# THE BLOCH-KATO CONJECTURE AND A THEOREM OF SUSLIN-VOEVODSKY

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Abstract. We give a new proof of the theorem of Suslin-Voevodsky which shows that the Bloch-Kato conjecture implies a portion of the Beilinson-Lichtenbaum conjectures. Our proof does not rely on resolution of singularities, and thereby extends the Suslin-Voevodsky theorem to positive characteristic.

## 1. Introduction

The purpose of this paper is to give an alternate proof of the main result of [25] along the lines of [13]. Our proof does not rely on resolution of singularities, hence extends the results of [25] to varieties over fields of arbitrary characteristic. The new ingredient which enables us to apply the techniques of [13] to motivic cohomology is the surjectivity result of [8, Corollary 4.4].

Let $F$ be a field, $q \geq 0$ an integer, and $m > 1$ an integer prime to the characteristic of $F$. We have the Milnor $K$-group $K_n^M(F)$, and the *Galois symbol*

$$\vartheta_{q,F} : K_q^M(F)/m \to H^q_{\text{ét}}(F, \mu_m^{\otimes q}). \tag{1.1}$$

The *Bloch-Kato conjecture* [3] asserts that the map $\vartheta_{q,F}$ is an isomorphism for all $F$, $q$ and $m$. For $q = 1$, this follows from the definition of $\vartheta_{1,F}$ via the Kummer sequence, and is the theorem of Merkurjev-Suslin [18] for $q = 2$. For $m$ a power of 2, the Bloch-Kato conjecture for arbitrary $q$ and $F$ has been proven by Voevodsky [28]

Let $X$ be a localization of a smooth scheme of finite type over $k$; we call such a scheme *essentially smooth* over $k$. We use Bloch's higher Chow groups as our definition of motivic cohomology $H^p(X, \mathbb{Z}(q))$ for essentially smooth $k$-schemes $X$. We sheafify Bloch's cycle complexes to define the *weight $q$ motivic complex* $\Gamma_X(q)$, as a complex of Zariski sheaves on $X$.

Let $m$ be prime to the characteristic of $k$, let $X$ be essentially smooth over $k$, and let $\epsilon : X_{\text{ét}} \to X_{\text{Zar}}$ be the change of topology map. We construct a natural *cycle class map* (in the derived category of Zariski sheaves)

$$\mathrm{cl}^a : \Gamma_X(a) \otimes^L \mathbb{Z}/m \to \tau_{\leq a} R\epsilon_*(\mu_m^{\otimes a}). \tag{1.2}$$

Our main result is

**Theorem 1.1.** *Let $k$ be a field, and let $m$ be an integer prime to the characteristic of $k$. Suppose that the maps $\vartheta_{q,F} : K_q^M(F)/m \to H^q_{\text{ét}}(F, \mu_m^{\otimes q})$ are surjective for*

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*all finitely generated field extensions $F$ of $k$. Then the cycle class map* (1.2) *is an isomorphism for all essentially smooth $X$ over $k$ and all $a$ with $0 \leq a \leq q$.*

As one consequence, we have

**Corollary 1.2.** *Let $k$ be a field, and let $m$ be an integer prime to the characteristic of $k$. Suppose that the maps $\vartheta_{q,F} : K_q^M(F)/m \to H^q_{\text{ét}}(F, \mu_m^{\otimes q})$ are surjective for all finitely generated field extensions $F$ of $k$. Let $X$ be essentially smooth over $k$. Then the cycle class map* (1.2) *induces an isomorphism*

$$\mathrm{cl}^q : H^p(X, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(X, \mu_m^{\otimes q})$$

*for all $p \leq q$, and an injection for $p = q + 1$.*

The conclusion of Theorem 1.1 is a part of the *Beilinson-Lichtenbaum conjectures* (see e.g. [16]).

Using Voevodsky's verification of the Bloch-Kato conjecture for $m$ a power of 2 (*loc. cit.*), we may apply Theorem 1.1 to extend the consequences of [28] to characteristic $p > 0$:

**Corollary 1.3.** *For $m = 2^\nu$, the cycle class map* (1.2) *is an isomorphism for all $X$ essentially smooth over $k$.*

Similarly, the Merkurjev-Suslin theorem (*loc. cit.*) gives as in [25]:

**Corollary 1.4.** *Let $k$ be a field, and let $m$ be prime to* $\operatorname{char} k$. *Then the cycle class map* (1.2) *is an isomorphism for all $a$, $0 \leq a \leq 2$, and all $X$ essentially smooth over $k$.*

We also consider the étale sheafification $\Gamma_X(q)_{\text{ét}}$ of $\Gamma_X(q)$. The results of Suslin-Voevodsky [24] on the Suslin homology of varieties over an algebraically closed field give us the following unconditional result:

**Theorem 1.5.** *Let $k$ be a field, and let $m$ be an integer prime to* $\operatorname{char} k$. *Let $X$ be essentially smooth over $k$. Then the étale cycle class map*

$$\mathrm{cl}^q_{\text{ét}} : \Gamma_X(q)_{\text{ét}} \otimes^L \mathbb{Z}/m \to \mu_m^{\otimes q}$$

*is an isomorphism in $\mathbf{D}^-(\mathrm{Sh}(X_{\text{ét}}))$ for all $q \geq 0$.*

With the aid of this result, one can reformulate Theorem 1.1 entirely in terms of $\Gamma_X(q)$ and $\Gamma_X(q)_{\text{ét}}$, giving the equivalent result which avoid the use of the cycle class map.

**Theorem 1.6.** *Let $k$ be a field, and let $m$ be an integer prime to* $\operatorname{char} k$. *Suppose that the maps $\vartheta_{q,F} : K_q^M(F)/m \to H^q_{\text{ét}}(F, \mu_m^{\otimes q})$ are surjective for all finitely generated field extensions $F$ of $k$. Then the natural map*

$$\iota_X : \Gamma_X(a) \otimes^L \mathbb{Z}/m \to \tau_{\leq a} R\epsilon_* \Gamma_X(a)_{\text{ét}}$$

*is an isomorphism for $0 \leq a \leq q$ and for all $X$ essentially smooth over $k$.*

An outline of the paper is as follows: In §2 we recall the definition of the primary object of study, Bloch's cycle complex, and describe its fundamental properties. We also extend the definition of the cycle complex to the relative situation, and to normal crossing schemes. In §3, we give a definition of the cycle class map from motivic cohomology to étale cohomology, and we show in §4 that this defines a natural transformation of cohomology theories, compatible with Gysin sequences and products. These two sections are fairly technical, and could be skipped on the first reading. The main argument for the proof of Theorem 1.1 occurs in §5-§7. In §5, we introduce the semi-local $n$-cube, the relative motivic cohomology of which provides the dimension shifting which enables us to move from $H^q(\mathbb{Z}/m(q))$ to $H^p(\mathbb{Z}/m(q))$ for $p \le q$. The heart of this dimension shifting argument is given in §6 and §7. We prove Theorem 1.5 and show that Theorem 1.1 and Theorem 1.6 are equivalent at the end of §4. We have included an appendix on products for Bloch's cycle complexes, which fills a gap in the construction of products given in [1].

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## 2. Prelimiaries

We set up some notations and conventions, and recall some of the notations and results from [8] for the reader's convenience. We fix a field $k$.

2.1. **Conventions.** We will have occasion to work with both complexes for which the differential has degree $-1$ (*homological* complexes) and those for which the differential has degree $+1$ (*cohomological* complexes). Rather than work with two equivalent derived categories, we will always work in the derived category formed from cohomological complexes. If $A_*$ is a homological complex, the associated cohomological object will be the cohomological complex $A^*$ with $A^n := A_{-n}$, and with $d^n := d_{-n}$. Similarly, we will give the category of homological complexes the translation structure which is compatible with that of the derived category, i.e., if $(A_*, d^A_*)$ is a homological complex, then $(A[1], d^{A[1]})$ is the complex with

$$A[1]_n := A_{n-1},\ d^{A[1]}_n := -d^A_{n-1}.$$

We will sometimes abuse notation by saying that a sequence of complexes

$$S := [A \xrightarrow{i} B \xrightarrow{j} C]$$

*gives a distinguished triangle* if the image of $S$ in the derived category has an extension to a distinguished triangle in some standard way. For instance, if $S$ is degree-wise exact, with $i$ degree-wise injective and $j$ degree-wise surjective, then the canonical map $\mathrm{cone}(i) \to C$ is a quasi-isomorphism, hence $S$ defines a canonical distinguished triangle.

We let $[n]$ denote the ordered set $\{0, \dots, n\}$, with the standard order, and let $\Delta$ denote the category with objects the sets $[n]$, $n = 0, 1, \dots$, and with morphisms the order-preserving maps. For a simplicial abelian group $A : \Delta^{\mathrm{op}} \to \mathbf{Ab}$, we view the associated chain complex as a cohomological complex $A^*$ in negative degrees, i.e., $A^n = A([-n])$, with differential the usual alternating sum. Similarly, for a simplicial object $A : \Delta^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab})$, we have the double complex $A^{**}$, with $A^{a,b} = A^a([-b])$.

2.2. **Cycle complexes.** Let $\Delta^n$ be the affine space

$$\Delta^n := \mathrm{Spec}\, k[t_0, \dots, t_n]/\sum_{i=0}^{n} t_i - 1.$$

A subscheme of $\Delta^m$ defined by equations of the form $t_{i_1} = \ldots = t_{i_r} = 0$ is called a *face* of $\Delta^n$ (we consider $\Delta^n$ a face of $\Delta^n$ as well). The *vertices* $v^n_i$ of $\Delta^n$ are the faces $t_j = 0$, $j \ne i$. For a map $g : [n] \to [m]$ in $\Delta$, we have the map of schemes

$g : \Delta^n \to \Delta^m$ defined by taking the affine-linear extension of the map of vertices $v_i^n \mapsto v_{g(i)}^m$; via these maps, sending $n$ to $\Delta^n$ extends to form a cosimplicial scheme $\Delta^*$.

Let $X$ be a finite type $k$-scheme, locally equi-dimensional over $k$. We let $z^q(X)$ denote the group of codimension $q$ algebraic cycles on $X$. Bloch [1] has defined the *cycle complex* $z^q(X,*)$, with $z^q(X,p)$ the subgroup of the group $z^q(X \times \Delta^n)$ generated by codimension $q$ subvarieties $W$ of $X \times \Delta^p$ such that, for each face $F$ of $\Delta^p$, $W \cap X \times F$ has codimension $\geq q$ on $X \times F$. For each order-preserving map $g : [n] \to [m]$, and each $W \in z^q(X,m)$, the pull-back $(\mathrm{id} \times g)^*(W)$ via the induced map $\mathrm{id} \times g : X \times \Delta^n \to X \times \Delta^m$ is defined and is in $z^q(X,n)$. Thus, sending $m$ to $z^q(X,m)$ gives rise to a simplicial abelian group, and $z^q(X,*)$ is defined as the associated chain complex. The *higher Chow groups* of $X$, $\mathrm{CH}^q(X,p)$, are then defined as the homology

$$\mathrm{CH}^q(X,p) := H_p(z^q(X,*)).$$

In fact, the definition given in [1] requires $X$ to be quasi-projective, but this is not necessary. More generally, if $X$ is the localization of a locally equi-dimensional finite type $k$-scheme $Y$, then define

$$z^q(X,*) = \varinjlim_{X \subset U \subset Y} z^q(U,*),$$

where the limit is over open subschemes $U$ of $Y$. It is easy to see that $z^q(X,*)$ is independent of the choice of $Y$. We set as above $\mathrm{CH}^q(X,p) := H_p(z^q(X,*))$. In particular, all the notions described above make sense for a $k$-scheme $X'$ of the form $\mathrm{Spec}(\mathcal{O}_{X,v})$, where $\mathcal{O}_{X,v}$ is the local ring of a smooth quasi-projective variety $X$ at a finite set $v$ of closed subvarieties.

2.3. **Pull-back.** As we will see below, the higher Chow groups have the formal properties of Borel-Moore homology for schemes essentially of finite type over $k$, and, for essentially smooth $k$-schemes, the formal properties of cohomology, including contravariant functoriality. The complexes $z^q(X,*)$ are not, however, contravariantly functorial for arbitrary maps $f : Y \to X$, even if $X$ and $Y$ are smooth, due to the fact that the pull-back of an arbitrary cycle on $X \times \Delta^m$ is not in general defined. As a first step in the construction of a pull-back map $f^*$, we need to consider subcomplexes of $z^q(X,*)$ constructed by using cycles having good intersections with a given finite collection of closed subsets of $X$.

Let $X$ be a $k$-scheme, essentially of finite type and locally equi-dimensional over $k$, and let $\mathcal{S}$ be a finite set of closed subsets of $X$, including $X$ as an element. We have the subcomplex $z^q(X,*)_{\mathcal{S}}$ of $z^q(X,*)$ with $z^q(X,p)_{\mathcal{S}}$ being the subgroup of $z^q(X,p)$ generated by subvarieties $W$ which intersect $S \times F$ properly on $X \times \Delta^p$, for all $S$ in $\mathcal{S}$, and all faces $F$ of $\Delta^p$.

Let $f : Y \to X$ be a morphism of locally equi-dimensional $k$-schemes, essentially of finite type, and let

$$\begin{aligned} S_i(f) &= \{x \in X \mid \dim_{k(x)} f^{-1}(x) \geq i\}, \\ \mathcal{S}(f) &= \{S_i(f) \mid i = -1, 0, 1, \dots\}. \end{aligned} \tag{2.1}$$

Then each $S_i(f)$ is a closed subset of $Y$. Suppose that $X$ is regular and that $\mathcal{S}(f) \subset \mathcal{S}$. Then $(f \times \mathrm{id})^*(W)$ is well-defined and in $z^q(Y,p)$ for each $W \in z^q(X,p)_{\mathcal{S}}$, giving the map of complexes

$$f^*_{\mathcal{S}} : z^q(X,*)_{\mathcal{S}} \to z^q(Y,*). \tag{2.2}$$

If $X$ is smooth over $k$ and affine, then the proof of [14, Chap. II, Theorem 3.5.14], shows that the inclusion

$$i_{\mathcal{S}} : z^q(X,*)_{\mathcal{S}} \to z^q(X,*) \tag{2.3}$$

is a quasi-isomorphism. Setting $f^* = f^*_{\mathcal{S}} \circ i_{\mathcal{S}}^{-1}$ defines a pull-back map in $\mathbf{D}(\mathbf{Ab})$,

$$f^* : z^q(X,*) \to z^q(Y,*), \tag{2.4}$$

and the resulting map in cohomology $f^* : \mathrm{CH}^q(X,*) \to \mathrm{CH}^q(Y,*)$.

If, for example, $f$ is an inclusion $i_Y : Y \to X$ and $Y$ is in $\mathcal{S}$, the map (2.2) is defined. Let $\mathcal{S}(Y)$ be the subset of $\mathcal{S}$ consisting of those $S$ contained in $Y$, and write $z^q(Y,*)_{\mathcal{S}}$ for $z^q(Y,*)_{\mathcal{S}(Y)}$. One sees directly that $i_Y^*(z^q(X,p)_{\mathcal{S}}) \subset z^q(Y,p)_{\mathcal{S}}$, giving the map of complexes $i_Y^* : z^q(X,*)_{\mathcal{S}} \to z^q(Y,*)_{\mathcal{S}}$.

2.4. **External products.** We refer the reader to the Appendix for a description of natural external products

$$z^q(X,*) \otimes^L z^{q'}(Y,*) \to z^{q+q'}(X \times_k Y,*)$$

in $\mathbf{D}^-(\mathbf{Ab})$. We will show in §2.10 how the external products give rise to natural cup products for $X$ essentially smooth over $k$.

2.5. **Motivic cohomology.** The higher Chow groups have been incorporated into a general theory of *motivic cohomology* via the categorical constructions of Voevodsky [27] and the second author [14]. In case resolution of singularities holds for finite type $k$-schemes (e.g., $k$ has characteristic zero), the motivic cohomology of [27] and [14] are canonically isomorphic (see [14, Chap. IV, Theorem 2.5.5]); for smooth $X$ which is the localization of a quasi-projective $k$-scheme both theories agree with the higher Chow groups via a canonical isomorphism

$$\mathrm{CH}^q(X,p) \cong H^{2q-p}(X,\mathbb{Z}(q)). \tag{2.5}$$

Even without assuming resolution of singularities, the isomorphism (2.5) holds for the motivic cohomology of [14] (see [14, Chap. II, Theorem 3.6.6]).

We reindex Bloch's cycle complex to reflect the isomorphism (2.5), defining the cohomological cycle complex $\mathcal{Z}^q(X,*)$, for $X$ essentially smooth over $k$, by

$$\mathcal{Z}^q(X,p) := z^q(X,2q-p),$$

and take the cohomology $H^p(\mathcal{Z}^q(X,*))$ as the definition of the motivic cohomology $H^p(X,\mathbb{Z}(q))$. We define the subcomplex $\mathcal{Z}^q(X,*)_{\mathcal{S}}$ of $\mathcal{Z}^q(X,*)$ by the similar reindexing of $z^q(X,*)_{\mathcal{S}}$.

Let $m$ be an integer, and $X$ essentially smooth over $k$. The *mod $m$* motivic cohomology, $H^p(X,\mathbb{Z}/m(q))$, is defined by

$$H^p(X,\mathbb{Z}/m(q)) := H^p(\mathcal{Z}^q(X,*) \otimes \mathbb{Z}/m).$$

The mod $m$ motivic cohomology has a natural definition via the motivic categories of [27] and [14] as well, which agrees with the definition given here, subject to the restrictions described above.

2.6. **Localization.** Let $X$ be a locally equi-dimensional $k$-scheme, essentially of finite type over $k$, and let $W$ be a closed subset of $X$, with inclusion $i: W \to X$, and open complement $j: U \to X$. We set

$$z^q_W(X,*)_{\mathcal{S}} := \mathrm{cone}(j^*: z^q(X,*)_{\mathcal{S}} \to z^q(U,*)_{\mathcal{S}})[-1].$$

In case $X$ is smooth over $k$, we set

$$\mathcal{Z}^q_W(X,*) := \mathrm{cone}(j^*: \mathcal{Z}^q(X,*)_{\mathcal{S}} \to \mathcal{Z}^q(U,*)_{\mathcal{S}})[-1], \tag{2.6}$$

and define the motivic cohomology with support as

$$H^p_W(X,\mathbb{Z}(q)) := H^p(\mathcal{Z}^q_W(X,*)).$$

Suppose now that $W$ has pure codimension $d$ on $X$, giving us the push-forward $i_*: z^{q-d}(W,*) \to z^q(X,*)$. Since $j^* \circ i_* = 0$, we have the canonical map

$$i^W_*: z^{q-d}(W,*) \to z^q_W(X,*). \tag{2.7}$$

For $X$ quasi-projective, it is shown in [2] that $i^W_*$ is a quasi-isomorphism. In fact, the argument of [2] never uses the fact that $X$ is quasi-projective, and shows that $i^W_*$ is a quasi-isomorphism for $X$ of finite type over $k$; a limit argument shows that $i^W_*$ is a quasi-isomorphism for $X$ essentially of finite type over $k$ as well. For details on this extension of the results of [2], we refer the reader to [15].

Combining $i^W_*$ with the standard cone sequence gives us the distinguished triangle

$$z^{q-d}(W,*) \xrightarrow{i_*} z^q(X,*) \xrightarrow{j^*} z^q(U,*) \to z^{q-d}(W,*)[1].$$

If $X$ and $W$ are smooth, we have the distinguished triangle

$$\mathcal{Z}^{q-d}(W,*)[-2d] \xrightarrow{i_*} \mathcal{Z}^q(X,*) \xrightarrow{j^*} \mathcal{Z}^q(U,*) \to \mathcal{Z}^{q-d}(W,*)[-2d+1],$$

giving the *localization sequence* for motivic cohomology

$$\begin{aligned} \to H^{p-2d}(W,\mathbb{Z}(q-d)) \xrightarrow{i_*} H^p(X,\mathbb{Z}(q)) \xrightarrow{j^*} H^p(U,\mathbb{Z}(q)) \\ \xrightarrow{\partial} H^{p-2d+1}(W,\mathbb{Z}(q-d)) \to \end{aligned} \tag{2.8}$$

The isomorphism $i^W_*: H^{p-2d}(W,\mathbb{Z}(q-d)) \to H^p_W(X,\mathbb{Z}(q))$ resulting from the quasi-isomorphism (2.7) is the so-called *Gysin isomorphism* for motivic cohomology.

2.7. **Relative motivic cohomology.** We describe a relative version of motivic cohomology; for simplicity, we restrict ourselves to the affine case.

Let $X$ be a smooth affine $k$-scheme, essentially of finite type over $k$, and let $Z_1, \ldots, Z_n$ be closed subschemes. For an index $I \subset \{1, \ldots, n\}$, let $Z_I := \cap_{i\in I} Z_i$; we allow the case $I = \emptyset$: $Z_\emptyset = X$. We suppose that all the $Z_I$ are smooth over $k$.

Let $\mathcal{S}$ be a finite set of closed subsets of $X$, containing all the $Z_I$. Form the complex $\mathcal{Z}^q(X; Z_1, \ldots, Z_n, *)_{\mathcal{S}}$ as the total complex of the double complex

$$\mathcal{Z}^q(X,*)_{\mathcal{S}} \to \bigoplus_{i=1}^n \mathcal{Z}^q(Z_i,*)_{\mathcal{S}} \to \ldots \to \bigoplus_{\substack{I\subset\{1,\ldots,n\}\\ |I|=j}} \mathcal{Z}^q(Z_I,*)_{\mathcal{S}} \to \ldots \tag{2.9}$$

with the total degree of $\mathcal{Z}^q(Z_I,p)_{\mathcal{S}}$ being $p+|I|$. Here the differential

$$\bigoplus_{\substack{I\subset\{1,\ldots,n\}\\ |I|=j}} \mathcal{Z}^q(Z_I,*)_{\mathcal{S}} \to \bigoplus_{\substack{I\subset\{1,\ldots,n\}\\ |I|=j+1}} \mathcal{Z}^q(Z_I,*)_{\mathcal{S}}$$

is a signed sum of the pull-back maps

$$\mathcal{Z}^q(Z_I,*)_{\mathcal{S}} \to \mathcal{Z}^q(Z_{I\cup\{i\}},*)_{\mathcal{S}}; \quad i \notin I,$$

induced by the inclusions $Z_{I\cup\{i\}} \subset Z_I$; the sign is $(-1)^l$, where $l$ is the number of $j \in I$ with $j > i$.

The complexes $\mathcal{Z}^q(X;Z_1,\ldots,Z_n,*)_{\mathcal{S}}$ for varying $\mathcal{S}$ are all quasi-isomorphic, since the inclusion (2.3) is a quasi-isomorphism. We will often drop the $\mathcal{S}$ from the notation.

The *relative motivic cohomology* $H^p(X;Z_1,\ldots,Z_n,\mathbb{Z}(q))$ is defined as

$$H^p(X;Z_1,\ldots,Z_n,\mathbb{Z}(q)) := H^p(\mathcal{Z}^q(X;Z_1,\ldots,Z_n,*));$$

the mod $m$ version is defined similarly by

$$H^p(X;Z_1,\ldots,Z_n,\mathbb{Z}/m(q)) := H^p(\mathcal{Z}^q(X;Z_1,\ldots,Z_n,*)\otimes\mathbb{Z}/m).$$

The pull-back maps $\mathcal{Z}^q(Z_I,*)_{\mathcal{S}} \to \mathcal{Z}^q(Z_{I\cup\{n\}},*)_{\mathcal{S}}$ induce the map of complexes

$$i_n^*: \mathcal{Z}^q(X;Z_1,\ldots,Z_{n-1},*)_{\mathcal{S}} \to \mathcal{Z}^q(Z_n;Z_{1,n},\ldots,Z_{n-1,n},*)_{\mathcal{S}}$$

and we have the evident isomorphism

$$\mathcal{Z}^q(X;Z_1,\ldots,Z_n,*)_{\mathcal{S}} \cong \mathrm{cone}(i_n^*)[-1]. \tag{2.10}$$

This gives us the long exact *relativization sequence*

$$\begin{aligned} \to H^{p-1}(Z_n;Z_{1,n},\ldots,Z_{n-1,n},\mathbb{Z}(q)) &\to H^p(X;Z_1,\ldots,Z_n,\mathbb{Z}(q)) \\ \to H^p(X;Z_1,\ldots,Z_{n-1},\mathbb{Z}(q)) &\xrightarrow{i_n^*} H^p(Z_n;Z_{1,n},\ldots,Z_{n-1,n},\mathbb{Z}(q)) \to \end{aligned} \tag{2.11}$$

and similarly for the mod $m$ version.

2.8. **Normal crossing schemes.** We describe an extension of the cycle complexes $\mathcal{Z}^q(X,*)$ to the simplest type of singular schemes, the *normal crossing $k$-schemes*. Taking the component in degree $2q$ gives an extension to normal crossing schemes of the notion of a codimension $q$ cycle on a smooth variety.

Let $Y$ be a scheme, essentially of finite type over $k$, with irreducible components $Y_1,\ldots,Y_n$. For $\emptyset \neq I \subset \{1,\ldots,n\}$, set $Y_I := \cap_{i\in I}Y_i$. If $Y$ happens to be a closed subscheme of an essentially smooth $k$-scheme $X$, we may consider $Y_1,\ldots,Y_n$ as closed subschemes of $X$, and the notation $Y_I$ agrees with that given in §2.7.

**Definition 2.9.** (1) Let $Y$ be essentially of finite type and locally equi-dimensional over $k$, with irreducible components $Y_1,\ldots,Y_n$. For a point $x$ of $Y$, let $N_x$ be the dimension of $Y$ over $k$ at $x$. We call $Y$ a *normal crossing $k$-scheme* if each intersection $Y_I$, $I \neq \emptyset$, is smooth over $k$, and if, for each point $x$ of $Y$, a neighborhood of $x$ in $Y$ is locally isomorphic, in the étale topology, to a union of some coordinate hyperplanes in $\mathbb{A}_k^{N_x+1}$.
(2) Let $Y$ be a normal crossing $k$-scheme, and $Z \subset Y$ a closed subscheme. We call $Z$ a *normal crossing subscheme of* $Y$ if $Z$ is a normal crossing $k$-scheme, and $Z\cap Y_I$ is a normal crossing $k$-scheme for each $I \neq \emptyset$.

For example, a reduced normal crossing divisor $D$ in a smooth $k$-scheme $X$ is a normal crossing scheme if each irreducible component of $D$ is smooth over $k$.

Let $Y$ be a normal crossing $k$-scheme with irreducible components $Y_1,\ldots,Y_n$. Let $\mathcal{S}$ be a finite set of closed subsets of $Y$, containing all the $Y_I$, and let $\mathcal{S}_I \subset \mathcal{S}$ be the set of those closed subsets contained in $Y_I$. We assume that for $S \in \mathcal{S}$, each irreducible component of $S \cap Y_I$ is also in $\mathcal{S}$; we write $\mathcal{Z}^q(Y_I,*)_{\mathcal{S}}$ for $\mathcal{Z}^q(Y_I,*)_{\mathcal{S}_I}$.

Let $\mathcal{Z}^q(Y,*)_{\mathcal{S}}$ be the complex defined by the term-wise exactness of

$$0 \to \mathcal{Z}^q(Y,*)_{\mathcal{S}} \xrightarrow{\iota} \bigoplus_{1\le i\le n} \mathcal{Z}^q(Y_i,*)_{\mathcal{S}} \xrightarrow{\alpha} \bigoplus_{1\le i<j\le n} \mathcal{Z}^q(Y_{i,j},*)_{\mathcal{S}}, \tag{2.12}$$

where $\alpha$ is the sum of the maps

$$\mathcal{Z}^q(Y_i,*)_{\mathcal{S}} \to \mathcal{Z}^q(Y_{i,j},*)_{\mathcal{S}},$$

these in turn being the restriction map for $i<j$, and the negative of the restriction map for $i>j$. If we take $\mathcal{S}$ to be the set of irreducible components of all the $Y_I$, we write $\mathcal{Z}^q(Y,*)$ for $\mathcal{Z}^q(Y,*)_{\mathcal{S}}$. We write $z^q(Y)_{\mathcal{S}}$ for $\mathcal{Z}^q(Y,2q)_{\mathcal{S}}$.

Let $Y$ and $Y'$ be normal crossing $k$-schemes, let $f: Y\to Y'$ be a morphism of $k$-schemes, and take $\mathcal{S}$ and $\mathcal{S}'$ to be the minimal choices, i.e., $\mathcal{S}$ is the set of irreducible components of all the $Y_I$, and similarly for $\mathcal{S}'$. Assume that the restriction of $f$ to each $Y_I$ factors as

$$Y_I \xrightarrow{g} Y'_J \xrightarrow{i_J} Y'$$

for some index $J$ (depending on $I$), with $g$ flat, and $i_J$ the inclusion. For each $i$, choose an index $j(i)$ such that $f(Y_i)\subset Y'_{j(i)}$. Then each of the pull-back maps $f^*_{|Y_i,j(i)}: \mathcal{Z}^q(Y'_{j(i)},*)_{\mathcal{S}'} \to \mathcal{Z}^q(Y_i,*)_{\mathcal{S}}$ is defined. Composing $f^*_{|Y_i,j(i)}$ with the projection

$$\pi_{j(i)}: \bigoplus_j \mathcal{Z}^q(Y'_j,*)_{\mathcal{S}'} \to \mathcal{Z}^q(Y'_{j(i)},*)_{\mathcal{S}'}$$

gives the map

$$f_i^*: \bigoplus_j \mathcal{Z}^q(Y'_j,*)_{\mathcal{S}'} \to \mathcal{Z}^q(Y_i,*)_{\mathcal{S}'}.$$

The sum of $f_i$ composed with the inclusion $\iota$ gives the well-defined pull-back map

$$f^* := \sum_i f_i^*: \mathcal{Z}^q(Y',*) \to \mathcal{Z}^q(Y,*).$$

We note that the map $f^*$ is independent of the choice of the $j(i)$. Indeed, if $f(Y_i)\subset Y'_j$ and $f(Y_i)\subset Y'_{j'}$, then $f(Y_i)\subset Y'_{j,j'}$. Letting $\rho: Y'_{j,j'}\to Y'_j$, $\rho': Y'_{j,j'}\to Y'_{j'}$ be the inclusions, and $\bar f: Y_i \to Y'_{j,j'}$ the map induced by $f$, we have

$$f^*_{|Y_i,j}\circ\pi_j\circ\iota = \bar f^*\circ\rho^*\circ\pi_j\circ\iota = \bar f^*\circ\rho'^*\circ\pi_{j'}\circ\iota = f^*_{|Y_i,j'}\circ\pi_{j'}\circ\iota.$$

Thus, we have extended the assignment $Y\mapsto \mathcal{Z}^q(Y,*)$ to a functor on the category of normal crossing $k$-schemes, where the morphisms are those morphisms of $k$-schemes which admit a factorization as above.

If we enlarge the subset $\mathcal{S}$, we get a pull-back defined for more general maps than the ones considered above, but it is not clear that the inclusion $\mathcal{Z}^q(Y,*)_{\mathcal{S}}\to \mathcal{Z}^q(Y,*)$ is a quasi-isomorphism. The complexes $\mathcal{Z}^q(Y,*)_{\mathcal{S}}$ for normal crossing schemes $Y$ will play only a technical and auxiliary role in our argument, and we will never require that the cohomology $H^*(\mathcal{Z}^q(Y,*)_{\mathcal{S}})$ is independent of the choice of $\mathcal{S}$. If, however, $Y$ is essentially smooth and affine, we may use the quasi-isomorphism (2.3) to give a well-defined pull-back in $\mathbf{D}(\mathbf{Ab})$,

$$f^*: \mathcal{Z}^q(Y,*) \to \mathcal{Z}^q(Y',*), \tag{2.13}$$

as in (2.4).

Let $Z_1, \dots, Z_n$ be closed subschemes of $Y$. Suppose that all the $Z_I$ are normal crossing subschemes of $Y$. If $\mathcal{S}$ is chosen appropriately, we will have the relative cycle complex $\mathcal{Z}^q(Y; Z_1, \dots, Z_n)_{\mathcal{S}}$ defined as in §2.7, (2.9), for the smooth case.

2.10. **Motivic complexes.** The complexes $\mathcal{Z}^q(-,*)$ are functorial for flat maps; in particular, for an essentially smooth $k$-scheme $X$, we may form the complex of sheaves of abelian groups $U \mapsto \mathcal{Z}^q(U,*)$, which we denote by $\Gamma_X(q)$. More generally, let $\mathcal{S}$ be a finite set of closed subsets of $X$. For $U \subset X$ open in $X$, let $\mathcal{S}(U) = \{S \cap U | S \in \mathcal{S}\}$. Let $\Gamma_X(q)_{\mathcal{S}} \subset \Gamma_X(q)$ be the Zariski sheaf $U \mapsto \mathcal{Z}^q(U,*)_{\mathcal{S}(U)}$. If $W$ is a closed subset of $X$ with complement $j : V \to X$, we set

$$\Gamma_X^W(q)_{\mathcal{S}} := \mathrm{cone}(j^* : \Gamma_X(q)_{\mathcal{S}} \to j_*\Gamma_V(q)_{\mathcal{S}(V)})[-1].$$

Since the inclusion $\mathcal{Z}^q(U,*)_{\mathcal{S}(U)} \to \mathcal{Z}^q(U,*)$ is a quasi-isomorphism for $U$ affine and essentially smooth over $k$, [14, Chap. II, Theorem 3.5.14], the inclusion

$$\Gamma_X^W(q)_{\mathcal{S}} \to \Gamma_X^W(q) \tag{2.14}$$

is a quasi-isomorphism for all $X$ essentially smooth over $k$ and all $\mathcal{S}$. Let $f : Y \to X$ be a morphism of essentially smooth $k$-schemes, and let $W' \subset Y$ be a closed subset containing $f^{-1}(W)$. We have the stratification $\mathcal{S}(f)$ (2.1), and $f^* : \Gamma_X^W(q)_{\mathcal{S}} \to \Gamma_Y^{W'}(q)$ is a well-defined map of complexes of sheaves over the map $f$. Thus we have the map $f^* : \Gamma_X^W(q) \to Rf_*\Gamma_Y^{W'}(q)$ in $\mathbf{D}^-(\mathrm{Sh}(X_{\mathrm{Zar}}))$ defined by the diagram

$$\Gamma_X^W(q) \leftarrow \Gamma_X^W(q)_{\mathcal{S}} \xrightarrow{f^*} Rf_*\Gamma_Y^{W'}(q).$$

Let $\mathrm{Sm}/k$ denote the category of essentially smooth $k$-schemes. The pull-back maps $f^*$ extend the assignment $X \mapsto Rp_{X*}\Gamma_X(q)$ ($p_X : X \to \operatorname{Spec} k$ the structure morphism) to a functor

$$Rp_{-*}\Gamma_-(q) : \mathrm{Sm}/k^{\mathrm{op}} \to \mathbf{D}^-(\mathbf{Ab}). \tag{2.15}$$

Similarly, let $\mathrm{PSm}/k$ denote the category of pairs $(X, W)$, with $X$ in $\mathrm{Sm}/k$ and $W$ a closed subset of $X$, where a morphism $f : (X', W') \to (X, W)$ is a map $f : X' \to X$ with $f^{-1}(W) \subset W'$. Then the assignment $(X, W) \mapsto Rp_{X*}\Gamma_X^W(q)$ extends to a functor

$$Rp_{-*}\Gamma_-^-(q) : \mathrm{PSm}/k^{\mathrm{op}} \to \mathbf{D}^-(\mathbf{Ab}). \tag{2.16}$$

Let $p : X \to \operatorname{Spec} k$ be essentially of finite type and locally equi-dimensional over $k$. The localization property for the Zariski presheaf of complexes $U \mapsto \mathcal{Z}^q(U,*)$ described in §2.6 implies that the natural map in $\mathbf{D}^-(\mathbf{Ab})$

$$\mathcal{Z}_W^q(X,*) \to Rp_*\Gamma_X^W(q) \tag{2.17}$$

is an isomorphism. Combining the isomorphism (2.17) and the functor (2.16), we have the functor

$$\begin{aligned} \mathcal{Z}^q_{(-)}(-,*) : \mathrm{PSm}/k^{\mathrm{op}} &\to \mathbf{D}^-(\mathbf{Ab}). \\ (X,W) &\mapsto \mathcal{Z}_W^q(W,*). \end{aligned} \tag{2.18}$$

We may sheafify the diagram (8.1) on $X \times Y$, giving the map in $\mathbf{D}^-(\mathrm{Sh}(X \times_k Y_{\mathrm{Zar}}))$

$$p_1^*\Gamma_X(q) \otimes p_2^*\Gamma_Y(q') \xrightarrow{\cup_{X,Y}} \Gamma_{X\times_k Y}(q+q'). \tag{2.19}$$

Via the isomorphism (2.17), this product agrees with the product (8.2). Thus, by either taking the map on hypercohomology induced by (2.19), or the map on cohomology induced by (8.2), we have the external product

$$\cup_{X,Y} : H^p(X, \mathbb{Z}(q)) \otimes H^{p'}(Y, \mathbb{Z}(q')) \to H^{p+p'}(X \times_k Y, \mathbb{Z}(q+q')).$$

Reducing all the relevant complexes mod $m$ gives the external cup product

$$\cup_{X,Y} : H^p(X, \mathbb{Z}/m(q)) \otimes H^{p'}(Y, \mathbb{Z}/m(q')) \to H^{p+p'}(X \times_k Y, \mathbb{Z}/m(q+q')).$$

For $X$ essentially smooth, composing $\cup_{X,X}$ with $\Delta^*$ defines the natural cup products

$$\begin{aligned} \cup_X &: H^p(X, \mathbb{Z}(q)) \otimes H^{p'}(X, \mathbb{Z}(q')) \to H^{p+p'}(X, \mathbb{Z}(q+q')) \\ \cup_X &: H^p(X, \mathbb{Z}/m(q)) \otimes H^{p'}(X, \mathbb{Z}/m(q')) \to H^{p+p'}(X, \mathbb{Z}/m(q+q')), \end{aligned} \tag{2.20}$$

making $\oplus_{p,q} H^p(X, \mathbb{Z}(q))$ and $\oplus_{p,q} H^p(X, \mathbb{Z}/m(q))$ into bi-graded rings, natural in $X$.

## 3. The construction of cycle maps

We describe the cycle class map from motivic cohomology to étale cohomology, as well as a relative version. The idea is to apply the cycle class map with values in étale cohomology with supports (as defined in [5, Cycle]) to the cosimplicial scheme $X \times \Delta^*$, with support in "codimension $q$". The use of "functor complexes" as described in the next section, is a technical device required for the construction.

3.1. **Functor complexes.** Let $\mathrm{Sch}_k$ denote the category of $k$-schemes, essentially of finite type over $k$, and let $\mathcal{C}$ be a subcategory of $\mathrm{Sch}_k$.

Suppose we have a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab})$. For a cosimplicial scheme in $\mathcal{C}$, $X^* : \Delta \to \mathcal{C}$, we have the simplicial object $n \mapsto F(X^n)$ of $\mathbf{C}(\mathbf{Ab})$; we let $F(X^*)$ denote the total complex of the associated double complex:

$$F(X^*)^n := \bigoplus_{p+q=n} F^p(X^{-q}). \tag{3.1}$$

Sending $X^*$ to $F(X^*)$ defines a contravariant functor from the category of cosimplicial objects of $\mathcal{C}$ to $\mathbf{C}(\mathbf{Ab})$, natural in $F$. We have the weakly convergent spectral sequence

$$E_1^{p,q} = H^q(F(X^{-p})) \Longrightarrow H^{p+q}(F(X^*)). \tag{3.2}$$

In particular, if $\phi : F \to G$ is a natural transformation with $\phi(i)$ a quasi-isomorphism for each $i \in \mathcal{C}$ (for short, a quasi-isomorphism), then $\phi$ induces an isomorphism $H^n(F(X^*)) \to H^n(G(X^*))$. Also, if $X^*$ is a constant cosimplicial scheme with value $X$, then the augmentation induces a quasi-isomorphism $F(X) \to F(X^*)$.

Let $j : U^* \to X^*$ be an open cosimplicial subscheme in $\mathcal{C}$, and let $W^*$ denote the collection of closed complements $W^n = X^n \setminus U^n$; we call $W^*$ a *closed subset* of $X^*$. Suppose in addition we have closed cosimplicial subschemes $Y_1^*, \ldots, Y_n^*$ of $X^*$ (all relevant morphisms being in $\mathcal{C}$). Let $F(X^*; Y_1^*, \ldots, Y_n^*)$ be the total complex of the double complex

$$F(X^*) \to \bigoplus_{i=1}^{n} F(Y_i^*) \to \ldots \to \bigoplus_{|I|=j} F(Y_I^*) \to \ldots$$

with $Y_I^*$ the cosimplicial closed subscheme $n \mapsto Y_I^n$, and the maps in the above complex defined as in §2.7. We set

$$
\begin{aligned}
(3.3)\quad & F_{W^*}(X^*; Y_1^*, \dots, Y_n^*) := \\
& \quad \mathrm{cone}(j^* : F(X^*; Y_1^*, \dots, Y_n^*) \to F(U^*; U^* \cap Y_1^*, \dots, U^* \cap Y_n^*))[-1].
\end{aligned}
$$

As above, a quasi-isomorphism $F \to G$ induces an isomorphism

$$H^*(F_{W^*}(X^*; Y_1^*, \dots, Y_n^*)) \to H^*(G_{W^*}(X^*; Y_1^*, \dots, Y_n^*)).$$

*Remark* 3.2. A functor $F : \mathrm{Sch}_k^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab})$ is called *homotopy invariant* if the map

$$F(X) \to F(X \times \mathbb{A}^1)$$

induced by the projection is a quasi-isomorphism for all $X$. If $F$ is homotopy invariant, then, for $X^* = X \times \Delta^*$, the spectral sequence (3.2) has $E_2^{p,q} = 0$ for $p \neq 0$ and $E_2^{0,q} = H^q(F(X))$. One sees thereby that the augmentation $F(X) \to F(X \times \Delta^*)$ is a quasi-isomorphism.

*Examples* 3.3. (1) Let $m$ be an integer with $1/m \in k$. For a $k$-scheme $X$, we let $G^*(X, \mu_m^{\otimes q})$ denote the complex of abelian groups gotten by taking the global sections of the Godement resolution of the étale sheaf $\mu_m^{\otimes q}$ on $X$; for example, $G^0(X, \mu_m^{\otimes q})$ is the product over all geometric points $x$ of $X$ of $H^0_{\text{ét}}(k(x), \mu_m^{\otimes q})$. Then sending $X$ to $G^*(X, \mu_m^{\otimes q})$ defines the functor

$$G^*(-, \mu_m^{\otimes q}) : \mathrm{Sch}_k^{\mathrm{op}} \to \mathbf{C}^+(\mathbf{Ab}).$$

Since the Godement resolution of $\mu_m^{\otimes q}$ is a resolution by flasque sheaves, the natural map $G^*(X, \mu_m^{\otimes q}) \to R\Gamma(X_{\text{ét}}, \mu_m^{\otimes q})$ is an isomorphism in $\mathbf{D}(\mathbf{Ab})$; in particular, the cohomology $H^p(G^*(X, \mu_m^{\otimes q}))$ is canonically isomorphic to the étale cohomology $H^p_{\text{ét}}(X, \mu_m^{\otimes q})$. The above discussion thus defines étale cohomology, with coefficients $\mu_m^{\otimes q}$, for a cosimplicial scheme $X^*$, as well as the relative version, with support in a closed subset.

(2) Let $X$ be an essentially smooth $k$-scheme, $W \subset X$ a closed subscheme with complement $j : U \to X$. Let $\mathcal{C}$ be the subcategory of $\mathrm{Sch}_X$ with objects the schemes $X \times \Delta^m$, $U \times \Delta^m$, $m \geq 0$. The morphisms in $\mathcal{C}$ are compositions of maps of the form $j \times \mathrm{id}$ and $\mathrm{id} \times g$, with $g$ the map induced by a map $[m] \to [l]$ in $\Delta$.

Let $\mathcal{S}$ be a finite set of closed subsets of $X$, giving the set $\mathcal{S}(U) = \{U \cap S | S \in \mathcal{S}\}$. We have the functor $z^q_{(q)}(-)_{\mathcal{S}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ defined by

$$z^q_{(q)}(X \times \Delta^m)_{\mathcal{S}} := z^q(X, m)_{\mathcal{S}};\ z^q_{(q)}(U \times \Delta^m)_{\mathcal{S}} := z^q(U, m)_{\mathcal{S}(U)},$$

with $z^q_{(q)}(f)_{\mathcal{S}} = f^*$ for each morphism $f$ in $\mathcal{C}$. The cycle complex with support in $W$, $\mathcal{Z}^q_W(X, *)_{\mathcal{S}}$, of (2.6) is by definition the complex $z^q_{(q),W}(X \times \Delta^*)_{\mathcal{S}}[-2q]$.

(3) Let $X$ be an essentially smooth $k$-scheme, $Z_1, \dots, Z_n$ closed subschemes with $Z_I$ smooth for all $I$, and $\mathcal{S}$ a set of closed subsets of $X$, containing all the $Z_I$. Let $\mathcal{C}$ be the subcategory of $\mathrm{Sch}_X$ with objects the schemes $Z_I \times \Delta^m$, $m \geq 0$. The morphisms $Z_I \times \Delta^m \to Z_J \times \Delta^l$ are those of the form $i \times g$, with $i$ an inclusion, and $g$ the map induced by a map $[m] \to [l]$ in $\Delta$.

Via the pull-back maps described in §2.3, we have the functor $z^q_{(q)}(-)_{\mathcal{S}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Ab}$ defined by

$$z^q_{(q)}(Z_I \times \Delta^m)_{\mathcal{S}} := z^q(Z_I, m)_{\mathcal{S}};$$

the relative cycle complex, $\mathcal{Z}^q(X;Z_1,\ldots,Z_n,*)_{\mathcal{S}}$, of (2.9) is by definition the complex $z^q_{(q)}(X\times\Delta^*;Z_1\times\Delta^*,\ldots,Z_n\times\Delta^*)_{\mathcal{S}}[-2q]$.

(4) For a normal crossing $k$-scheme $Y$, with irreducible components $Y_1,\ldots,Y_n$, let $\mathcal{S}$ be a set of closed subsets containing all the $Y_I$. If we take $\mathcal{C}$ to be the subcategory of $\mathrm{Sch}_Y$ formed by the cosimplicial scheme $Y\times\Delta^*$, we form as in (2) the functor $z^q_{(q)}(-)_{\mathcal{S}}:\mathcal{C}^{\mathrm{op}}\to\mathbf{Ab}$, giving the identity

$$z^q_{(q)}(Y\times\Delta^*)_{\mathcal{S}}[-2q]=\mathcal{Z}^q(Y,*)_{\mathcal{S}},$$

where $\mathcal{Z}^q(Y,*)_{\mathcal{S}}$ is the cycle complex of (2.12). Similarly, if $Z_1,\ldots,Z_n$ are closed subschemes of $Y$ such that each $Z_I$ is a normal crossing subscheme of $Y$, let $\mathcal{C}$ be the category with the objects and morphisms defined as in (3) for the smooth case, and assume that $\mathcal{S}$ contains all the intersections $Y_J\cap Z_I$. We have the functor $z^q_{\mathcal{S}}:\mathcal{C}^{\mathrm{op}}\to\mathbf{Ab}$ as in (3), and the identity

$$z^q_{\mathcal{S}}(Y\times\Delta^*;Z_1\times\Delta^*,\ldots,Z_n\times\Delta^*)[-2q]=\mathcal{Z}^q(Y;Z_1,\ldots,Z_n,*)_{\mathcal{S}},$$

where $\mathcal{Z}^q(Y;Z_1,\ldots,Z_n,*)_{\mathcal{S}}$ is the relative version of (2.12) defined at the end of §2.8.

3.4. **Cohomology with support.** Let $k$ be a field, let $X$ be an essentially smooth $k$-scheme, and $W$ a closed subset of $X$ of codimension $\geq q$. We denote the group of codimension $q$ cycles on $X$ with support in $W$ by $z^q_W(X)$. Let $m$ be an integer prime to the characteristic of $k$. We recall from [5, Cycle] the construction of the natural isomorphism

$$\mathrm{cyc}^q_W:z^q_W(X)\otimes\mathbb{Z}/m\to H^{2q}_W(X,\mu_m^{\otimes q}),$$

where $H^p_W(X,\mu_m^{\otimes q})$ is the étale cohomology with support in $W$.

Suppose at first that $X$ is of finite type over $k$ and $W$ is smooth of codimension $q$. By the purity theorem for étale cohomology [19, Chap. VI, Theorem 5.1],

$$H^p_W(X,\mu_m^{\otimes q})\cong H^{2q-p}(W,\mathbb{Z}/m);$$

in particular, $H^p_W(X,\mu_m^{\otimes q})=0$ for $p<2q$. The group $H^0(W,\mathbb{Z}/m)$ is just the free $\mathbb{Z}/m$-module on the irreducible components of $W$, which in turn is isomorphic to the $\mathbb{Z}/m$-module $z^q_W(X)\otimes\mathbb{Z}/m$. The construction of [5, Cycles, Chapter 2] gives a canonical choice $\mathrm{cyc}^q_W$ of the isomorphism $z^q_W(X)\otimes\mathbb{Z}/m\cong H^{2q}_W(X,\mu_m^{\otimes q})$.

Now suppose that $k$ is perfect and $W$ is an arbitrary closed subset of codimension $\geq q$, with singular locus $W_{sing}$. By noetherian induction, the localization sequence for étale cohomology with support [4] gives the isomorphism

$$H^{2q}_W(X,\mu_m^{\otimes q})\cong H^{2q}_{W\setminus W_{sing}}(X\setminus W_{sing},\mu_m^{\otimes q}),$$

which gives the isomorphism $\mathrm{cyc}^q_W$ for general $W$ in $X$. Similarly, $H^p_W(X,\mu_m^{\otimes q})=0$ for $p<2q$.

The isomorphisms $\mathrm{cyc}^q_W$ are compatible with flat pull-back (see e.g. Lemma 3.5 below), hence extend to $X$ essentially smooth over $k$. We now extend the isomorphisms $\mathrm{cyc}^q_W$ to the case of a not necessarily perfect base field $k$.

Let $k_0$ be the prime field in $k$; $k_0$ is thus a perfect field. Let $X$ be an essentially smooth $k$-scheme, and $W$ a codimension $\geq q$ closed subset. Then the pair $(X,W)$ is a filtered projective limit of pairs $(X_\alpha,W_\alpha)$, where each $X_\alpha$ is a smooth $k_\alpha$-scheme for some subfield $k_\alpha$ of $k$ which is finitely generated over $k_0$, $W_\alpha$ is a closed codimension $\geq q$ closed subset of $X_\alpha$, and the transition maps in the projective system are flat and affine. Since motivic cohomology and étale cohomology with supports transform filtered projective limits with flat affine transition maps to inductive limits, and since we have compatibility of $\mathrm{cyc}^q_W$ with respect to flat pull-back when defined, it suffices to extend the isomorphisms $\mathrm{cyc}^q_W$ to the case in which $k$ is finitely generated over $k_0$. In this case, each essentially smooth $X$ over $k$ is the localization of a smooth $k_0$-scheme, so the isomorphisms $\mathrm{cyc}^q_W$ are defined. This completes the construction of the isomorphisms $\mathrm{cyc}^q_W$ for general base fields; the same argument shows that $H^p_W(X, \mu_m^{\otimes q}) = 0$ for $p < 2q$.

**Lemma 3.5.** *(1) The maps $\mathrm{cyc}^q_W$ are natural with respect to pull-back: Let $f : Y \to X$ be a morphism of essentially smooth $k$-schemes, let $W \subset X$ be a codimension $\geq q$ closed subset of $X$, and let $T$ be a closed subset of $Y$ such that $f^{-1}(W) \subset T$ and $T$ has codimension $\geq q$ on $Y$. Then*

$$f^* \circ \mathrm{cyc}^q_W = \mathrm{cyc}^q_T \circ f^*.$$

*(2) The maps $\mathrm{cyc}^q_W$ are natural with respect to proper push-forward, with the appropriate shift in the codimension: Let $f : Y \to X$ be a proper morphism of essentially smooth $k$-schemes, of relative dimension $d$, let $W \subset X$ and $T \subset Y$ be closed subsets with $f(T) \subset W$. Suppose that $T$ has codimension $\geq q + d$ on $Y$ and $W$ has codimension $\geq q$ on $X$. Then*

$$\mathrm{cyc}^q_W \circ f_* = f_* \circ \mathrm{cyc}^{q+d}_T.$$

*(3) The maps $\mathrm{cyc}^q_W$ are compatible with external products: Let $X$ and $Y$ be essentially smooth $k$-schemes, $W \subset X$ and $W' \subset Y$ closed subsets of codimension $\geq q$ and $\geq q'$, respectively, and $Z \in z^q_W(X)$, $Z' \in z^{q'}_{W'}(Y)$. Then*

$$p_1^* \mathrm{cyc}^q_W(Z) \cup_{X,Y} p_2^* \mathrm{cyc}^{q'}_{W'}(Z') = \mathrm{cyc}^{q+q'}_{W \times_k W'}(Z \times Z').$$

*Proof.* These properties all follow from [5, Cycles, Chapter 2]. □

We now extend the definition of the cycle map to normal crossing schemes. Let $Y$ be a normal crossing scheme with irreducible components $Y_1, \ldots, Y_n$. Let $W$ be a closed subset of $Y$ with complement $j : U \to Y$. For each index $J$, set $W_J := Y_J \cap W$.

**Lemma 3.6.** *Suppose that $\mathrm{codim}_{Y_J}(W_J) \geq q$ for all indices $J$. Then*

*(1) $H^p_W(Y, \mu_m^{\otimes q}) = 0$ for $p < 2q$.*

*(2) The sequence*

$$0 \to H^{2q}_W(Y, \mu_m^{\otimes q}) \to \bigoplus_{1 \leq j \leq r} H^{2q}_{W_j}(Y_j, \mu_m^{\otimes q}) \xrightarrow{\alpha} \bigoplus_{1 \leq j < l \leq r} H^{2q}_{W_{j,l}}(Y_{j,l}, \mu_m^{\otimes q})$$

*is exact, where $\alpha$ is defined as in §2.7*

*Proof.* For each $i = 0, \ldots, n$, let $Y^{(i)}$ be the disjoint union of the $Y_I$ with $|I| = i$, and let $p^{(i)} : Y^{(i)} \to Y$ be the evident morphism. This gives us the standard resolution of the sheaf $\mu_m^{\otimes q}$ on $Y$:

$$0 \to \mu_m^{\otimes q} \to p^{(1)}_* \mu_m^{\otimes q} \to \ldots \to p^{(i)}_* \mu_m^{\otimes q} \to \ldots$$

(to see that this is indeed a resolution, reduce to the case of the union of $n$ hyperplanes in $\mathbb{A}^{N+1}$, and use induction on $n$ and $N$). The corresponding hypercohomology spectral sequence gives the Mayer-Vietoris spectral sequence

$$E_1^{a,b} = \bigoplus_{|J|=a+1} H^b_{W_J}(Y_J, \mu_m^{\otimes q}) \Longrightarrow H^{a+b}_W(Y, \mu_m^{\otimes q}).$$

By purity, we have $E_1^{a,b} = 0$ for $b < 2q$; since $E_1^{a,b} = 0$ for $a < 0$, $E_1^{a,b} = 0$ for $a + b < 2q$, whence (1). Similarly, the only non-zero terms with $a + b \leq 2q + 1$ and $b \leq 2q$ are $E_1^{0,2q}$, $E_1^{1,2q}$. Thus

$$E_2^{0,2q} = E_\infty^{0,2q} = H^{2q}_W(Y, \mu_m^{\otimes q}).$$

The map $\alpha$ is $d_1^{0,2q}$, proving (2). □

Recall from §2.8 the cycle complex $\mathcal{Z}^q(Y, *)_\mathcal{S}$, and the group of cycles on $Y$, $z^q(Y)_\mathcal{S} := \mathcal{Z}^q(Y, 2q)_\mathcal{S}$. Explicitly, $z^q(Y)_\mathcal{S}$ is the kernel of the map

$$\bigoplus_{1\leq i\leq n} z^q(Y_i)_\mathcal{S} \xrightarrow{\alpha} \bigoplus_{1\leq i<j\leq n} z^q(Y_{i,j})_\mathcal{S}$$

with $\alpha$ as in §2.8. If $W$ is a closed subset of $Y$, we let $z^q_W(Y)_\mathcal{S}$ be the subgroup of $z^q(Y)_\mathcal{S}$ consisting of cycles with support in $W$. We omit the $\mathcal{S}$ from the notation if $\mathcal{S}$ is the minimal choice, i.e., the set of all the $Y_I$.

We have the exact sequence

$$0 \to z^q_W(Y)_\mathcal{S} \to \bigoplus_{1\leq i\leq n} z^q_{W_i}(Y_i)_\mathcal{S} \to \bigoplus_{1\leq i<j\leq n} z^q_{W_{i,j}}(Y_{i,j})_\mathcal{S}, \tag{3.4}$$

with the maps defined as above. If we suppose that $W$ satisfies the conditions of Lemma 3.6, there therefore is a unique homomorphism

$$\mathrm{cyc}^q_W : z^q_W(Y) \to H^{2q}_W(Y, \mu_m^{\otimes q})$$

which makes the diagram

$$\begin{tikzcd}
z^q_W(Y) \arrow[r] \arrow[d, "\mathrm{cyc}^q_W"'] & \bigoplus_{1\leq i\leq n} z^q_{W_i}(Y_i) \arrow[d, "\oplus\,\mathrm{cyc}^q_{W_i}"] \\
H^{2q}_W(Y, \mu_m^{\otimes q}) \arrow[r] & \bigoplus_{1\leq i\leq n} H^{2q}_{W_i}(Y_i, \mu_m^{\otimes q})
\end{tikzcd}$$

commute. Using Lemma 3.5(1) we see that the $\mathrm{cyc}^q_W$ are functorial for maps of normal crossing schemes $f : (Y', W') \to (Y, W)$ (satisfying the factorization conditions of §2.8) in case $W' \supset f^{-1}(W)$ and $W'$ satisfies the conditions of Lemma 3.6.

3.7. **The cycle class map.** We first construct the cycle class map as a map

$$\mathrm{cl}^q : \mathcal{Z}(X, *)_\mathcal{S} \otimes^L \mathbb{Z}/m \to G^*(X, \mu_m^{\otimes q})$$

in $\mathbf{D}(\mathbf{Ab})$, where $X$ is an essentially smooth $k$-scheme, $\mathcal{S}$ is a finite set of closed subsets of $X$, and $G^*(X, \mu_m^{\otimes q})$ is the Godement resolution of Example 3.3(1). We then extend this to a version for the relative cycle complexes for a normal crossing $k$-scheme. In the next section, we give a sheafified version of $\mathrm{cl}^q$, which, among other things, enables us to show that $\mathrm{cl}^q$ is natural.

For a category $\mathcal{C}$, we let $\mathrm{Func}(\mathcal{C}, \mathbf{Ab})$ denote the abelian category of functors $\mathcal{C} \to \mathbf{Ab}$. We may also form the category of complexes $\mathbf{C}^*(\mathrm{Func}(\mathcal{C}, \mathbf{Ab}))$ ($*$ a boundedness condition), and the derived category $\mathbf{D}^*(\mathrm{Func}(\mathcal{C}, \mathbf{Ab}))$.

Let $X$ be a reduced $k$-scheme, and $\mathcal{S}$ a finite set of closed subsets of $X$, with $X \in \mathcal{S}$. Let $\mathcal{S}_0$ be a subset of $\mathcal{S}$ consisting of (reduced) schemes locally equidimensional over $k$. Let $\mathcal{C}(\mathcal{S}_0)$ be the subcategory of $\mathrm{Sch}_X$ with objects the schemes of the form $S \times \Delta^m$, with $S \in \mathcal{S}_0$, where a morphism $S \times \Delta^m \to T \times \Delta^l$ is a map of the form $i \times g$, with $i : S \to T$ an inclusion, and with $g : \Delta^m \to \Delta^l$ the affine-linear map induced from some $\bar{g} : [m] \to [l]$ in $\Delta$.

For example, when we use $\mathcal{C}(\mathcal{S}_0)$ to define the cycle class map for the relative complex $\mathcal{Z}^q(X; Z_1, \ldots, Z_n, *)_{\mathcal{S}}$, we take $\mathcal{S}_0$ to be the set of irreducible components of the $Z_I$. The set $\mathcal{S}$ is needed to show the naturality of the cycle class map.

For $S \in \mathcal{S}_0$, let $(S \times \Delta^m)^{(q)}_{\mathcal{S}}$ be the set of closed subsets $W$ of $S \times \Delta^m$ such that, for each $T \in \mathcal{S}$, $T \subset S$, and each face $F$ of $\Delta^m$, the intersection $W \cap (T \times F)$ has codimension $\geq q$ on $T \times F$. We set

$$G^*_{(q)}(S \times \Delta^m, \mu_m^{\otimes q})_{\mathcal{S}} := \varinjlim_{W \in (S \times \Delta^m)^{(q)}_{\mathcal{S}}} G^*_W(S \times \Delta^m, \mu_m^{\otimes q}).$$

Sending $S \times \Delta^m$ to $G^*_{(q)}(S \times \Delta^m, \mu_m^{\otimes q})_{\mathcal{S}}$ defines the functor

$$G^*_{(q)}(-, \mu_m^{\otimes q})_{\mathcal{S}} : \mathcal{C}(\mathcal{S}_0)^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab}).$$

Assume that each $S \in \mathcal{S}_0$ is a normal crossing scheme. Let $z^q_{(q)}(-)_{\mathcal{S}} : \mathcal{C}(\mathcal{S}_0)^{\mathrm{op}} \to \mathbf{Ab}$ be the functor

$$z^q_{(q)}(S \times \Delta^m)_{\mathcal{S}} = \varinjlim_{W \in (S \times \Delta^m)^{(q)}_{\mathcal{S}}} z^q_W(S \times \Delta^m),$$
$$z^q_{(q)}(f)_{\mathcal{S}} = f^*;$$

the conditions on $S$ and $W$ imply that the relevant cycle intersection products are all defined, so that $z^q_{(q)}(-)_{\mathcal{S}}$ is well-defined. In addition, for each $S \in \mathcal{S}_0$, the restriction of $z^q_{(q)}(-)_{\mathcal{S}}$ to the subcategory $\mathcal{C}(\{S\})^{\mathrm{op}}$ of $\mathcal{C}(\mathcal{S}_0)^{\mathrm{op}}$ agrees with the functor $z^q_{(q)}(-)_{\mathcal{S}}$ defined in Example 3.3(4), i.e.,

$$z^q_{(q)}(S \times \Delta^m)_{\mathcal{S}} = z^q(S, m)_{\mathcal{S}}. \tag{3.5}$$

As noted in the Examples 3.3(2)-(4), we have

$$z^q_{(q)}(S \times \Delta^*)_{\mathcal{S}}[-2q] = \mathcal{Z}^q(S, *)_{\mathcal{S}} \tag{3.6}$$

for each $S \in \mathcal{S}_0$.

**Lemma 3.8.** *Suppose that*

1. *Each $S \in \mathcal{S}_0$ is a normal crossing scheme.*
2. *If $S \in \mathcal{S}_0$ has irreducible components $S_1, \ldots, S_m$, then each $S_J$ is in $\mathcal{S}_0$.*

*Then there is a unique isomorphism in* $\mathrm{Func}(\mathcal{C}(\mathcal{S}_0)^{\mathrm{op}}, \mathbf{Ab})$,

$$\mathrm{cyc}^q_{(q)} : z^q_{(q)}(-)_{\mathcal{S}} \otimes \mathbb{Z}/m \to H^{2q}(G^*_{(q)}(-, \mu_m^{\otimes q})_{\mathcal{S}}),$$

*such that, for $S \in \mathcal{S}_0$ smooth, and $W \in (S \times \Delta^m)^{(q)}_{\mathcal{S}}$, the diagram*

$$\begin{tikzcd} z^q_W(S\times\Delta^m)_{\mathcal{S}}\otimes\mathbb{Z}/m \arrow[r, "\mathrm{cyc}^q_W"] \arrow[d] & H^{2q}(G^*_W(S\times\Delta^m,\mu_m^{\otimes q})_{\mathcal{S}}) \arrow[d] \\ z^q_{(q)}(S\times\Delta^m)_{\mathcal{S}}\otimes\mathbb{Z}/m \arrow[r, "\mathrm{cyc}^q_{(q)}"'] & H^{2q}(G^*_{(q)}(S\times\Delta^m,\mu_m^{\otimes q})_{\mathcal{S}}) \end{tikzcd}$$

*commutes. In addition, we have*

$$H^p(G^*_{(q)}(-,\mu_m^{\otimes q})_{\mathcal{S}}) = 0$$

*for $p < 2q$.*

*Proof.* Since $z^q_{(q)}(S\times\Delta^m)_{\mathcal{S}}$ is the direct limit of the $z^q_W(S\times\Delta^m)$, the uniqueness of $\mathrm{cyc}^q_{(q)}$ is clear. The existence follows directly from Lemma 3.5, Lemma 3.6 and the comments following Lemma 3.6; the cohomology vanishing for $p < 2q$ follows from Lemma 3.6. □

We apply the lemma to construct the cycle class map from motivic cohomology to étale cohomology. We first illustrate the most basic case, that of an essentially smooth $k$-scheme $X$. Take $\mathcal{S}_0 = \{X\}$. We have the diagram

(3.7)
$$\begin{tikzcd} z^q_{(q)}(-)_{\mathcal{S}}[-2q]\otimes\mathbb{Z}/m \arrow[r, "{\mathrm{cyc}^q_{(q)}[-2q]}"] & H^{2q}(G^*_{(q)}(-,\mu_m^{\otimes q})_{\mathcal{S}})[-2q] \\ G^*(-,\mu_m^{\otimes q}) & \tau_{\le 2q}G^*_{(q)}(-,\mu_m^{\otimes q})_{\mathcal{S}}, \arrow[l, "\beta"] \arrow[u, "\alpha"'] \end{tikzcd}$$

in $\mathbf{C}(\mathrm{Func}(\mathcal{C}(\{X\})^{\mathrm{op}},\mathbf{Ab}))$, where $\alpha$ is the canonical map, and $\beta$ is the composition of the canonical inclusion $\tau_{\le 2q}G^*_{(q)}(-,\mu_m^{\otimes q})_{\mathcal{S}} \to G^*_{(q)}(-,\mu_m^{\otimes q})_{\mathcal{S}}$, followed by the "forget the support" map $G^*_{(q)}(-,\mu_m^{\otimes q})_{\mathcal{S}} \to G^*(-,\mu_m^{\otimes q})$. By Lemma 3.8, the map $\alpha$ is a quasi-isomorphism. Thus, the diagram (3.7) defines a map

(3.8)
$$z^q_{(q)}(-)_{\mathcal{S}}[-2q]\otimes\mathbb{Z}/m \to G^*(-,\mu_m^{\otimes q})$$

in $\mathbf{D}(\mathrm{Func}(\mathcal{C}(\{X\})^{\mathrm{op}},\mathbf{Ab}))$.

We now apply the map (3.8) to the cosimplicial scheme $X\times\Delta^*$, in the sense of §3.1. As noted in (3.6), we have

$$z^q_{(q)}(X\times\Delta^*)_{\mathcal{S}}[-2q] = \mathcal{Z}^q(X,*)_{\mathcal{S}}.$$

By Remark 3.2, and the homotopy property for étale cohomology, the augmentation to $X$ induces a quasi-isomorphism $G^*(X,\mu_m^{\otimes q}) \to G^*(X\times\Delta^*,\mu_m^{\otimes q})$. Thus, we have constructed the map in $\mathbf{D}(\mathbf{Ab})$

(3.9)
$$\mathrm{cl}^q : \mathcal{Z}^q(X,*)_{\mathcal{S}}\otimes\mathbb{Z}/m \to G^*(X,\mu_m^{\otimes q}).$$

From the construction of $\mathrm{cl}^q$, it is obvious that, for $\mathcal{S}' \subset \mathcal{S}$, the diagram

(3.10)
$$\begin{tikzcd} \mathcal{Z}^q(X,*)_{\mathcal{S}}\otimes\mathbb{Z}/m \arrow[r, "i"] \arrow[dr, "\mathrm{cl}^q"'] & \mathcal{Z}^q(X,*)_{\mathcal{S}'}\otimes\mathbb{Z}/m \arrow[d, "\mathrm{cl}^q"] \\ & G^*(X,\mu_m^{\otimes q}) \end{tikzcd}$$

commutes, where $i$ is the evident inclusion of complexes. Taking $\mathcal{S} = \{X\}$ and taking cohomology, we have the *mod m cycle class map*

$$\mathrm{cl}^q : H^*(X, \mathbb{Z}/m(q)) \to H^*_{\text{ét}}(X, \mu_m^{\otimes q}). \tag{3.11}$$

*Remark* 3.9. Assuming $X$ affine, we may enlarge the set $\mathcal{S}$ without changing the cohomology of $\mathcal{Z}^q(X,*)_{\mathcal{S}}$, since the map (2.3) is a quasi-isomorphism. From this, one easily sees that $\mathrm{cl}^q$ is natural for affine $X$. We will discuss the naturality in general in the next section.

We may modify the construction by restricting the support throughout to a fixed closed subset $W$ of $X$. This gives the version with support in $W$,

$$\mathrm{cl}^q : \mathcal{Z}^q_W(X,*)_{\mathcal{S}} \otimes \mathbb{Z}/m \to G^*_W(X, \mu_m^{\otimes q}), \tag{3.12}$$

and on taking cohomology

$$\mathrm{cl}^q : H^*_W(X, \mathbb{Z}/m(q)) \to H^*_W(X, \mu_m^{\otimes q}). \tag{3.13}$$

A similar construction gives an extension of the cycle class map to the setting of relative cohomology of normal crossing schemes (including the case of relative cohomology for smooth schemes). Indeed, let $Y$ be a normal crossing scheme with irreducible components $Y_1, \ldots, Y_m$, and let $Z_1, \ldots, Z_n$ closed normal crossing subschemes of $Y$ such that each $Z_I$ is a closed normal crossing subscheme of $Y$. Let $\mathcal{S}_0$ be the set of irreducible components of all the $Z_I \cap Y_J$, and let $\mathcal{S} \supset \mathcal{S}_0$ be a finite set of closed subsets of $Y$. One easily sees that $\mathcal{S}_0$ satisfies the conditions of Lemma 3.8. We have the diagram (3.7) in $\mathbf{C}(\mathrm{Func}(\mathcal{C}(\mathcal{S}_0)^{\mathrm{op}}, \mathbf{Ab}))$, with $\alpha$ a quasi-isomorphism. We thus get a similar diagram in $\mathbf{C}(\mathbf{Ab})$ upon evaluation (in the sense of §3.1) at $(Y \times \Delta^*; Z_1 \times \Delta^*, \ldots, Z_n \times \Delta^*)$. The homotopy property for étale cohomology gives, as above, the quasi-isomorphism

$$G^*(Y; Z_1, \ldots, Z_n, \mu_m^{\otimes q}) \to G^*(Y \times \Delta^*; Z_1 \times \Delta^*, \ldots, Z_n \times \Delta^*, \mu_m^{\otimes q}).$$

Using the identity (3.6), we have

$$z^q_{(q)}(Y \times \Delta^*; Z_1 \times \Delta^*, \ldots, Z_n \times \Delta^*)_{\mathcal{S}}[-2q] = \mathcal{Z}^q(Y; Z_1, \ldots, Z_n, *)_{\mathcal{S}},$$

giving the map $\mathrm{cl}^q : \mathcal{Z}^q(Y; Z_1, \ldots, Z_n)_{\mathcal{S}} \to G^*(Y; Z_1, \ldots, Z_n, \mu_m^{\otimes q})$ in $\mathbf{D}(\mathbf{Ab})$. If we take $\mathcal{S}$ to be the minimal choice, or if we assume that $Y$ and all the $Z_I$ are smooth and affine, taking cohomology gives us the cycle class map

$$\mathrm{cl}^q : H^*(Y; Z_1, \ldots, Z_n, \mathbb{Z}/m(q)) \to H^*_{\text{ét}}(Y; Z_1, \ldots, Z_n, \mu_m^{\otimes q}). \tag{3.14}$$

3.10. **Cycle classes for motivic complexes.** We give a refinement of the cohomological cycle class map described above to a natural map in the derived category $\mathbf{D}^-(\mathrm{Sh}(X_{\mathrm{Zar}}))$ of Zariski sheaves on an essentially smooth $k$-scheme $X$, as well as a variant for complexes with support. As a tool, we construct a category $\mathcal{C}/k$ out of pairs consisting of an essentially smooth $k$-scheme $X$ and a finite set of closed subsets of $X$, so that the operation of taking the codimension $q$ cycles on $(X, \mathcal{S})$ becomes functorial for pull-back.

As in §3.7, let $X$ be an essentially smooth $k$-scheme, and let $\mathcal{S}$ be a finite set of closed subsets of $X$, with $X \in \mathcal{S}$. For $S \in \mathcal{S}$, we let $S*$ be the union of the $S' \in \mathcal{S}$ which contain no generic point of $S$, and let $S^0$ denote the open subset $S \setminus S*$ of $S$.

Let $\mathcal{C}/k$ be the following category: Objects are pairs $(T, \mathcal{S}(T))$ consisting of an essentially smooth $k$-scheme $T$, and a finite set of closed subsets of $T$, $\mathcal{S}(T)$. A morphism $(T,\mathcal{S}(T))\to(T',\mathcal{S}(T'))$ in $\mathcal{C}/k$ is a morphism of $k$-schemes, $g:T'\to T$, such that such that the following condition holds:

(3.15)
For each $S'\in\mathcal{S}(T')$, there is an $S\in\mathcal{S}(T)$ such that $S'^0\subset f^{-1}(S^0)$ and $f: S'^0\to S^0$ is equi-dimensional.

Recall the category $\mathcal{C}(\mathcal{S}_0)$ defined in §3.7; in particular, for an essentially smooth $k$-scheme $T$, we have the category $\mathcal{C}(\{T\})$. Let $\mathcal{C}(\{*\})$ be the category with

$$\mathrm{Obj}\ \mathcal{C}(\{*\}) := \coprod_{(T,\mathcal{S}(T))\in\mathcal{C}/k} \mathrm{Obj}\ \mathcal{C}(\{T\}).$$

For $T'\times\Delta^m$ in the component $(T',\mathcal{S}(T'))$ and $T\times\Delta^l$ in the component $(T,\mathcal{S}(T))$, a morphism $T'\times\Delta^m\to T\times\Delta^l$ is a map of $k$-schemes of the form $f\times g: T'\times\Delta^m\to T\times\Delta^l$, with $f:(T,\mathcal{S}(T))\to(T',\mathcal{S}(T'))$ a morphism in $\mathcal{C}/k$, and $\mathrm{id}_T\times g$ a morphism in $\mathcal{C}(\{T\})$. Sending $T\times\Delta^m$ to $(T,\mathcal{S}(T))$ and $f\times g$ to $f$ makes $\mathcal{C}(\{*\})$ a category over $\mathcal{C}/k$, with fiber $\mathcal{C}(\{T\})$ over $(T,\mathcal{S}(T))$.

For each $(T,\mathcal{S}(T))\in\mathcal{C}/k$, we have the functors defined in §3.7

$$z^q_{(q)}(-)_{\mathcal{S}(T)}, G^*_{(q)}(-,\mu_m^{\otimes q})_{\mathcal{S}(T)}:\mathcal{C}(\{T\})^{\mathrm{op}}\to\mathbf{C}(\mathbf{Ab}).$$

For each $f:(T',\mathcal{S}(T'))\to(T,\mathcal{S}(T))$ in $\mathcal{C}/k$ and for each $W\in(T\times\Delta^n)^{(q)}_{\mathcal{S}(T)}$, the condition (3.15) implies that the cycle $(f\times\mathrm{id})^*(W)$ is defined and is in $(T'\times\Delta^n)^{(q)}_{\mathcal{S}(T')}$. Thus the assignments

$$T\times\Delta^n\mapsto z^q_{(q)}(T\times\Delta^n)_{\mathcal{S}(T)}$$
$$T\times\Delta^n\mapsto G^*_{(q)}(T\times\Delta^n,\mu_m^{\otimes q})_{\mathcal{S}(T)}$$

extend to functors

$$z^q_{(q)}(-)_{\mathcal{S}}, G^*_{(q)}(-,\mu_m^{\otimes q})_{\mathcal{S}}:\mathcal{C}(\{*\})^{\mathrm{op}}\to\mathbf{C}(\mathbf{Ab}).$$

The construction of §3.7 yields the diagram (3.7) in $\mathbf{C}(\mathrm{Func}(\mathcal{C}(\{*\})^{\mathrm{op}},\mathbf{Ab}))$, and the natural map

$$z^q_{(q)}(-)_{\mathcal{S}}[-2q]\otimes\mathbb{Z}/m\to G^*(-,\mu_m^{\otimes q}) \tag{3.16}$$

in $\mathbf{D}(\mathrm{Func}(\mathcal{C}(\{*\})^{\mathrm{op}},\mathbf{Ab}))$. Applying (3.16) to $T\times\Delta^*$ for $(T,\mathcal{S}(T))$ in $\mathcal{C}/k$ and composing with the inverse of the quasi-isomorphism $G^*(T,\mu_m^{\otimes q})\to G^*(T\times\Delta^*,\mu_m^{\otimes q})$ gives us the natural map in $\mathbf{D}(\mathrm{Func}(\mathcal{C}/k^{\mathrm{op}},\mathbf{Ab}))$,

$$\mathrm{cl}^q:\mathcal{Z}^{(q)}(-,*)_{\mathcal{S}}\otimes\mathbb{Z}/m\to G^*(-,\mu_m^{\otimes q}), \tag{3.17}$$

where $\mathcal{Z}^{(q)}(-,*)_{\mathcal{S}}$ is the functor $(T,\mathcal{S}(T))\mapsto\mathcal{Z}^{(q)}(T,*)_{\mathcal{S}(T)}$.

Call a map $f:(T',\mathcal{S}(T'))\to(T,\mathcal{S}(T))$ étale if the underlying map of schemes $f:T'\to T$ is étale, and if $\mathcal{S}(T')=f^{-1}(\mathcal{S}(T))$. We make $\mathcal{C}/k$ into a site $\mathcal{C}/k_{\text{ét}}$ by defining a covering family of $(T,\mathcal{S}(T))$ to be a collection of étale maps such that the underlying family of maps of schemes is an étale cover. The Zariski site $\mathcal{C}/k_{\mathrm{Zar}}$ is defined analogously. Clearly, for each $(T,\mathcal{S}(T))$, the restriction of $\mathcal{C}/k_{\text{ét}}$ (resp. $\mathcal{C}/k_{\mathrm{Zar}}$) to $(T,\mathcal{S}(T))$ is isomorphic to the small étale site $T_{\text{ét}}$ (resp. small Zariski site $T_{\mathrm{Zar}}$).

Let $\hat\epsilon:\mathcal{C}/k_{\text{ét}}\to\mathcal{C}/k_{\mathrm{Zar}}$ be the change of topology map. For each essentially smooth $k$-scheme $T$, $G^*(T,\mu_m^{\otimes q})$ is a functorial representative in $\mathbf{C}^+(\mathbf{Ab})$

of $R\Gamma(T_{\text{ét}}, \mu_m^{\otimes q})$. Letting $\mathcal{G}(\mu_m^{\otimes q})_T$ denote the complex of Zariski sheaves associated to the presheaf

$$(U \subset T) \mapsto G^*(U, \mu_m^{\otimes q}),$$

it follows that $T \mapsto \mathcal{G}(\mu_m^{\otimes q})_T$ is a functorial complex of sheaves representing $R\hat{\epsilon}_*\mu_m^{\otimes q}$.

The map (3.17), together with the quasi-isomorphism (2.14), thus gives us the map in $\mathbf{D}(\mathrm{Sh}(\mathcal{C}/k_{\mathrm{Zar}}))$

$$\mathrm{cl}^q : \Gamma_{(-)}(q) \otimes^L \mathbb{Z}/m \to R\hat{\epsilon}_*(\mu_m^{\otimes q}), \tag{3.18}$$

The natural map $\tau_{\leq q}\Gamma_X(q) \to \Gamma_X(q)$ is a quasi-isomorphism, since $H^p(R, \mathbb{Z}(q)) = 0$ if $p > q$, for a regular local $k$-algebra $R$. Thus (3.18) defines the map in $\mathbf{D}^-(\mathrm{Sh}(\mathcal{C}/k_{\mathrm{Zar}}))$

$$\mathrm{cl}^q : \Gamma_{(-)}(q) \otimes^L \mathbb{Z}/m \to \tau_{\leq q}R\hat{\epsilon}_*(\mu_m^{\otimes q}). \tag{3.19}$$

If we evaluate (3.19) at $(X, \{X\})$, we obtain the map in $\mathbf{D}^-(\mathrm{Sh}(X_{\mathrm{Zar}}))$

$$\mathrm{cl}^q : \Gamma_X(q) \otimes^L \mathbb{Z}/m \to \tau_{\leq q}R\epsilon_{X*}(\mu_m^{\otimes q}), \tag{3.20}$$

giving us the sheafified version of (3.9).

Similarly, if $W$ is a closed subset of $X$ with complement $j : U \to X$, we may apply (3.19) to the map $j$ (i.e., take cone$(j^*)[-1]$), giving the map for the complexes with support

$$\mathrm{cl}^q : \Gamma_X^W(q) \otimes^L \mathbb{Z}/m \to R\epsilon_{X*}Ri_W^!(\mu_m^{\otimes q}). \tag{3.21}$$

## 4. Properties of the cycle class map

4.1. **Naturality of the cycle class map.** We proceed to show that the cycle class maps defined in the previous section are natural. For this, we show that the sheaf-theoretic cycle class map (3.21) is natural, and that it gives rise to the "naive" cycle class maps defined in §3.7 by passing to hypercohomology.

Recall the category of pairs PSm$/k$ defined in §2.10. Let $(X, W)$ be in PSm$/k$. The natural isomorphism $R\Gamma(X_{\mathrm{Zar}}, -) \circ R\epsilon_* \cong R\Gamma(X_{\text{ét}}, -)$ gives us the natural isomorphism $\eta : R\Gamma(X_{\mathrm{Zar}}, R\epsilon_{X*}Ri_W^!\mu_m^{\otimes q}) \to R\Gamma(X_{\text{ét}}, Ri_W^!\mu_m^{\otimes q})$. We have the natural isomorphisms

$$\phi : \mathcal{Z}_W^q(X, *) \to R\Gamma(X_{\mathrm{Zar}}, \Gamma_X^W(q)),$$
$$\phi_{\text{ét}} : G_W^*(X, \mu_m^{\otimes q}) \to R\Gamma(X_{\text{ét}}, Ri_W^!\mu_m^{\otimes q})$$

in $\mathbf{D}(\mathbf{Ab})$, the first one being the isomorphism (2.17), and the second coming from the fact that the functor $Y \mapsto G^*(Y, \mu_m^{\otimes q})$ of Example 3.3(1) represents the functor $Y \mapsto R\Gamma(Y_{\text{ét}}, \mu_m^{\otimes q})$ from $\mathrm{Sch}_k^{\mathrm{op}}$ to $\mathbf{D}(\mathbf{Ab})$.

Thus, the construction of $\mathrm{cl}^q$ gives us the commutative diagram in $\mathbf{D}(\mathbf{Ab})$

$$\begin{tikzcd} \mathcal{Z}_W^q(X, *) \otimes^L \mathbb{Z}/m \arrow[r, "\phi", "\sim"'] \arrow[d, "\mathrm{cl}^q"'] & R\Gamma(X_{\mathrm{Zar}}, \Gamma_X^W(q) \otimes^L \mathbb{Z}/m) \arrow[d, "\eta\circ R\Gamma(\mathrm{cl}_X^q)"] \\ G_W^*(X, \mu_m^{\otimes q}) \arrow[r, "\sim", "\phi_{\text{ét}}"'] & R\Gamma(X_{\text{ét}}, Ri_W^!\mu_m^{\otimes q}). \end{tikzcd} \tag{4.1}$$

The cycle class map (3.13) for $X$ with supports in $W$ is thus the map on $\mathbb{H}^*(X_{\mathrm{Zar}}, -)$ induced by the sheafified version (3.21).

This discussion gives the naturality of the cycle class maps defined in the previous section.

**Proposition 4.2.** *(1) The cycle class maps* (3.9)*,* (3.11) *and* (3.20) *are natural on* $\mathrm{Sm}/k$*.*

*(2) The cycle class maps with support* (3.12)*,* (3.13) *and* (3.21) *are natural on* $\mathrm{PSm}/k$*.*

*(3) The cycle class map for relative cohomology* (3.14) *is natural on the category of tuples* $(X, Z_1, \dots, Z_n)$ *of essentially smooth affine* $k$*-schemes* $X$*, with closed subschemes* $Z_j$ *such that each* $Z_I$ *is essentially smooth over* $k$*, where a morphism*

$$f : (X', Z'_1, \dots, Z'_n) \to (X, Z_1, \dots, Z_n)$$

*is a morphism* $f : X' \to X$ *such that* $f(Z'_i) \subset Z_i$ *for all* $i$*.*

*(4) The cycle class map for relative cohomology* (3.14) *is natural on the category of tuples* $(Y, Z_1, \dots, Z_n)$ *of normal crossing* $k$*-schemes* $Y$*, with closed subschemes* $Z_j$ *such that each* $Z_I$ *is a normal crossing subscheme of* $Y$*, where the morphisms are as described in §2.8.*

*(5) The cycle class map* (3.14) *(for* $n = 0$*) is natural for arbitrary maps* $f : Y \to X$*, where* $X$ *is affine and essentially smooth over* $k$*,* $Y$ *is a normal crossing* $k$*-scheme, and* $f^* : \mathcal{Z}^q(X, *) \to \mathcal{Z}^q(Y, *)$ *is the map* (2.13)*.*

*Proof.* (1) is clearly a special case of (2). For (2), let $f : (X', W') \to (X, W)$ be a morphism in $\mathrm{PSm}/k$. We have the set $\mathcal{S}(f)$ (2.1) of closed subsets of $X$, giving us the lifting of the commutative diagram in $\mathrm{Sm}/k$

$$\begin{tikzcd}
X' \setminus W' \arrow[r, "j'"] \arrow[d, "f"'] & X' \arrow[d, "f"] \\
X \setminus W \arrow[r, "j"'] & X
\end{tikzcd}$$

to the commutative diagram in $\mathcal{C}/k$

$$\begin{tikzcd}
(X' \setminus W', \{X' \setminus W'\}) \arrow[r, "j'"] \arrow[d, "f"'] & (X', \{X'\}) \arrow[d, "f"] \\
(X \setminus W, \mathcal{S}(f_{|X' \setminus W'})) \arrow[r, "j"'] & (X, \mathcal{S}(f)).
\end{tikzcd}$$

Applying the morphism (3.19) to this diagram, dropping the truncation, and taking the cones with respect to the horizontal maps gives the commutative diagram in $\mathbf{D}(\mathrm{Sh}(X_{\mathrm{Zar}}))$

$$\begin{tikzcd}
\Gamma_X^W \arrow[r, "f^*"] \arrow[d, "\mathrm{cl}^q"'] & Rf_*\Gamma_{X'}^{W'} \arrow[d, "Rf_*\,\mathrm{cl}^q"] \\
R\epsilon_{X*}Ri^!_W\mu_m^{\otimes q} \arrow[r, "f^*"] & Rf_*R\epsilon_{X'*}Ri^!_{W'}\mu_m^{\otimes q}.
\end{tikzcd}$$

Applying $R\Gamma(X_{\mathrm{Zar}}, -)$ to this diagram and using the commutativity of the diagram (4.1) proves (2).

For (3), the naturality of the diagram (3.7) gives the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{Z}^q(X; Z_1, \ldots, Z_n, *) \arrow[drr, "\mathrm{cl}^q"] & & \\
\mathcal{Z}^q(X; Z_1, \ldots, Z_n, *)_{\mathcal{S}(f)} \arrow[u, "\mathrm{id}^*"] \arrow[rr, "\mathrm{cl}^q"] \arrow[d, "f^*"'] & & G^*(X; Z_1, \ldots, Z_n, \mu_m^{\otimes q}) \arrow[d, "f^*"] \\
\mathcal{Z}^q(X'; Z'_1, \ldots, Z'_n, *) \arrow[rr, "\mathrm{cl}^q"] & & G^*(X'; Z'_1, \ldots, Z'_n, \mu_m^{\otimes q})
\end{tikzcd}$$

in $\mathbf{D}(\mathbf{Ab})$, which gives the desired naturality on taking cohomology. The proof of (4) and (5) are similar, and are left to the reader. ☐

4.3. **Compatibility with Gysin maps.** We prove that the maps $\mathrm{cl}^q$ are compatible with the localization/Gysin sequences for motivic cohomology/étale cohomology. For a scheme $X$, we let $\mathrm{Sm}/X$ denote the category of schemes smooth and essentially of finite type over $X$.

Let $i : Y \to X$ be a smooth codimension $d$ closed subscheme of an essentially smooth $k$-scheme $X$. Let $T \to X$ be a smooth $X$-scheme, and let $T_Y \to Y$ denote the fiber product $T \times_X Y$, with inclusion $i_T : T_Y \to T$. The canonical Gysin isomorphisms $\mu_m^{\otimes q} \cong Ri^!_{T_Y}\mu_m^{\otimes q+d}[2d]$ in $\mathbf{D}(\mathrm{Sh}(T_{Y\acute{e}t}))$ (see [19, VI, Theorem 6.1]) give rise to the isomorphism in $\mathbf{D}(\mathrm{Func}(\mathrm{Sm}/X^{\mathrm{op}}, \mathbf{Ab}))$

$$\iota : [T \mapsto G^*(T_Y, \mu_m^{\otimes q})] \to \big[T \mapsto G^*_{T_Y}(T, \mu_m^{\otimes q+d})[2d]\big].$$

Let $i_W : W \to T_Y$ be a closed subset of $T_Y$, with complement $j_Y : V_Y \to T_Y$ in $T_Y$ and $j : V \to T$ in $T$. Evaluating $\iota$ at $T$ and $V$ gives the canonical map of cones

$$\begin{aligned}\mathrm{cone}[G^*(T_Y, \mu_m^{\otimes q}) &\xrightarrow{j_Y^*} G^*(V_Y, \mu_m^{\otimes q})][-1] \\ &\xrightarrow{\iota_W(T)} \mathrm{cone}[G^*_{T_Y}(T, \mu_m^{\otimes q+d}) \xrightarrow{j^*} G^*_{V_Y}(V, \mu_m^{\otimes q+d})][2d-1].\end{aligned}$$

Since $W = T_Y \setminus V_Y = T \setminus V$, it follows that

$$\mathrm{cone}[G^*(T_Y, \mu_m^{\otimes q}) \xrightarrow{j_Y^*} G^*(V_Y, \mu_m^{\otimes q})][-1] = G^*_W(T_Y, \mu_m^{\otimes q}),$$

and that the evident map

$$\mathrm{cone}[G^*_{T_Y}(T, \mu_m^{\otimes q+d}) \xrightarrow{j^*} G^*_{V_Y}(V, \mu_m^{\otimes q+d})][-1] \to G^*_W(T, \mu_m^{\otimes q+r})$$

is a homotopy equivalence. In addition, $G^*_W(T_Y, \mu_m^{\otimes q})$ and $G^*_W(T, \mu_m^{\otimes q+r})$ are complexes representing $R\Gamma(T_Y, Ri^!_W(\mu_m^{\otimes q}))$ and $R\Gamma(T, R(i_T \circ i_W)^!(\mu_m^{\otimes q+d}))$, respectively. Thus the map

$$\iota_W(T) : G^*_W(T_Y, \mu_m^{\otimes q}) \to G^*_W(T, \mu_m^{\otimes q+r})[2d] \tag{4.2}$$

represents the natural isomorphism in $\mathbf{D}(\mathbf{Ab})$

$$\iota_W : R\Gamma(T_Y, Ri^!_W(\mu_m^{\otimes q})) \to R\Gamma(T, R(i_T \circ i_W)^!(\mu_m^{\otimes q+d})[2d]), \tag{4.3}$$

which in turn induces the Gysin isomorphism $H^p_W(T_Y, \mu_m^{\otimes q}) \cong H^{p+2d}_W(T, \mu_m^{\otimes q+d})$ on taking cohomology.

**Lemma 4.4.** *Let $i : Y \to X$ be a closed codimension $d$ embedding of essentially smooth $k$-schemes. Then the diagram in $\mathbf{D}^-(\mathrm{Sh}(X_{\mathrm{Zar}}))$*

$$\begin{tikzcd}
\Gamma_Y(q) \arrow[r, "\mathrm{cl}^q_Y"] \arrow[d, "\alpha"'] & R\epsilon_{Y*}\mu_m^{\otimes q} \arrow[d, "\beta"] \\
\Gamma_X^Y(q+d)[2d] \arrow[r, "\mathrm{cl}^{q+d}_X"'] & R\epsilon_{X*}Ri^!_Y(\mu_m^{\otimes q+d})[2d]
\end{tikzcd}$$

*commutes, where $\alpha$ and $\beta$ are the respective Gysin isomorphisms.*

*Proof.* Let $U$ be an open subscheme of $X$, $U_Y = Y \cap U$. Applying the Gysin isomorphism (4.3) for $T = U \times \Delta^m$, and for $W$ in $(U_Y \times \Delta^m)^{(q)}$, we have the following diagram in $\mathbf{D}(\mathbf{Ab})$:

$$\begin{tikzcd}
z^q_W(U_Y \times \Delta^m) \otimes \mathbb{Z}/m[-2q] \arrow[r, equal] \arrow[d, "\mathrm{cyc}^q[-2q]"'] & z^{q+d}_W(U \times \Delta^m) \otimes \mathbb{Z}/m[-2q] \arrow[d, "\mathrm{cyc}^{q+d}[-2q]"] \\
H^{2q}_W(U_Y \times \Delta^m, \mu_m^{\otimes q})[-2q] \arrow[r, "\iota_1"] & H^{2(q+d)}_W(U \times \Delta^m, \mu_m^{\otimes q+d})[-2q] \\
\tau_{\le 2q} G^*_W(U_Y \times \Delta^m, \mu_m^{\otimes q}) \arrow[r, "\iota_2"] \arrow[u, "p"] \arrow[d, "q"'] & \tau_{\le 2q}\left[G^*_W(U \times \Delta^m, \mu_m^{\otimes q+d})[2d]\right] \arrow[u, "p'"'] \arrow[d, "q'"] \\
G^*(U_Y \times \Delta^m, \mu_m^{\otimes q}) \arrow[r, "\iota_3"] & G^*_{U_Y \times \Delta^m}(U \times \Delta^m, \mu_m^{\otimes q+d})[2d] \\
G^*(U_Y, \mu_m^{\otimes q}) \arrow[r, "\iota_4"] \arrow[u, "p_1^*"] & G^*_{U_Y}(U, \mu_m^{\otimes q+d})[2d] \arrow[u, "p_1^*"']
\end{tikzcd} \tag{4.4}$$

Here the left-hand column is the diagram used in defining $\mathrm{cl}^q$ in §3.7, the right-hand column is the diagram used in defining the version of $\mathrm{cl}^q$ for $U$ with supports in $U_Y$, and the maps $\iota_i$ are induced by the appropriate Gysin map (4.2). In particular, all the vertical maps, except for $q$ and $q'$, are quasi-isomorphisms, and all the horizontal maps are isomorphisms in $\mathbf{D}(\mathbf{Ab})$. It follows from Lemma 3.5 that the top square commutes; the next two squares commute in $\mathbf{D}(\mathbf{Ab})$ since $\iota_2 = \tau_{\le 2q}\iota_3$ by definition, and $\iota_1$ is the map on cohomology induced by $\iota_2$. The bottom square commutes in $\mathbf{D}(\mathbf{Ab})$ by the naturality of the Gysin isomorphism.

Since the $\iota_i$ come from isomorphisms in $\mathbf{D}(\mathrm{Func}(\mathrm{Sm}/X^{\mathrm{op}}, \mathbf{Ab}))$, (4.4) defines a commutative diagram in $\mathbf{D}(\mathrm{Func}(\mathrm{Sm}/X^{\mathrm{op}}, \mathbf{Ab}))$. Taking the limit over $W$, this gives us the commutative diagram in the derived category of presheaves on $X$:

$$U \mapsto \left(\begin{tikzcd} z^q(U_Y,*)\otimes\mathbb{Z}/m[-2q] \arrow[r, "\iota''"] \arrow[d, "\mathrm{cyc}^q"'] & z^{q+d}_{U_Y}(U,*)\otimes\mathbb{Z}/m[-2q] \arrow[d, "\mathrm{cyc}^{q+d}"] \\ G^*(U_Y\times\Delta^*,\mu_m^{\otimes q}) \arrow[r, "\iota'"] & G^*_{U_Y\times\Delta^*}(U\times\Delta^*,\mu_m^{\otimes q+d})[2d] \\ G^*(U_Y,\mu_m^{\otimes q}) \arrow[r, "\iota"] \arrow[u, "p_1^*"] & G^*_{U_Y}(U,\mu_m^{\otimes q+d})[2d] \arrow[u, "p_1^*"'] \end{tikzcd}\right) \tag{4.5}$$

As $\mathrm{cl}^q = (p_1^*)^{-1}\circ\mathrm{cyc}^q$ and $\mathrm{cl}^{q+d} = (p_1^*)^{-1}\circ\mathrm{cyc}^{q+d}$, sheafifying (4.5) gives the desired commutative diagram in $\mathbf{D}^-(\mathrm{Sh}(X_{\mathrm{Zar}}))$. □

**Proposition 4.5.** *(1) The cycle class map for relative cohomology is compatible with the maps in the long exact relativization sequences for motivic cohomology and étale cohomology.*

*(2) Let $i : Y \to X$ be a closed codimension $d$ embedding of essentially smooth $k$-schemes with complement $j : U \to X$. Then the diagram*

$$\begin{tikzcd} H^{p-1}(U,\mathbb{Z}/m(q)) \arrow[r, "\partial"] \arrow[d, "\mathrm{cl}^q"'] & H^{p-2d}(Y,\mathbb{Z}/m(q-d)) \arrow[r, "i_*"] \arrow[d, "\mathrm{cl}^{q-d}"'] & H^p(X,\mathbb{Z}/m(q)) \arrow[d, "\mathrm{cl}^q"'] \\ H^{p-1}_{\text{ét}}(U,\mu_m^{\otimes q}) \arrow[r, "\partial"'] & H^{p-2d}_{\text{ét}}(Y,\mu_m^{\otimes q-d}) \arrow[r, "i_*"'] & H^p_{\text{ét}}(X,\mu_m^{\otimes q}) \end{tikzcd}$$

*commutes.*

*Proof.* (1) follows immediately from the naturality of the map (3.8) and the definition of relative motivic cohomology (resp. relative étale cohomology) via cones.

For (2), the same reasoning as in (1) gives us the map of distinguished triangles in $\mathbf{D}(\mathbf{Ab})$:

$$\begin{tikzcd} \mathcal{Z}^q_Y(X,*) \arrow[r] \arrow[d, "\mathrm{cl}^q"'] & \mathcal{Z}^q(X,*) \arrow[r, "j^*"] \arrow[d, "\mathrm{cl}^q"'] & \mathcal{Z}^q(U,*) \arrow[d, "\mathrm{cl}^q"'] \\ G^*_Y(X,\mu_m^q) \arrow[r] & G^*(X,\mu_m^q) \arrow[r, "j^*"'] & G^*(U,\mu_m^q) \end{tikzcd}$$

which in turn gives the commutative diagram

$$\begin{tikzcd} H^{p-1}(U,\mathbb{Z}/m(q)) \arrow[r, "\partial"] \arrow[d, "\mathrm{cl}^q"'] & H^p_Y(X,\mathbb{Z}/m(q)) \arrow[r, "i_*"] \arrow[d, "\mathrm{cl}^q"'] & H^p(X,\mathbb{Z}/m(q)) \arrow[d, "\mathrm{cl}^q"'] \\ H^{p-1}_{\text{ét}}(U,\mu_m^{\otimes q}) \arrow[r, "\partial"'] & H^p_Y(X,\mu_m^{\otimes q}) \arrow[r, "i_*"'] & H^p_{\text{ét}}(X,\mu_m^{\otimes q}) \end{tikzcd} \tag{4.6}$$

Since the diagram in (2) is derived from (4.6) by replacing $H^p_Y(X,\mathbb{Z}/m(q))$ and $H^p_Y(X,\mu_m^{\otimes q})$ with $H^{p-2d}(Y,\mathbb{Z}/m(q-d))$ and $H^{p-2d}_{\text{ét}}(Y,\mu_m^{\otimes q-d})$ via the respective Gysin isomorphisms, Lemma 4.4 together with the commutativity of (4.6) proves (2). □

4.6. **Products.** We proceed to check that the cycle class map is compatible with products.

**Proposition 4.7.** *Let $X$ and $Y$ be $k$-schemes, of finite type and locally equidimensional over $k$. The diagram*

$$\begin{array}{ccc} H^p(X,\mathbb{Z}/m(q))\otimes H^{p'}(Y,\mathbb{Z}/m(q')) & \xrightarrow{\cup} & H^{p+p'}(X\times_k Y,\mathbb{Z}/m(q+q')) \\ \downarrow{\scriptstyle \mathrm{cl}^q\otimes\mathrm{cl}^{q'}} & & \downarrow{\scriptstyle \mathrm{cl}^{q+q'}} \\ H^p_{\text{ét}}(X,\mu_m^{\otimes q})\otimes H^{p'}_{\text{ét}}(Y,\mu_m^{\otimes q'}) & \xrightarrow[\cup]{} & H^{p+p'}_{\text{ét}}(X\times_k Y,\mu_m^{\otimes q+q'}) \end{array}$$

*is commutative. In particular, for $X$ essentially smooth over $k$, the map*

$$\mathrm{cl}^*:\oplus_{p,q}H^p(X,\mathbb{Z}/m(q))\to\oplus_{p,q}H^p_{\text{ét}}(X,\mu_m^{\otimes q})$$

*is a ring homomorphism.*

*Proof.* Following [9, Chapter 6], a map of sheaves $\mu:\mathcal{F}\otimes\mathcal{G}\to\mathcal{H}$ on a topological space $X$ induces a functorial map (over $\mu$) on the Godement resolutions

$$G(\mu):G^*(X,\mathcal{F})\otimes G^*(X,\mathcal{G})\to G^*(X,\mathcal{H}).$$

We apply the methods of §3.7, which, together with Lemma 3.5(3), gives us the commutative diagram in $\mathbf{D}^-(\mathbf{Mod}_{\mathbb{Z}/m})$

$$\begin{array}{ccc} \mathcal{Z}^q(X,*)/m\otimes^L_{\mathbb{Z}/m}\mathcal{Z}^{q'}(Y,*)/m & \xrightarrow{\cup} & \mathcal{Z}^{q+q'}(X\times Y,*)/m \\ \downarrow{\scriptstyle \mathrm{cl}^q\otimes\mathrm{cl}^{q'}} & & \downarrow{\scriptstyle \mathrm{cl}^{q+q'}} \\ G^*(X,\mu_m^{\otimes q})\otimes^L_{\mathbb{Z}/m}G^*(Y,\mu_m^{\otimes q'}) & \xrightarrow[\cup']{} & G^*(X\times Y,\mu_m^{\otimes q+q'}) \end{array}$$

Here, the map $\cup'$ is defined by the diagram analogous to (8.1):

$$\begin{aligned} G^*(X,\mu_m^{\otimes q})\otimes^L G^*(Y,\mu_m^{\otimes q'}) &\xrightarrow{p_1^*\otimes p_1^*} G^*(X\times\Delta^*,\mu_m^{\otimes q})\otimes^L G^*(Y\times\Delta^*,\mu_m^{\otimes q'}) \\ &\xrightarrow{G(\boxtimes)} G^*(X\times Y\times\Delta^*\times\Delta^*,\mu_m^{\otimes q+q'}) \\ &\xrightarrow{T} G^*(X\times Y\times\Delta^*,\mu_m^{\otimes q+q'})\leftarrow G^*(X\times Y,\mu_m^{\otimes q+q'}). \end{aligned}$$

It thus suffices to see that $\cup'$ agrees with the product

$$\cup:G^*(X,\mu_m^{\otimes q})\otimes^L_{\mathbb{Z}/m}G^*(Y,\mu_m^{\otimes q'})\to G^*(X\times Y,\mu_m^{\otimes q+q'})$$

induced by the product of sheaves $\mu_m^{\otimes q}\otimes\mu_m^{\otimes q'}\to\mu_m^{\otimes q+q'}$. Since the augmentations

$$G^*(X,\mu_m^{\otimes q})\to G^*(X\times\Delta^*,\mu_m^{\otimes q})$$
$$G^*(Y,\mu_m^{\otimes q'})\to G^*(Y\times\Delta^*,\mu_m^{\otimes q'})$$
$$G^*(X\times Y,\mu_m^{\otimes q+q'})\to G^*(X\times Y\times\Delta^*,\mu_m^{\otimes q+q'})$$

are quasi-isomorphisms, this follows from the fact that the map

$$G^*(X\times Y\times\Delta^*\times\Delta^*,\mu_m^{\otimes q+q'})\xrightarrow{T}G^*(X\times Y\times\Delta^*,\mu_m^{\otimes q+q'})$$

is a map over the identity on $G^*(X\times Y,\mu_m^{\otimes q+q'})$. □

4.8. **The case of fields.** We discuss some additional properties of the cycle class map for $X = \mathrm{Spec}\, F$, $F$ a field.

From [1, Theorem 6.1], the map sending $u \in F \setminus \{0,1\}$ to the cycle $[u] := 1 \cdot (\frac{1}{1-u}, \frac{u}{u-1})$ of $\Delta^1_F$ gives rise to an isomorphism $F^* \to H^1(F, \mathbb{Z}(1))$. Additionally, $H^p(F, \mathbb{Z}(1)) = 0$ for $p \neq 1$, hence we have the isomorphism $F^*/F^{*m} \cong H^1(F, \mathbb{Z}/m(1))$. The Kummer sequence gives $H^1_{\text{ét}}(F, \mu_m) \cong F^*/F^{*m}$ for all $m$ prime to $\mathrm{char}\, F$.

**Proposition 4.9.** *Let $u$ be in $F \setminus \{0,1\}$. The cycle class map*

$$\mathrm{cl}^1 : H^1(F, \mathbb{Z}/m(1)) \to H^1_{\text{ét}}(F, \mu_m) \cong F^*/F^{*m}$$

*sends the class of $[u]$ to the class of $u \mod F^{*m}$, hence is an isomorphism.*

.

*Proof.* Let $U = \Delta^1_F \setminus \{[u]\}$. We have the Gysin sequence

$$H^1_{\text{ét}}(U, \mu_m) \xrightarrow{\partial} H^2_{[u]}(\Delta^1_F, \mu_m) \to H^2_{\text{ét}}(\Delta^1_F, \mu_m).$$

By definition of the étale cycle class of a divisor [5, Cycle, Définition 2.1.2], it follows that the cycle class of $[u]$ in $H^2_{[u]}(\Delta^1_F, \mu_m)$ is given by $\mathrm{cyc}^1_{[u]}([u]) = \partial([f])$, where $f$ is a regular function on $\Delta^1_F$ with $\mathrm{div}\, f = [u]$, and $[f]$ is the class in $H^1_{\text{ét}}(U, \mu_m)$ corresponding to $f \in H^0_{\text{ét}}(U, \mathbb{G}_m)$ via the Kummer sequence.

We claim that

$$\mathrm{cl}^1([u]) = i^*_{(1,0)}([f]) - i^*_{(0,1)}([f]) \in H^1_{\text{ét}}(F, \mu_m). \tag{4.7}$$

To see this, let $\{[f]\}$ be a cocycle in $G^1(U, \mu_m)$ representing $[f]$. Let $\tilde{G}^p_{[u]}(\Delta^1_F, \mu_m)$ be the kernel of the surjection $G^p(\Delta^1_F, \mu_m) \xrightarrow{j^*} G^p(U, \mu_m)$, giving the term-wise exact sequence of complexes

$$0 \to \tilde{G}^*_{[u]}(\Delta^1_F, \mu_m) \xrightarrow{i_*} G^*(\Delta^1_F, \mu_m) \xrightarrow{j^*} G^*(U, \mu_m) \to 0. \tag{4.8}$$

The canonical map $\tilde{G}^*_{[u]}(\Delta^1_F, \mu_m) \to G^*_{[u]}(\Delta^1_F, \mu_m)$ is a homotopy equivalence, and the Gysin sequence for the inclusion of $[u]$ in $\Delta^1_F$ is the long exact cohomology sequence resulting from (4.8). Thus, if $\phi \in G^1(\Delta^1_F, \mu_m)$ is a cochain lifting $\{[f]\}$, then $d\phi = i_*\eta$, $\eta \in \tilde{G}^2_{[u]}(\Delta^1_F, \mu_m)$, and $\eta$ is a cocycle representing $\mathrm{cyc}^1_{[u]}([u])$. On the other hand, it follows from the construction of the map $\mathrm{cl}^1$ that the element $\mathrm{cl}^1([u]) \in H^1_{\text{ét}}(F, \mu_m)$ is given as follows: Since $i^*_{(1,0)} \circ i_* = i^*_{(0,1)} \circ i_* = 0$, the map $i_*$ extends to the map of complexes

$$\tilde{i}_* : \tilde{G}^*_{[u]}(\Delta^1_F, \mu_m) \to G^*(\Delta^*_F, \mu_m),$$

and $\mathrm{cl}^1([u])$ is the element of $H^1_{\text{ét}}(F, \mu_m)$ corresponding to $\tilde{i}_*(\mathrm{cyc}^1_{[u]}([u]))$ via the quasi-isomorphism $G^*_{\text{ét}}(F, \mu_m) \cong G^*(\Delta^*_F, \mu_m)$ induced by the augmentation $\Delta^*_F \to F$. Tracing through this quasi-isomorphism, we see that $\mathrm{cl}^1([u])$ is the class of $i^*_{(1,0)}(\psi) - i^*_{(0,1)}(\psi)$, where $\psi$ is any element of $G^1(\Delta^1_F, \mu_m)$ with $d\psi = i_*\eta$. Since we may take $\psi = \phi$, and since $i^*_p(\phi) = i^*_p(\{[f]\})$ for all $p \in U$, it follows that $\mathrm{cl}^1([u])$ is the class of $i^*_{(1,0)}(\{[f]\}) - i^*_{(0,1)}(\{[f]\})$, which verifies (4.7).

Since we may take $f$ to be the function $f(t_0, t_1) = t_0 - \frac{1}{1-u}$, we have

$$\mathrm{cl}^1([u]) = \frac{1 - \frac{1}{1-u}}{-\frac{1}{1-u}} \mod F^{*m} = u \mod F^{*m}.$$

□

This result has the following consequence for the relation of the Bloch-Kato conjecture and the cycle class map:

**Lemma 4.10.** *Let $F$ be a field, $m$ an integer prime to the characteristic of $F$. If $\vartheta_{q,F} : K_q^M(F)/m \to H^q_{\text{ét}}(F, \mu_m^{\otimes q})$ is surjective, then the cycle class map $\mathrm{cl}^q$ : $H^q(F, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(F, \mu_m^{\otimes q})$ is surjective as well.*

*Proof.* The Galois symbol for $q = 1$ is by definition the Kummer isomorphism $F^*/F^{*m} \cong H^1_{\text{ét}}(F, \mu_m)$, and the map $\vartheta_{q,F}$ is the multiplicative extension of $\vartheta_{1,F}$. Thus, $\vartheta_{q,F}$ is surjective if and only if the cup product map

$$H^1_{\text{ét}}(F, \mu_m)^{\otimes q} \to H^q_{\text{ét}}(F, \mu_m^{\otimes q})$$

is surjective. From Proposition 4.9 and Proposition 4.7, this implies the surjectivity of $\mathrm{cl}^q$. □

*Remark* 4.11. One could also prove Lemma 4.10 by showing that the isomorphisms $K_n^M(F) \to H^n(F, \mathbb{Z}(n))$ described in [20] or [26] are multiplicative. The map described in [26] uses a cubical version of the higher Chow groups, and the issue of multiplicativity is not addressed in [20]; we have used the somewhat indirect argument above to avoid checking this detail.

Using Proposition 4.9, one can show that $\mathrm{cl}^1 : H^0(F, \mathbb{Z}/m(1)) \to H^0_{\text{ét}}(F, \mu_m)$ is an isomorphism as well. We are indebted to the referee for the argument.

**Proposition 4.12.** *Let $F$ be a field, and $m$ an integer prime to* $\mathrm{char}\, F$. *Then the cycle class map $\mathrm{cl}^1 : \mathcal{Z}^1(F, *) \otimes \mathbb{Z}/m \to G^*(F, \mu_m)$ is an isomorphism. In particular, the map $\mathrm{cl}^1 : H^0(F, \mathbb{Z}/m(1)) \to H^0_{\text{ét}}(F, \mu_m)$ is an isomorphism.*

*Proof.* We may assume that $m$ is a prime power $l^\nu$. As the map

$$\mathrm{cl}^1 : \mathcal{Z}^1(X, *) \otimes \mathbb{Z}/m \to G^*(X, \mu_m)$$

is defined by a finite zigzag diagram of natural maps of complexes, $\mathrm{cl}^1$ is compatible with inverse systems of schemes with flat affine transition maps, and induces a map on the corresponding direct limit of complexes. Since both $\mathcal{Z}^1(X, *)$ and $G^*(X, \mu_m)$ are compatible with such inverse systems, it suffices to prove the result for $F$ finitely generated over the prime field.

We now consider the inverse system of maps $\mathrm{cl}^1$ for $m = l^n$, $n = 1, 2, \ldots$. Since the Godement resolution transforms surjective maps of sheaves to term-wise surjective maps of complexes, the complexes in the system of zigzag diagrams defining $\mathrm{cl}^1$ for $m = l^n$, $n = 1, 2, \ldots$, satisfy the Mittag-Leffler conditions term-wise. Therefore, $\mathrm{cl}^1$ induces a map on the derived inverse limit

$$\mathrm{cl}^1 : R \varprojlim_n \mathcal{Z}^1(F, *) \otimes \mathbb{Z}/l^n \to R \varprojlim_n G^*(F, \mu_{l^n}), \tag{4.9}$$

and we have the commutative diagram for each $N$

$$\begin{tikzcd} R\varprojlim_n \mathcal{Z}^1(F,*)\otimes\mathbb{Z}/l^n \arrow[r] \arrow[d, "\mathrm{cl}^1"'] & \mathcal{Z}^1(F,*)\otimes\mathbb{Z}/l^N \arrow[d, "\mathrm{cl}^1"] \\ R\varprojlim_n G^*(F,\mu_{l^n}) \arrow[r] & G^*(F,\mu_{l^N}) \end{tikzcd} \tag{4.10}$$

where the horizontal arrows are the canonical maps.

Both $\mathcal{Z}^1(F,*)\otimes\mathbb{Z}/l^n$ and $G^*(F,\mu_{l^n})$ have the same cohomology, namely, $F^*/(F^*)^{l^n}$ in degree one, and $\mu_{l^n}(F)$ in degree zero. Since $F$ is finitely generated, both $R\varprojlim \mathcal{Z}^1(F,*)\otimes\mathbb{Z}/l^n$ and $R\varprojlim G^*(F,\mu_{l^n})$ have cohomology only in degree one, with that cohomology being $\varprojlim F^*/(F^*)^{l^n}$. By Proposition 4.9 and the commutative diagram (4.10), the map (4.9) is an isomorphism.

The commutativity of the diagram (4.10) identifies $\mathrm{cl}^1:\mathcal{Z}^1(F,*)\otimes\mathbb{Z}/l^N\to G^*(F,\mu_{l^N})$ with the map on the derived tensor product

$$\mathrm{cl}^1\otimes\mathrm{id}:[R\varprojlim_n \mathcal{Z}^1(F,*)\otimes\mathbb{Z}/l^n]\otimes^L\mathbb{Z}/l^N\to[R\varprojlim_n G^*(F,\mu_{l^n})]\otimes^L\mathbb{Z}/l^N.$$

As this latter map is an isomorphism, the proposition is proved. □

4.13. **The étale sheafification of $\Gamma_X(q)$.** In this section, we prove Theorem 1.5, and show that Theorem 1.1 and Theorem 1.6 are equivalent.

Let $\Gamma_X(q)_{\text{ét}}$ denote the sheafification of $\Gamma_X(q)$ for the étale topology on $X$. Sheafifying the cycle class map (3.20) for the étale topology on $X$ as in §3.10 gives the étale cycle class map

$$\mathrm{cl}^q_{\text{ét}}:\Gamma_X(q)_{\text{ét}}\otimes^L\mathbb{Z}/m\to\mu_m^{\otimes q}$$

in $\mathbf{D}^-(\mathrm{Sh}(X_{\text{ét}}))$, natural in $X$. We have the natural map $\Gamma_X(q)\to R\epsilon_*\Gamma_X(q)_{\text{ét}}$; truncating gives us the natural map

$$\iota_X:\Gamma_X(q)\to\tau_{\le q}R\epsilon_*\Gamma_X(q)_{\text{ét}}.$$

*Proof of Theorem 1.5.* Let $\pi:X\to\operatorname{Spec}k$ be the structure morphism. By the rigidity result of [1, Lemma 11.1], the natural map

$$\pi^*\Gamma_k(q)_{\text{ét}}\otimes^L\mathbb{Z}/m\to\Gamma_X(q)_{\text{ét}}\otimes^L\mathbb{Z}/m$$

is a quasi-isomorphism, reducing us to the case of $X=\operatorname{Spec}k$, with $k$ separably closed.

Suppose that $\operatorname{char}k=0$. By the Suslin-Voeovodsky theorem relating Suslin homology and étale homology [24, Corollary 7.8] and Suslin's comparison of Suslin homology and motivic cohomology [23], we have

$$H^p(k,\mathbb{Z}/m(q))=\begin{cases}0 & \text{for } p\neq 0\\ \mu_m^{\otimes q}(k) & \text{for } p=0.\end{cases}$$

In case $\operatorname{char}k>0$, the same result follows by using de Jong's resolution of singularities [11] to extend the results of [24] to positive characteristic (see e.g. [12] or [7] for further details).

By Proposition 4.12, the map $\mathrm{cl}^1 : H^0(k, \mathbb{Z}/m(1)) \to H^0_{\text{ét}}(k, \mu_m)$ is an isomorphism. By Proposition 4.7, we have the commutative diagram

$$\begin{tikzcd} H^0(k,\mathbb{Z}/m(1))^{\otimes q} \arrow[r, "\cup"] \arrow[d, "(\mathrm{cl}^1)^{\otimes q}"'] & H^0(k,\mathbb{Z}/m(q)) \arrow[d, "\mathrm{cl}^q"] \\ H^0_{\text{ét}}(k,\mu_m)^{\otimes q} \arrow[r, "\cup"'] & H^0_{\text{ét}}(k,\mu_m^{\otimes q}) \end{tikzcd}$$

from which we see that the cup product $H^0(k, \mathbb{Z}/m(1))^{\otimes q} \to H^0(k, \mathbb{Z}/m(q))$ is injective. From the computation of $H^0(k, \mathbb{Z}/m(q))$ given above, this shows that the product map $H^0(k, \mathbb{Z}/m(1))^{\otimes q} \to H^0(k, \mathbb{Z}/m(q))$ is an isomorphism. Thus the cycle class map induces a quasi-isomorphism

$$\mathrm{cl}^q : \Gamma_k(q) \otimes^L \mathbb{Z}/m \to \mu_m^{\otimes q}(k),$$

for $k$ separably closed, completing the proof. □

We now show that Theorem 1.1 and Theorem 1.6 are equivalent.

*Proof.* By Theorem 1.5, we have the isomorphism

$$\mathrm{cl}^a_{\text{ét}} : \Gamma_X(a)_{\text{ét}} \otimes^L \mathbb{Z}/m \to \mu_m^{\otimes a}$$

in $\mathbf{D}^-(\mathrm{Sh}(X_{\text{ét}}))$ for all $a \geq 0$. Applying $R\epsilon_*$ and truncating gives the isomorphism in $\mathbf{D}^-(\mathrm{Sh}(X_{\mathrm{Zar}}))$

$$\tau_{\leq a} R\epsilon_* \,\mathrm{cl}^a_{\text{ét}} : \tau_{\leq a} R\epsilon_* \Gamma_X(a)_{\text{ét}} \otimes^L \mathbb{Z}/m \to \tau_{\leq a} R\epsilon_* \mu_m^{\otimes a}.$$

By the adjunction property of $R\epsilon_*$, the cycle class map factors through $\tau_{\leq a} R\epsilon_* \,\mathrm{cl}^a_{\text{ét}}$, giving the commutative diagram

$$\begin{tikzcd} \Gamma_X(a) \otimes^L \mathbb{Z}/m \arrow[r, "\iota_X"] \arrow[dr, "\mathrm{cl}^a"'] & \tau_{\leq a} R\epsilon_* \Gamma_X(a)_{\text{ét}} \otimes^L \mathbb{Z}/m \arrow[d, "\tau_{\leq a} R\epsilon_* \mathrm{cl}^a_{\text{ét}}"] \\ & \tau_{\leq a} R\epsilon_* \mu_m^{\otimes a} \end{tikzcd}$$

with $\tau_{\leq a} R\epsilon_* \,\mathrm{cl}^a_{\text{ét}}$ an isomorphism. Thus $\mathrm{cl}^a$ is an isomorphism if and only if $\iota_X$ is an isomorphism, showing that Theorem 1.1 and Theorem 1.6 are equivalent. □

## 5. The semi-local $n$-cube

We relate the Bloch-Kato conjecture to the Beilinson-Lichtenbaum conjectures by taking motivic cohomology and étale cohomology of the semi-local $n$-cube, relative to its faces.

5.1. **The semi-local $n$-cube.** We write $\square_n$ for the affine space $\mathrm{Spec}\, k[t_1, \ldots, t_n]$. Let $v$ be the set of points $(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_j \in \{0, 1\}$, of $\square_n$, $R_n$ the semi-local ring $\mathcal{O}_{\square_n, v}$ and $\hat{\square}_n$ the semi-local scheme $\mathrm{Spec}\, R_n$.

Let $\hat{\square}_n^{i;\epsilon}$ be the subscheme of $\hat{\square}_n$ defined by the ideal $(t_i - \epsilon)$, $\epsilon \in \{0, 1\}$. We define the set of subschemes $T_n$ of $\hat{\square}_n$ by

$$T_n := \{\hat{\square}_n^{i;\epsilon} \mid i = 1, \ldots, n;\ \epsilon \in \{0, 1\}\}.$$

We order the elements of $T_n$ by setting

$$\hat{\square}_{n,s} := \begin{cases} \hat{\square}_n^{s;1} & \text{for } s \leq n \\ \hat{\square}_n^{s-n;0} & \text{for } s > n, \end{cases}$$

and let $T_n^s$ be the subset of $T_n$ consisting of the first $s$ subschemes. Let

$$S_n := T_n^{2n-1} = T_n - \{\hat{\square}_n^{n;0}\}.$$

For $r < s$, we let $T_{n,s}^r$ denote the set of subschemes of $\hat{\square}_{n,s}$ of the form $\hat{\square}_{n,i} \cap \hat{\square}_{n,s}$, $i = 1, \ldots, r$.

As in §2.7, we write $\hat{\square}_{n,I}$ for the intersection $\cap_{s \in I} \hat{\square}_{n,s}$; we call $\hat{\square}_{n,I}$ a *face* of $\hat{\square}_n$. Using the coordinates $t_j$, $j \notin I$, in the standard order identifies $\hat{\square}_{n,I}$ with $\hat{\square}_{n-|I|}$. Identifying the face $\hat{\square}_{n,s}$ of $\hat{\square}_n$ with $\hat{\square}_{n-1}$ in this way, we have the identification

$$T_{n,s}^r = \begin{cases} T_{n-1}^r & \text{for } r < s \leq n \text{ or } r + n < s, \\ T_{n-1}^{r-1} & \text{for } 1 \leq s - n \leq r < s. \end{cases}$$

5.2. **Relativization sequences.** For $1 \leq s \leq 2n$ we have the fundamental relativization sequence

$$\to H^{p-1}(\hat{\square}_{n,s}; T_{n,s}^{s-1}, \mathbb{Z}/m(q)) \to H^p(\hat{\square}_n; T_n^s, \mathbb{Z}/m(q)) \to \\ H^p(\hat{\square}_n; T_n^{s-1}, \mathbb{Z}/m(q)) \to H^p(\hat{\square}_{n,s}; T_{n,s}^{s-1}, \mathbb{Z}/m(q)) \to .$$

We have similar sequences for étale cohomology. Taking $s = 2n$ and identifying $\hat{\square}_{n,2n}$ with $\hat{\square}_{n-1}$ as described above gives the most important case

$$\to H^{p-1}(\hat{\square}_{n-1}; T_{n-1}, \mathbb{Z}/m(q)) \to H^p(\hat{\square}_n; T_n, \mathbb{Z}/m(q)) \to \\ H^p(\hat{\square}_n; S_n, \mathbb{Z}/m(q)) \to H^p(\hat{\square}_{n-1}; T_{n-1}, \mathbb{Z}/m(q)) \to .$$

We will rely on the fundamental surjectivity property, proved in [8, Corollary 4.4]:

**Lemma 5.3.** *For all $n \geq 1$ and all $q \geq 0$, the restriction map*

$$H^q(\hat{\square}_n; S_n, \mathbb{Z}/m(q)) \to H^q(\hat{\square}_{n-1}; T_{n-1}, \mathbb{Z}/m(q))$$

*is surjective.*

*Remark* 5.4. We have not been able to prove the analog of Lemma 5.3 for the semi-local scheme of the vertices in $\Delta^n$, which is why we need to introduce the $n$-cube $\hat{\square}_n$. Lemma 5.3 is similar to the result [25, Corollary 9.7](revised version), and plays the same crucial role in the argument.

The combinatorics of $\hat{\square}_n$ and its faces are similar to that of $\Delta^n$ and its faces. In the next few sections, we discuss these combinatorics, leading to the splitting result Proposition 5.7.

5.5. **Relative complexes.** Let $\mathcal{C}$ be a subcategory of $\mathrm{Sch}_k$, and $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab})$ a functor. As in §3.1, if we have a $k$-scheme $X$ with closed subschemes $Y_1, \ldots, Y_m$ such that all the inclusions $Y_I \to Y_J$, $J \subset I \subset \{1, \ldots, n\}$, are in $\mathcal{C}$, we form the relative complex $F(X; Y_1, \ldots, Y_n)$, which is defined as the total complex of the double complex

$$F(X) \to \bigoplus_{i=1}^n F(Y_i) \to \ldots \to \bigoplus_{|I|=j} F(Y_I) \to \ldots$$

with $F^i(Y_I)$ in total degree $i+|I|$. The relative complexes $F(X;Y_1,\dots,Y_n)$ are natural in $F$, and have the functorialities described in §2.8. We have as well the subcomplex

$$F(X;Y_1,\dots,Y_n)^{\ker} := \ker\big(F(X)\to\bigoplus_{i=1}^n F(Y_i)\big)$$

of $F(X;Y_1,\dots,Y_n)$.

5.6. **The category $\mathcal{C}(n)$.** For $1\le s\le n$ let

$$i_s:(t_1,\dots,t_{n-1})\mapsto(t_1,\dots,t_{s-1},0,t_s,\dots,t_{n-1})$$
$$j_s:(t_1,\dots,t_{n-1})\mapsto(t_1,\dots,t_{s-1},1,t_s,\dots,t_{n-1})$$

be the inclusion of the faces $t_s=0$ and $t_s=1$ into $\hat{\square}_n$. We have the identities

(5.1)
$$i_ti_s=\begin{cases}i_{s+1}i_t & s\ge t\\ i_si_{t-1} & s<t;\end{cases}\qquad j_ti_s=\begin{cases}i_{s+1}j_t & s\ge t\\ i_sj_{t-1} & s<t;\end{cases}\qquad j_tj_s=\begin{cases}j_{s+1}j_t & s\ge t\\ j_sj_{t-1} & s<t.\end{cases}$$

Similarly, we define projection maps for $1\le s\le n$ and $1\le s<n$, respectively,

$$p_s:(t_1,\dots,t_n)\mapsto(t_1,\dots,t_{s-1},t_{s+1},\dots,t_n)$$
$$q_s:(t_1,\dots,t_n)\mapsto(t_1,\dots,t_{s-1},1-(t_s-1)(t_{s+1}-1),t_{s+2},\dots,t_n).$$

The following identities hold

$$p_sj_t=\begin{cases}j_{t-1}p_s & t>s\\ \mathrm{id} & t=s\\ j_tp_{s-1} & t<s;\end{cases}\qquad p_si_t=\begin{cases}i_{t-1}p_s & t>s\\ \mathrm{id} & t=s\\ i_tp_{s-1} & t<s,\end{cases}\tag{5.2}$$

$$q_sj_t=\begin{cases}j_{t-1}q_s & t>s+1\\ j_sp_s & t=s,s+1\\ j_tq_{s-1} & t<s;\end{cases}\qquad q_si_t=\begin{cases}i_{t-1}q_s & t>s+1\\ \mathrm{id} & t=s,s+1\\ i_tq_{s-1} & t<s.\end{cases}\tag{5.3}$$

Let $I$ be a subset of $\{1,\dots,n\}$ (possibly empty), and $s_1,\dots,s_r$ elements of the complement of $I$. We write $\partial_I^{s_1,\dots,s_r}\hat{\square}_n$ for the subscheme $\cup_{i=1}^r\hat{\square}_{n,I\cup\{s_i\}}$ of $\hat{\square}_{n,I}$. We write $\partial^{s_1,\dots,s_r}\hat{\square}_n$ for $\partial_\emptyset^{s_1,\dots,s_r}\hat{\square}_n$; we also write $\partial\hat{\square}_n$ for $\partial^{1,\dots,2r}\hat{\square}_n$. We allow the case $r=0$, i.e., $\partial_I\hat{\square}_n=\hat{\square}_{n,I}$. We note that the $\partial_I^{s_1,\dots,s_r}\hat{\square}_n$ are all normal crossing subschemes of $\hat{\square}_n$.

A *face* of $\partial_I^{s_1,\dots,s_r}\hat{\square}_n$ is a face $\hat{\square}_{n,J}$ of $\hat{\square}_n$ which is contained in $\partial_I^{s_1,\dots,s_r}\hat{\square}_n$. We let $\mathcal{C}(n)$ be the subcategory of $\mathrm{Sch}_k$ with objects the subschemes of $\hat{\square}_n$ of the form $\partial_I^{s_1,\dots,s_r}\hat{\square}_n$. A morphism $f:\partial_I^{s_1,\dots,s_r}\hat{\square}_n\to\partial_{I'}^{s'_1,\dots,s'_{r'}}\hat{\square}_n$ is a map of $k$-schemes such that, for each face $\hat{\square}_{n,J}$ of $\partial_I^{s_1,\dots,s_r}\hat{\square}_n$, there is a face $\hat{\square}_{n,J'}$ of $\partial_{I'}^{s'_1,\dots,s'_{r'}}\hat{\square}_n$ such that the restriction of $f$ to $\hat{\square}_{n,J}$ factors as

$$\hat{\square}_{n,J}\xrightarrow{f_J}\hat{\square}_{n,J'}\xrightarrow{i_{J'}}\partial_{I'}^{s'_1,\dots,s'_{r'}}\hat{\square}_n,$$

with $f_J$ flat, and $i_{J'}$ the inclusion.

**Proposition 5.7.** *Let $F:\mathcal{C}(n)^{\mathrm{op}}\to\mathbf{C}(\mathbf{Ab})$ be a functor. Then for all $s<2n$:*

1. *The canonical maps*
$$F(\hat{\square}_n, T_n^s)^{\ker} \to F(\hat{\square}_n, T_n^s); \quad F(\partial\hat{\square}_n, T_n^s)^{\ker} \to F(\partial\hat{\square}_n, T_n^s)$$
*are quasi-isomorphisms.*
2. *The inclusions*
$$F(\hat{\square}_n, T_n^s)^{\ker} \subseteq F(\hat{\square}_n, T_n^{s-1})^{\ker}; \quad F(\partial\hat{\square}_n, T_n^s)^{\ker} \subseteq F(\partial\hat{\square}_n, T_n^{s-1})^{\ker}$$
*are split; the splittings are natural in $F$.*

*Proof.* We can assume that the proposition holds for functors $\mathcal{C}(n-1)^{\mathrm{op}} \to \mathbf{C}(\mathbf{Ab})$ and proceed by induction on $s$. We will prove the proposition for $\partial\hat{\square}_n$; the proof for $\hat{\square}_n$ is exactly the same, replacing $\partial\hat{\square}_n$ with $\hat{\square}_n$, and is left to the reader.

By contravariant functoriality, there are maps
$$i_s^*, j_s^* : F(\partial\hat{\square}_n) \to F(\hat{\square}_{n-1,s})$$
$$p_s^*, q_s^* : F(\hat{\square}_{n-1,s}) \to F(\partial\hat{\square}_n).$$
For $s < 2n$, consider the following commutative diagram of complexes:

$$\begin{tikzcd}
F(\partial\hat{\square}_n, T_n^s)^{\ker} \arrow[r, "\mathrm{incl}"] \arrow[d, "\alpha"'] & F(\partial\hat{\square}_n, T_n^{s-1})^{\ker} \arrow[r, "\phi"] \arrow[d, "\beta"'] & F(\hat{\square}_{n,s}, T_{n,s}^{s-1})^{\ker} \arrow[d, "\gamma"'] \\
F(\partial\hat{\square}_n, T_n^s) \arrow[r] & F(\partial\hat{\square}_n, T_n^{s-1}) \arrow[r] & F(\hat{\square}_{n,s}, T_{n,s}^{s-1}),
\end{tikzcd} \tag{5.4}$$

where the vertical arrows are the natural inclusions, and $\phi$ is the restriction map. By definition, $F(\partial\hat{\square}_n, T_n^s)^{\ker} = \ker(\phi)$.

The bottom row of (5.4) defines a distinguished triangle via the isomorphism (2.10). If we can show that the map $\phi$ is naturally split surjective, then (2) follows, and the top row of (5.4) defines a distinguished triangle as well. Since $\beta$ and $\gamma$ are quasi-isomophisms by induction, $\alpha$ will be a quasi-isomorphism as well, proving (1).

We consider the following diagram, where $\iota_s = j_s^*$ for $s \leq n$ and $\iota_s = i_{s-n}^*$ for $s > n$, and where $\tilde{\beta}$ and $\tilde{\gamma}$ are the evident inclusions:

$$\begin{tikzcd}
0 \arrow[r] & F(\partial\hat{\square}_n, T_n^{s-1})^{\ker} \arrow[r, "\tilde{\beta}"] \arrow[d, "\phi"'] & F(\partial\hat{\square}_n) \arrow[r, "\prod_{t\leq s-1}\iota_t"] \arrow[d, "\iota_s"'] & \bigoplus_{t\leq s-1} F(\hat{\square}_{n,t}) \arrow[d, "\oplus\iota_s"'] \\
0 \arrow[r] & F(\hat{\square}_{n,s}, T_{n,s}^{s-1})^{\ker} \arrow[r, "\tilde{\gamma}"'] & F(\hat{\square}_{n,s}) \arrow[r, "\prod_{t\leq s-1}\iota_t"'] & \bigoplus_{t\leq s-1} F(\hat{\square}_{n,t,s}).
\end{tikzcd}$$

The rows are the degree-wise exact sequences defining the complexes $F(\partial\hat{\square}_n, T_n^{s-1})^{\ker}$ and $F(\partial\hat{\square}_n, T_n^{s-1})^{\ker}$, respectively.

Let $\rho_s : F(\hat{\square}_{n,s}) \to F(\partial\hat{\square}_n)$ be the map $p_s^*$ in case $s \leq n$, or $q_{n-s}^*$ in case $n < s < 2n$. Let $x$ be an element of $F(\hat{\square}_{n,s}, T_{n,s}^{s-1})^{\ker}$. Then $x$ can be viewed as an element of $F(\hat{\square}_{n,s})$ mapping to zero under $\iota_t$ for $t < s$. Map $x$ to $\bar{x}$ in $F(\partial\hat{\square}_n)$ by taking $\bar{x} = \rho_s(x)$.

We note that $\iota_s \circ \rho_s = \mathrm{id}$, by (5.2) for $s \leq n$ or (5.3) for $n < s < 2n$. Thus, to show that $\rho_s$ defines our desired splitting, we need only show that $\bar{x}$ lies in the kernel of $\iota_t$ for $t < s$. But this follows from (5.2) for $t < s \leq n$:
$$\iota_t\bar{x} = j_t^* p_s^* x = p_{s-1}^* j_t^* x = 0.$$

For $s > t > n$ this follows from (5.3):

$$\iota_t \bar{x} = i^*_{t-n} q^*_{s-n} x = q^*_{s-n-1} i^*_{t-n} x = 0,$$

and for $t \leq n < s$,

$$\iota_t \bar{x} = j^*_t q^*_{s-n} x = \begin{cases} q^*_{s-n} j^*_{t-1} x = 0 & t > s-n+1 \\ p^*_{s-n} j^*_t x = 0 & t = s-n, s-n+1 \\ q^*_{s-n-1} j^*_t x = 0 & t < s-n. \end{cases}$$

□

## 6. The Beilinson-Lichtenbaum conjectures and cycle maps for $\hat{\square}_n$

In the main result of this section, Proposition 6.5, we show how the surjectivity of certain relative cycle class maps for semi-local $n$-cubes implies a part of the Beilinson-Lichtenbaum conjectures for fields.

We have the following version of a part of the Beilinson-Lichtenbaum conjectures [16]:

**Conjecture 6.1** ($\mathrm{BL}_a$). *Let $m$ be an integer prime to the characteristic of $k$. The cycle class map*

$$\mathrm{cl}^a : H^p(F, \mathbb{Z}/m(a)) \to H^p_{\text{ét}}(F, \mu_m^{\otimes a})$$

*is an isomorphism for all $p \leq a$, and for all finitely generated field extensions $F$ of $k$.*

Write $\mathbb{A}$ for $\mathbb{A}^n_k$, and let $\mathbb{A}^{(b)'}$ denote the set of codimension $b$ points of $\mathbb{A}$ which are not in $\hat{\square}_n$. The standard constructions of Bloch-Ogus [4] give the augmented Gersten complex for motivic cohomology

$$(6.1)\quad 0 \to H^p(\mathbb{A}, \mathbb{Z}/m(q)) \xrightarrow{\epsilon} H^p(\hat{\square}_n, \mathbb{Z}/m(q)) \to \bigoplus_{x \in \mathbb{A}^{(1)'}} H^{p-1}(k(x), \mathbb{Z}/m(q-1)) \to \dots \to \bigoplus_{x \in \mathbb{A}^{(r)'}} H^{p-r}(k(x), \mathbb{Z}/m(q-r)) \to \dots$$

and for étale cohomology

$$(6.2)\quad 0 \to H^p_{\text{ét}}(\mathbb{A}, \mu_m^{\otimes q}) \xrightarrow{\epsilon} H^p_{\text{ét}}(\hat{\square}_n, \mu_m^{\otimes q}) \to \bigoplus_{x \in \mathbb{A}^{(1)'}} H^{p-1}_{\text{ét}}(k(x), \mu_m^{\otimes q-1}) \to \dots \to \bigoplus_{x \in \mathbb{A}^{(r)'}} H^{p-r}_{\text{ét}}(k(x), \mu_m^{\otimes q-r}) \to \dots$$

To fix the notation, we take $H^p(\mathbb{A}, \mathbb{Z}/m(q))$ and $H^p_{\text{ét}}(\mathbb{A}, \mu_m^{\otimes q})$ in degree $-1$.

**Lemma 6.2.** *The complexes* (6.1) *and* (6.2) *are exact.*

*Proof.* We discuss the case of motivic cohomology; the argument for étale cohomology is exactly the same.

For a scheme $X$, we let $X^{(r)}$ denote the set of codimension $r$ points of $X$. We have the standard Gersten complex for $\mathbb{A}$:

$$
\begin{aligned}
(6.3)\quad 0 \to H^p(\mathbb{A}, \mathbb{Z}/m(q)) \to H^p(k(\mathbb{A}), \mathbb{Z}/m(q)) \to \bigoplus_{x\in\mathbb{A}^{(1)}} H^{p-1}(k(x), \mathbb{Z}/m(q-1)) \\
\to \ldots \to \bigoplus_{x\in\mathbb{A}^{(r)}} H^{p-r}(k(x), \mathbb{Z}/m(q-r)) \to \ldots,
\end{aligned}
$$

and for $\hat{\square}_n$:

$$
\begin{aligned}
(6.4)\quad 0 \to H^p(\hat{\square}_n, \mathbb{Z}/m(q)) \to H^p(k(\hat{\square}_n), \mathbb{Z}/m(q)) \to \bigoplus_{x\in\hat{\square}_n^{(1)}} H^{p-1}(k(x), \mathbb{Z}/m(q-1)) \\
\to \ldots \to \bigoplus_{x\in\hat{\square}_n^{(r)}} H^{p-r}(k(x), \mathbb{Z}/m(q-r)) \to \ldots .
\end{aligned}
$$

Sherman [21] has considered the analog of (6.3), where one replaces motivic cohomology with $K$-cohomology $H^*(-, \mathcal{K}_*)$, and shows that the resulting complex is exact. The same argument works for any Bloch-Ogus twisted duality theory, in particular, the complex (6.3) is exact. Similarly, the "classical" Gersten's lemma [4, Theorem 4.2] for a Bloch-Ogus cohomology theory shows that (6.4) is exact.

We define the map of complexes $\psi : (6.3) \to (6.4)$ by the restriction map

$$\psi^{-1} : H^p(\mathbb{A}, \mathbb{Z}/m(q)) \to H^p(\hat{\square}_n, \mathbb{Z}/m(q)),$$

in degree $-1$ and the evident projection

$$\psi^r : \bigoplus_{x\in\mathbb{A}^{(r)}} H^{p-r}(k(x), \mathbb{Z}/m(q-r)) \to \bigoplus_{x\in\hat{\square}^{(r)}} H^{p-r}(k(x), \mathbb{Z}/m(q-r))$$

in degree $r \geq 0$. It is easy to check that the complex (6.1) is homotopy equivalent to $\mathrm{cone}(\psi)[-1]$, which shows that (6.1) is exact. □

We write 1 for the point $(1, \ldots, 1)$ of $\hat{\square}_n$.

**Lemma 6.3.** *Suppose* $\mathrm{BL}_a$ *is true for* $0 \leq a < q$. *Then the cycle class map*

$$\mathrm{cl}^q : H^p(\hat{\square}_n; 1, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(\hat{\square}_n; 1, \mu_m^{\otimes q})$$

*is an isomorphism for all* $p \leq q$ *and all* $n \geq 1$.

*Proof.* By Lemma 6.2, the Gersten complexes (6.1) and (6.2) are exact.

Let $\pi : \mathbb{A} \to \operatorname{Spec} k$ be the structure morphism, and $i : 1 \to \hat{\square}_n$, $j : \hat{\square}_n \to \mathbb{A}$ the inclusions. Since

$$H^p(k, \mathbb{Z}/m(q)) \xrightarrow{\pi^*} H^p_{\text{ét}}(\mathbb{A}, \mathbb{Z}/m(q)); \quad H^p_{\text{ét}}(k, \mu_m^{\otimes q}) \xrightarrow{\pi^*} H^p_{\text{ét}}(\mathbb{A}, \mu_m^{\otimes q})$$

are isomorphisms, and $\pi \circ j \circ i = \mathrm{id}$, we may split off the first term in both (6.1) and (6.2), giving the exact Gersten complexes

$$
\begin{aligned}
(6.5)\quad 0 \to H^p(\hat{\square}_n; 1, \mathbb{Z}/m(q)) \to \bigoplus_{x\in\mathbb{A}^{(1)\prime}} H^{p-1}(k(x), \mathbb{Z}/m(q-1)) \to \ldots \\
\to \bigoplus_{x\in\mathbb{A}^{(r)\prime}} H^{p-r}(k(x), \mathbb{Z}/m(q-r)) \to \ldots
\end{aligned}
$$

and

$$(6.6)\quad 0 \to H^p_{\text{ét}}(\hat{\square}_n; 1, \mu_m^{\otimes q}) \to \bigoplus_{x\in\mathbb{A}^{(1)\prime}} H^{p-1}_{\text{ét}}(k(x), \mu_m^{\otimes q-1}) \to \dots$$
$$\to \bigoplus_{x\in\mathbb{A}^{(r)\prime}} H^{p-r}_{\text{ét}}(k(x), \mu_m^{\otimes q-r}) \to \dots$$

The compatibility of the cycle class maps with localization (Proposition 4.5(2)) implies that the various cycle class maps give a map of complexes $\mathrm{cl}^* : (6.5) \to (6.6)$. This together with our hypothesis proves the lemma. □

**Lemma 6.4.** *Suppose that* $\mathrm{BL}_a$ *is true for* $0 \le a < q$. *Then the cycle class map*

$$\mathrm{cl}^q : H^p(\hat{\square}_n; S_n, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(\hat{\square}_n; S_n, \mu_m^{\otimes q})$$

*is an isomorphism for all* $p \le q$.

*Proof.* We prove more generally that the cycle class map

$$\mathrm{cl}^q : H^p(\hat{\square}_n; T_n^s, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(\hat{\square}_n; T_n^s, \mu_m^{\otimes q})$$

is an isomorphism for all $p \le q$ and for $1 \le s < 2n$. We proceed by induction on $s$ and $n$.

By Proposition 4.5(1), we have the commutative diagram

$$\begin{array}{ccccccc}
\to H^p(\hat{\square}_n; T_n^s, \mathbb{Z}/m(q)) & \to & H^p(\hat{\square}_n; T_n^{s-1}, \mathbb{Z}/m(q)) & \to & H^p(\hat{\square}_{n,s}; T_{n,s}^{s-1}, \mathbb{Z}/m(q)) \to \\
\downarrow{\scriptstyle \mathrm{cl}^q} & & \downarrow{\scriptstyle \mathrm{cl}^q} & & \downarrow{\scriptstyle \mathrm{cl}^q} \\
\to H^p_{\text{ét}}(\hat{\square}_n; T_n^s, \mu_m^{\otimes q}) & \longrightarrow & H^p_{\text{ét}}(\hat{\square}_n; T_n^{s-1}, \mu_m^{\otimes q}) & \longrightarrow & H^p_{\text{ét}}(\hat{\square}_{n,s}; T_{n,s}^{s-1}, \mu_m^{\otimes q}) \to,
\end{array}$$

where the rows are the respective relativization sequences. Induction reduces us to the case $s = 1$.

We have the relativization sequence

$$\to H^p(\hat{\square}_n; T_n^1, 1, \mathbb{Z}/m(q)) \to H^p(\hat{\square}_n; T_n^1, \mathbb{Z}/m(q)) \to H^p(1; 1, \mathbb{Z}/m(q)) \to;$$

since $\mathcal{Z}^q(1; 1, *)$ is the cone on the identity map, we thus have the isomorphism

$$H^p(\hat{\square}_n; T_n^1, 1, \mathbb{Z}/m(q)) \cong H^p(\hat{\square}_n; T_n^1, \mathbb{Z}/m(q)),$$

and similarly for relative étale cohomology. Comparing the relativization sequences for motivic cohomology and étale cohomology using Proposition 4.5(1) gives the commutative diagram

$$\begin{array}{ccccccc}
\to H^p(\hat{\square}_n; T_n^1, 1, \mathbb{Z}/m(q)) & \to & H^p(\hat{\square}_n; 1, \mathbb{Z}/m(q)) & \to & H^p(\hat{\square}_{n-1}; 1, \mathbb{Z}/m(q)) \to \\
\downarrow{\scriptstyle \mathrm{cl}^q} & & \downarrow{\scriptstyle \mathrm{cl}^q} & & \downarrow{\scriptstyle \mathrm{cl}^q} \\
\to H^p_{\text{ét}}(\hat{\square}_n; T_n^1, 1, \mu_m^{\otimes q}) & \longrightarrow & H^p_{\text{ét}}(\hat{\square}_n; 1, \mu_m^{\otimes q}) & \longrightarrow & H^p_{\text{ét}}(\hat{\square}_{n-1}; 1, \mu_m^{\otimes q}) \to .
\end{array}$$

Using Lemma 6.3, this shows that the cycle class map

$$\mathrm{cl}^q : H^p(\hat{\square}_n; T_n^1, 1, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(\hat{\square}_n; T_n^1, 1, \mu_m^{\otimes q})$$

is an isomorphism in the desired range. □

**Proposition 6.5.** *Suppose that* $\mathrm{BL}_a$ *is true for* $0 \le a < q$. *Suppose further that, for all finitely generated field extensions* $F$ *of* $k$, *the cycle class maps*

$$\mathrm{cl}^q : H^q(\hat{\square}_n \times_k F; T_n \times_k F, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(\hat{\square}_n \times_k F; T_n \times_k F, \mu_m^{\otimes q})$$

*are surjective for all* $n \ge 0$. *Then* $\mathrm{BL}_q$ *is true.*

*Proof.* We first reduce $\mathrm{BL}_q$ to showing

(6.7)
*For all finitely generated field extensions* $F$ *of* $k$, *the cycle class maps*

$$\mathrm{cl}^q : H^p(\hat{\square}_n \times_k F; T_n \times_k F, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(\hat{\square}_n \times_k F; T_n \times_k F, \mu_m^{\otimes q})$$

*are isomorphisms for all* $n \ge 1$ *and all* $p \le q$.

Indeed, suppose that (6.7) is true. If $k'$ is a finitely generated field extension of $k$, (6.7) remains true with $k'$ replacing $k$. Thus, it suffices to prove that

$$\mathrm{cl}^q : H^p(\operatorname{Spec} k, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(\operatorname{Spec} k, \mu_m^{\otimes q})$$

is an isomorphism for $p \le q$.

We have the commutative diagram relating the motivic and étale relativization sequences

$$\begin{tikzcd}
\to H^p(\hat{\square}_1; T_1, \frac{\mathbb{Z}}{m}(q)) \arrow[r] \arrow[d, "\mathrm{cl}^q"'] & H^p(\hat{\square}_1; S_1, \frac{\mathbb{Z}}{m}(q)) \arrow[r] \arrow[d, "\mathrm{cl}^q"'] & H^p(k, \frac{\mathbb{Z}}{m}(q)) \arrow[r] \arrow[d, "\mathrm{cl}^q"'] & H^{p+1}(\hat{\square}_1; T_1, \frac{\mathbb{Z}}{m}(q)) \to \arrow[d, "\mathrm{cl}^q"'] \\
\to H^p_{\text{ét}}(\hat{\square}_1; T_1, \mu_m^{\otimes q}) \arrow[r] & H^p_{\text{ét}}(\hat{\square}_1; S_1, \mu_m^{\otimes q}) \arrow[r] & H^p_{\text{ét}}(k, \mu_m^{\otimes q}) \arrow[r] & H^{p+1}_{\text{ét}}(\hat{\square}_1; T_1, \mu_m^{\otimes q}) \to
\end{tikzcd}$$

With this, (6.7) together with Lemma 6.4 and (for $p = q$) Lemma 5.3 implies that $\mathrm{cl}^q$ is an isomorphism in the desired range.

To prove (6.7), we may replace $F$ with $k$. We consider commutative diagram of relativization sequences

$$\begin{tikzcd}
\to H^q(\hat{\square}_{n+1}; T_{n+1}, \frac{\mathbb{Z}}{m}(q)) \arrow[r] \arrow[d, "\mathrm{cl}^q"'] & H^q(\hat{\square}_{n+1}; S_{n+1}, \frac{\mathbb{Z}}{m}(q)) \arrow[r] \arrow[d, "\mathrm{cl}^q"'] & H^q(\hat{\square}_n; T_n, \frac{\mathbb{Z}}{m}(q)) \arrow[r] \arrow[d, "\mathrm{cl}^q"'] & 0 \\
\to H^q_{\text{ét}}(\hat{\square}_{n+1}; T_{n+1}, \mu_m^{\otimes q}) \arrow[r] & H^q_{\text{ét}}(\hat{\square}_{n+1}; S_{n+1}, \mu_m^{\otimes q}) \arrow[r] & H^q_{\text{ét}}(\hat{\square}_n; T_n, \mu_m^{\otimes q}) \arrow[r] & {}
\end{tikzcd}$$

where the surjectivity for motivic cohomology is given by Lemma 5.3. The map

$$\mathrm{cl}^q : H^q(\hat{\square}_n; S_n, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(\hat{\square}_n; S_n, \mu_m^{\otimes q})$$

is an isomorphism by Lemma 6.4, hence surjectivity of

$$\mathrm{cl}^q : H^q(\hat{\square}_{n+1}; T_{n+1}, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(\hat{\square}_{n+1}; T_{n+1}, \mu_m^{\otimes q})$$

implies injectivity of $\mathrm{cl}^q : H^q(\hat{\square}_n; T_n, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(\hat{\square}_n; T_n, \mu_m^{\otimes q})$.

We now proceed by descending induction on the cohomology degree. Fix an integer $p < q$, and suppose that $\mathrm{cl}^q : H^a(\hat{\square}_n; T_n, \mathbb{Z}/m(q)) \to H^a_{\text{ét}}(\hat{\square}_n; T_n, \mu_m^{\otimes q})$ is an isomorphism for all $a$ with $p + 1 \le a \le q$ and for all $n$. We have the commutative diagram whose columns are the respective relativization sequences

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
H^p(\hat{\square}_{n+1}; T_{n+1}, \mathbb{Z}/m(q)) & \xrightarrow{\mathrm{cl}^q} & H^p_{\text{ét}}(\hat{\square}_{n+1}; T_{n+1}, \mu_m^{\otimes q}) \\
\downarrow & & \downarrow \\
H^p(\hat{\square}_{n+1}; S_{n+1}, \mathbb{Z}/m(q)) & \xrightarrow{\mathrm{cl}^q} & H^p_{\text{ét}}(\hat{\square}_{n+1}; S_{n+1}, \mu_m^{\otimes q}) \\
\downarrow & & \downarrow \\
H^p(\hat{\square}_{n}; T_{n}, \mathbb{Z}/m(q)) & \xrightarrow{\mathrm{cl}^q} & H^p_{\text{ét}}(\hat{\square}_{n}; T_{n}, \mu_m^{\otimes q}) \\
\downarrow & & \downarrow \\
H^{p+1}(\hat{\square}_{n+1}; T_{n+1}, \mathbb{Z}/m(q)) & \xrightarrow{\mathrm{cl}^q} & H^{p+1}_{\text{ét}}(\hat{\square}_{n+1}; T_{n+1}, \mu_m^{\otimes q}) \\
\downarrow & & \downarrow \\
H^{p+1}(\hat{\square}_{n+1}; S_{n+1}, \mathbb{Z}/m(q)) & \xrightarrow{\mathrm{cl}^q} & H^{p+1}_{\text{ét}}(\hat{\square}_{n+1}; S_{n+1}, \mu_m^{\otimes q}) \\
\downarrow & & \downarrow
\end{array}$$

As we have the isomorphism (Lemma 6.4)

$$\mathrm{cl}^q : H^a(\hat{\square}_{n+1}; S_{n+1}, \mathbb{Z}/m(q)) \to H^a_{\text{ét}}(\hat{\square}_{n+1}; S_{n+1}, \mu_m^{\otimes q})$$

for all $a \leq q$, the map $\mathrm{cl}^q : H^p(\hat{\square}_n; T_n, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(\hat{\square}_n; T_n, \mu_m^{\otimes q})$ is surjective; as $n$ is arbitrary, the map

$$\mathrm{cl}^q : H^p(\hat{\square}_{n+1}; T_{n+1}, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(\hat{\square}_{n+1}; T_{n+1}, \mu_m^{\otimes q})$$

is surjective as well, hence $\mathrm{cl}^q : H^p(\hat{\square}_n; T_n, \mathbb{Z}/m(q)) \to H^p_{\text{ét}}(\hat{\square}_n; T_n, \mu_m^{\otimes q})$ is injective. $\square$

## 7. The Bloch-Kato conjecture and surjectivity

We now complete the discussion, by showing how the Bloch-Kato conjecture for fields implies the surjectivity condition of Proposition 6.5.

**Proposition 7.1.** *Let $X = \operatorname{Spec} R$ be a semi-local $k$-scheme, where $R$ is a localization of a $k$-algebra of finite type. Then, for each element $\eta$ of $H^p_{\text{ét}}(X, \mu_m^{\otimes q})$, there is a $k$-morphism $i : X \to Y$, with $Y$ a smooth finite-type semi-local $k$-scheme, and an element $\tau$ of $H^p_{\text{ét}}(Y, \mu_m^{\otimes q})$, such that $\eta = i^*\tau$.*

*Proof.* Take a closed embedding of $X$ into the semi-localization $A$ of an affine space $\mathbb{A}^N_k$, and let $X_h$ be the henselization of $A$ along $X$. Let $i_h : X \to X_h$ be the inclusion and let $\mathcal{F}$ be a torsion étale sheaf on $X_h$. It follows from [6, Theorem 1] (see also [22]) that $i_h^* : H^p_{\text{ét}}(X_h, \mathcal{F}) \to H^p_{\text{ét}}(X, i_h^*\mathcal{F})$ is an isomorphism. Since $X$ and $A$ are affine, $X_h$ is a filtered projective limit of smooth semi-local $k$-schemes of finite type, with flat affine transition maps. Since $H^p_{\text{ét}}(-, \mu_m^{\otimes q})$ maps filtered projective limits of this type to filtered inductive limits, the result follows. $\square$

**Proposition 7.2.** *Suppose that, for all fields $F$ finitely generated over $k$, the cycle class map $\mathrm{cl}^q : H^q(F, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(F, \mu_m^{\otimes q})$ is a surjection and the cycle class map $\mathrm{cl}^{q-1} : H^{q-1}(F, \mathbb{Z}/m(q-1)) \to H^{q-1}_{\text{ét}}(F, \mu_m^{\otimes q-1})$ is an injection. Then the cycle class map*

$$\mathrm{cl}^q : H^q(\hat{\square}_n \times_k F; T_n \times_k F, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(\hat{\square}_n \times_k F; T_n \times_k F, \mu_m^{\otimes q})$$

*is surjective for all fields $F$ finitely generated over $k$ and for all $n$.*

*Proof.* As in the proof of Proposition 6.5, it suffices to prove the result for $F = k$. By a limit argument, similar to the argument of §3.4, we may assume that $k$ is perfect.

Let $R$ be the semi-local ring of finitely many smooth points on a algebraic variety $X$ over $k$. By Bloch-Ogus theory [4, Theorem 4.2], we have the exact Gersten complex

$$
\begin{aligned}
(7.1)\quad & 0 \to H^p(R, \mathbb{Z}/m(q)) \to H^p(k(X), \mathbb{Z}/m(q)) \to \dots \\
& \qquad \to \bigoplus_{x \in (\operatorname{Spec} R)^{(a)}} H^{p-a}(k(x), \mathbb{Z}/m(q-a)) \to \dots
\end{aligned}
$$

and a similar exact Gersten complex for étale cohomology

$$
\begin{aligned}
(7.2)\quad & 0 \to H^p_{\text{ét}}(R, \mu_m^{\otimes q}) \to H^p_{\text{ét}}(k(X), \mu_m^{\otimes q}) \to \dots \\
& \qquad \to \bigoplus_{x \in (\operatorname{Spec} R)^{(a)}} H^{p-a}_{\text{ét}}(k(x), \mu_m^{\otimes q-a}) \to \dots
\end{aligned}
$$

By the compatibility of the cycle class maps with localization (Proposition 4.5(2)), the maps $\mathrm{cl}^{q-a}$ define the map of complexes $\mathrm{cl}^* : (7.1) \to (7.2)$.

Taking $p = q$ and using our assumption, this shows that the cycle class map $\mathrm{cl}^q : H^q(R, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(R, \mu_m^{\otimes q})$ is surjective. By Proposition 7.1, and the naturality of the cycle class map (Proposition 4.2(5)), the map $\mathrm{cl}^q : H^q(\partial\hat{\square}_{n+1}, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(\partial\hat{\square}_{n+1}, \mu_m^{\otimes q})$ is surjective for all $n$. We have the commutative diagram

$$
(7.3)\qquad
\begin{tikzcd}
H^q(\partial\hat{\square}_{n+1}, \mathbb{Z}/m(q)) \arrow[r, "\mathrm{cl}^q"] \arrow[d, "s"] & H^q_{\text{ét}}(\partial\hat{\square}_{n+1}, \mu_m^{\otimes q}) \arrow[d, "s"] \\
H^q(\partial\hat{\square}_{n+1}; S_{n+1}, \mathbb{Z}/m(q)) \arrow[r, "\mathrm{cl}^q"] \arrow[d, "i^*"] & H^q_{\text{ét}}(\partial\hat{\square}_{n+1}; S_{n+1}, \mu_m^{\otimes q}) \arrow[d, "i^*"] \\
H^q(\hat{\square}_n; T_n, \mathbb{Z}/m(q)) \arrow[r, "\mathrm{cl}^q"] & H^q_{\text{ét}}(\hat{\square}_n; T_n, \mu_m^{\otimes q})
\end{tikzcd}
$$

where $s$ is the natural splitting given by Lemma 5.7, and $i$ is the inclusion of the face $t_{n+1} = 0$. Thus, the map

$$
(7.4)\qquad \mathrm{cl}^q : H^q(\partial\hat{\square}_{n+1}; S_{n+1}, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(\partial\hat{\square}_{n+1}; S_{n+1}, \mu_m^{\otimes q})
$$

is surjective.

We recall that étale cohomology satisfies excision for unions of closed subschemes. Indeed, suppose $Z = Z_1 \cup Z_2$, with $Z_i$ closed in $Z$. The Mayer-Vietoris property (see e.g. the proof of Lemma 3.6) implies we have the distinguished triangle

$$
G^*(Z, \mu_m^{\otimes q}) \to G^*(Z_1, \mu_m^{\otimes q}) \oplus G^*(Z_2, \mu_m^{\otimes q}) \to G^*(Z_1 \cap Z_2, \mu_m^{\otimes q}) \to
$$

from which it follows that the natural map

$$
\operatorname{cone}(G^*(Z, \mu_m^{\otimes q}) \to G^*(Z_2, \mu_m^{\otimes q})) \to \operatorname{cone}(G^*(Z_1, \mu_m^{\otimes q}) \to G^*(Z_1 \cap Z_2, \mu_m^{\otimes q}))
$$

is a quasi-isomorphism. Thus, the restriction morphism

$$
i^*_{Z_1} : H^*_{\text{ét}}(Z, Z_2, \mu_m^{\otimes q}) \to H^*_{\text{ét}}(Z_1, Z_1 \cap Z_2, \mu_m^{\otimes q})
$$

is an isomorphism.

In particular, the map $i^* : H^q_{\text{ét}}(\partial\hat{\square}_{n+1}; S_{n+1}, \mu_m^{\otimes q}) \to H^q_{\text{ét}}(\hat{\square}_n; T_n, \mu_m^{\otimes q})$ is an isomorphism. The surjectivity of (7.4) and the commutativity of (7.3) thus prove the proposition. $\square$

We conclude this sequence of results on surjectivity with the following elementary but useful result:

**Lemma 7.3.** *The surjectivity of* $\mathrm{cl}^q_F : H^q(F, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(F, \mu_m^{\otimes q})$ *for all fields $F$ finitely generated over $k$ implies the surjectivity of* $\mathrm{cl}^a_F : H^a(F, \mathbb{Z}/m(a)) \to H^a_{\text{ét}}(F, \mu_m^{\otimes a})$ *for all fields $F$ finitely generated over $k$ and for all $a$ with $0 \le a \le q$.*

*Proof.* We proceed by downward induction on $a$. Let $R$ be the local ring $F[X]_{(X)}$, let $\eta$ be in $H^a_{\text{ét}}(F, \mu_m^{\otimes a})$, and assume that $\mathrm{cl}^{a+1}_{F(X)}$ is surjective. Let $p^* : H^a_{\text{ét}}(F, \mu_m^{\otimes a}) \to H^a_{\text{ét}}(F(X), \mu_m^{\otimes a})$ be the map induced by the inclusion $F \to F(X)$, and let $\omega = \mathrm{cl}^1(X) \cup p^*\eta$. By Proposition 4.5, we have the commutative diagram

$$\begin{CD}
H^{a+1}(F(X), \mathbb{Z}/m(a+1)) @>{\partial}>> H^a(F, \mathbb{Z}/m(a))\\
@V{\mathrm{cl}^{a+1}_{F(X)}}VV @VV{\mathrm{cl}^a_F}V\\
H^{a+1}_{\text{ét}}(F(X), \mu_m^{\otimes a+1}) @>>{\partial}> H^a_{\text{ét}}(F, \mu_m^{\otimes a}),
\end{CD}$$

where the maps $\partial$ are the boundary maps in the localization/Gysin sequence for the open complement $\operatorname{Spec} F(X)$ of $\operatorname{Spec} F$ in $\operatorname{Spec} F[X]_{(X)}$. For $a = 0$, we have canonical isomorphisms

$$H^0(F, \mathbb{Z}/m(0)) \cong \mathbb{Z}/m \cong H^0_{\text{ét}}(F, \mathbb{Z}/m),$$

and $\mathrm{cl}^0_F$ is the identity. In addition, the map

$$\partial : H^1(F(X), \mathbb{Z}(1)) \cong F(X)^\times \to H^0(F, \mathbb{Z})$$

is just the classical divisor map, hence

$$\partial(\mathrm{cl}^1(X)) = \mathrm{cl}^0(\partial(X)) = \mathrm{cl}^0(1) = 1.$$

Since the boundary map is a $H^*_{\text{ét}}(F, \mu_m^{\otimes *})$-module homomorphism, this gives the identity $\eta = \partial(\omega)$. By assumption, there is an element $z \in H^{a+1}(F(X), \mathbb{Z}/m(a+1))$ with $\mathrm{cl}^{a+1}(z) = \omega$, giving $\mathrm{cl}^a_F(\partial z) = \partial(\mathrm{cl}^{a+1}_{F(X)}(z)) = \partial(\omega) = \eta$. $\square$

7.4. **The main theorem.** We can now give the proof of our main result Theorem 1.1. By Lemma 4.10, it suffices to prove

**Theorem 7.5.** *Suppose that the maps* $\mathrm{cl}^q : H^q(F, \mathbb{Z}/m(q)) \to H^q_{\text{ét}}(F, \mu_m^{\otimes q})$ *are surjective for all fields $F$ finitely generated over $k$. The the cycle class map* (1.2) *is an isomorphism for all essentially smooth $X$ over $k$ and all $a$ with $0 \le a \le q$.*

Let $R$ be the local ring of a smooth point on a $k$-variety $X$ of finite type. To prove Theorem 7.5, it suffices to show that the map

$$\mathrm{cl}^a : H^p(R, \mathbb{Z}/m(a)) \to H^p_{\text{ét}}(R, \mu_m^{\otimes a})$$

is an isomorphism for all $p \le a \le q$. We first reduce to the case of $R$ a field which is finitely generated over $k$.

As in the proof of Proposition 7.2, we have the exact (augmented) Gersten complex for motivic cohomology (7.1), the exact (augmented) Gersten complex for étale cohomology (7.2), and the cycle class maps give the map of complexes $\mathrm{cl}^* : (7.1) \to (7.2)$. As this map in degree $a \geq 0$ is

$$\mathrm{cl}^{q-a} : \bigoplus_{x \in \mathrm{Spec}\, R^{(a)}} H^{p-a}(k(x), \mathbb{Z}/m(q-a)) \to H^{p-a}_{\text{ét}}(k(x), \mu_m^{\otimes q-a}),$$

the result for fields which are finitely generated over $k$ yields the general case.

We now handle the case of fields which are finitely generated over $k$. We proceed by induction on $q$, the case $q = 0$ being trivially satisfied.

By Lemma 7.3, the hypothesis in Theorem 7.5 implies that

$$\mathrm{cl}^a : H^a(F, \mathbb{Z}/m(a)) \to H^a_{\text{ét}}(F, \mu_m^{\otimes a})$$

is surjective for all fields $F$ finitely generated over $k$ and for all $a$ with $0 \leq a \leq q$. Using our induction hypothesis, this implies that

$$\mathrm{cl}^a : H^p(F, \mathbb{Z}/m(a)) \to H^p_{\text{ét}}(F, \mu_m^{\otimes a}) \tag{7.5}$$

is an isomorphism for all fields $F$ finitely generated over $k$, and for $p \leq a < q$. Proposition 6.5 and Proposition 7.2 together with the isomorphisms (7.5) complete the proof.

*Remark* 7.6. The idea of using Gabber's rigidity theorem to pass from a version of the Bloch-Kato conjecture for singular schemes to the usual version can be traced back to R. Hoobler [10], who used an argument of this type to extend the Merkurjev-Suslin theorem (which is the Bloch-Kato conjecture for weight two) to arbitrary semi-local rings.

7.7. We conclude by proving Corollary 1.2. Let $\mathcal{H}^b(\mathbb{Z}/m(q))$ be the Zariski sheaf of motivic cohomology on $X$, and $\mathcal{H}^b_{\text{ét}}(\mu_m^{\otimes q})$ the Zariski sheaf of étale cohomology on $X$. We have the spectral sequences

$$E_2^{a,b} = H^a(X_{\mathrm{Zar}}, \mathcal{H}^b(\mathbb{Z}/m(q))) \Longrightarrow H^{a+b}(X, \mathbb{Z}/m(q))$$
$$E_{2,\text{ét}}^{a,b} = H^a(X_{\mathrm{Zar}}, \mathcal{H}^b_{\text{ét}}(\mu_m^{\otimes q})) \Longrightarrow H^{a+b}_{\text{ét}}(X, \mu_m^{\otimes q}).$$

The cycle class map gives a map of spectral sequences $\mathrm{cl}^q : E^{**} \to E^{**}_{\text{ét}}$. If the hypotheses of Corollary 1.2 are satisfied, then, by Theorem 1.1, $\mathrm{cl}^q : \mathcal{H}^b(\mathbb{Z}/m(q)) \to \mathcal{H}^b_{\text{ét}}(\mu_m^{\otimes q})$ is an isomorphism for $b \leq q$, hence $\mathrm{cl}^q : E_2^{a,b} \to E_{2,\text{ét}}^{a,b}$ is an isomorphism for $b \leq q$. Since $E_2^{a,b} = E_{2,\text{ét}}^{a,b} = 0$ for $a < 0$, this implies that $\mathrm{cl}^q : E_\infty^{a,b} \to E_{\infty,\text{ét}}^{a,b}$ is an isomorphism for $a + b \leq q$, and for $a + b = q + 1$, $b \leq q$. Since $\mathcal{H}^b(\mathbb{Z}/m(q)) = 0$ for $b > q$, this shows that $\mathrm{cl}^q : H^n(X, \mathbb{Z}/m(q)) \to H^n_{\text{ét}}(X, \mu_m^{\otimes q})$ is an isomorphism for $n \leq q$ and an injection for $n = q + 1$, completing the proof.

## 8. Appendix–Products

The cycle complexes have a natural external product which can be constructed along the lines described in [1]. There is a gap in the construction given in *loc. cit.*, in that a part of the construction (Lemma (5.0)) relies on the incorrect proof of the "moving lemma" [1, Theorem (3.3)]. Although the results of [2] give a proof of [1, Theorem (3.3)], it is not clear that the argument given in [2] can be used to prove Lemma (5.0). Therefore, in this Appendix, we give a construction of the product on the cycle complexes.

Let $X$ be a scheme, essentially of finite type over $k$. Let $z^q(X, p, p')$ be the free abelian group on the irreducible codimension $q$ closed subsets $W$ of $X \times \Delta^p \times \Delta^{p'}$

such that each irreducible component of $W \cap X \times A \times B$ has codimension $q$ on $X \times A \times B$ for each face $A$ of $\Delta^p$ and $B$ of $\Delta^{p'}$. The $z^q(X, p, p')$ form in the evident way a bisimplicial abelian group; let $z^q(X, *, *)$ be the associated double complex, and let $\mathrm{Tot}\, z^q(X, *, *)$ be the total complex of $z^q(X, *, *)$.

We give the product set $[p] \times [q]$ the product partial order,

$$(a, b) \le (a', b') \iff a \le a' \text{ and } b \le b',$$

and identify $[p] \times [q]$ with the vertices of $\Delta^p \times \Delta^q$ in the evident way. For an order-preserving map $g : [n] \to [p] \times [q]$, the unique affine-linear extension of $g$ gives the map

$$\Delta(g) : \Delta^n \to \Delta^p \times \Delta^q.$$

A *face* of $\Delta^p \times \Delta^q$ is a subscheme of the form $\Delta(g)(\Delta^n)$ for some injective $g$. Let $z^q(X, p, p')_{\mathcal{T}}$ be the subgroup of $z^q(X, p, p')$ generated by the irreducible $W \subset X \times \Delta^p \times \Delta^{p'}$ such that, for each face $A$ of $\Delta^p \times \Delta^{p'}$, each irreducible component of $W \cap X \times A$ has codimension $q$ on $X \times A$. The $z^q(X, p, p')_{\mathcal{T}}$ form a sub-double complex $z^q(X, *, *)_{\mathcal{T}}$ of $z^q(X, *, *)$.

If $g = (g_1, g_2) : [p+q] \to [p] \times [q]$ is injective, $g$ determines a $p$-$q$-shuffle by sending $i \in \{1, \dots, p+q\}$ to $g_1(i)$ if $g_1(i-1) < g_1(i)$, and to $g_2(i) + p$ if $g_2(i-1) < g_2(i)$. Taking the sign of this permutation defines the sign $\mathrm{sgn}(g)$. Let

$$T_{p,q} : z^r(X, p, q)_{\mathcal{T}} \to z^r(X, p+q)$$

be the map $\sum_g \mathrm{sgn}(g)\Delta(g)^*$, where the sum is over $g$ as above. The map

$$T := \sum_{p,q} T_{p,q} : \mathrm{Tot}\, z^r(X, *, *)_{\mathcal{T}} \to z^r(X, *),$$

is a well-defined map of complexes.

**Lemma 8.1.** *The inclusion $i : \mathrm{Tot}\, z^q(X, *, *)_{\mathcal{T}} \to \mathrm{Tot}\, z^q(X, *, *)$ is a quasi-isomorphism.*

*Proof.* We have the standard $E^1$ spectral sequence (of homological type) for each of the double complexes $z^q(X, *, *)_{\mathcal{T}}$, $z^q(X, *, *)$,

$$E^1_{a,b} = H_b(z^q(X, a, *)) \Longrightarrow H_{a+b}(\mathrm{Tot}\, z^q(X, *, *)),$$
$${}_{\mathcal{T}}E^1_{a,b} = H_b(z^q(X, a, *)_{\mathcal{T}}) \Longrightarrow H_{a+b}(\mathrm{Tot}\, z^q(X, *, *)_{\mathcal{T}});$$

the inclusion $i$ induces the map of spectral sequences ${}_{\mathcal{T}}E \to E$. It thus suffices to show that the inclusion $i_a : z^q(X, a, *)_{\mathcal{T}} \to z^q(X, a, *)$ is a quasi-isomorphism for each $a$. The rest of the argument is similar to the proof of [1, Theorem 2.1]; we now proceed to give the necessary modifications.

The complex $z^q(X, a, *)$ is evidently a subcomplex of $z^q(X \times \Delta^a, *)$. It thus suffices to show that the two inclusions $z^q(X, a, *) \to z^q(X \times \Delta^a, *)$, $z^q(X, a, *)_{\mathcal{T}} \to z^q(X \times \Delta^a, *)$, are quasi-isomorphisms.

The coordinates $t_1, \dots, t_a$ on $\Delta^a$ give an isomorphism of $\Delta^a$ with $\mathbb{A}^a$, and thereby define an action of the group scheme $(\mathbb{A}^a, +, 0)$ on $\Delta^a$ by translation. Let $t$ be the generic point of $\mathbb{A}^a$, $K$ the field $k(t)$, and $\phi_t : \Delta^a_K \to \Delta^a_K$ the translation by $t$. Let $\Phi : \Delta^a_K \times \Delta^1 \to \Delta^a_K$ be the map $\Phi(x, (u_0, u_1)) = \phi_{u_1 t}(x) = x + u_1 t$. We let $\pi : X \times \Delta^a_K \to X \times \Delta^a$ be the projection, and $i_0, i_1 : X \times \Delta^a_K \to X \times \Delta^a_K \times \Delta^1$ the inclusions $i_0(y) = (y, (1, 0))$, $i_1(y) = (y, (0, 1))$. Note that $\Phi \circ i_0 = \mathrm{id}$, $\Phi \circ i_1 = \phi_t$.

If $F$ is a face of $\Delta^a \times \Delta^p$, then the orbit $\mathbb{A}^a \cdot F$ is of the form $\Delta^a \times F'$ for some face $F'$ of $\Delta^p$. The argument of [1, Lemma 2.2] shows that the composition

$$\Phi^* \circ \pi^* : z^q(X \times \Delta^a, *) \to z^q(X \times \Delta^a \times \Delta^1_K, *)$$

has image in the subcomplex $z^q(X \times \Delta^a_K, 1, *)_{\mathcal{T}}$ of $z^q(X \times \Delta^a \times \Delta^1_K, *)$, and that the composition

$$z^q(X \times \Delta^a, *) \xrightarrow{\Phi^* \circ \pi^*} z^q(X \times \Delta^a_K, 1, *)_{\mathcal{T}} \xrightarrow{i_1^*} z^q(X \times \Delta^a_K, *),$$

which is just $\phi_t^* \circ \pi^*$, has image in $z^q(X_K, a, *)_{\mathcal{T}}$.

We have the map $T^1 : z^q(X \times \Delta^a_K, 1, *)_{\mathcal{T}} \to z^q(X \times \Delta^a_K, *+1)$ induced by $T$. The argument of [1, Lemma 2.2] shows in addition that $T^1 \circ \Phi^* \circ \pi^*$ gives a homotopy between the compositions

$$\pi^*, \phi_t^* \circ \pi^* : z^q(X \times \Delta^a, *) \to z^q(X \times \Delta^a_K, *),$$

the compositions

$$\pi^*, \phi_t^* \circ \pi^* : z^q(X, a, *) \to z^q(X_K, a, *),$$

and the compositions

$$\pi^*, \phi_t^* \circ \pi^* : z^q(X, a, *)_{\mathcal{T}} \to z^q(X_K, a, *)_{\mathcal{T}}.$$

This implies that the maps

$$\bar\pi^* : z^q(X \times \Delta^a, *)/z^q(X, a, *) \to z^q(X_K \times \Delta^a, *)/z^q(X_K, a, *)$$
$$\bar\pi^* : z^q(X \times \Delta^a, *)/z^q(X, a, *)_{\mathcal{T}} \to z^q(X_K \times \Delta^a, *)/z^q(X_K, a, *)_{\mathcal{T}}$$

induce zero on homology. Since $\operatorname{Spec} K$ is a filtered inverse limit of finite type $k$-schemes $U$ with $U(k) \neq \emptyset$, and since the functors

$$U \mapsto z^q(X \times U \times \Delta^a, *),$$
$$U \mapsto z^q(X \times U, a, *)$$
$$U \mapsto z^q(X \times U, a, *)_{\mathcal{T}}$$

transform filtered inverse limits of schemes with flat transition maps to direct limits of complexes, it follows that the maps $\bar\pi^*$ are injective on homology: for each such $U$, the choice of a $k$-point of $U$ gives a left inverse to the maps induced by the projection $X \times U \to X$. Thus $z^q(X \times \Delta^a, *)/z^q(X, a, *)$ and $z^q(X \times \Delta^a, *)/z^q(X, a, *)_{\mathcal{T}}$ are acyclic, as desired. $\square$

Now let $X$ and $Y$ be schemes, essentially of finite type over $k$. Sending $W \in z^q(X \times \Delta^p)$ and $W' \in z^{q'}(Y \times \Delta^{p'})$ to the "product" cycle $W \times W' \in z^q(X \times_k Y \times \Delta^p \times \Delta^{p'})$ defines the map of complexes

$$\boxtimes : z^q(X, *) \otimes z^{q'}(Y, *) \to \operatorname{Tot} z^{q+q'}(X \times_k Y, *, *).$$

We have the diagram

$$\begin{aligned} (8.1) \quad z^q(X, *) \otimes z^{q'}(Y, *) \xrightarrow{\boxtimes} \operatorname{Tot} z^{q+q'}(X \times_k Y, *, *) \xleftarrow{i} \operatorname{Tot} z^{q+q'}(X \times_k Y, *, *)_{\mathcal{T}} \\ \xrightarrow{T} z^{q+q'}(X \times_k Y, *). \end{aligned}$$

Via (8.1) and Lemma 8.1, the composition $T \circ i^{-1} \circ \boxtimes$ defines the external product map in $\mathbf{D}^-(\mathbf{Ab})$

$$(8.2) \qquad \cup_{X,Y} : z^q(X, *) \otimes^L z^{q'}(Y, *) \to z^{q+q'}(X \times_k Y, *).$$

The associativity and commutativity of $\cup_{*,*}$ in $\mathbf{D}^{-}(\mathbf{Ab})$ follow easily from well-known associativity and commutativity properties of the triangulation $T$ (see e.g. [17, Chap. 3]); we leave the details to the reader.

## References

[1] Bloch, S. *Algebraic cycles and higher $K$-theory*, Adv. in Math. **61** (1986), 267-304
[2] Bloch, S. *The moving lemma for higher Chow groups* J. Algebraic Geom. **3** (1994), no. 3, 537-568.
[3] Bloch, S., Kato, K. *$p$-adic étale cohomology*, Inst. Hautes Études Sci. Publ. Math. **63** (1986), 107-152.
[4] Bloch, S., Ogus, A. *Gersten's conjecture and the homology of schemes*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 181-201 (1975).
[5] Deligne, P. **Cohomologie étale**. Sèminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$. Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. Lecture Notes in Mathematics, **569**, Springer-Verlag, Berlin-New York, 1977.
[6] Gabber, O. *Affine analog of the proper base change theorem*, Israel J. Math. **87** (1994), no. 1-3, 325-335.
[7] Geisser, T. *Resolution of Singularities*, In tribute to Oscar Zariski. Eds.: H.Hauser, J.Lipman, F.Oort, A.Quiros. Progress in Math. Birkhäuser, 1999
[8] Geisser, T. and Levine, M. *The $K$-theory of fields in characteristic $p$*, to appear in Inv. Math.
[9] Godement, R. **Topologie algébrique et théorie des faisceaux**. Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII. Hermann, Paris, 1973.
[10] Hoobler, R. *The Merkuriev-Suslin theorem for arbitrary semi-local rings*, preprint (1996).
[11] de Jong, A. J. *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. **83** (1996), 51-93.
[12] Levine, M. *Homology of algebraic varieties: An introduction to the works of Suslin and Voevodsky*, Bull. AMS **34** (1997) no. 3, 293-312.
[13] Levine, M. *Relative Milnor $K$-theory*, $K$-theory **6** (1992), 113-175.
[14] Levine, M. **Mixed Motives**, Mathematical Surveys and Monographs, **57**. American Mathematical Society, Providence, RI, 1998.
[15] Levine, M. *Techniques of localization in the theory of algebraic cycles*, preprint (1998).
[16] Lichtenbaum, S. *Motivic complexes*. Motives (Seattle, WA, 1991), 303-313, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994.
[17] May, J. P. **Simplicial objects in algebraic topology**. Reprint of the 1967 original. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.
[18] Merkurjev, A., Suslin, A. *$K$-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 1011-1046, 1135-1136.
[19] Milne, J. **Étale cohomology**. Princ. Univ. Press 1980.
[20] Nesterenko, Y. and Suslin, A. *Homology of the general linear group over a local ring, and Milnor's $K$-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146.
[21] Sherman, C. *$K$-cohomology of regular schemes*, Com.Alg. **7** (1979), 999-1027.
[22] Strano, R. *On the étale cohomology of Hensel rings*, Comm. Algebra **12** (1984), no. 17-18, 2195-2211.
[23] Suslin, A. *Higher Chow groups of affine varieties and étale cohomology*, preprint (1995).
[24] Suslin, A. and Voevodsky, V. *Singular homology of abstract algebraic varieties*, Invent. Math. **123** (1996) 61-94.
[25] Suslin, A. and Voevodsky, V. *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, preprint (1995), revised version (1999).
[26] Totaro, B. *Milnor $K$-theory is the simplest part of algebraic $K$-theory*, $K$-Theory **6** (1992), no. 2, 177–189.
[27] Voevodsky, V., *Triangulated categories of motives over a field*, Cycles, Transfers and Motivic Homology Theories, ed. V. Voevodsky, A. Suslin, E. Friedlander. To appear, Ann. of Math Studies, Princeton Univ. Press.
[28] Voevodsky, V., *The Milnor Conjecture*, preprint (1996).

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