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The Bohnenblust-Hille inequality for homogeneous polynomials is hypercontractive

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Abstract

The Bohnenblust-Hille inequality says that the $\ell^{\frac{2m}{m+1}}$-norm of the coefficients of an m-homogeneous polynomial P on $\mathbb{C}^n$ is bounded by $\|P\|_\infty$ times a constant independent of n, where $\|\cdot\|_\infty$ denotes the supremum norm on the polydisc $\mathbb{D}^n$. The main result of this paper is that this inequality is hypercontractive, i.e., the constant can be taken to be C^m for some $C > 1$. Combining this improved version of the Bohnenblust-Hille inequality with other results, we obtain the following: The Bohr radius for the polydisc $\mathbb{D}^n$ behaves asymptotically as $\sqrt{(\log n)/n}$ modulo a factor bounded away from 0 and infinity, and the Sidon constant for the set of frequencies $\{\log n : n \text{ a positive integer } \le N\}$ is $\sqrt{N}\exp\{(-1/\sqrt{2}+o(1))\sqrt{\log N \log\log N}\}$ as $N \to \infty$.

1. Introduction and statement of results

In 1930, Littlewood [23] proved the following, often referred to as Littlewood's 4/3-inequality: For every bilinear form $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ we have

$$\left(\sum_{i,j} |B(e^{(i)}, e^{(j)})|^{4/3} \right)^{3/4} \le \sqrt{2} \sup_{z^{(1)}, z^{(2)} \in \mathbb{D}^n} |B(z^{(1)}, z^{(2)})| ,$$

where $\mathbb{D}^n$ denotes the open unit polydisc in $\mathbb{C}^n$ and $\{e^{(i)}\}_{i=1,\dots,n}$ is the canonical base of $\mathbb{C}^n$. The exponent 4/3 is optimal, meaning that for smaller exponents it will not be possible to replace $\sqrt{2}$ by a constant independent of n. H. Bohnenblust and E. Hille immediately realized the importance of this result, as well as the techniques used in its proof, for what was known as Bohr's absolute convergence problem: Determine the maximal width T of the vertical strip in

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which a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly but not absolutely. The problem was raised by H. Bohr [7] who in 1913 showed that $T \leq 1/2$. It remained a central problem in the study of Dirichlet series until 1931, when Bohnenblust and Hille [6] in an ingenious way established that $T = 1/2$.

A crucial ingredient in [6] is an m-linear version of Littlewood's 4/3-inequality: For each m there is a constant $C_m \geq 1$ such that for every m-linear form $B : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ we have

$$\left(\sum_{i_1,\ldots,i_m} |B(e^{(i_1)}, \ldots, e^{(i_m)})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \sup_{z^{(i)} \in \mathbb{D}^n} |B(z^{(1)}, \ldots, z^{(m)})| , \tag{1}$$

and again the exponent $\frac{2m}{m+1}$ is optimal. Moreover, if C_m stands for the best constant, then the original proof gives that $C_m \leq m^{\frac{m+1}{2m}} (\sqrt{2})^{m-1}$. This inequality was long forgotten and rediscovered more than forty years later by A. Davie [11] and S. Kaijser [21]. The proofs in [11] and [21] are slightly different from the original one and give the better estimate

$$C_m \leq (\sqrt{2})^{m-1} . \tag{2}$$

In order to solve Bohr's absolute convergence problem, Bohnenblust and Hille needed a symmetric version of (1). For this purpose, they in fact invented polarization and deduced from (1) that for each m there is a constant $D_m \geq 1$ such for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_m \sup_{z \in \mathbb{D}^n} \left| \sum_{|\alpha|=m} a_\alpha z^\alpha \right| ; \tag{3}$$

they showed again, through a highly nontrivial argument, that the exponent $\frac{2m}{m+1}$ cannot be improved. Let us assume that D_m in (3) is optimal. By an estimate of L. A. Harris [18] for the polarization constant of ℓ^∞, getting from (2) to

$$D_m \leq (\sqrt{2})^{m-1} \frac{m^{\frac{m}{2}} (m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}}$$

is now quite straightforward; see e.g. [17, §4]. Using Sawa's Khinchine-type inequality for Steinhaus variables, H. Queffélec [25, Th. III-1] obtained the slightly better estimate

$$D_m \leq \left(\frac{2}{\sqrt{\pi}} \right)^{m-1} \frac{m^{\frac{m}{2}} (m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}} . \tag{4}$$

Our main result is that the Bohnenblust-Hille inequality (3) is in fact hypercontractive, i.e., $D_m \leq C^m$ for some $C \geq 1$:

THEOREM 1. *Let m and n be positive integers larger than 1. Then we have*

$$\Big(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le \Big(1+\frac{1}{m-1}\Big)^{m-1}\sqrt{m}(\sqrt{2})^{m-1}\sup_{z\in\mathbb{D}^n}\Big|\sum_{|\alpha|=m} a_\alpha z^\alpha\Big| \tag{5}$$

for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$.

Before presenting the proof of this theorem, we mention some particularly interesting consequences that serve to illustrate its applicability and importance.

We begin with the Sidon constant $S(m,n)$ for the index set

$$\{\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) : \ |\alpha| = m\},$$

which is defined in the following way. Let

$$P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$$

be an m-homogeneous polynomial in n complex variables. We set

$$\|P\|_\infty = \sup_{z\in\mathbb{D}^n} |P(z)| \qquad \text{and} \qquad |||P|||_1 = \sum_{|\alpha|=m} |a_\alpha|;$$

then $S(m,n)$ is the smallest constant C such that the inequality $|||P|||_1 \le C\|P\|_\infty$ holds for every P. It is plain that $S(1,n)=1$ for all n, and this case is therefore excluded from our discussion. Since the dimension of the space of m-homogeneous polynomials in $\mathbb{C}^n$ is $\binom{n+m-1}{m}$, an application of Hölder's inequality to (5) gives:

COROLLARY 1. *Let m and n be positive integers larger than 1. Then*

$$S(m,n) \le \Big(1+\frac{1}{m-1}\Big)^{m-1}\sqrt{m}(\sqrt{2})^{m-1}\binom{n+m-1}{m}^{\frac{m-1}{2m}}. \tag{6}$$

Note that the Sidon constant $S(m,n)$ coincides with the unconditional basis constant of the monomials z^α of degree m in $H^\infty(\mathbb{D}^n)$, which is defined as the best constant $C \ge 1$ such that for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{D}^n$ and any choice of scalars ε_α with $|\varepsilon_\alpha| \le 1$ we have

$$\sup_{z\in\mathbb{D}^n}\Big|\sum_{|\alpha|=m} \varepsilon_\alpha a_\alpha z^\alpha\Big| \le C \sup_{z\in\mathbb{D}^n}\Big|\sum_{|\alpha|=m} a_\alpha z^\alpha\Big|.$$

This and similar unconditional basis constants were studied in [13], where it was established that $S(m,n)$ is bounded from above and below by $n^{\frac{m-1}{2}}$ times constants depending only on m. The more precise estimate

$$S(m,n) \le C^m n^{\frac{m-1}{2}}, \tag{7}$$

with C an absolute constant, can be extracted from [15].

Note that we also have the following trivial estimate:

$$S(m,n) \leq \sqrt{\binom{n+m-1}{m}}, \tag{8}$$

which is a consequence of the Cauchy-Schwarz inequality along with the fact that the number of different monomials of degree m in n variables is $\binom{n+m-1}{m}$. Comparing (6) and (8), we see that our estimate gives a nontrivial result only in the range $\log n > m$. Using the Salem-Zygmund inequality for random trigonometric polynomials (see [20, p. 68]), one may check that we have obtained the right value for $S(m,n)$, up to a factor less than c^m with $c > 1$ an absolute constant (for a different argument see [16, (4.4)]).

We will use our estimate for $S(m,n)$ to find the precise asymptotic behavior of the n-dimensional Bohr radius, which was introduced and studied by H. Boas and D. Khavinson [5]. Following [5], we now let K_n be the largest positive number r such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ satisfy

$$\sup_{z\in r\mathbb{D}^n} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z\in \mathbb{D}^n} \left|\sum_\alpha a_\alpha z^\alpha\right|.$$

The classical Bohr radius K_1 was studied and estimated by H. Bohr [9] himself, and it was shown independently by M. Riesz, I. Schur, and F. Wiener that $K_1 = 1/3$. In [5], the two inequalities

$$\frac{1}{3}\sqrt{\frac{1}{n}} \leq K_n \leq 2\sqrt{\frac{\log n}{n}} \tag{9}$$

were established for $n > 1$. The paper of Boas and Khavinson aroused new interest in the Bohr radius and has been a source of inspiration for many subsequent papers. For some time (see for instance [4]) it was thought that the left-hand side of (9) could not be improved. However, using (7), A. Defant and L. Frerick [15] showed that

$$K_n \geq c\sqrt{\frac{\log n}{n \log\log n}}$$

holds for some absolut constant $c > 0$.

Using Corollary 1, we will prove the following estimate which in view of (9) is asymptotically optimal.

Theorem 2. *The n-dimensional Bohr radius K_n satisfies*

$$K_n \geq \gamma\sqrt{\frac{\log n}{n}}$$

for an absolute constant $\gamma > 0$.

Combining this result with the right inequality in (9), we conclude that

$$K_n = b(n)\sqrt{\frac{\log n}{n}} \tag{10}$$

with $\gamma \le b(n) \le 2$. We will in fact obtain

$$b(n) \ge \frac{1}{\sqrt{2}} + o(1)$$

when $n \to \infty$ as a lower estimate; see the concluding remark of Section 4, which contains the proof of Theorem 2.

Using a different argument, Defant and Frerick have also computed the right asymptotics for the Bohr radius for the unit ball in $\mathbb{C}^n$ with the ℓ^p norm. This result will be presented in the forthcoming paper [14].

Another interesting point is that Theorem 1 yields a refined version of a striking theorem of S. Konyagin and H. Queffeléc [22, Th. 4.3] on Dirichlet polynomials, a result that was recently sharpened by R. de la Bretèche [12]. To state this result, we define the Sidon constant $S(N)$ for the index set

$$\Lambda(N) = \Big\{\log n : \ n \text{ a positive integer } \ \le N\Big\}$$

in the following way. For a Dirichlet polynomial

$$Q(s) = \sum_{n=1}^{N} a_n n^{-s},$$

we set $\|Q\|_\infty = \sup_{t\in\mathbb{R}} |Q(it)|$ and $|||Q|||_1 = \sum_{n=1}^{N} |a_n|$. Then $S(N)$ is the smallest constant C such that the inequality $|||Q|||_1 \le C\|Q\|_\infty$ holds for every Q.

THEOREM 3. *We have*

$$S(N) = \sqrt{N}\exp\left\{\left(-\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log N \operatorname{loglog} N}\right\} \tag{11}$$

when $N \to \infty$.

The inequality

$$S(N) \ge \sqrt{N}\exp\left\{\left(-\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log N \operatorname{loglog} N}\right\}$$

was established by R. de la Bretèche [12] combining methods from analytic number theory with probabilistic arguments. It was also shown in [12] that the inequality

$$S(N) \le \sqrt{N}\exp\left\{\left(-\frac{1}{2\sqrt{2}} + o(1)\right)\sqrt{\log N \log\log N}\right\}$$

follows from an ingenious method developed by Konyagin and Queffélec in [22]. The same argument, using Theorem 1 instead of the weaker inequality (4), gives (11). More precisely, following Bohr, we set $z_j = p_j^{-s}$, where $p_1, p_2, \ldots$

denote the prime numbers ordered in the usual way, and make accordingly a translation of Theorem 1 into a statement about Dirichlet polynomials; we then replace Lemme 2.4 in [12] by this version of Theorem 1 and otherwise follow the arguments in Section 2.2 of [12] step by step.

Theorem 3 enables us to make a nontrivial remark on Bohr's absolute convergence problem. To this end, we recall that a theorem of Bohr [8] says that the abscissa of uniform convergence equals the abscissa of boundedness and regularity for a given Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$; the latter is the infimum of those σ_0 such that the function represented by the Dirichlet series is analytic and bounded in $\Re s = \sigma > \sigma_0$. When discussing Bohnenblust and Hille's solution of Bohr's problem, it is therefore quite natural to introduce the space $\mathscr{H}^\infty$, which consists of those bounded analytic functions f in $\mathbb{C}_+ = \{s = \sigma + \mathrm{i}\, t : \ \sigma > 0\}$ such that f can be represented by an ordinary Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ in some half-plane.

COROLLARY 2. *The supremum of the set of real numbers c such that*

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\frac{1}{2}} \exp\Big\{c\sqrt{\log n \log\log n}\Big\} < \infty \tag{12}$$

for every $\sum_{n=1}^{\infty} a_n n^{-s}$ in $\mathscr{H}^\infty$ equals $1/\sqrt{2}$.

This result is a refinement of a theorem of R. Balasubramanian, B. Calado, and H. Queffélec [1, Th. 1.2], which implies that (12) holds for every $\sum_{n=1}^{\infty} a_n n^{-s}$ in $\mathscr{H}^\infty$ if c is less than $1/(2\sqrt{2})$. We will present the deduction of Corollary 2 from Theorem 3 in Section 5 below.

An interesting consequence of the theorem of Balasubramanian, Calado, and Queffélec is that the Dirichlet series of an element in $\mathscr{H}^\infty$ converges absolutely on the vertical line $\sigma = 1/2$. But Corollary 2 gives a lot more; it adds a level precision that enables us to extract much more precise information about the absolute values $|a_n|$ than what is obtained from the solution of Bohr's absolute convergence theorem.

2. **Preliminaries on multilinear forms**

We begin by fixing some useful index sets. For two positive integers m and n, both assumed to be larger than 1, we define

$$M(m,n) = \Big\{ i = (i_1, \dots, i_m) : \quad i_1, \dots, i_m \in \{1, \dots, n\} \Big\}$$

and

$$J(m,n) = \Big\{ j = (j_1, \dots, j_m) \in M(m,n) : \quad j_1 \le \dots \le j_m \Big\}.$$

For indices $i, j \in M(m,n)$, the notation $i \sim j$ will mean that there is a permutation σ of the set $\{1, 2, \dots, m\}$ such that $i_{\sigma(k)} = j_k$ for every $k = 1, \dots, m$. For a given index i, we denote by $[i]$ the equivalence class of all indices j such that $i \sim j$. Moreover, we let $|i|$ denote the cardinality of $[i]$ or in other words the number of different indices belonging to $[i]$. Note that for each $i \in M(m,n)$ there is a unique $j \in J(m,n)$ with $[i]=[j]$. Given an index i in $M(m,n)$, we set $i^k = (i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_m)$, which is then an index in $M(m-1,n)$.

The transformation of a homogeneous polynomial to a corresponding multilinear form will play a crucial role in the proof of Theorem 1. We denote by B an m-multilinear form on $\mathbb{C}^n$; i.e., given m points $z^{(1)}, \dots, z^{(m)}$ in $\mathbb{C}^n$, we set

$$B(z^{(1)}, \dots, z^{(m)}) = \sum_{i \in M(m,n)} b_i z^{(1)}_{i_1} \cdots z^{(m)}_{i_m}.$$

We may express the coefficients as $b_i = B(e^{(i_1)}, \dots, e^{(i_m)})$. The form B is symmetric if for every permutation σ of the set $\{1, 2, \dots, m\}$, $B(z^{(1)}, \dots, z^{(m)}) = B(z^{(\sigma(1))}, \dots, z^{(\sigma(m))})$. If we restrict a symmetric multilinear form to the diagonal $P(z) = B(z, \dots, z)$, then we obtain a homogeneous polynomial. The converse is also true: Given a homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ of degree m, by polarization, we may define the symmetric m-multilinear form $B : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ so that $B(z, \dots, z) = P(z)$. In what follows, B will denote the symmetric m-multilinear form obtained in this way from P.

It will be important for us to be able to relate the norms of P and B. It is plain that $\|P\|_\infty = \sup_{z \in \mathbb{D}^n} |P(z)|$ is smaller than $\sup_{\mathbb{D}^n \times \cdots \times \mathbb{D}^n} |B|$. On the other hand, it was proved by Harris [18] that we have, for nonnegative integers $m_1, \dots, m_k$ with $m_1 + \cdots + m_k = m$,

$$|B(\underbrace{z^{(1)}, \dots, z^{(1)}}_{m_1}, \dots, \underbrace{z^{(k)}, \dots, z^{(k)}}_{m_k})| \leq \frac{m_1! \cdots m_k!}{m_1^{m_1} \cdots m_k^{m_k}} \frac{m^m}{m!} \, \|P\|_\infty. \tag{13}$$

Given an m-homogeneous polynomial in n variables $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$, we will write it as

$$P(z) = \sum_{j \in J(m,n)} c_j z_{j_1} \cdots z_{j_m}.$$

For every i in $M(m,n)$, we set $c_{[i]} = c_j$ where j is the unique element of $J(m,n)$ with $i \sim j$. Observe that in this representation the coefficient b_i of the multilinear form B associated to P can be computed from its corresponding coefficient: $b_i = c_{[i]}/|i|$.

3. **Proof of Theorem 1**

For the proof of Theorem 1, we will need two lemmas. The first is due to R. Blei [3, Lemma 5.3]:

LEMMA 1. *For all families $(c_i)_{i \in M(m,n)}$ of complex numbers, we have*

$$\left(\sum_{i \in M(m,n)} |c_i|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq \prod_{1 \leq k \leq m} \left[\sum_{i_k=1}^{n} \left(\sum_{i^k \in M(m-1,n)} |c_i|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{m}}.$$

We now let μ^n denote normalized Lebesgue measure on $\mathbb{T}^n$; the second lemma is a result of F. Bayart [2, Th. 9], whose proof relies on an inequality first established by A. Bonami [10, Th. 7, Ch. III].

LEMMA 2. *For every m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$, we have*

$$\Big(\sum_{|\alpha|=m} |a_\alpha|^2\Big)^{\frac{1}{2}} \le (\sqrt{2})^m \Big\| \sum_{|\alpha|=m} a_\alpha z^\alpha \Big\|_{L^1(\mu^n)}.$$

We note also that Lemma 2 is a special case of a variant of Bayart's theorem found in [19], relying on an inequality in D. Vukotic's paper [26]. The latter inequality, giving the best constant in an inequality of Hardy and Littlewood, appeared earlier in a paper of M. Mateljević [24].

Proof of Theorem 1. We write the homogeneous polynomial P as

$$P(z) = \sum_{j\in J(m,n)} c_j z_{j_1} \cdots z_{j_m}.$$

We now get

$$\sum_{j\in J(m,n)} |c_j|^{\frac{2m}{m+1}} = \sum_{i\in M(m,n)} |i|^{-\frac{1}{m+1}} \left(\frac{|c_{[i]}|}{|i|^{\frac{1}{2}}}\right)^{\frac{2m}{m+1}} \le \sum_{i\in M(m,n)} \left(\frac{|c_{[i]}|}{|i|^{\frac{1}{2}}}\right)^{\frac{2m}{m+1}}.$$

Using Lemma 1 and the estimate $|i|/|i^k| \le m$, we therefore obtain

$$\begin{aligned}\Big(\sum_{j\in J(m,n)} |c_j|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} &\le \prod_{k=1}^m \left[\sum_{i_k=1}^n \Big(\sum_{i^k\in M(m-1,n)} \frac{|c_{[i]}|^2}{|i|}\Big)^{\frac{1}{2}}\right]^{\frac{1}{m}} \\ &\le \sqrt{m} \prod_{k=1}^m \left[\sum_{i_k=1}^n \Big(\sum_{i^k\in M(m-1,n)} |i^k| \frac{|c_{[i]}|^2}{|i|^2}\Big)^{\frac{1}{2}}\right]^{\frac{1}{m}}.\end{aligned}$$

Thus it suffices to prove that

$$\sum_{i_k=1}^n \Big(\sum_{i^k\in M(m-1,n)} |i^k| \frac{|c_{[i]}|^2}{|i|^2}\Big)^{\frac{1}{2}} \le \left(1+\frac{1}{m-1}\right)^{m-1} (\sqrt{2})^{m-1} \|P\|_\infty \tag{14}$$

for $k = 1, 2, \ldots, m$.

We observe that if we write $P_k(z) = B(z, \ldots, z, e^{(i_k)}, z, \ldots, z)$, then we have

$$\Big(\sum_{i^k\in M(m-1,n)} |i^k| \frac{|c_{[i]}|^2}{|i|^2}\Big)^{\frac{1}{2}} = \Big(\sum_{i^k\in M(m-1,n)} |i^k| |b_i|^2\Big)^{\frac{1}{2}} = \|P_k\|_2.$$

Hence, applying Lemma 2 to P_k, we get

$$\Bigg(\sum_{i^k\in M(m-1,n)} |i^k|\frac{|c_{[i]}|^2}{|i|^2}\Bigg)^{\frac{1}{2}} \le (\sqrt{2})^{m-1}\int_{\mathbb{T}^n} |B(z,\dots,z,e^{(i_k)},z,\dots,z)|\,d\mu^n(z).$$

It is clear that we may replace $e^{(i_k)}$ by $\lambda_{i_k}(z)e^{(i_k)}$ with $\lambda_{i_k}(z)$ any point on the unit circle. If we choose $\lambda_{i_k}(z)$ such that $B(z,\dots,z,\lambda_{i_k}(z)e^{(i_k)},z,\dots,z)>0$ and write $\tau_k(z)=\sum_{i_k=1}^n \lambda_{i_k}(z)e^{(i_k)}$, then we obtain

$$\sum_{i_k=1}^n\Bigg(\sum_{i^k\in M(m-1,n)} |i^k|\frac{|c_{[i]}|^2}{|i|^2}\Bigg)^{\frac{1}{2}} \le (\sqrt{2})^{m-1}\int_{\mathbb{T}^n} B(z,\dots,z,\tau_k(z),z,\dots,z)\,d\mu(z).$$

We finally arrive at (14) by applying (13) to the right-hand side of this inequality. □

4. **Proof of Theorem 2**

We now turn to multidimensional Bohr radii. In [16, Th. 2.2], a basic link between Bohr radii and unconditional basis constants was given. Indeed, we have

$$\frac{1}{3\sup_m \sqrt[m]{C_m}} \le K_n \le \min\Bigg(\frac{1}{3},\ \frac{1}{\sup_m \sqrt[m]{C_m}}\Bigg),$$

where C_m is the unconditional basis constant of the monomials of degree m in $H^\infty(\mathbb{D}^n)$. Thus the estimates for unconditional basis constants for m-homogeneous polynomials always lead to estimates for multidimensional Bohr radii.

We still choose to present a direct proof of Theorem 2, as this leads to a better estimate on the asymptotics of the quantity $b(n)$ in (10). We need the following lemma of F. Wiener (see [5]).

LEMMA 3. *Let P be a polynomial in n variables and $P=\sum_{m\ge 0}P_m$ its expansion in homogeneous polynomials. If $\|P\|_\infty\le 1$, then $\|P_m\|_\infty \le 1-|P_0|^2$ for every $m>0$.*

Proof of Theorem 2. We assume that $\sup_{\mathbb{D}^n}\left|\sum a_\alpha z^\alpha\right|\le 1$. Observe that for all z in $r\mathbb{D}^n$,

$$\sum |a_\alpha z^\alpha| \le |a_0| + \sum_{m>1} r^m \sum_{|\alpha|=m}|a_\alpha|.$$

If we take into account the estimates

$$\frac{(\log n)^m}{n}\le m! \qquad \text{and} \qquad \binom{n+m-1}{m}\le e^m\Big(1+\frac{n}{m}\Big)^m,$$

then Corollary 1 and Lemma 3 give

$$\sum_{m>1} r^m\sum_{|\alpha|=m}|a_\alpha| \le \sum_{m>1} r^m e\sqrt{m}(2\sqrt{e})^m\Bigg(\frac{n}{\log n}\Bigg)^{m/2}(1-|a_0|^2).$$

Choosing $r \le \varepsilon\sqrt{\frac{\log n}{n}}$ with ε small enough, we obtain

$$\sum |a_\alpha z^\alpha| \le |a_0| + (1 - |a_0|^2)/2 \le 1$$

whenever $|a_0| \le 1$. Thus the theorem is proved with $\gamma = \varepsilon$. □

A closer examination of this proof shows that we get a better constant if in the range $m > \log n$ we use (8) instead of Corollary 1. By this approach, we get

$$b(n) \ge \frac{1}{\sqrt{2}} + o(1)$$

when $n \to \infty$.

5. **Proof of Corollary 2**

We need the following auxiliary result [1, Lemma 1.1].

LEMMA 4. *If $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ belongs to $\mathscr{H}^\infty$, then we have*

$$\left\|\sum_{n=1}^{N} a_n n^{-s}\right\|_\infty \le C \log N \sup_{\sigma>0} |f(\sigma + it)| \tag{15}$$

for an absolute constant C and every $N \ge 2$.

Proof of Corollary 2. For this proof, we will use the notation $n_k = 2^k$. Assume first that $c < 1/\sqrt{2}$, and suppose we are given an arbitrary element $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ in $\mathscr{H}^\infty$. Then we have

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\frac{1}{2}} \exp\Big\{c\sqrt{\log n \log\log n}\Big\}$$
$$\le \sum_{k=0}^{\infty} n_k^{-\frac{1}{2}} \exp\Big\{c\sqrt{\log n_k \log\log n_k}\Big\} \sum_{n=1}^{n_{k+1}} |a_n|.$$

Applying Theorem 3 and Lemma 4 to each of the sums $\sum_{n=1}^{n_{k+1}} |a_n|$, we see that the right-hand is finite.

On the other hand, assume instead that $c > 1/\sqrt{2}$. By Theorem 3, we may find a positive constant δ and a sequence of Dirichlet polynomials

$$Q_k(s) = \sum_{n=1}^{n_{2k}-1} a_n^{(k)} n^{-s}$$

such that $\|Q_k\|_\infty = 1$ and

$$\sum_{n=1}^{n_{2k}-1} |a_n^{(k)}| \ge \delta n_{2k}^{\frac{1}{2}} \exp\Big\{-c\sqrt{\log n_{2k} \operatorname{loglog} n_{2k}}\Big\}$$

for $k = 1, 2, ...$ In fact, by the construction in [12, §2.1], we may assume that

$$\sum_{n=n_{2(k-1)}}^{n_{2k}-1} |a_n^{(k)}| \geq \delta n_{2k}^{\frac{1}{2}} \exp\Big\{-c\sqrt{\log n_{2k} \operatorname{loglog} n_{2k}}\Big\} \tag{16}$$

for $k = 1, 2, \ldots$. We observe that the function

$$f(s) = \sum_{k=1}^{\infty} \exp\Big\{-\varepsilon\sqrt{\log n_{2k} \operatorname{loglog} n_{2k}}\Big\}\ Q_k(s)$$

is an element in $\mathscr{H}^\infty$ for every positive ε. Setting $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and assuming again that Q_k has been constructed as in [12, §2.1], we get that

$$\sum_{n=n_{2(k-1)}}^{n_{2k}-1} |a_n| \geq C \sum_{n=n_{2(k-1)}}^{n_{2k}-1} |a_n^{(k)}| \exp\Big\{-\varepsilon\sqrt{\log n_{2k} \operatorname{loglog} n_{2k}}\Big\}$$

for some constant C independent of k and ε. (Here the point is that $a_n^{(j)}$ decays sufficiently fast when j grows because $n_{2(j+1)} = 4n_{2j}$.) Combining this estimate with (16), we see that

$$\sum_{n=1}^{\infty} |a_n| n^{-\frac{1}{2}} \exp\Big\{\Big(c+\varepsilon\Big)\sqrt{\log n \log\log n}\Big\} = \infty.$$

Since this can be achieved for arbitrary $c > 1/\sqrt{2}$ and $\varepsilon > 0$, the result follows. □

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