Math. 344(10) (2021), 112543.
[14] P. Stănică, Using double Weil sums in finding the c-boomerang connectivity table for monomial functions on finite fields, Appl. Algebra Eng. Commun. Comput., (2021). https://doi.org/10.1007/s00200-021-00520-9
[15] P. Stănică, A. Geary, The c-differential behaviour of the inverse function under the EA-equivalence, Cryptogr. Commun. 13 (2021), 295-306.
[16] P. Stănică, C. Riera, A. Tkachenko, Characters, Weil sums and c-differential uniformity with an application to the perturbed Gold function, Cryptogr. Commun. (2021), https://doi.org/10.1007/s12095-021-00485-z
[17] D. Wagner, The boomerang attack, In: L. R. Knudsen (ed.) Fast Software Encryption-FSE 1999. LNCS 1636, Springer, Berlin, Heidelberg, pp. 156-170, 1999.
[18] Z. Zha, L. Hu, Some classes of power functions with low c-differential uniformity over finite fields, Des. Codes Cryptogr. (2021), https://doi.org/10.1007/s10623-021-00866-8.

## Appendix A.

Let $G(x)=x^{5}+g x^{17}=\left(x^{4}+g x\right) \circ x^{17} \in \mathbb{F}_{2^{6}}[x]$, where $g$ is a root of the primitive polynomial $y^{6}+y^{4}+y^{3}+y+1$ over $\mathbb{F}_{2}$. The following Table 1 gives the set of the $c$-BCT entries for $x^{17}$ as well as $G(x)=x^{5}+g x^{17}$ for all $c \in \mathbb{F}_{2^{n}} \backslash \mathbb{F}_{2}$. In view of the discussion in Section 7, it is sufficient to compute the set of $c$-BCT entries for either of $c$ or $c^{-1}$ as they are going to be exactly the same.

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| $c$ | Set of $c$-BCT entries of $x^{17}$ | Set of $c$-BCT entries of $G(x)$ |
| :---: | :---: | :---: |
| $g$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{2}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{3}$ | $\{0,1,2,3,5\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{4}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{5}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{6}$ | $\{0,1,2,3,5\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{7}$ | $\{0,1,2,3,4,5\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{8}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{9}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{10}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{11}$ | $\{0,1,2,3\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{12}$ | $\{0,1,2,3,5\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{13}$ | $\{0,1,2,3\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{14}$ | $\{0,1,2,3,4,5\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{15}$ | $\{0,1,2,3,5\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{16}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{17}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{18}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{19}$ | $\{0,1,2,3\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{20}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{21}$ | $\{0,1,4\}$ | $\{0,1,4\}$ |
| $g^{22}$ | $\{0,1,2,3\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{23}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{24}$ | $\{0,1,2,3,5\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{23}$ | $\{0,1,2,3\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{26}$ | $\{0,1,2,3\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{27}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{28}$ | $\{0,1,2,3,4,5\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{29}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5\}$ |
| $g^{30}$ | $\{0,1,2,3,5\}$ | $\{0,1,2,3,4,5,6\}$ |
| $g^{31}$ | $\{0,1,2,3,4\}$ | $\{0,1,2,3,4,5\}$ |

Table 1. $c$-BCT entries of $x^{17}$ and $x^{5}+g x^{17}$


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    Key words and phrases. Finite fields, double Weil sums, boomerang uniformity, $c$-boomerang uniformity.

