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The Binding Number of a Zero Divisor Graph

S. Sankeetha and J. Ravi Sankar

Department of Mathematics, Saradha Gangadharan College
Puducherry, India - 605 004
sangeetha.sivarajp@gmail.com, ravisankar.maths@gmail.com

R. Vasanthakumari

Principal, Kasthurba College for Women, Villianur
Puducherry, India - 605 110
vasunthara1@gmail.com

S. Meena

Department of Mathematics, Government Arts College
Chidambaram, India - 608 104
meenasaravanan14@gmail.com

Abstract

In this paper, we evaluate $b(\Gamma(Z_n))$. Our main result is, we give maximum value of b($\Gamma(Z_n)$) is 0.99999999979642762648923624307266l, where n is any positive integer upto fiftieth million.

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1 Introduction

Let R be a commutative ring and let Z(R) be its set of zero-divisors. We associate a graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, the set of non-zero zero divisors of R and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. Throughout this paper, consider the commutative ring R as Z_n and zero divisor graph $\Gamma(R)$ as $\Gamma(Z_n)$. The binding number of $\Gamma(Z_n)$, denoted by $b(\Gamma(Z_n))$ is defined by, $\Gamma(Z_n) = \{\frac{|N(S)|}{|S|}$, where $S \subseteq V(\Gamma(Z_n))$, $S \neq \phi, N(S) \neq V(\Gamma(Z_n))\}$ which satisfies the following conditions; (i) $N(S) \cup S = V(\Gamma(Z_n))$ (ii) $N(S) \cap S = \phi$ (iii) $d(u) \leq d(v)$ for $u \in S$ and $v \in N(S)$ (iv) no two vertices in S are adjacent. For notation and graph theory terminology, we generally follow $[1, 2, 3, 5, 6]$.

2 Binding Number of a Zero Divisor Graph

Lemma 2.1 [4] *A graph $\Gamma(Z_n)$ has a domination set iff $\Gamma(Z_n)$ is connected and n is a composite number.*

Theorem 2.2 *For any prime $p>2$, then $b(\Gamma(Z_{2p})) = \frac{1}{p-1}$.*

Proof: *The vertex set of $\Gamma(Z_{2p})$ is $\{2, 4, 6, ...2(p-1), p\}$. Using theorem (4.4) in [4], $\Gamma(Z_{2p})$ is a star graph $K_{1,p-1}$. Let S be a non-empty subset of the vertex set $V(\Gamma(Z_{2p}))$, then for any $x \in S$, such that $d(x) < d(y)$, where $y \in V - S$. Clearly, all the vertices are of minimum degree except p, then $S = \{2, 4, 6, ...2(p-1)\}$, that is $|S| = p-1$ and the neighbourhood of the set $S = N(S)$ and $|N(S)| = p - (p-1) = 1$. Hence, $b(\Gamma(Z_{2p})) = \frac{|N(S)|}{|S|} = \frac{1}{p-1}$.*

Theorem 2.3 *For any prime p, $b(\Gamma(Z_{p^2})) = \frac{1}{p-2}$.*

Proof: *The vertex set of $\Gamma(Z_{2p})$ is $\{p, 2p, 3p, ...p(p-1)\}$. Any two vertices in $b(\Gamma(Z_{p^2}))$ are adjacent. Clearly, $b(\Gamma(Z_{p^2}))$ is a complete graph namely K_{p-1}. Let S be a non-empty maximun subset of $b(\Gamma(Z_{p^2}))$ then $\{p, 2p, 3p, ...p(p-2)\} \in S$ implies $|S| = p-2$ and the neighbourhood of the set S contains only one point $\{p(p-1)\}$ that is $|N(S)| = 1$. Clearly, $b(\Gamma(Z_{p^2})) = \frac{|N(S)|}{|S|} = \frac{1}{p-2}$.*

Theorem 2.4 *If p and q are distinct prime numbers with $p < q$, then $b(\Gamma(Z_{pq})) = \frac{p-1}{q-1}$.*

Proof: *The proof is by the method of induction on p and q. The vertex set of $\Gamma(Z_{pq})$ is $\{p, 2p, 3p, ..., p(q-1), q, 2q, 3q, ..., (p-1)q\}$. Let S and $N(S)$ be the minimum degree set and the neighbourhood of S respectively.*

Case(i): *Let $p = 2$, q is any prime > 2.*

Using theorem (2.1), $b(\Gamma(Z_{2q})) = \frac{1}{q-1} = \frac{p-1}{q-1}$.

Case(ii): *Let $p = 3$, q is any prime > 3.*

The vertex set of $\Gamma(Z_{3q})$ is $\{3, 6, 9, ..., 3(q-1), q, 2q\}$. Let $u = q$ and $v = 2q$ be two vertices in $\Gamma(Z_{3q})$ with maximum degree then there exist any other vertex $w \neq q$ and $w \neq 2q$ in $\Gamma(Z_{3q})$ such that w is adjacent to both u and v. That is, $uw = vw = 0$. But $uv \neq 0$. Therefore u and v are non-adjacent vertices. Then *the vertex set* $V(\Gamma(Z_{3q}))$ *can be partitioned into two parts* S *and* $N(S)$ *such that* $S = \{3, 6, 9, ..., 3(q-1)\}$ *and* $N(S) = \{u, v\} = \{q, 2q\}$. *Clearly* $|S| = q-1$ *and* $|N(S)| = 2$, *then* $|V(\Gamma(Z_{3q}))| = |S| + |N(S)| = q-1+2 = q+1$. *Note that the vertices in the set* S *have the smallest degree compared to the set* $N(S)$. *Clearly, any two vertices in* S *are non-adjacent. Moreover* $V(\Gamma(Z_{3q})) = S \cup N(S)$ *and* $S \cap N(S) = \phi$ *and* $d(u) \leq d(v)$ *for all* $u \in S$ *and* $v \in N(S)$.

Then, $b(\Gamma(Z_{3p})) = \frac{|N(S)|}{|S|} = \frac{2}{q-1} = \frac{p-1}{q-1}$, *where* $p = 3$ *and* $q > 3$.

Case(iii): *Let* $p < q$.

The vertex set of $\Gamma(Z_{pq})$ *is* $\{p, 2p, 3p, ..., p(q-1), q, 2q, 3q, ..., (p-1)q\}$. *Using the above cases, the vertex set* $V(\Gamma(Z_{pq}))$ *can be partitioned into two parts* S *and* $N(S)$ *which implies that the vertex* p, *multiples of* p *are in* S *and* q, *multiples of* q *are in* $N(S)$. *Clearly, every vertices in* S *are non-adjacent which holds for* $N(S)$. *Then,* $|V(\Gamma(Z_{pq}))| = |S| + |N(S)| = p-1+q-1 = p+q-2$. *That is* $S = \{p, 2p, .., p(q-1)\}$ *and* $N(S) = \{q, 2q, .., (p-1)q\}$. *Clearly,* $d(u) < d(v)$ *where* $u \in S$ *and* $v \in N(S)$. *We note that, every vertex in* S *are adjacent to all the vertices in* $N(S)$. *Using all the above cases,* $b(\Gamma(Z_{pq})) = \frac{|N(S)|}{|S|} = \frac{p-1}{q-1}$.

Theorem 2.5 *For any graph* $\Gamma(Z_{2^n})$, *where* $n > 2$ *is a positive integer then,*

a) If n *is even,* $b(\Gamma(Z_{2^n})) = \frac{2^{n-1} - 2^{\frac{n}{2}} \sum_{i=0}^{\frac{n-4}{2}} 2^i - 2}{2^{\frac{n}{2}} \sum_{i=0}^{\frac{n-4}{2}} 2^i + 1}$.

b) If n *is odd,* $b(\Gamma(Z_{2^n})) = \frac{2^{\frac{n-1}{2}} (2^{\frac{n-1}{2}} - \sum_{i=0}^{\frac{n-3}{2}} 2^i) - 1}{2^{\frac{n-1}{2}} \sum_{i=0}^{\frac{n-3}{2}} 2^i}$.

Proof: *The vertex set of* $\Gamma(Z_{2^n})$ *is* $\{2, 4, ., 2(2^{n-1} - 1)\}$ *and* $|V(\Gamma(Z_{2^n}))| = 2^{n-1} - 1$. *The proof is by the method of induction on* n.

Case(a): *When* n *is even.*

Subcase(i): *Let* $n = 4$. *The vertex set of* $\Gamma(Z_{2^4})$ *is* $\{2, 4, 6, 8, 10, 12, 14\}$. *Let* S *be a vertex subset of* V *such that* $d(u) \leq d(v)$, *where* $u \in S$ *and* $v \in N(S)$. *Let* P *be a set of all pendant vertices in* $\Gamma(Z_{2^4})$. *Clearly,* $P = \{2, 6, 10, 14\}$ *with* $d(u) = 1$, *for all* $u \in P$. *It seems that* $P \subseteq S$. *Let* $v = 2^{n-1} = 2^{4-1} = 8$ *and* $w = 2^4 - 2$ *be any other vertex in* $\Gamma(Z_{2^4})$ *then* $vw = 8 \times (2^4 - 2) = 112$. *Clearly,* 2^4 *must divides* 112. *Thus, the vertex* v *is adjacent to all vertices in* $\Gamma(Z_{2^4})$ *which implies* $v = 8 \in N(S)$. *Let* $x = 4$ *and* $y = 12$ *be the remaining vertices in* V *such that* $xv = yv = 0$. *That is,* x, y *and* v *are adjacent vertices. Clearly, either* $x = 4 \in S$ *or* $y = 12 \in S$. *Suppose,* $x, y \in S$, *we get a contradiction to our definition that no two vertices in* S *are adjacent. Finally we conclude that* $S = \{2, 4, 6, 10, 14\}$ *or* $S = \{2, 6, 10, 12, 14\}$ *and* $N(S) = \{8, 12\}$ *or* $N(S) = \{4, 8\}$, *respectively. That is* $|S| = 5$ *and* $|N(S)| = 2$. *Clearly,* $V(\Gamma(Z_{2^4})) = S \cup N(S)$ *and* $S \cap N(S) = \phi$. *Since, degree of any vertex in* S *is less than or equal to degree of any vertex in* $N(S)$ *and* $|N(S)| = V(\Gamma(Z_{2^4})) - |S| = 7 - 5 = 2$.

Then, $b(\Gamma(Z_{2^4})) = \frac{|N(S)|}{|S|} = \frac{2}{5} = \frac{2^{4-1} - 1 - 2^{\frac{4}{2}} \sum_{i=0}^{\frac{4-4}{2}} 2^i - 1}{= 2^2 + 2^0} = \frac{2^{4-1} - 2^{\frac{4}{2}} \sum_{i=0}^{\frac{4-4}{2}} 2^i - 2}{= 2^{4/2}(2^0) + 2^0}$

$$= \frac{2^{n-1}-2^{\frac{n}{2}}\sum_{i=0}^{\frac{n-4}{2}} 2^i-2}{2^{\frac{n}{2}}\sum_{i=0}^{\frac{n-4}{2}} 2^i+1}, \text{where } n = 4.$$

Subcase(ii): *Let $n = 6$.*

The vertex set of $\Gamma(Z_{2^6})$ is $\{2, 4, 6, ..., 62\}$. That is $|V(\Gamma(Z_{2^6}))| = 31$. Let S be a vertex subset of V such that $d(u) \leq d(v)$, where $u \in S$ and $v \in N(S)$. Let P be the set of all pendant vertices in $\Gamma(Z_{2^6})$. Clearly, $P = \{2, 6, ..., (2^6 - 2)\}$ with $d(u) = 1$, for all $u \in P$. It seems that $P \subseteq S$. Using subcase (i), let $v = 2^{n-1} = 2^{6-1} = 32$ and $w = 2^6 - 2$ be any other vertex in $\Gamma(Z_{2^6})$ such that 2^6 must divides $vw = 32 \times (2^6 - 2) = 1984$. Thus, the vertex v is adjacent to all the vertices in $\Gamma(Z_{2^6})$ which implies $v = 32 \in N(S)$. Similarly, 2^4 and 3×2^4 are adjacent to all the vertices in $\Gamma(Z_{2^6})$ except P, then $\{16, 48\} \in N(S)$.

Let U be a vertex subset of V with $U = \{4, 12, 20, ..., (2^6 - 4)\}$. Clearly, no two vertices in U is adjacent and every vertex in U are adjacent to $\{16, 32, 48\}$. It seems that $d(U) < d(N(S))$ which implies that $U \subseteq S$.

Let $W = V - (P \cup U \cup N(S)) = \{8, 24, 40, 56\}$ be a vertex subset of V. Finally, the vertices in W make a complete subgraph, namely K_4 and all the vertices in W are adjacents to $N(S)$. Using theorem (2.4), any one of the vertex in W is in S. Otherwise, if any two vertices in W belongs to S, then we get a contradiction that no two vertices are adjacent in S. Hence, $|S| = |P| + |U| +$ any one vertex in $W = 25$ and $|N(S)| = V(\Gamma(Z_{2^6})) - |S| = 31 - 25 = 6$. Then,

$$b(\Gamma(Z_{2^6})) = \frac{|N(S)|}{|S|} = \frac{6}{25} = \frac{2^{6-1}-1-2^{\frac{6}{2}}\sum_{i=0}^{\frac{6-4}{2}} 2^i-1}{2^4+2^3+2^0} = \frac{2^{6-1}-2^{\frac{6}{2}}\sum_{i=0}^{\frac{6-4}{2}} 2^i-2}{2^{\frac{6}{2}}2^3(2^1+2^0)+1}$$

$$= \frac{2^{n-1}-2^{\frac{n}{2}}\sum_{i=0}^{\frac{n-4}{2}} 2^i-2}{2^{\frac{n}{2}}\sum_{i=0}^{\frac{n-4}{2}} 2^i+1}, \text{ where } n=6.$$

Subcase(iii): *Let $n > 6$ is even.*

The vertex set of $\Gamma(Z_{2^n})$ is $\{2, 4, ..., 2(2^{n-1} - 1)\}$ and $|V(\Gamma(Z_{2^n}))| = 2^{n-1} - 1$. Since P is a pendant vertex set with $|P| = 2^{n-2}$. Using above cases, $|S| = 2^{\frac{n}{2}}(2^0 + ... + 2^{\frac{n}{2}-1}) + 2^0 = 2^{\frac{n}{2}}\sum_{i=0}^{\frac{n-4}{2}} 2^i + 1$, and $|N(S)| = V(\Gamma(Z_{2^n})) - |S| = 2^{n-1} - 1 - 2^{\frac{n}{2}}\sum_{i=0}^{\frac{n-4}{2}} 2^i - 1 = 2^{n-1} - 2^{\frac{n}{2}}\sum_{i=0}^{\frac{n-4}{2}} 2^i - 2$. Then,

$$b(\Gamma(Z_{2^n})) = \frac{|N(S)|}{|S|} = \frac{2^{n-1}-2^{\frac{n}{2}}\sum_{i=0}^{\frac{n-4}{2}} 2^i-2}{2^{\frac{n}{2}}\sum_{i=0}^{\frac{n-4}{2}} 2^i+1}, \text{ where } n \text{ is even.}$$

Case(b): *When n is odd.*

Subcase(i): *Let $n = 3$. The vertex set of $\Gamma(Z_{2^3})$ is $\{2, 4, 6\}$. Let S be a vertex subset of V and let P be the set of all pendant vertices in $\Gamma(Z_{2^3})$. Clearly, $P = \{2, 6\}$ with $d(u) = 1$, for all $u \in P$. It seems that $P \subseteq S$. Let $v = 6$ and $w = 2$ be any other vertex in $\Gamma(Z_{2^3})$ then $vw = 0$. Thus, the vertex v is adjacent to all the vertices in $\Gamma(Z_{2^3})$ which implies $v = 4 \in N(S)$. Let $x = 2$ and $y = 6$ be the remaining vertices in V such that $xv = yv = 0$ and $xy \neq 0$. Finally we conclude that $S = \{2, 6\}$ and $N(S) = \{4\}$. Hence,*

$$b(\Gamma(Z_{2^3})) = \frac{|N(S)|}{|S|} = \frac{1}{2} = \frac{2^{3-1}-1-2^{\frac{3-1}{2}}\sum_{i=0}^{\frac{3-3}{2}} 2^i}{2^{(3-1)/2}(2^0)} = \frac{2^{\frac{n-1}{2}}(2^{\frac{n-1}{2}}-\sum_{i=0}^{\frac{n-3}{2}} 2^i)-1}{2^{\frac{n-1}{2}}\sum_{i=0}^{\frac{n-3}{2}} 2^i}.$$

Subcase(ii): *Let $n = 5$.*

The vertex set of $\Gamma(Z_{2^5})$ is $\{2, 4, ..., 30\}$. Let P be the set of all pendant vertices in $\Gamma(Z_{2^5})$. Clearly, $P = \{2, 6, ..., 30\}$ with $d(u) = 1$. It seems that $P \subseteq S$. Let $v = 16$ and $w = 2$ be any other vertex in $\Gamma(Z_{2^5})$ then $vw = 32 = 0$. Clearly, 2^5 must divides 32. Thus, the vertex v is adjacent to all vertices in $\Gamma(Z_{2^5})$ which implies $v = 16 \in N(S)$. Let U be a vertex subset of V with $U = \{4, 8, 12, 20, 24, 28\}$. Since, U has a induced subgraph $K_{2,4}$. Clearly, $d(4) = d(12) = d(20) = d(28) < d(8) = d(24)$ implies that the vertices $8, 24 \in N(S)$ and the remaining vertices belongs to S. Therefore the set $S = \{2, 4, 6, 10, 12, 14, 18, 20, 22, 26, 28, 30\}$ with $|S| = 12$ and $|N(S)| = 3$. Then,

$$b(\Gamma(Z_{2^5})) = \frac{|N(S)|}{|S|} = \frac{3}{12} = \frac{2^{5-1}-1-2^{\frac{5-1}{2}}\sum_{i=0}^{\frac{5-3}{2}} 2^i}{2^2(2^1+2^0)} = \frac{2^{\frac{5-1}{2}}(2^{\frac{5-1}{2}}-\sum_{i=0}^{\frac{5-3}{2}} 2^i)-1}{2^{(5-1)/2}(2^1+2^0)}$$

$$= \frac{2^{\frac{n-1}{2}}(2^{\frac{n-1}{2}}-\sum_{i=0}^{\frac{n-3}{2}} 2^i)-1}{2^{\frac{n-1}{2}}\sum_{i=0}^{\frac{n-3}{2}} 2^i}, \text{ where } n = 5.$$

Subcase(iii): *Let $n > 5$ is any odd number*

The vertex set of $\Gamma(Z_{2^n})$ is $\{2, 4, ..., 2^{n-1}, 2(2^{n-1} - 1)\}$ and $|V(\Gamma(Z_{2^n}))| = 2^{n-1} - 1$. Using the above subcases, $|S| = 2^{\frac{n-1}{2}}(2^0 + 2^1 + ... + 2^{\frac{n-3}{2}}) = 2^{\frac{n-1}{2}}\sum_{i=0}^{\frac{n-3}{2}} 2^i$ and $|N(S)| = V(\Gamma(Z_{2^n})) - |S| = 2^{n-1} - 1 - 2^{\frac{n-1}{2}}\sum_{i=0}^{\frac{n-3}{2}} 2^i = 2^{\frac{n-1}{2}}(2^{\frac{n-1}{2}} - \sum_{i=0}^{\frac{n-3}{2}} 2^i) - 1$. Then, $b(\Gamma(Z_{2^n})) = \frac{|N(S)|}{|S|} = \frac{2^{\frac{n-1}{2}}(2^{\frac{n-1}{2}}-\sum_{i=0}^{\frac{n-3}{2}} 2^i)-1}{2^{\frac{n-1}{2}}\sum_{i=0}^{\frac{n-3}{2}} 2^i}$.

Theorem 2.6 *If $p > 4$ is any prime, then $(\Gamma(Z_{4p})) = \frac{3}{2(p-1)}$.*

Proof: *The proof is by the method of induction on p. Let $P, S, N(S)$ be the pendant set, minimum degree set, neighbourhood of S, respectivily.*

Case(i): *Let $p = 5$.*

The vertex set of $\Gamma(Z_{20})$ is $\{2, 4, .., 2(10 - 1), 5, 10, 15\}$ with $|V(\Gamma(Z_{20}))|= 11$. Clearly, the vertex $v = 2p = 10$ is adjacent to all the vertices in $V(\Gamma(Z_{20}))$ except 5 and 15, then $10 \in N(S)$. Let $x = 4$ and $y = 24$, then 96 is not divisible by 20 which implies x and y are non adjacent vertices. Then, the pendant set $P = \{2, 6, 14, 18\}$ with degree of any vertex in P is 1 and $P \subseteq S$.

Let $U = \{4, 8, 12, 16\}$ be the vertex subset of $V(\Gamma(Z_{20}))$. Clearly no two vertices in U are adjacent. That is 20 does not divide $32(= 4 \times 8)$. But, the vertices in U are adjacent to the vertices $5, 10,$ and 15 with $d(4) = d(8) = d(12) = d(16) < d(5) = d(15)$. Clearly, $U \subseteq S$ and the vertices $5, 15 \in N(S)$ then $N(S) = \{5, 10, 15\}$. Clearly, $|S| = |P| + |U| = 4 + 4 = 8$. Hence, $b(\Gamma(Z_{20})) = \frac{|N(S)|}{|S|} = \frac{3}{8} = \frac{3}{2\times5-2} = \frac{3}{2(p-1)}$, where $p = 5$.

Case(ii): *Let $p = 7$*

The vertex set of $\Gamma(Z_{28})$ is $\{2, 4, .., 2(14 - 1), 7, 14, 21\}$.Clearly, the vertex $v = 2p = 14$ is adjacent to all the vertices in $V(\Gamma(Z_{28}))$ except 7 and 21, then $14 \in N(S)$. Let $x = 6$ and $y = 18$ then 108 is not divisible by 28 which implies x and y are non adjacent vertices. Then, the pendant set $P = \{2, 6, 10, 18, 22, 26\}$ with degree of any vertex in P is 1 and $P \subseteq S$.

Let $U = \{4, 8, 12, 16, 20, 24\}$ be a vertex subset of $V(\Gamma(Z_{20}))$. Clearly no two vertices in U is adjacent. But, the vertices in U are adjacent to the vertices $7, 14,$ and 21. Clearly, $U \subseteq S$ and the vertices $7, 21 \in N(S)$ then $N(S) = \{7, 14, 21\}$. Then, $|S| = |P| + |U| = 6 + 6 = 12$. Hence,

$$b(\Gamma(Z_{42})) = \frac{|N(S)|}{|S|} = \frac{3}{12} = \frac{3}{2\times 7-2} = \frac{3}{2(p-1)}, \text{ where } p = 7.$$

Case(iii): *Let $p > 7$*

The vertex set of $\Gamma(Z_{4p})$ is $\{2, 4, ..., 2(2p-1), p, 2p, 3p\}$ with $|V(\Gamma(Z_{4p}))| = 2p + 1$. Since,the vertex $v = 2p$ is adjacent to all the vertices in $V(\Gamma(Z_{4p}))$ except p and $3p$, then $v = 2p \in N(S)$. Let P be the pendant vertex set and using above cases, $P = \{2, 6, ..., 2(p-2), 2(p+2), ..., 2(2p-1)\}$. Similarly, Let $U = \{4, ..., 4(p-1)\}$. Since, no two vertices in U are adjacent. But, the vertices in U are adjacent to the vertices $p, 2p$ and $3p$. Clearly, $U \subseteq S$ and the vertices $p, 3p \in N(S)$ then $|N(S)| = 3$. Hence, $|S| = |V(\Gamma(Z_{4p}))| - |N(S)| = 2p + 1 - 3 = 2(p-1)$. Thus,

$$b(\Gamma(Z_{4p})) = \frac{|N(S)|}{|S|} = \frac{3}{2\times p-2} = \frac{3}{2(p-1)}, \text{ where } p \text{ is any prime} > 4.$$

Theorem 2.7 *In $\Gamma(Z_{8p})$, $b(\Gamma(Z_{8p})) = \frac{7}{4(p-1)}$ where p is any prime > 8.*

Proof: *Since, the vertex set of $\Gamma(Z_{8p})$ is $\{2, ..., 2(4p-1), p, 2p, .., 7p\}$ with $|V(\Gamma(Z_{8p}))| = 4p + 3$. Using theorem (2.6), $N(S) = \{p, 2p, 3p, .., 7p\}$ and $|N(S)| = 7$. Hence, $|S| = |V(\Gamma(Z_{8p}))| - |N(S)| = 4p + 3 - 7 = 4(p-1)$. Then, $b(\Gamma(Z_{8p})) = \frac{|N(S)|}{|S|} = \frac{7}{4(p-1)}$, where p is any prime > 8.*

Theorem 2.8 *In $\Gamma(Z_{2^n p})$ where p is any prime $> 2^n$ and n is any positive integer, then $b(\Gamma(Z_{2^n p})) = \frac{2^n-1}{2^{n-1}(p-1)}$.*

Proof: *The vertex set of $\Gamma(Z_{2^n p})$ is $\{2, .., 2(2^{n-1}p-1), p, 2p, ..., (2^n-1)p\}$ with $|V(\Gamma(Z_{2^n p}))| = 2^{n-1}p + 2^{n-1} - 1$. Using theorems (2.6) and (2.7), $N(S) = \{p, 2p, ..., (2^n-1)p\}$ then $|N(S)| = (2^n - 1)$. Then, $|S| = |V(\Gamma(Z_{2^n p}))| - |N(S)| = 2^{n-1}p + 2^{n-1} - 1 - (2^n - 1) = 2^{n-1}(p-1)$.*

$$\text{Hence, } b(\Gamma(Z_{2^n p})) = \frac{|N(S)|}{|S|} = \frac{2^n-1}{2^{n-1}(p-1)}.$$

Theorem 2.9 *For any prime $p > 3$, $b(\Gamma(Z_{3^n})) = \frac{7}{3^{n-1}-8}$.*

Proof: *The vertex set of $\Gamma(Z_{3^n})$ is $\{3, 6...., 3(3^{n-1}-1)\}$ and $|V(\Gamma(Z_{3^n}))| = 3^{n-1} - 1$.The proof is by the method of induction.*

Case(i): *Let $n = 4$.*

The vertex set of $\Gamma(Z_{81})$ is $\{3, 6, ..78\}$ and $|V(\Gamma(Z_{81}))| = 26$. Let S be the vertex subset of V and $N(S)$ be the neibourhood of S such that $d(u) < d(v)$ where $u \in S$ and $v \in N(S)$. Let $x = 27$,$y = 54$ and $u = 3$ then $ux = uy = 0$. This implies that the vertices 27 and 54 are adjacent to all the remaining vertices of $\Gamma(Z_{81})$. Clearly. $27, 54 \in N(S)$. Consider another vertex set $X = \{9, 18, 36, 45, 63, 72\}$ which is the next maximum degree compared to the vertices $27, 54$. Let $u = 18$ and $v = 72$ then uv is divided by 81 that is u and v are adjacent. Since, X has a subgraph K_6 implies that any five *vertices* $\in N(S)$. *Thus,* $N(S) = \{9, 18, 27, 36, 45, 54, 63, 72\}$. *Then,* $|S| = |V(\Gamma(Z_{81}))| - |N(S)| = 19$. *Hence,* $b(\Gamma(Z_{81})) = \frac{|N(S)|}{|S|} = \frac{7}{19} = \frac{7}{3^{4-1}-8} = \frac{7}{3^{n-1}-8}$.

Case(ii): *Let* $n = 5$.

The vertex set of $\Gamma(Z_{243})$ *is* $\{3, 6, ..240\}$ *and* $|V(\Gamma(Z_{243}))| = 80$. *Using case*$(i)$*, the vertex set* $X = \{81, 162\}$. *Since, the vertices in* X *has highest degree then* $X \in N(S)$. *The vertex set* $Y = \{27, 54, 108, 135, 189, 216\}$ *is the next maximum degree compared to the vertex set* X. *Let* $u = 27$ *and* $v = 216$ *in* Y *then* uv *is divided by* 243 *that is* u *and* v *are adjacent. Using case*(i)*, any five vertices in* Y *belongs to* $N(S)$. *Thus,* $N(S) = \{27, 54, 81, 108, 135, 162, 189\}$. *Then* $|S| = |V(\Gamma(Z_{243}))| - |N(S)| = 80 - 7 = 73$.

Hence, $b(\Gamma(Z_{243})) = \frac{|N(S)|}{|S|} = \frac{7}{73} = \frac{7}{3^{5-1}-8} = \frac{7}{3^{n-1}-8}$.

Case(iii): *Let* $n > 5$.

In general, $V(\Gamma(Z_{3^n}))$ *is* $\{3, 6, .., 3(3^{n-1} - 1)\}$ *and* $|V(\Gamma(Z_{3^n}))| = 3^{n-1} - 1$. *Clearly,* $N(S) = \{1.3^{n-2}, 2.3^{n-2}, .., 7.3^{n-2}\}$ *then* $|S| = |V(\Gamma(Z_{3^n}))| - |N(S)| = 3^{n-1} - 1 - 7 = 3^{n-1} - 8$. *Hence,* $b(\Gamma(Z_{243})) = \frac{|N(S)|}{|S|} = \frac{7}{3^{n-1}-8}$.

3 Main Result

The value of the binding number of $\Gamma(Z_n)$ for some positive integer n forms an inequalities that $\Gamma(Z_{2p}) \leq \Gamma(Z_{4p}) \leq \Gamma(Z_{8p}) \leq \Gamma(Z_{pq})$ where p and q are any distinct primes with $p < q$ and $\Gamma(Z_{3^n}) \leq \Gamma(Z_{2^n}) \leq \Gamma(Z_{2^n p}) \leq \Gamma(Z_{pq})$ where n is any positive integer ≥ 2. Using the above two inequalities, we conclude that the maximum value of the binding number is $\Gamma(Z_{pq})$. Since $b(\Gamma(Z_{pq})) = \frac{p-1}{q-1}$. That is the numerator is greater when compared to the other prime number with respect to the denominator. The last two twin prime numbers of fiftieth million are $p = 982451579$ and $q = 982451581$. The maximum value of the $(\Gamma(Z_n))$ is 0.99999999796427626489236243072661 for some positive integer n upto fiftieth million.

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