THE BINET FORMULA AND REPRESENTATIONS OF k-GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

We consider a generalization of Fibonacci sequence, which is called the k-generalized Fibonacci sequence for a positive integer $k \geq 2$. The *k-generalized Fibonacci sequence* $\{g_n^{(k)}\}$ is defined as

$$g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, \;\; g_{k-1}^{(k)} = g_k^{(k)} = 1$$

and, for $n > k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

We call $g_n^{(k)}$ the n^{th} *k-generalized Fibonacci number*. For example, if $k = 2$, then $\{g_n^{(2)}\}$ is a Fibonacci sequence and, if $k = 5$, then $g_1^{(5)} = g_2^{(5)} = g_3^{(5)} = 0$, $g_4^{(5)} = g_5^{(5)} = 1$, and then the 5-generalized Fibonacci sequence is

$$0, 0, 0, 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, \ldots.$$

Let I_{k-1} be the identity matrix of order $k-1$ and let E be a $1 \times (k-1)$ matrix whose entries are 1's. For any $k \geq 2$, the fundamental recurrence relation $g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}$ can be defined by the vector recurrence relation

$$\begin{bmatrix} g_{n+1}^{(k)} \\ g_{n+2}^{(k)} \\ \vdots \\ g_{n+k}^{(k)} \end{bmatrix} = Q_k \begin{bmatrix} g_n^{(k)} \\ g_{n+1}^{(k)} \\ \vdots \\ g_{n+k-1}^{(k)} \end{bmatrix},$$

where

$$Q_k = \begin{bmatrix} 0 & I_{k-1} \\ 1 & E \end{bmatrix}_{k \times k}. \tag{1}$$

The matrix Q_k is said to be a *k-generalized Fibonacci matrix*. In [4] and [5], we gave the relationships between the k-generalized Fibonacci sequences and their associated matrices.

In 1843, Binet found a formula giving F_n in terms of n. It is a very complicated-looking expression, and the formula is

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are eigenvalues of Q_2. In [6], Levesque gave a Binet formula for the Fibonacci sequence by using a generating function.

In this paper, we derive a generalized Binet formula for the k-generalized Fibonacci sequence by using the determinant and we give several combinatorial representations of k-generalized Fibonacci numbers.

2. GENERALIZED BINET FORMULA

Let $\{g_n^{(k)}\}$ be a k-generalized Fibonacci sequence. Throughout the paper we will use $g_n = g_{n+k-2}^{(k)}$, $n = 1, 2, \ldots$, and $G_k = (g_1, g_2, g_3, \ldots)$ for notational convenience.

For example, if $k = 2$, $G_2 = (1, 1, 2, 3, \ldots)$, and if $k \geq 3$, $G_k = (1, 1, 2, 4, \ldots)$. For G_k, $k \geq 2$, since $g_1 = g_2 = 1$, we can replace the matrix Q_k in (1) with

$$Q_k = \begin{bmatrix} 0 & g_1 & 0 & \cdots & 0 \\ 0 & 0 & g_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & g_1 \\ g_1 & g_1 & \cdots & g_1 & g_2 \end{bmatrix}.$$

Then we can find the following matrix in [3]:

$$Q_k^n = \begin{bmatrix} g_{n-(k-1)} & g_{1,2}^{\dagger} & g_{1,3}^{\dagger} & \cdots & g_{1,k-1}^{\dagger} & g_{n-(k-2)} \\ g_{n-(k-2)} & g_{2,2}^{\dagger} & g_{2,3}^{\dagger} & \cdots & g_{2,k-1}^{\dagger} & g_{n-(k-3)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ g_{n-1} & g_{k-1,2}^{\dagger} & g_{k-1,3}^{\dagger} & \cdots & g_{k-1,k-1}^{\dagger} & g_n \\ g_n & g_{k,2}^{\dagger} & g_{k,3}^{\dagger} & \cdots & g_{k,k-1}^{\dagger} & g_{n+1} \end{bmatrix}, \tag{2}$$

where

$$\begin{aligned} g_{i,2}^{\dagger} &= g_{n-(k-i)} + g_{n-(k-(i-1))} \\ g_{i,3}^{\dagger} &= g_{n-(k-i)} + g_{n-(k-(i-1))} + g_{n-(k-(i-2))} \\ &\vdots \\ g_{i,k-1}^{\dagger} &= g_{n-(k-i)} + g_{n-(k-(i-1))} + g_{n-(k-(i-2))} + \cdots + g_{n-(k-(i-(k-2)))}, \end{aligned}$$

$i = 1, 2, \ldots, k$. Since $Q_k^n Q_k^m = Q_k^{n+m}$, $g_{n+m} = (Q_k^{n+m})_{k,1}$; hence, we have the following theorem.

Theorem 2.1 (see [3]): Let $G_k = (g_1, g_2, \ldots)$. Then, for any positive integers n and m,

$$\begin{aligned} g_{n+m} = g_n g_{m-(k-1)} &+ (g_n + g_{n-1}) g_{m-(k-2)} \\ &+ (g_n + g_{n-1} + g_{n-2}) g_{m-(k-3)} + \cdots \\ &+ (g_n + g_{n-1} + g_{n-2} + \cdots + g_{n-(k-2)}) g_{m-1} + g_{n+1} g_m. \end{aligned}$$

Note that $g_{n+m} = (Q_k^{n+m})_{k,1} = (Q_k^{n+m})_{k-1,k}$. Then we have the following corollary.

Corollary 2.2: Let $G_k = (g_1, g_2, \ldots)$. Then, for any positive integers n and m,

$$\begin{aligned} g_{n+m} = g_{n-1} g_{m-(k-2)} &+ (g_{n-1} + g_{n-2}) g_{m-(k-3)} \\ &+ (g_{n-1} + g_{n-2} + g_{n-3}) g_{m-(k-4)} + \cdots \\ &+ (g_{n-1} + g_{n-2} + g_{n-3} + \cdots + g_{n-(k-1)}) g_m + g_n g_{m+1}. \end{aligned}$$

Now we are going to find the generalized Binet formula for the k-generalized Fibonacci sequence.

Lemma 2.3: Let $b_k = \frac{2^{k+1}}{k+1}\left(\frac{k}{k+1}\right)^k$. Then $b_k < b_{k+1}$ for $k \geq 2$.

Proof: Since $\frac{k+1}{k+2} > \frac{k}{k+1}$ and $k \geq 2$,

$$\left(\frac{k+1}{k+2}\right)^{k+1} > \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \frac{2^{k+2}}{k+2} \geq \frac{2^{k+1}}{k}.$$

Therefore,

$$b_{k+1} = \frac{2^{k+2}}{k+2}\left(\frac{k+1}{k+2}\right)^{k+1} > \frac{2^{k+1}}{k+1}\left(\frac{k}{k+1}\right)^{k} = b_k$$

for each positive integer k. □

Lemma 2.4: The equation $z^{k+1} - 2z^k + 1 = 0$ does not have multiple roots for $k \geq 2$.

Proof: Let $f(z) = z^k - z^{k-1} - \cdots - z - 1$ and let $g(z) = (z-1)f(z)$. Then $g(z) = z^{k+1} - 2z^k + 1$. So 1 is a root but not a multiple root of $g(z) = 0$, since $k \geq 2$ and $f(1) \neq 0$. Suppose that α is a multiple root of $g(z) = 0$. Note that $\alpha \neq 0$ and $\alpha \neq 1$. Since α is a multiple root, $g(z) = \alpha^{k+1} - 2\alpha^k + 1 = 0$ and $g'(\alpha) = (k+1)\alpha^k - 2k\alpha^{k-1} = \alpha^{k-1}((k+1)\alpha - 2k) = 0$. Thus, $\alpha = \frac{2k}{k+1}$, and hence

$$\begin{aligned} 0 &= -\alpha^{k+1} + 2\alpha^k - 1 = \alpha^k(2-\alpha) - 1 \\ &= \left(\frac{2k}{k+1}\right)^k\left(2 - \frac{2k}{k+1}\right) - 1 = \left(\frac{2k}{k+1}\right)^k\left(\frac{2k+2-2k}{k+1}\right) - 1 \\ &= \frac{2^{k+1}}{k+1}\left(\frac{k}{k+1}\right)^k - 1 = b_k - 1. \end{aligned}$$

Since, by Lemma 2.3, $b_2 = \left(\frac{2}{3}\right)^3 \times 2^2 = \frac{2^5}{3^3} > 1$ and $b_k < b_{k+1}$ for $k \geq 2$, $b_k \neq 1$, a contradiction.

Therefore, the equation $g(z) = 0$ does not have multiple roots. □

Let $f(\lambda)$ be the characteristic polynomial of the k-generalized Fibonacci matrix Q_k. Then $f(\lambda) = \lambda^k - \lambda^{k-1} - \cdots - \lambda - 1$, which is a well-known fact. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of Q_k. Then, by Lemma 2.4, $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. Let Λ be a $k \times k$ Vandermonde matrix as follows:

$$\Lambda = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{bmatrix}.$$

Set $V = \Lambda^T$. Let

$$\mathbf{d}_i = \begin{bmatrix} \lambda_1^{n+i-1} \\ \lambda_2^{n+i-1} \\ \vdots \\ \lambda_k^{n+i-1} \end{bmatrix}$$

and let $V_j^{(i)}$ be a $k \times k$ matrix obtained from V by replacing the j^{th} column of V by $\mathbf{d}_i$. Then we have the generalized Binet formula as the following theorem.

Theorem 2.5: Let $\{g_n^{(k)}\}$ be a k-generalized Fibonacci sequence. Then

$$g_n = \frac{\det(V_1^{(k)})}{\det(V)}, \tag{3}$$

where $g_n = g_{n+k-2}^{(k)}$.

Proof: Since the eigenvalues of Q_k are distinct, Q_k is diagonalizable. It is easy to show that $Q_k\Lambda = \Lambda D$, where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Since Λ is invertible, $\Lambda^{-1}Q_k\Lambda = D$. Thus, Q_k is similar to D. So we have $Q_k^n\Lambda = \Lambda D^n$. Let $Q_k^n = [q_{ij}]_{k\times k}$. Then we have the following linear system of equations:

$$\begin{aligned} q_{i1} + q_{i2}\lambda_1 + \cdots + q_{ik}\lambda_1^{k-1} &= \lambda_1^{n+i-1} \\ q_{i1} + q_{i2}\lambda_2 + \cdots + q_{ik}\lambda_2^{k-1} &= \lambda_2^{n+i-1} \\ \vdots \qquad\qquad & \quad \vdots \\ q_{i1} + q_{i2}\lambda_k + \cdots + q_{ik}\lambda_k^{k-1} &= \lambda_k^{n+i-1}. \end{aligned}$$

And, for each $j = 1, 2, \ldots, k$, we get

$$q_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}.$$

Therefore, by (2), we have the explicit form

$$q_{k1} = g_n = \frac{\det(V_1^{(k)})}{\det(V)}. \quad \square$$

We note that, if $k = 2$, then (3) is the Binet formula for the Fibonacci sequence.

3. COMBINATORIAL REPRESENTATIONS OF k-GENERALIZED FIBONACCI NUMBERS

In this section, we consider some combinatorial representations of $g_n = g_{n+k-2}^{(k)}$ for $k \geq 2$. Let S_k be a $k \times k$ (0, 1)-matrix as follows:

$$S_k = \begin{bmatrix} E & 1 \\ I_{k-1} & 0 \end{bmatrix}.$$

Then, by (2),

$$S_k^n = [s_{ij}] = \begin{bmatrix} g_{n+1} & g_{k,k-1}^{\dagger} & \cdots & g_{k,3}^{\dagger} & g_{k,2}^{\dagger} & g_n \\ g_n & g_{k-1,k-1}^{\dagger} & \cdots & g_{k-1,3}^{\dagger} & g_{k-1,2}^{\dagger} & g_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n-(k-3)} & g_{2,k-1}^{\dagger} & \cdots & g_{2,3}^{\dagger} & g_{2,2}^{\dagger} & g_{n-(k-2)} \\ g_{n-(k-2)} & g_{1,k-1}^{\dagger} & \cdots & g_{1,3}^{\dagger} & g_{1,2}^{\dagger} & g_{n-(k-1)} \end{bmatrix}. \tag{4}$$

In [1], we can find the following lemma.

Lemma 3.1 (see [1]):

$$s_{ij} = \sum_{(m_1,\ldots,m_k)} \frac{m_j + m_{j+1} + \cdots + m_k}{m_1 + \cdots + m_k} \times \binom{m_1 + \cdots + m_k}{m_1, \ldots, m_k},$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + km_k = n - i + j$ and defined to be 1 if $n = i - j$.

Corollary 3.2: Let $\{g_n^{(k)}\}$ be the k-generalized Fibonacci sequence. Then

$$g_n = \sum_{(m_1, \dots, m_k)} \frac{m_k}{m_1 + \cdots + m_k} \times \binom{m_1 + \cdots + m_k}{m_1, \dots, m_k},$$

where the summation is over nonnegative integers satisfying $m_1 + 2m_2 + \cdots + km_k = n - 1 + k$.

Proof: From Lemma 3.1, if $i = 1$ and $j = k$, then the conclusion can be derived directly from (4). □

Let $A = [a_{ij}]$ be an $n \times n$ (0, 1)-matrix. The *permanent* of A is defined by

$$\text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where σ runs over all permutations of the set $\{1, 2, \dots, n\}$. A matrix A is called *convertible* if there is an $n \times n$ $(1, -1)$-matrix H such that $\text{per} A = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of A and H. Such a matrix H is called a *converter* of A.

Let $\mathcal{F}^{(n,k)} = [f_{ij}] = T_n + B_n$, where $T_n = [t_{ij}]$ is the $n \times n$ $(0, 1)$-matrix defined by $t_{ij} = 1$ if and only if $|i - j| \le 1$, and $B_n = [b_{ij}]$ is the $n \times n$ $(0, 1)$-matrix defined by $b_{ij} = 1$ if and only if $2 \le j - i \le k - 1$. In [4] and [5], the following theorem gave a representation of $g_n^{(k)}$.

Theorem 3.3 (see [4], [5]): Let $\{g_n^{(k)}\}$ be the k-generalized Fibonacci sequence. Then

$$g_n = \text{per}\, \mathcal{F}^{(n-1, k)},$$

where $g_n = g_{n+k-2}^{(k)}$.

Let H be a $(1, -1)$-matrix of order $n - 1$, defined by

$$H = \begin{bmatrix} 1 & -1 & 1 & \cdots & 1 \\ 1 & 1 & -1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then the following theorem holds.

Theorem 3.4: Let $\{g_n^{(k)}\}$ be the k-generalized Fibonacci sequence. Then

$$g_n = \det(\mathcal{F}^{(n-1, k)} \circ H),$$

where $g_n = g_{n+k-2}^{(k)}$.

Proof: Since the matrix $\mathcal{F}^{(n-1, k)}$ is a convertible matrix with converter H, we have

$$\text{per}\, \mathcal{F}^{(n-1, k)} = \det(\mathcal{F}^{(n-1, k)} \circ H)$$

and, by Theorem 3.3, the proof is complete. □

Now we consider the generating function of the k-generalized Fibonacci sequence. We can easily find the characteristic polynomial, $x^k - x^{k-1} - \cdots - x - 1$, of the k-Fibonacci matrix Q_k. It follows that all of the eigenvalues of Q_k satisfy $x^k = x^{k-1} + x^{k-2} + \cdots + x + 1$. And we can find the following fact in [5]:

$$\begin{aligned} x^n = g_{n-k+2}x^{k-1} &+ (g_{n-k+1} + g_{n-k} + \cdots + g_{n-2k+3})x^{k-2} \\ &+ (g_{n-k+1} + g_{n-k} + \cdots + g_{n-2k+4})x^{k-3} \\ &+ \cdots + (g_{n-k+1} + g_{n-k})x + g_{n-k+1}. \end{aligned} \tag{5}$$

Let $G_k(x) = g_1 + g_2x + g_3x^2 + \cdots + g_{n+1}x^n + \cdots$. Then

$$G_k(x) - xG_k(x) - x^2G_k(x) - \cdots - x^kG_k(x) = (1 - x - x^2 - \cdots - x^k)G_k(x).$$

Using equation (5), we have $(1 - x - x^2 - \cdots - x^k)G_k(x) = g_1 = 1$. Thus,

$$G_k(x) = (1 - x - x^2 - \cdots - x^k)^{-1}$$

for $0 \le x + x^2 + \cdots + x^k < 1$.

Let $f_k(x) = x + x^2 + \cdots + x^k$. Then $0 \le f_k(x) < 1$ and we have the following lemma.

Lemma 3.5: For positive integers p and n, the coefficient of x^n in $(f_k(x))^p$ is

$$\sum_{l=0}^{p} (-1)^l \binom{p}{l} \binom{n-kl-1}{n-kl-p}, \ \frac{n}{k} \le p \le n.$$

Proof:

$$\begin{aligned} (f_k(x))^p &= (x + x^2 + \cdots + x^k)^p = x^p(1 + x + x^2 + \cdots + x^{k-1})^p \\ &= x^p\left(\frac{1-x^k}{1-x}\right)^p = x^p\left((1-x^k)\left(\frac{1}{1-x}\right)\right)^p \\ &= x^p\left(\left(\sum_{l=0}^{p}\binom{p}{l}(-1)^l x^{kl}\right)\left(\sum_{i=0}^{\infty}\binom{p+i-1}{i}x^i\right)\right). \end{aligned}$$

In the above equation, we consider the coefficient of x^n. Since the first term on the right is x^p, we have $kl + i = n - p$, that is, $i = n - kl - p$. If $l = q$ for any $q = 0, 1, \ldots, p$, then the second term on the right is

$$\left((-1)^q \binom{p}{q}\binom{n-kq-1}{n-kq-p}\right)x^{n-p}.$$

So the coefficient of x^n is

$$\sum_{l=0}^{p} (-1)^l \binom{p}{l} \binom{n-kl-1}{n-kl-p}, \ \frac{n}{k} \le p \le n. \ \square$$

Theorem 3.6: For positive integers p and n,

$$g_{n+1} = \sum_{\frac{n}{k} \le p \le n} \ \sum_{l=0}^{p} (-1)^l \binom{p}{l} \binom{n-kl-1}{n-kl-p}. \tag{6}$$

Proof: Since

$$G_k(x) = g_1 + g_2x + g_3x^2 + \cdots + g_{n+1}x^n + \cdots = \frac{1}{1 - x - x^2 - \cdots - x^k},$$

the coefficient of x^n is the $(n+k-1)^{\text{st}}$ Fibonacci number, that is, g_{n+1} in G_k. And

$$
\begin{aligned}
G_k(x) &= \frac{1}{1-x-x^2-\cdots-x^k} = \frac{1}{1-f_k(x)} \\
&= 1+f_k(x)+(f_k(x))^2+\cdots+(f_k(x))^n+\cdots \\
&= 1+f_k(x)+x^2\sum_{l=0}^{n}\binom{2}{l}(-1)^l x^{kl}\sum_{i=0}^{\infty}\binom{i+1}{i}x^i \\
&\quad +\cdots+x^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l x^{kl}\sum_{i=0}^{\infty}\binom{n+i-1}{i}x^i+\cdots.
\end{aligned}
\tag{7}
$$

Since we need the coefficient of x^n, we only need the first $n+1$ terms on the right and the $(p+1)^{\text{st}}$ term in (7) such that

$$x^p\sum_{l=0}^{p}\binom{p}{l}(-1)^l x^{kl}\sum_{i=0}^{\infty}\binom{p+i-1}{i}x^i.$$

So $kl+i=n-p$, as we see in (6), and $\frac{n}{k}\le p\le n$. Thus, by Lemma 3.5, we have the theorem. □

From the above theorems, we have five representations for g_n, $g_n=g_{n+k-2}^{(k)}$. That is,

$$
\begin{aligned}
g_n &= \operatorname{per}\mathcal{F}^{(n-1,k)} = \det(\mathcal{F}^{(n-1,k)}\circ H) = \frac{\det(V_1^{(k)})}{\det(V)} \\
&= \sum_{\frac{n-1}{k}\le p\le n-1}\sum_{l=0}^{p}(-1)^l\binom{p}{l}\binom{n-kl-2}{n-kl-p-1} \\
&= \sum_{(m_1,\ldots,m_k)}\frac{m_k}{m_1+\cdots+m_k}\times\binom{m_1+\cdots+m_k}{m_1,\ldots,m_k},
\end{aligned}
$$

where the summation is over nonnegative integers satisfying $m_1+2m_2+\cdots+km_k=n-1+k$.

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REFERENCES

1. W. Y. C. Chen & J. D. Louck. "The Combinatorial Power of the Companion Matrix." *Linear Algebra Appl.* **232** (1996):261-78.
2. R. A. Horn & C. R. Johnson. *Matrix Analysis.* New York: Cambridge Univ. Press, 1985.
3. G.-Y. Lee. "A Completeness on Generalized Fibonacci Sequences." *Bull. Korean Math. Soc.* **32.2** (1995):239-49.
4. G.-Y. Lee & S.-G. Lee. "A Note on Generalized Fibonacci Numbers." *The Fibonacci Quarterly* **33.3** (1995):273-78.
5. G.-Y. Lee, S.-G. Lee, & H.-G. Shin. "On the k-Generalized Fibonacci Matrix Q_k." *Linear Algebra Appl.* **251** (1997):73-88.
6. Claude Levesque. "On m^{th}-Order Linear Recurrences." *The Fibonacci Quarterly* **23.4** (1985):290-93.

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