

THE BOND-BASED PERIDYNAMIC SYSTEM WITH DIRICHLET-TYPE VOLUME CONSTRAINT

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ABSTRACT. In this paper, the bond-based peridynamic system is analyzed as a nonlocal boundary value problem with volume constraint. The study extends earlier works in the literature on nonlocal diffusion and nonlocal peridynamic models to include non-positive definite kernels. We prove the well-posedness of both linear and nonlinear variational problems with volume constraints. The analysis is based on some nonlocal Poincaré type inequalities and compactness of the associated nonlocal operators. It also offers careful characterizations of the associated solution spaces such as compact embedding, separability and completeness. In the limit of vanishing nonlocality, the convergence of the peridynamic system to the classical Navier equations of elasticity is demonstrated.

Keywords: peridynamic model, nonlocal diffusion, nonlocal operator, fractional Sobolev spaces, nonlocal Poincaré inequality, well-posedness

AMS Subject Classifications: 45A05, 45K05, 47G10, 74G65, 74H20, 74H25

1. INTRODUCTION

Nonlocal models are becoming ubiquitous in describing physical and social phenomena. The monograph [3] presents results on the nonlocal equations that approximate, for instance, equations of porous media flow. The recent paper [22] models self-organized dynamics using nonlocal equations. A study of nonlocal operators with application to image processing is given in [17] and [4]. Other areas of application include, to name a few, modeling wave propagation, pattern formation and population aggregation.

Of interest to us is the recent nonlocal reformulation of the basic equations of motion in a continuous body, originally proposed by Silling [23]. In its core this nonlocal continuum model, called peridynamics (PD), uses integration in lieu of differentiation to compute the force on a material particle by summing up interactions with other near-by particles. The model completely avoids spacial derivatives and is found to be effective in modeling problems related to the spontaneous formation of discontinuities in solids

The PD system of equations of motion, for a bond-based materials [23], is

$$(1.1) \quad m(\mathbf{x})\mathbf{u}_{tt}(\mathbf{x}, t) = \int_{B_\delta(\mathbf{x}) \cap \Omega} \mathbf{f}(\mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}', t) - \mathbf{u}(\mathbf{x}, t))d\mathbf{x}' + \mathbf{b}(\mathbf{x}, t)$$

The research is supported in part by the U.S. Department of Energy grant DE-SC0005346 and the U.S. NSF grant DMS-1016073.

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where the body has a mass density $m(\mathbf{x})$ and occupies the bounded domain Ω . The vector valued function $\mathbf{f}(\boldsymbol{\xi}, \boldsymbol{\eta})$ is a pairwise force density function that contains all constitutive relations. The function \mathbf{u} is the displacement field and \mathbf{b} is a given loading force density function. The parameter $\delta > 0$ is called the horizon and specifies the extent of the nonlocal interaction. The first expression on the right hand side of (1.1) represents the force per unit reference volume at a particle due to interaction with other particles.

Focusing on the case when the relative displacement $|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|$ is uniformly small, our goal in this paper is to closely study the linearized equilibrium equations corresponding to (1.1). For a class of bond-based materials, namely microelastic and isotropic, the linearized PD equilibrium system takes the form

$$(1.2) \quad - \int_{B_\delta(\mathbf{x}) \cap \Omega} \mathbb{C}(\mathbf{x}' - \mathbf{x})(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}' = \mathbf{b}(\mathbf{x}),$$

where the matrix function $\mathbb{C}(\boldsymbol{\xi})$, called the micromodulus tensor, is

$$(1.3) \quad \mathbb{C}(\boldsymbol{\xi}) = 2 \frac{\rho(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^2} \boldsymbol{\xi} \otimes \boldsymbol{\xi} + 2F_0(|\boldsymbol{\xi}|)\mathbb{I}$$

where \mathbb{I} is the identity matrix and ρ and F_0 are given radial functions. Equation (1.2) will be accompanied by a "boundary" condition, imposed as a volumetric constraint. This is in contrast to local problems where conditions are prescribed on the boundary of the domain. We will study the linear system (1.2) as a nonlocal boundary value problem with Dirichlet-type volumetric boundary conditions. We prove well posedness of the system after establishing some structural properties of the function space associated with it, and in the event of vanishing of nonlocality we show that (1.2) approximates the Navier equations of linear elasticity.

We briefly describe the content of this paper and, along the way, the contributions of our work.

First, to apply standard variational techniques, we present a careful study of the function space associated with (1.2):

$$\mathcal{S}(\Omega) = \left\{ \mathbf{u} \in L^2(\Omega; \mathbb{R}^d) : \int_{\Omega} \int_{\Omega} \rho(|\mathbf{x}' - \mathbf{x}|) \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} < \infty \right\}.$$

To begin with, if $\rho(|\boldsymbol{\xi}|) \in L^1_{loc}(\mathbb{R}^d)$, then we will show that the space $\mathcal{S}(\Omega)$ is precisely $L^2(\Omega; \mathbb{R}^d)$. In the absence of this condition however, $\mathcal{S}(\Omega)$ is a proper subset of $L^2(\Omega; \mathbb{R}^d)$ and obtaining desirable structural properties of $\mathcal{S}(\Omega)$ is the key issue. One way to get such properties is to determine kernels ρ where $\mathcal{S}(\Omega)$ coincides with well known spaces, such as the fractional Sobolev spaces. This has been done in [14, 27] for cases where $\Omega = \mathbb{R}^d$, or for a subspace of functions satisfying certain periodic boundary conditions, where Fourier transform/expansion can be used to characterize $\mathcal{S}(\Omega)$ and the underlying nonlocal operator. Indeed, when ρ is comparable to $|\boldsymbol{\xi}|^{-d-2s}$ for $s \in (0, 1)$ and $\Omega = \mathbb{R}^d$, the space $\mathcal{S}(\Omega)$ coincides with $H^s(\Omega; \mathbb{R}^d)$. For bounded domains, however, it is not obvious that $\mathcal{S}(\Omega)$ coincides with fractional Sobolev spaces, partly due to the lack of Korn-type characterization of fractional Sobolev spaces. As such, basic structural properties such as completeness, compactness and separability need to be shown from scratch using the space $\mathcal{S}(\Omega)$ as is. In Section 2, we study $\mathcal{S}(\Omega)$ and the energy space corresponding to the nonlocal Dirichlet-type boundary value problem. We establish the completeness

and separability in an appropriate norm and present conditions on ρ that guarantee compactness in $L^2(\Omega; \mathbb{R}^d)$. A nonlocal Poincaré-type inequality that holds in the energy space can then be proved and in turn it is used to establish the weak coercivity of the underlying peridynamic operator. We should mention that the analysis carried out benefits from the works Bourgain, Brézis and Mironescu [5], Brézis [7] and the vector extension [20].

Second, although analytical and numerical aspects of the linear model (1.2) have been studied by [1, 11, 14, 15, 16, 18, 27], they have done so under the assumption that $F_0 \equiv 0$. However, Silling in [23] argued that this assumption is too restrictive to impose in modeling real materials, and that not only F_0 is not identically zero but also must change sign. This is the physical motivation for our work of analyzing (1.2) with ρ nonnegative but F_0 sign changing. The extra requirement creates the challenge of having a micromodulus tensor $\mathbb{C}(\boldsymbol{\xi})$ that has sign changing eigenvalues $\rho(|\boldsymbol{\xi}|) + F_0(|\boldsymbol{\xi}|)$ and $F_0(|\boldsymbol{\xi}|)$. This difficulty is circumvented here by interpreting (1.2) as a small or compact perturbation of a well behaved system, thus extending similar results obtained in [1, 11, 14, 15, 16, 18, 27] for physically more realistic cases involving indefinite kernels. In Section 3 we define, and gather necessary properties for, both the nonlocal "leading" and the peridynamic operators. In Section 4 we prove the well posedness (1.2) as a nonlocal constrained value problem with Dirichlet-type volume constraint. In the same section we also discuss some related nonlinear variational problems.

Finally, in Section 5 we demonstrate that in the limit of vanishing nonlocality, solutions to the nonlocal problems approximate the classical Navier systems. In the absence of the perturbation due to the presence of F_0 the result is already proved in [14, 27, 12, 15]. With proper scaling, we show, the solutions of the nonlocal system (1.2) strongly converge in $L^2(\Omega; \mathbb{R}^d)$ to that of the same Navier equation even with the presence of F_0 . This is significant in the sense that while the 'large scale' Navier system does not see the effect of the addition of F_0 , the 'small scale' peridynamic system does indeed detect the effect. The convergence is possible due to the compactness result proved in [5, 20] and a tighter version of Poincaré-type inequality whose proofs uses arguments similar to that of A. Ponce used in [25] for scalar functions.

2. THE ENERGY SPACE

2.1. Definition and notation. Throughout the paper we take Ω to be a connected bounded domain with sufficiently smooth boundary. We assume that:

$$(H) \quad \begin{cases} \rho \text{ is nonnegative, radial, compactly supported and} \\ \hat{\rho}(|\boldsymbol{\xi}|) = |\boldsymbol{\xi}|^2 \rho(|\boldsymbol{\xi}|) \in L^1_{loc}(\mathbb{R}^d), \\ \text{and there exists a constant } \sigma > 0, \text{ such that } (0, \sigma) \subset \text{supp}(\rho). \end{cases}$$

The radial function ρ is not necessarily locally integrable rather we assume that it has finite second moments, a property that is necessary to ensure well-defined elastic moduli [23]. As we proceed, additional necessary conditions on ρ will be provided.

We also assume that F_0 is a radial locally integrable function in \mathbb{R}^d . Unlike ρ , F_0 may be sign changing with additional conditions being imposed later.

As we discussed in the introduction the space of vector fields that will contain our equilibrium solution is

$$\mathcal{S}(\Omega) = \left\{ \mathbf{u} \in L^2(\Omega; \mathbb{R}^d) : \int_{\Omega} \int_{\Omega} \rho(|\mathbf{x}' - \mathbf{x}|) \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} < \infty \right\}.$$

Obviously $\mathcal{S}(\Omega)$ is a subspace of $L^2(\Omega; \mathbb{R}^d)$. Let the bilinear form $((\cdot, \cdot)) : \mathcal{S}(\Omega) \times \mathcal{S}(\Omega) \rightarrow \mathbb{R}$ be defined by

$$(2.1) \quad ((\mathbf{u}, \mathbf{w})) = \int_{\Omega} \int_{\Omega} \rho(|\mathbf{x}' - \mathbf{x}|) \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})) d\mathbf{x}' d\mathbf{x}.$$

Denoting the L^2 inner product by (\cdot, \cdot) , the space $(\mathcal{S}(\Omega), (\cdot, \cdot)_s)$ is then a real inner product space with the inner product $(\cdot, \cdot)_s$ defined as

$$(2.2) \quad (\mathbf{u}, \mathbf{w})_s = (\mathbf{u}, \mathbf{w}) + ((\mathbf{u}, \mathbf{w}))$$

We use the notation $\|\mathbf{u}\|$ to denote the L^2 norm of \mathbf{u} and $|\mathbf{u}|_s$ to denote the seminorm $\sqrt{((\mathbf{u}, \mathbf{u}))}$ of \mathbf{u} in $\mathcal{S}(\Omega)$ and $\|\cdot\|_s$ to denote the norm on $\mathcal{S}(\Omega)$:

$$\|\mathbf{u}\|_s^2 = \|\mathbf{u}\|_{L^2}^2 + |\mathbf{u}|_s^2.$$

We remark that, using the function $\hat{\rho}$ introduced in (H), the above bilinear form can be equivalently expressed as

$$((\mathbf{u}, \mathbf{w})) = \int_{\Omega} \int_{\Omega} \hat{\rho}(|\mathbf{x} - \mathbf{x}'|) \left(\frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \right) \left(\frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} \cdot (\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})) \right) d\mathbf{x}' d\mathbf{x}.$$

which enables us make connections between $((\cdot, \cdot))$ and standard (local) bilinear forms. Indeed, for smooth \mathbf{u} , say $\mathbf{u} \in C^2(\bar{\Omega}; \mathbb{R}^d)$, we have

$$\frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \approx \langle e(\nabla \mathbf{u}), \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|}, \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \rangle \quad \text{for all } \mathbf{x}', \mathbf{x} \in \Omega$$

where $e(\nabla \mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the symmetrized gradient. Thus one may think of $((\mathbf{u}, \mathbf{w}))$ given by (2.1) as a nonlocal counterpart of the local inner product $(e(\nabla \mathbf{u}), e(\nabla \mathbf{w}))$.

2.2. Some structural properties of \mathcal{S} . We prove in this section, among other things, that function space $\mathcal{S}(\Omega)$ actually is a complete inner product space with the inner product given (2.2). As discussed earlier if $\rho(|\boldsymbol{\xi}|)$ satisfies the additional condition

$$(2.3) \quad \rho(|\boldsymbol{\xi}|) \in L^1_{loc}(\mathbb{R}^d),$$

then the space $\mathcal{S}(\Omega)$ is precisely $L^2(\Omega; \mathbb{R}^d)$ and in this case there is nothing to prove. But the completeness of the space $\mathcal{S}(\Omega)$ in the event that (2.3) does not hold requires justification. Related results in [14, 27] are applicable when Ω is the whole space or when the functions in $\mathcal{S}(\Omega)$ satisfy certain periodicity conditions. In these special cases, the applicability of the Fourier transform or Fourier series expansion in characterizing the function space proves to be convenient. In this work, our argument for the completeness of $\mathcal{S}(\Omega)$ is more direct and general as it is applicable for some nonnegative radial ρ on any connected and bounded and measurable domain Ω .

Theorem 2.1. *Assume that ρ satisfies (H). Then $(\mathcal{S}(\Omega), (\cdot, \cdot)_s)$ is a Hilbert space.*

Proof. It suffices to check that the space $\mathcal{S}(\Omega)$ is complete since we already know it is an inner product space. To that end, suppose that $\{\mathbf{u}_n\} \in \mathcal{S}(\Omega)$ is a Cauchy sequence in $\mathcal{S}(\Omega)$. Then the sequence $\{\mathbf{u}_n\}$ is also Cauchy in $L^2(\Omega; \mathbb{R}^d)$ and therefore converges to some $\{\mathbf{u}\}$ strongly in $L^2(\Omega; \mathbb{R}^d)$. We claim that $\{\mathbf{u}_n\}$ actually converges to \mathbf{u} in $\mathcal{S}(\Omega)$. To show that we only need to check that $|\mathbf{u}_n - \mathbf{u}|_s \rightarrow 0$ as $n \rightarrow \infty$. Let $\epsilon > 0$. Choose K large such that for all $n, m \geq K$

$$|\mathbf{u}_n - \mathbf{u}_m|_s^2 \leq \epsilon^2.$$

For all $\tau > 0$, we denote $\rho_\tau(r) = \rho(r)\chi_{[\tau, \infty)}(r)$. Then for all n, m

$$\int_{\Omega} \int_{\Omega} \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot ((\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x}') - (\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} \leq |\mathbf{u}_n - \mathbf{u}_m|_s^2.$$

Since the kernel is integrable the left hand side can be written as

$$-2 \int_{\Omega} \left(\int_{\Omega} \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \frac{(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x}') - (\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x}) d\mathbf{x}' \right) (\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x}) d\mathbf{x}.$$

Note that for a fixed n , and all $\mathbf{x} \in \Omega$,

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_{\Omega} \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \frac{(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} [(\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x}') - (\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x})] d\mathbf{x}' \\ = \int_{\Omega} \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \frac{(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} [(\mathbf{u}_n - \mathbf{u})(\mathbf{x}') - (\mathbf{u}_n - \mathbf{u})(\mathbf{x})] d\mathbf{x}' \end{aligned}$$

and therefore by dominated convergence theorem, for all $\tau > 0$ and for all $n \geq K$,

$$\begin{aligned} -2 \int_{\Omega} \left(\int_{\Omega} \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \frac{(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} [(\mathbf{u}_n - \mathbf{u})(\mathbf{x}') - (\mathbf{u}_n - \mathbf{u})(\mathbf{x})] d\mathbf{x}' \right) (\mathbf{u}_n - \mathbf{u})(\mathbf{x}) d\mathbf{x} \\ = \lim_{m \rightarrow \infty} -2 \int_{\Omega} \left(\int_{\Omega} \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \frac{(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} [(\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x}') - (\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x})] d\mathbf{x}' \right) (\mathbf{u}_n - \mathbf{u}_m)(\mathbf{x}) d\mathbf{x} \leq \epsilon^2. \end{aligned}$$

That is, for all $\tau > 0$ and for all $n \geq K$,

$$-2 \int_{\Omega} \left(\int_{\Omega} \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \frac{(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} [(\mathbf{u}_n - \mathbf{u})(\mathbf{x}') - (\mathbf{u}_n - \mathbf{u})(\mathbf{x})] d\mathbf{x}' \right) (\mathbf{u}_n - \mathbf{u})(\mathbf{x}) d\mathbf{x} \leq \epsilon^2.$$

Rewriting the left hand expression, which we can do since the kernel is integrable, we have for each $\tau > 0$ and $n \geq K$

$$\int_{\Omega} \int_{\Omega} \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \left| \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot [(\mathbf{u}_n - \mathbf{u})(\mathbf{x}') - (\mathbf{u}_n - \mathbf{u})(\mathbf{x})] \right|^2 d\mathbf{x}' d\mathbf{x} \leq \epsilon^2.$$

Now applying Fatou's lemma, we obtain that for all $n \geq K$,

$$\int_{\Omega} \int_{\Omega} \rho(|\mathbf{x} - \mathbf{x}'|) \left| \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot ((\mathbf{u}_n - \mathbf{u})(\mathbf{x}') - (\mathbf{u}_n - \mathbf{u})(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} \leq \epsilon^2,$$

proving that for any $\epsilon > 0$, there exists K large such that

$$|\mathbf{u}_n - \mathbf{u}|_s \leq \epsilon, \quad \forall n \geq K,$$

This completes the proof of the theorem. \square

The next result gives a relationship between the standard Sobolev space $H^1(\Omega; \mathbb{R}^d)$ and $\mathcal{S}(\Omega)$: namely $H^1(\Omega; \mathbb{R}^d)$ is continuously embedded in $\mathcal{S}(\Omega)$. This result is precisely [20, Lemma 2.1]. For functions see [5, Theorem 1]. For completeness we state the theorem as a lemma but refer to [20] for the proof.

Lemma 2.2. *Assume that (H) holds. Then the Sobolev space $H^1(\Omega; \mathbb{R}^d)$ is continuously embedded in $\mathcal{S}(\Omega)$. In particular, there exists a positive constant $C = C(\Omega)$ such that for all $\mathbf{u} \in H^1(\Omega; \mathbb{R}^d)$,*

$$|\mathbf{u}|_s \leq C |e(\nabla \mathbf{u})|_{L^2} \|\hat{\rho}\|_{L^1(\mathbb{R}^d)}$$

where $e(\nabla \mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$.

2.3. A special subspace. As our goal is to study the solvability of the peridynamic equation with Dirichlet-type boundary conditions, we now define the energy space in which we expect the solution to belong. Clearly this special space will be a subspace of $\mathcal{S}(\Omega)$. It is well known by now that (1.2) together with a Dirichlet data on the boundary of Ω , $\partial\Omega$, does not have to be well posed, as the underlying operator is a nonlocal one. Thus the energy space must contain elements of $\mathcal{S}(\Omega)$ that satisfy certain volumetric constraints as opposed to surface constraints. As a consequence the Dirichlet-type condition we are imposing must be imposed on a subspace of Ω with positive volume measure.

To that end, for a given Ω' that is compactly contained in Ω , denoted as $\Omega' \Subset \Omega$, define $V_0(\Omega')$ to be the closure in $\mathcal{S}(\Omega)$ of $C_c^\infty(\Omega'; \mathbb{R}^d)$;

(2.4)

$$V_0(\Omega') = \{\mathbf{u} \in \mathcal{S}(\Omega) : \mathbf{u}_n \rightarrow \mathbf{u} \text{ in } \mathcal{S}(\Omega) \text{ as } n \rightarrow \infty \text{ for some } \mathbf{u}_n \in C_c^\infty(\Omega'; \mathbb{R}^d)\}.$$

The set $V_0(\Omega')$ contains essentially all $\mathbf{u} \in \mathcal{S}(\Omega)$ that vanish on the subset of Ω outside of Ω' , which has a positive volume measure.

A special case is when $\Omega' = \Omega_\delta = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) > \delta\}$, the set of points in Ω that are δ distance away from the boundary, where δ is the horizon. This enable us to impose "boundary" conditions on a boundary layer of thickness δ , mimicking the local Dirichlet boundary value problems. We denote this energy space by $V_0^\delta = V_0(\Omega_\delta)$, the set of elements of $\mathcal{S}(\Omega)$ that vanish on a boundary layer of thickness δ .

Theorem 2.3. *Assume that (H) holds. Then the following claims hold.*

- i) $H_0^1(\Omega'; \mathbb{R}^d) \subset V_0(\Omega') \subset L^2(\Omega'; \mathbb{R}^d)$.
- ii) $V_0(\Omega')$ is a closed subspace of $\mathcal{S}(\Omega)$. As a consequence, it is a separable Hilbert space with the inner product $(\cdot, \cdot)_s$.
- iii) $V_0(\Omega')$ is a dense and closed subspace of $L^2(\Omega'; \mathbb{R}^d)$.

Proof. Observe that elements of $V_0(\Omega')$ are supported on the fixed set Ω' , and we understand the inclusions as restrictions on Ω' . The proof of i) follows from the definition. If $\mathbf{u} \in H_0^1(\Omega'; \mathbb{R}^d) \subset H^1(\Omega; \mathbb{R}^d)$, then there exists a sequence $\mathbf{u}_n \in C_c^\infty(\Omega'; \mathbb{R}^d)$ such that $\mathbf{u}_n \rightarrow \mathbf{u}$ in $H^1(\Omega'; \mathbb{R}^d)$. Applying Lemma 2.2,

$$\|\mathbf{u}_n - \mathbf{u}\|_{\mathcal{S}(\Omega)} \leq C \|\mathbf{u}_n - \mathbf{u}\|_{H^1(\Omega; \mathbb{R}^d)} \leq C \|\mathbf{u}_n - \mathbf{u}\|_{H^1(\Omega'; \mathbb{R}^d)} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

obtaining that $\mathbf{u} \in V_0(\Omega')$.

The set $V_0(\Omega')$ is closed under addition and scalar multiplication, and thus it is a subspace of $\mathcal{S}(\Omega)$. Moreover, $V_0(\Omega')$ is a closed subspace of $\mathcal{S}(\Omega)$. Indeed, suppose that $\mathbf{v}_n \in V_0(\Omega')$ converges to \mathbf{v} in $\mathcal{S}(\Omega)$. For each n , choose $\mathbf{u}_n \in C_c^\infty(\Omega'; \mathbb{R}^d)$ such

that $\|\mathbf{u}_n - \mathbf{v}_n\|_s \leq 1/n$. Then

$$\|\mathbf{u}_n - \mathbf{v}\|_{\mathcal{S}(\Omega)} \leq \|\mathbf{u}_n - \mathbf{v}_n\|_{\mathcal{S}(\Omega)} + \|\mathbf{v}_n - \mathbf{v}\|_{\mathcal{S}(\Omega)} \leq 1/n + \|\mathbf{v}_n - \mathbf{v}\|_{\mathcal{S}(\Omega)} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

implying that the limit $\mathbf{v} \in V_0(\Omega')$. Separability follows from the separability of $H_0^1(\Omega'; \mathbb{R}^d)$, part i) and Lemma 2.2. Indeed, the countable dense subset of $H_0^1(\Omega'; \mathbb{R}^d)$ will also be dense in $V_0(\Omega')$ with the norm $\|\cdot\|_s$. Part iii) is obvious. \square

2.3.1. Nonlocal Poincaré-type inequality. The subspace $V_0(\Omega')$ supports a nonlocal Poincaré-type inequality that help us control the L^2 norm of an element by its semi-norm, $|\cdot|_s$. To obtain such an inequality we first give a characterization of the zero set of the seminorm $|\cdot|_s$. We use the notation Π to denote this set which is precisely the set of rigid deformations:

$$\Pi = \{\mathbf{u} : \mathbf{u} = \mathbb{Q}\mathbf{x} + \mathbf{b}, \mathbb{Q} \in \mathbb{R}^{d \times d}, \mathbb{Q}^T = -\mathbb{Q}, \mathbf{b} \in \mathbb{R}^d\}$$

The following lemma provides a nonlocal means of identifying elements of Π , See [20, Corollary 3.2] or [12] for proof.

Lemma 2.4. *Suppose that ρ satisfies (H). Then*

$$\mathbf{u} \in \Pi \iff \int_{\Omega} \int_{\Omega} \hat{\rho}(|\mathbf{x}' - \mathbf{x}|) \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} = 0.$$

Remark 2.1. *It is not difficult to observe that there are no nontrivial rigid deformation defined on Ω that vanishes in a subset of positive volume measure. As a consequence,*

$$V_0(\Omega') \cap \Pi = \{\mathbf{0}\}.$$

We will also need the following compactness result for the proof of the nonlocal Poincaré inequality. The lemma is precisely [20, Theorem 5.3] adapted to our setting.

Lemma 2.5. *Suppose that \mathbf{u}_n is a bounded sequence in $L^2(\Omega; \mathbb{R}^d)$ with compact support in Ω . Then if*

$$\lim_{n \rightarrow \infty} \int_{\Omega} \int_{\Omega} \hat{\rho}(|\mathbf{x}' - \mathbf{x}|) \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} = 0,$$

then \mathbf{u}_n is precompact in $L^2(\Omega; \mathbb{R}^d)$.

We now state and prove the nonlocal Poincaré-type inequality. We should mention that for functions nonlocal Poincaré-type inequalities are available in the literature, see for example, [25, 20, 12, 18].

Proposition 2.6 (Nonlocal Poincaré Inequality). *Suppose that ρ satisfies (H). Then there exists $C = C(\rho, \Omega)$ such that*

$$\|\mathbf{u}\|_{L^2}^2 \leq C \int_{\Omega} \int_{\Omega} \hat{\rho}(|\mathbf{x}' - \mathbf{x}|) \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x}, \quad \forall \mathbf{u} \in V_0(\Omega').$$

As a consequence, by taking $\kappa = \sqrt{C} |\text{diam}(\Omega)|$

$$\|\mathbf{u}\| \leq \kappa |\mathbf{u}|_s, \quad \text{for all } \mathbf{u} \in V_0(\Omega').$$

Proof. We will show that

$$0 < m = \inf_{\mathbf{u} \in V_0(\Omega'), \|\mathbf{u}\|_{L^2} = 1} \left(\int_{\Omega} \int_{\Omega} \hat{\rho}(|\mathbf{x}' - \mathbf{x}|) \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \right|^2 dx' dx \right).$$

Clearly $m \geq 0$. Suppose $m = 0$. Then there exist $\mathbf{u}_n \in V_0(\Omega')$ such that for all n , $\|\mathbf{u}_n\|_{L^2(\Omega)} = 1$, and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \int_{\Omega} \hat{\rho}(|\mathbf{x}' - \mathbf{x}|) \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \right|^2 dx' dx = 0.$$

We can now apply Lemma 2.5 to conclude that \mathbf{u}_n is precompact in $L^2(\Omega; \mathbb{R}^d)$. Let $\mathbf{u} \in V_0(\Omega')$ be a strong limit (up to a subsequence) of \mathbf{u}_n in $L^2(\Omega; \mathbb{R}^d)$. Then on the one hand,

$$\|\mathbf{u}\|_{L^2(\Omega; \mathbb{R}^d)} = 1.$$

On the other hand rewriting the expression, which is possible since $\hat{\rho} \in L^1_{loc}$, and letting $n \rightarrow \infty$, we obtain that

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\Omega} \int_{\Omega} \frac{\hat{\rho}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \right|^2 dx' dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} \left(-2 \int_{\Omega} \frac{\hat{\rho}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) dx' \right) \cdot \mathbf{u}_n(\mathbf{x}) dx \\ &= \int_{\Omega} \left(-2 \int_{\Omega} \frac{\hat{\rho}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) dx' \right) \cdot \mathbf{u}(\mathbf{x}) dx \\ &= \int_{\Omega} \int_{\Omega} \hat{\rho}(|\mathbf{x}' - \mathbf{x}|) \left| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \right|^2 dx' dx. \end{aligned}$$

By Lemma 2.4 and Remark 2.1, we have $\mathbf{u} \in V_0(\Omega') \cap \Pi = \{\mathbf{0}\}$. But this is a contradiction as $\|\mathbf{u}\|_{L^2} = 1$. That completes the proof. \square

Remark 2.2. *Later we may need to know the details of the dependence of the constant κ in Proposition 2.6 in terms of the size of the support of ρ . That requires us to give a tighter version of the nonlocal Poincaré-type inequality, and this is accomplished in Section 5.*

2.3.2. Compact embedding. The nonlocal Poincaré-type inequalities proved above imply that $V_0(\Omega')$ with the norm $|\cdot|_s$ is continuously embedded in $L^2(\Omega; \mathbb{R}^d)$. In applications a stronger than continuous embedding is necessary, namely compact embedding. In this subsection we state condition to expect compactness or not. The following lemma says if ρ is locally integrable we cannot expect compactness.

Lemma 2.7. *Suppose that in addition to (H), ρ satisfies (2.3). Then $\mathcal{S} = L^2(\Omega; \mathbb{R}^d)$. Moreover, then there exists constants $0 < c \leq C$ such that*

$$c\|\mathbf{u}\|_{L^2(\Omega)} \leq |\mathbf{u}|_s \leq C\|\mathbf{u}\|_{L^2(\Omega)} \quad \text{for all } \mathbf{u} \in V_0(\Omega').$$

Proof. The proof of the Lemma follows from the observation that when $\rho(|\xi|)$ is locally integrable, the seminorm $|\mathbf{u}|_s^2$ can be written as

$$|\mathbf{u}|_s^2 = \int_{\Omega} \mathbf{u} \cdot \left(-2 \int_{\Omega} \frac{\rho(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) dx' \right) dx$$

and the function

$$\mathbf{x} \mapsto -2 \int_{\Omega} \frac{\rho(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}'$$

is in $L^2(\Omega; \mathbb{R}^d)$. The latter part of the lemma uses the nonlocal Poincaré's inequalities proved in the previous section. \square

When condition (2.3) fails, in general, the space \mathcal{S} may become a proper subset of $L^2(\Omega; \mathbb{R}^d)$. We next present a condition on ρ such that the closed subspace $V_0(\Omega')$ is compactly embedded in $L^2(\Omega; \mathbb{R}^d)$.

Lemma 2.8. *Suppose that in addition to (H), $\hat{\rho}(|\xi|)$ is nonincreasing in $|\xi|$ and satisfying the density condition*

$$(2.5) \quad \frac{\epsilon^2}{\int_{B_\epsilon(0)} \hat{\rho}(|\xi|) d\xi} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Then $V_0(\Omega')$ is compactly embedded in $L^2(\Omega; \mathbb{R}^d)$.

Proof. Suppose that $\mathbf{u}_n \in V_0(\Omega')$ is bounded in $\mathcal{S}(\Omega)$. This implies that \mathbf{u}_n is bounded in $L^2(\Omega; \mathbb{R}^d)$ and that

$$\sup_{n \geq 1} \int_{\Omega} \int_{\Omega} \rho(\mathbf{x}' - \mathbf{x}) \left| \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} < \infty.$$

Extending \mathbf{u}_n to be 0 outside of Ω , it is not difficult to show that

$$(2.6) \quad \sup_{n \geq 1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(\mathbf{x}' - \mathbf{x}) \left| \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} < \infty.$$

Thus after some scaling we may assume that $\Omega = \mathbb{R}^d$ and that $\text{supp}(\mathbf{u}_n) \subset B$, the unit ball. Let

$$\mathbb{P}(\mathbf{z}) = \frac{d}{\omega_d} \frac{\mathbf{z} \otimes \mathbf{z}}{|\mathbf{z}|^2} \chi_B(\mathbf{z})$$

and define $\mathbb{P}^\epsilon(\mathbf{z}) = \frac{1}{\epsilon^d} \mathbb{P}(\frac{\mathbf{z}}{\epsilon})$. Then with inequality (2.6) and the boundedness in $L^2(\Omega; \mathbb{R}^d)$ at hand, according to [20, Lemma 5.4] (see also [5]), it suffices to show that

$$(2.7) \quad \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \|\mathbf{u}_n - \mathbb{P}^\epsilon * \mathbf{u}_n\|_{L^2(\mathbb{R}^d)} = 0.$$

Following the exact argument as in [5, 20], we obtain that

$$(2.8) \quad \int_{\mathbb{R}^d} |\mathbf{u}_n(\mathbf{x}) - \mathbb{P}^\epsilon * \mathbf{u}_n(\mathbf{x})|^2 d\mathbf{x} \leq \frac{d^2}{|B_\epsilon|} \int_0^\epsilon r^{d-1} F_n(r) dr.$$

where $(\mathbb{P}^\epsilon * \mathbf{u}_n)_i = \sum_j^d \mathbb{P}_{ij}^\epsilon * (\mathbf{u}_n)_j$ and

$$F_n(r) = \int_{S^{d-1}} \int_{\mathbb{R}^d} |s \cdot (\mathbf{u}_n(\mathbf{x} + rs) - \mathbf{u}_n(\mathbf{x}))|^2 d\mathbf{x} \sigma(s).$$

The rest of the proof is similar to the proof of the compactness theorem in [5]. The function F_n enjoys similar properties as its counterpart in [5]. The most important properties being that

$$F(2r) \leq 2^2 F(r),$$

and that the estimate (2.6) can be expressed as

$$(2.9) \quad \sup_n \int_0^1 r^{d-1} \frac{F_n(r)}{r^2} \hat{\rho}(r) dr \leq C.$$

Moreover, from application of [5, Lemma 2] (with $g(r) = F_n(r)/r^2$ and $h(t) = \hat{\rho}(t)$) we obtain that

$$|B_\epsilon|^{-1} \int_0^\epsilon r^{d-1} \frac{F_n(r)}{r^2} dr \leq C \left(\int_0^\epsilon r^{d-1} \frac{F_n(r)}{r^2} \hat{\rho}(r) dr \right) / \left(\int_{B_\epsilon} \hat{\rho}(\mathbf{x}) d\mathbf{x} \right)$$

Then by (2.9) we obtain that for any $\epsilon > 0$

$$|B_\epsilon|^{-1} \int_0^\epsilon r^{d-1} F_n(r) dr \leq C \epsilon^2 |B_\epsilon|^{-1} \int_0^\epsilon r^{d-1} \frac{F_n(r)}{r^2} dr \leq C \frac{\epsilon^2}{\int_{B_\epsilon} \hat{\rho}(\mathbf{x}) d\mathbf{x}}, \quad \forall n \geq n_\epsilon.$$

Now we take the limit first in n and then in ϵ , to obtain (2.7) by (2.5). That completes the proof. \square

Remark 2.3. Note that for ρ satisfying (2.5), the local integrability condition (2.3) is no longer valid. Moreover, a simple calculation shows that $\rho(|\boldsymbol{\xi}|)$ that are comparable to $|\boldsymbol{\xi}|^{-d-2s}$ for any $s \in (0, 1)$, near the origin satisfy (2.5). Another kernel, not included in the above class is the radial function

$$\rho(|\boldsymbol{\xi}|) = \begin{cases} \frac{-\ln(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^d} & \text{when } |\boldsymbol{\xi}| \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $d \geq 2$.

3. THE NONLOCAL OPERATOR

3.1. Definition. The integral operator on the left hand side of (1.2) exists at all points $\mathbf{x} \in \Omega$, and for all $\mathbf{u} \in L^2(\Omega; \mathbb{R}^d)$ if \mathbb{C} is locally integrable. But this is not so in general in the absence of the integrability assumption on ρ . We, therefore, have to make sense of underlying peridynamic operator corresponding to (1.2). First we define and understand the "leading operator" \mathcal{L} corresponding to the kernel ρ . Denoting the dual space of $\mathcal{S}(\Omega)$ by $\mathcal{S}'(\Omega)$, define the operator $-\mathcal{L} : \mathcal{S}(\Omega) \rightarrow \mathcal{S}'(\Omega)$ by

$$(3.1) \quad \langle -\mathcal{L}\mathbf{u}, \mathbf{w} \rangle = ((\mathbf{u}, \mathbf{w}))$$

for any $\mathbf{u}, \mathbf{w} \in \mathcal{S}(\Omega)$, where the bilinear form is as defined in (2.1). Then we see that \mathcal{L} is a bounded linear operator and satisfies the estimate

$$|\langle -\mathcal{L}\mathbf{u}, \mathbf{w} \rangle| \leq |\mathbf{u}|_s |\mathbf{w}|_s.$$

We would like to understand this operator better. To that end, let us look at the simpler case when ρ satisfies (2.3). In this case $\mathcal{S} = L^2(\Omega; \mathbb{R}^d)$ and $\mathcal{L} : L^2(\Omega; \mathbb{R}^d) \rightarrow L^2(\Omega; \mathbb{R}^d)$ is a linear and bounded operator. Moreover, since the duality pairing is just the $L^2(\Omega; \mathbb{R}^d)$ inner product, for any \mathbf{u} and \mathbf{w} in $L^2(\Omega; \mathbb{R}^d)$,

$$\begin{aligned} (-\mathcal{L}\mathbf{u}, \mathbf{w}) &= ((\mathbf{u}, \mathbf{w})) \\ &= \int_\Omega \mathbf{w}(\mathbf{x}) \cdot \left(-2 \int_\Omega \frac{\rho(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}' \right) d\mathbf{x}, \end{aligned}$$

obtaining the value of \mathcal{L} at $\mathbf{u} \in L^2(\Omega; \mathbb{R}^d)$ by the closed form

$$(3.2) \quad \mathcal{L}\mathbf{u}(\mathbf{x}) = 2 \int_\Omega \frac{\rho(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}', \quad \text{a.e. } \mathbf{x} \in \Omega.$$

However, without the additional integrability assumption on ρ , the operator \mathcal{L} is an unbounded operator on $L^2(\Omega; \mathbb{R}^d)$, and worse yet, the integral in (3.2) may diverge for some $\mathbf{u} \in L^2(\Omega; \mathbb{R}^d)$. The next proposition states that with the more general assumption (H) of ρ one can understand \mathcal{L} and give a closed form in some generalized sense. We begin by introducing the sequence of operator

$$L_\tau \mathbf{u} = 2 \int_{\Omega} \frac{\rho_\tau(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}',$$

where as defined earlier $\rho_\tau(r) = \rho(r)\chi_{[\tau, \infty)}(r)$. From our discussion above, for all $\tau > 0$, $-L_\tau$ is a linear bounded operators on $L^2(\Omega; \mathbb{R}^d)$, and therefore for each \mathbf{u} , $-L_\tau \mathbf{u} \in \mathcal{S}'(\Omega)$.

Proposition 3.1. *Suppose that ρ satisfies (H). Then*

a) *For each $\mathbf{u} \in \mathcal{S}(\Omega)$, and as $\tau \rightarrow 0$, $-L_\tau \mathbf{u} \rightarrow -\mathcal{L}\mathbf{u}$, in $\mathcal{S}'(\Omega)$. That is,*

$$\mathcal{L}\mathbf{u} = (\text{P.V.}) 2 \int_{\Omega} \frac{\rho(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}) (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}' \quad \text{in } \mathcal{S}'(\Omega).$$

b) *Let $\mathbf{u} \in C_c^\infty(\Omega; \mathbb{R}^d)$, and $h = \text{dist}(\text{Supp}(\mathbf{u}), \partial\Omega) > 0$. Then*

i) $\sup_{0 < \tau < h/2} \sup_{\mathbf{x} \in \Omega} |L_\tau(\mathbf{u})| \leq C$, $C = C(\mathbf{u}, \rho, h)$, constant

ii) *As $\tau \rightarrow 0$, $-L_\tau \mathbf{u} \rightarrow -\mathcal{L}\mathbf{u}$ strongly in $L^2(\Omega; \mathbb{R}^d)$.*

iii) *The integral formula for $-\mathcal{L}\mathbf{u}$ in part a) holds pointwise for $\mathbf{x} \in \Omega$.*

Proof. Part a). For $\mathbf{u}, \mathbf{w} \in \mathcal{S}(\Omega)$,

$$\langle -L_\tau \mathbf{u}, \mathbf{w} \rangle = \int_{\Omega} \int_{\Omega} \frac{\rho_\tau(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) (\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})) d\mathbf{x}' d\mathbf{x}$$

Using the fact that $\rho_\tau \rightarrow \rho$ pointwise, we may apply dominated convergence theorem, to see that the last double integral converges to $((\mathbf{u}, \mathbf{w})) = \langle -\mathcal{L}\mathbf{u}, \mathbf{w} \rangle$. That is precisely the convergence in $\mathcal{S}'(\Omega)$.

Part b). The proof of i) is as follows. Write the operator $L_\tau \mathbf{u}$ as

$$\begin{aligned} L_\tau \mathbf{u}(\mathbf{x}) &= 2 \int_{\Omega} \chi_{[0, h/2]}(|\mathbf{x}' - \mathbf{x}|) \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \mathbf{x}' - \mathbf{x} d\mathbf{x}' \\ &\quad + 2 \int_{\Omega} (1 - \chi_{[0, h/2]}(|\mathbf{x}' - \mathbf{x}|)) \rho_\tau(|\mathbf{x}' - \mathbf{x}|) \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \mathbf{x}' - \mathbf{x} d\mathbf{x}', \end{aligned}$$

where $h = \text{dist}(\text{Supp}(\mathbf{u}), \partial\Omega)$. Then using the Taylor expansion

$$\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) = \nabla \mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x}) + \frac{1}{2} D^2 \mathbf{u}(\zeta)(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}), \quad \text{for some } \zeta,$$

we obtain that for any $0 < \tau < h/2$ and $\mathbf{x} \in \Omega_{h/2} = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) \geq h/2\}$ we have

$$\begin{aligned} |L_\tau \mathbf{u}(\mathbf{x})| &\leq \left(2 \int_{\Omega} (\chi_{[0, h/2]}(|\mathbf{x}' - \mathbf{x}|)) \rho(|\mathbf{x}' - \mathbf{x}|) |\mathbf{x}' - \mathbf{x}|^2 d\mathbf{x}' \right) \|D^2 \mathbf{u}\|_{L^\infty(\Omega)} \\ &\quad + 4 \left(\int_{\Omega} (1 - \chi_{[0, h/2]}(|\mathbf{x}' - \mathbf{x}|)) \rho(|\mathbf{x}' - \mathbf{x}|) d\mathbf{x}' \right) \|\mathbf{u}\|_{L^\infty(\Omega)} \end{aligned}$$

while for $\mathbf{x} \in \Omega \setminus \Omega_{h/2}$, since $B(\mathbf{x}, h/2) \subset \Omega \setminus \Omega_h$,

$$|L_\tau \mathbf{u}(\mathbf{x})| \leq 4 \left(\int_\Omega (1 - \chi_{[0, h/2]}(|\mathbf{x}' - \mathbf{x}|)) \rho(|\mathbf{x}' - \mathbf{x}|) d\mathbf{x}' \right) \|\mathbf{u}\|_{L^\infty(\Omega)}.$$

To prove *ii*) define the the vector field $L_0 \mathbf{u}$ at $\mathbf{x} \in \Omega$ as

$$L_0 \mathbf{u}(\mathbf{x}) := \lim_{\tau \rightarrow 0} L_\tau \mathbf{u}(\mathbf{x}).$$

Then it is not difficult to see that the limit exist for all $\mathbf{x} \in \Omega$. Using part *i*) and dominated convergence theorem,

$$L_\tau \mathbf{u}(\mathbf{x}) \longrightarrow L_0 \mathbf{u}$$

strongly in $L^2(\Omega; \mathbb{R}^d)$. In addition, for any $\mathbf{w} \in \mathcal{S}$

$$\langle -L_0 \mathbf{u}, \mathbf{w} \rangle = \langle -L_0 \mathbf{u}, \mathbf{w} \rangle = \lim_{\tau \rightarrow 0} \langle -L_\tau \mathbf{u}, \mathbf{w} \rangle = \langle (\mathbf{u}, \mathbf{w}) \rangle = \langle -\mathcal{L} \mathbf{u}, \mathbf{w} \rangle,$$

proving that $\mathcal{L} \mathbf{u} = L_0 \mathbf{u}$, when $\mathbf{u} \in C_c^\infty(\Omega; \mathbb{R}^d)$.

Part *iii*) is an easy consequence of *ii*) since $\mathcal{L} \mathbf{u} \in L^2(\Omega; \mathbb{R}^d)$ \square

We note that the operator \mathcal{L} has been discussed in earlier works such as [11, 12] so that nonlocal peridynamic models can be reformulated via nonlocal divergence and gradients. Briefly, given an anti-symmetric mapping $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\mathbf{x}, \mathbf{x}')$ from $\mathbb{R}^3 \times \mathbb{R}^3$ to \mathbb{R}^3 , e.g., $\boldsymbol{\alpha}(\mathbf{x}, \mathbf{x}') = (\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'| = -\boldsymbol{\alpha}(\mathbf{x}', \mathbf{x})$ as in our case, for vector valued functions $\boldsymbol{\Psi} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\mathbf{v} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, the *nonlocal divergence operator* $\mathcal{D} \boldsymbol{\Psi} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ for tensors and its adjoint nonlocal gradient operator $\mathcal{D}^* \mathbf{v} : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ are defined as

$$(3.3a) \quad \mathcal{D}(\boldsymbol{\Psi})(\mathbf{x}) = \int_{\mathbb{R}^3} (\boldsymbol{\Psi}(\mathbf{x}, \mathbf{x}') + \boldsymbol{\Psi}(\mathbf{x}', \mathbf{x})) \cdot \boldsymbol{\alpha}(\mathbf{x}, \mathbf{x}') d\mathbf{x}' \quad \text{for } \mathbf{x} \in \mathbb{R}^3,$$

$$(3.3b) \quad \mathcal{D}^*(\mathbf{v})(\mathbf{x}, \mathbf{x}') = (\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})) \otimes \boldsymbol{\alpha}(\mathbf{x}', \mathbf{x}) \quad \text{for } \mathbf{x}, \mathbf{x}' \in \mathbb{R}^3.$$

Then, we have formally $\mathcal{L} = -\mathcal{D}(\rho(\mathcal{D}^*)^T)$. The discussion given above puts such definitions on a rigorous footing even when $\rho = \rho(|\mathbf{x} - \mathbf{x}'|)$ is not in $L^1_{loc}(\mathbb{R})$. Moreover, we see that (3.1) is simply the relation between adjoint operators:

$$\langle \mathcal{D}(\rho(\mathcal{D}^*)^T) \mathbf{u}, \mathbf{w} \rangle = \langle \rho \mathcal{D}^* \mathbf{u}, \mathcal{D}^* \mathbf{w} \rangle.$$

By building upon our understanding of the operator associated with (1.2), we interpret the integral operator

$$\mathcal{P} \mathbf{u} = \int_\Omega \mathbb{C}(\mathbf{x}' - \mathbf{x})(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')) d\mathbf{x}',$$

as a perturbation of the leading operator \mathcal{L} . To make this precise, corresponding to F_0 define the nonlocal operator

$$\mathcal{F} \mathbf{u}(\mathbf{x}) = 2 \int_\Omega F_0(|\mathbf{x}' - \mathbf{x}|)(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}'.$$

Using the integrability assumption on F_0 , we see that \mathcal{F} is a linear bounded operator on $L^2(\Omega; \mathbb{R}^d)$. Moreover, for any $\mathbf{u}, \mathbf{w} \in L^2(\Omega; \mathbb{R}^d)$,

$$(-\mathcal{F} \mathbf{u}, \mathbf{w}) = \int_\Omega \int_\Omega F_0(|\mathbf{x}' - \mathbf{x}|)(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \cdot (\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})) d\mathbf{x}' d\mathbf{x}.$$

Noting that $\mathbb{C}(\boldsymbol{\xi}) = 2 \frac{\rho(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2} \boldsymbol{\xi} \otimes \boldsymbol{\xi} + 2F_0(\boldsymbol{\xi}) \mathbb{I}$, we can now write

$$(3.4) \quad \mathcal{P} = \mathcal{L} + \mathcal{F}.$$

Observe that we are not imposing any assumption on the sign of F_0 . As a consequence, the perturbation of \mathcal{L} by \mathcal{F} may result in a nonlocal integral operator \mathcal{P} with an indefinite kernel. Indeed, the kernel matrix for \mathcal{P} is $\mathbb{C}(\boldsymbol{\xi})$ and for any $\mathbf{v} \in \mathbb{R}^d$

$$(\mathbb{C}(|\boldsymbol{\xi}|)\mathbf{v}, \mathbf{v}) = 2\frac{\rho(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}|\boldsymbol{\xi} \cdot \mathbf{v}|^2 + 2F_0(|\boldsymbol{\xi}|)|\mathbf{v}|^2,$$

could change sign.

3.2. Properties of the nonlocal operator. From the nonlocal Poincaré-type inequality we notice that the seminorm $|\cdot|_s$ defines an equivalent norm in $V_0(\Omega')$. It is possible to pose variational problems in $V_0(\Omega')$. A standard application of Riesz representation theorem yields the following result. $V'_0(\Omega')$ denotes the dual space of $V_0(\Omega')$

Lemma 3.1. *Assume that ρ satisfies (H). For a given $\mathbf{b} \in V'_0(\Omega')$, there exists a unique $u \in V_0(\Omega')$ such that*

$$\int_{\Omega} \int_{\Omega} \frac{\rho(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} (\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) (\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})) d\mathbf{x}' d\mathbf{x} = \langle \mathbf{b}, \mathbf{w} \rangle,$$

for all $\mathbf{w} \in V_0(\Omega')$. Moreover, $|\mathbf{u}|_s = |\mathbf{b}|_{V'_0(\Omega')}$.

Utilizing the operator \mathcal{L} defined in the previous section, it follows then that

$$-\mathcal{L} : V_0(\Omega') \rightarrow V'_0(\Omega'); \quad \mathbf{u} \mapsto -\mathcal{L}\mathbf{u}$$

is an isometry with $\|-\mathcal{L}\mathbf{u}\|_{V'_0(\Omega')} = |\mathbf{u}|_s$ for any $\mathbf{u} \in V_0(\Omega')$ and $|(-\mathcal{L})^{-1}\mathbf{b}|_s = |\mathbf{b}|_{V'_0(\Omega')}$ for any $\mathbf{b} \in V'_0(\Omega')$. In case, $\mathbf{b} \in L^2(\Omega; \mathbb{R}^d)$, there exists a constant C , independent of \mathbf{b} such that $|\mathcal{L}^{-1}\mathbf{b}|_s \leq C\|\mathbf{b}\|_{L^2}$. This follows from the continuous embedding of $L^2(\Omega; \mathbb{R}^d)$ into $V'_0(\Omega')$.

Our tool in proving the well posedness of the peridynamic equation is the Fredholm Alternative Theorem. When applying the theorem we will encounter the operator $\mathcal{K} = \mathcal{L}^{-1}\mathcal{F}$. Anticipating this, we obtain some useful properties of the composite operator $\mathcal{K} = \mathcal{L}^{-1}\mathcal{F} : L^2(\Omega; \mathbb{R}^d) \rightarrow L^2(\Omega; \mathbb{R}^d)$ defined as

$$\mathbf{v} = \mathcal{K}\mathbf{u} \text{ if and only if } \langle -\mathcal{L}\mathbf{v}, \mathbf{w} \rangle = \langle -\mathcal{F}\mathbf{u}, \mathbf{w} \rangle, \quad \text{for all } \mathbf{w} \in V_0(\Omega').$$

Observe that the range of \mathcal{K} is contained in $V_0(\Omega')$. The following lemma gives conditions such that the operator \mathcal{K} is a compact operator.

Lemma 3.2. *Suppose that in addition to (H), $\hat{\rho}(r)$ is nonincreasing and satisfies (2.5). Then \mathcal{K} is a self adjoint compact operator on $L^2(\Omega; \mathbb{R}^d)$.*

Proof. The proof follows from the facts that under the assumption on ρ , $\text{Range}(\mathcal{K}) \subset V_0(\Omega')$ and the subspace $V_0(\Omega')$ is compactly embedded into $L^2(\Omega; \mathbb{R}^d)$, by Lemma 2.8. \square

Corollary 3.3. *Under the assumption of Lemma 3.2, \mathcal{K} has at most countably many eigenvalues, $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$.*

We now examine the case where (2.5) is not satisfied. Under the assumption (2.3), we have the following necessary and sufficient condition for \mathcal{K} to be a compact operator. The main idea used here is that the operator \mathcal{F} is made to act like the convolution operator on elements of the subspace. Recall that $V_0^\delta = V_0(\Omega_\delta)$, where $\Omega_\delta = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) \geq \delta\}$.

Theorem 3.4. *Suppose that (H) and (2.3) hold. Suppose also that $F_0(r)$ is a function supported on $(0, \delta)$ for $\delta > 0$ small. Denote*

$$m_0 = 2 \int_{B_\delta(0)} F_0(|\xi|) d\xi.$$

Then the operator $\mathcal{K} : L^2(\Omega; \mathbb{R}^d) \rightarrow V_0^\delta \subset L^2(\Omega; \mathbb{R}^d)$ is compact if and only if $m_0 = 0$.

Proof. Now sufficiency follows from the fact that when $m_0 = 0$, then the operator \mathcal{F} is the convolution operator, and hence a compact operator. Indeed, when $m_0 = 0$, for any $\mathbf{u} \in L^2(\Omega; \mathbb{R}^d)$ and $\mathbf{x} \in \Omega_\delta$, $\mathcal{F}\mathbf{u}(\mathbf{x}) = 2F_0 * \bar{\mathbf{u}}(x)$, $\bar{\mathbf{u}}$ being the zero extension of \mathbf{u} to \mathbb{R}^d , and $*$ is the standard convolution operator. It follows then that the linear operator \mathcal{K} is essentially a composition of a compact and a bounded operator on $L^2(\Omega; \mathbb{R}^d)$ and therefore compact, [6].

Let us now prove the necessity by contradiction. Suppose that \mathcal{K} is compact. and yet $m_0 \neq 0$. Recalling the definition of \mathcal{K} , we have

$$\mathbf{v} = \mathcal{K}\mathbf{u} \quad \text{if and only if } (\mathcal{L}\mathbf{v}, \mathbf{w}) = (\mathcal{F}\mathbf{u}, \mathbf{w}) \text{ for all } \mathbf{w} \in V_0^\delta.$$

It follows then that if $\mathbf{v} = \mathcal{K}\mathbf{u}$,

$$\mathcal{L}(\mathcal{K}\mathbf{u})(\mathbf{x}) = \mathcal{L}\mathbf{v}(\mathbf{x}) = \mathcal{F}\mathbf{u}(\mathbf{x}) = 2F_0 * \bar{\mathbf{u}} - m_0\mathbf{u}(\mathbf{x}) \quad \text{for almost every } \mathbf{x} \in \Omega_\delta.$$

Then identifying V_0^δ by $L^2(\Omega_\delta; \mathbb{R}^d)$, the last equation, together with the assumption on \mathcal{K} and m_0 , enable us to write the identity operator as a sum of two compact operators:

$$\mathbf{u}(\mathbf{x}) = \frac{1}{m_0} (-\mathcal{L}(\mathcal{K}\mathbf{u}) + 2F_0 * \bar{\mathbf{u}}) \quad \text{for all } \mathbf{u} \in L^2(\Omega_\delta; \mathbb{R}^d),$$

which obviously is a contradiction. \square

4. THE PERIDYNAMIC EQUILIBRIUM EQUATION

In this section we study the well-posedness of the linear peridynamic equilibrium equation as a nonlocal constrained value problem:

$$(4.1) \quad \begin{cases} - \int_{\Omega} \mathbb{C}(\mathbf{x}' - \mathbf{x})(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x} = \mathbf{b}(\mathbf{x}), & \mathbf{x} \in \Omega' \\ \mathbf{u}(\mathbf{x}) = 0 & \mathbf{x} \in \Omega \setminus \Omega' \end{cases}$$

where $\mathbb{C}(\xi) = 2 \frac{\rho(|\xi|)}{|\xi|^2} \xi \otimes \xi + 2F_0(|\xi|)\mathbb{I}$ is as given in (1.3). Note that unlike local boundary value problems, the necessary "boundary condition" is in fact a volume constraint. In a generalized sense we may write the constrained value problem (4.1), as given $\mathbf{b} \in V_0'(\Omega')$ find $\mathbf{u} \in V_0(\Omega')$ such that

$$-\mathcal{P}\mathbf{u} = \mathbf{b},$$

where \mathcal{P} is given by (3.4). To be precise, we define the weak solution to (4.1) as follows.

Definition 4.1. *Let $\mathbf{b} \in V_0'(\Omega')$. We say $\mathbf{u} \in V_0(\Omega')$ is a solution to (4.1) if*

$$\langle -\mathcal{P}\mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{b}, \mathbf{w} \rangle \quad \text{for all } \mathbf{w} \in V_0(\Omega').$$

Note that if $\mathbf{u} \in V_0(\Omega')$ is a weak solution corresponding to $\mathbf{b} \in L^2(\Omega; \mathbb{R}^d)$, then it satisfies (4.1) for almost all $\mathbf{x} \in \Omega'$. Indeed, by definition for any $\phi \in C_c^\infty(\Omega', \mathbb{R}^d) \subset V_0(\Omega')$,

$$\langle -\mathcal{P}\mathbf{u}, \phi \rangle = \int_{\Omega} \mathbf{b}(\mathbf{x}) \cdot \phi(\mathbf{x}) d\mathbf{x}$$

implying that the distribution $-\mathcal{P}\mathbf{u} \in L^2(\Omega'; \mathbb{R}^d)$ and that

$$-\mathcal{P}\mathbf{u} = \mathbf{b} \quad \text{a.e. } \mathbf{x} \in \Omega',$$

which is precisely (4.1) since $\mathbf{u} \in V_0(\Omega')$.

Now we prove the existence of a weak solution corresponding to a given $\mathbf{b} \in V_0'(\Omega')$. A solution $\mathbf{u} \in V_0(\Omega')$ solves the peridynamic equilibrium equation (4.1) if and only if,

$$(4.2) \quad \langle -\mathcal{L}\mathbf{u}, \mathbf{w} \rangle + \langle -\mathcal{F}\mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{b}, \mathbf{w} \rangle \quad \text{for all } \mathbf{w} \in V_0(\Omega').$$

The later in turn is true if and only if

$$\langle -\mathcal{L}\mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{b} + \mathcal{F}\mathbf{u}, \mathbf{w} \rangle \quad \text{for all } \mathbf{w} \in V_0(\Omega').$$

The last equation can be rewritten to read as the variational equation

$$(4.3) \quad (\mathcal{I} + \mathcal{L}^{-1}\mathcal{F})\mathbf{u} = -\mathcal{L}^{-1}\mathbf{b}.$$

In Section 3 we have already introduced and studied the composite operator $\mathcal{K} = \mathcal{L}^{-1}\mathcal{F}$ as a bounded linear operator on $L^2(\Omega; \mathbb{R}^d)$. Note that we have its range contained in $V_0(\Omega')$ with the estimate

$$|\mathcal{K}\mathbf{u}| \leq C_1 \|\mathcal{F}\mathbf{u}\| \leq C_2 \|\mathbf{u}\|_{L^2}, \quad \forall \mathbf{u} \in L^2(\Omega; \mathbb{R}^d).$$

Moreover, depending on the integrability of ρ , we have shown that \mathcal{K} is a compact operator. We have now the right set up to apply Fredholm Alternative Theorem to determine the solvability of (4.3).

The case when ρ satisfies (2.5). In this case by Lemma 3.2, \mathcal{K} is compact. Thus we may apply the Fredholm Alternative Theorem to obtain the following well posedness result.

Theorem 4.2. *Suppose that in addition to (H), ρ is nonincreasing and satisfies (2.5). Suppose also g is a compactly supported locally integrable radial function. Then either*

$$-\mathcal{P}\mathbf{u} = 0 \quad \text{has a nontrivial solution in } V_0(\Omega')$$

or

$$-\mathcal{P}\mathbf{u} = \mathbf{b} \quad \text{has a unique solution in } V_0(\Omega') \text{ for any } \mathbf{b} \in V_0'(\Omega').$$

To see clearly the above result indeed allows micromodulus tensors that are indefinite, we use the well known property of compact operators: their spectrum is discrete. To that end, assume that $F = \lambda F_0$, where λ is a parameter and F_0 is given compactly supported locally integrable radial function. Denoting $\mathcal{P}_\lambda = \mathcal{L} + \lambda\mathcal{F}$, we have the following result.

Theorem 4.3. *Suppose that in addition to (H), $\hat{\rho}(r)$ is nonincreasing and satisfies (2.5). Suppose also F_0 is a compactly supported locally integrable radial function. Then the variational equation*

$$(4.4) \quad -\mathcal{P}_\lambda = \mathbf{b}, \quad \mathbf{u} \in V_0(\Omega')$$

has a unique solution for $\mathbf{b} \in V_0'(\Omega')$ if and only if $\lambda \notin \Sigma$, the spectrum of $\mathcal{K} = \mathcal{L}^{-1}\mathcal{F}$. Moreover the following holds.

(1) For each $\lambda \notin \Sigma$, there exists a constant C , such that if \mathbf{u} is a solution, then

$$|\mathbf{u}|_s \leq C \|\mathbf{b}\|_{V_0'(\Omega')}.$$

(2) There exists $\lambda_0 > 0$ such that for all $|\lambda| < \lambda_0$ and any $\mathbf{b} \in V_0'(\Omega')$ the unique solution $\mathbf{u} \in V_0(\Omega')$ minimizes the potential energy

$$(4.5) \quad \mathbf{E}_\lambda(\mathbf{u}) = \frac{1}{2} \langle -\mathcal{L}\mathbf{u}, \mathbf{u} \rangle + \frac{\lambda}{2} \langle -\mathcal{F}\mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{b}, \mathbf{u} \rangle.$$

Proof. The only part that needs proof is part 2). Since \mathbf{E}_λ is a quadratic energy and $V_0(\Omega')$ is a closed subspace of $\mathcal{S}(\Omega)$, by direct method of calculus of variations, it suffices to prove that the functional \mathbf{E}_λ is coercive. That will depend on λ and we will find conditions on λ so that

$$|\mathbf{E}_\lambda(\mathbf{u})| \rightarrow \infty, \text{ when } |\mathbf{u}|_s \rightarrow \infty, \text{ and } \mathbf{u} \in V_0(\Omega').$$

But this follows from the estimate, for all $\mathbf{u} \in V_0(\Omega')$

$$\frac{\mathbf{E}_\lambda(\mathbf{u})}{\|\mathbf{u}\|_s^2} \geq \frac{1}{2(1+\kappa)} (1 - \kappa|\lambda|(2\|F_0\|_{L^1(\mathbb{R}^d)} + \|f_0(\mathbf{x})\|_{L^\infty(\Omega)})) - \|b\| \frac{\|\mathbf{u}\|}{\|\mathbf{u}\|_s^2},$$

where $\kappa = \kappa(V_0(\Omega'))$ is the nonlocal Poincaré constant corresponding to $V_0(\Omega')$, and $f_0(\mathbf{x})$ is the continuous function given by $f_0(\mathbf{x}) = 2 \int_\Omega F_0(|\mathbf{x}' - \mathbf{x}|) d\mathbf{x}'$. To complete the proof, we now take λ_0 to be $\frac{1}{\lambda_0} = \kappa(2\|F_0\|_{L^1(\mathbb{R}^d)} + \|f_0(\mathbf{x})\|_{L^\infty(\Omega)})$. \square

Remark 4.1. We observe that the micromodulus tensor associated with the equation (4.4) is given by

$$\mathbb{C}_\lambda(\boldsymbol{\xi}) = 2 \frac{\rho(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2} \boldsymbol{\xi} \otimes \boldsymbol{\xi} + \lambda F_0(|\boldsymbol{\xi}|) \mathbb{I}.$$

and it is now clear that for a given F_0 , any $\mathbf{v} \in \mathbb{R}^d$, one can find $\lambda \notin \Sigma$ large enough such that $(\mathbb{C}_\lambda(\boldsymbol{\xi})\mathbf{v}, \mathbf{v})$ is negative and yet (4.4) has a unique solution.

The case when ρ satisfies (2.3). In this case Lemma 3.4 proves the compactness of \mathcal{K} . Again application of the Fredholm Alternative Theorem, we have the following result.

Theorem 4.4. Suppose that ρ satisfies (H) and (2.3). Under the assumption that the function $F_0(r)$ has a support contained in $[0, \delta)$, $m_0 = 2 \int_{\mathbb{R}^d} F_0(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = 0$, either

$$-\mathcal{P}\mathbf{u} = 0 \quad \text{has a nontrivial solution in } V_0^\delta$$

or

$$-\mathcal{P}\mathbf{u} = \mathbf{b} \quad \text{has a unique solution in } V_0^\delta \text{ for any } \mathbf{b} \in V_0^{\delta'}.$$

A nonlinear nonlocal constrained value problem. The operator \mathcal{P} is taken to be the perturbation of the nonlocal operator \mathcal{L} by the linear operator \mathcal{F} that essentially act the same way as \mathcal{L} . More generally one may consider a potentially non-linear, however compact, perturbation of \mathcal{L} . To illustrate this we minimize the nonlinear potential energy

$$\mathbf{E}_u = \frac{1}{2} \langle -\mathcal{L}\mathbf{u}, \mathbf{u} \rangle + \psi(\mathbf{u}) - \langle \mathbf{b}, \mathbf{u} \rangle,$$

for $\mathbf{b} \in V_0'(\Omega')$ by imposing conditions on ψ , depending on the integrability of ρ . We then have the following theorem.

Theorem 4.5. *Suppose that ρ satisfies (H), Assume that ψ is bounded on $V_0(\Omega')$, and satisfies*

$$(4.6) \quad \psi(\mathbf{u}) \geq C - \theta |\mathbf{u}|_s^2, \quad \forall \mathbf{u} \in V_0(\Omega').$$

Assume that

- i) when ρ is nonincreasing and satisfies (2.5), ψ is continuous on $V_0(\Omega')$ with respect to the L^2 norm and
- ii) when ρ satisfies (2.3), ψ is convex in $V_0(\Omega')$.

Then for sufficiently small θ , the functional \mathbf{E}_u has a minimizer \mathbf{u} in $V_0(\Omega')$. Moreover, \mathbf{u} solves the equation

$$\langle -\mathcal{L}\mathbf{u}, \mathbf{v} \rangle + D_{\mathbf{v}}\psi(\mathbf{u}) = \langle \mathbf{b}, \mathbf{v} \rangle, \quad \text{for all } \mathbf{v} \in V_0(\Omega'),$$

as long as $D_{\mathbf{v}}\psi(\mathbf{u})$, the Gâteaux derivative of ψ at \mathbf{u} in the direction of \mathbf{v} , exists for all $\mathbf{v} \in V_0(\Omega')$.

Proof. The proof is similar to that of part 2) of Theorem 4.3 and standard direct method of calculus of variations. \square

An example of a functional ψ satisfying (4.6) and the continuity condition is

$$\psi(\mathbf{u}) = \int_{\Omega} \int_{\Omega} F_0(|\mathbf{x}' - \mathbf{x}|) |\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^2 d\mathbf{x}' d\mathbf{x} + \int_{\Omega} h_{\alpha}(\mathbf{x}, \mathbf{u}(\mathbf{x})) d\mathbf{x},$$

where $F_0 \in L^1_{loc}$, and $h(\mathbf{x}, \boldsymbol{\eta})$ is continuous in $\boldsymbol{\eta}$, and has the growth condition

$$|h_{\alpha}(\mathbf{x}, \boldsymbol{\eta})| \leq c(k(\mathbf{x}) + |\boldsymbol{\eta}|^{\alpha}),$$

with $0 < \alpha < 2$ and $k(\mathbf{x}) \in L^1(\Omega)$. Note that using this functional, existence of a minimizer of \mathbf{E}_u in $V_0(\Omega')$ is guaranteed if $\|F_0\|_{L^1} + \|\int_{\Omega} F_0(\mathbf{x}' - \mathbf{x}) d\mathbf{x}'\|_{L^{\infty}(\Omega)}$ is sufficiently small.

5. THE LIMITING BEHAVIOR FOR VANISHING NON-LOCALITY

So far we have not discussed about the significance of the horizon δ , the extent of the nonlocal interaction between material points in a body. The aim of this section is to demonstrate that with appropriate scaling when the horizon approaches to zero, the solution to the nonlocal equation approximates the solution of the classical Navier system. Approximations of this type are already known to hold, see for example, [14, 27, 12, 15], although the case of indefinite kernels is not studied. To that end, assume that $\hat{\rho}$ is a nonnegative nonincreasing radial function satisfying

$$\hat{\rho}(|\boldsymbol{\xi}|) > 0, \text{ near } \boldsymbol{\xi} = 0, \quad \text{Supp}(\hat{\rho}) \subset B(0, 1), \text{ and } \int_{B(0,1)} \hat{\rho}(|\boldsymbol{\xi}|) d\boldsymbol{\xi} = 1.$$

We also assume that F_0 is a radial function supported in $B(0, 1)$ such that F_0 and $|F_0|$ are in $L^1(B(0, 1))$. We denote

$$\hat{\rho}_{\delta}(|\boldsymbol{\xi}|) = \frac{1}{\delta^d} \hat{\rho}\left(\frac{|\boldsymbol{\xi}|}{\delta}\right), \quad \rho^{\delta}(|\boldsymbol{\xi}|) = |\boldsymbol{\xi}|^{-2} \hat{\rho}_{\delta}(|\boldsymbol{\xi}|), \quad \text{and } F_0^{\delta} = \frac{1}{\delta^d} F_0\left(\frac{|\boldsymbol{\xi}|}{\delta}\right).$$

Given $\mathbf{b} \in L^2(\Omega; \mathbb{R}^d)$, we would like to study the limiting behavior of the solution to the nonlocal equation as the nonlocality $\delta \rightarrow 0$. We pick δ from a sequence of positive numbers converging to zero. That is, $\delta \in I = \{\delta_n : \delta_n \rightarrow 0, \text{ as } n \rightarrow \infty\}$.

Denote $\Sigma(n)$ to be the spectrum of $\mathcal{K}^n = (\mathcal{L}^{\delta_n})^{-1}\mathcal{F}^{\delta_n}$ (see Corollary 3.3) where we used the notation \mathcal{L}^δ , and \mathcal{F}^δ corresponding to ρ^δ and F^δ , respectively. Choose $\lambda \notin \Sigma = \cup_{i=1}^n \Sigma(n)$, such that for each $\delta \in I$ the nonlocal constrained value problem

$$\begin{cases} - \int_{\Omega} \mathbb{C}_{\lambda}^{\delta}(\mathbf{x}' - \mathbf{x})(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))d\mathbf{x}' = \mathbf{b} & \mathbf{x} \in \Omega_{\delta} \\ \mathbf{u} = 0 & \text{on } \Omega \setminus \Omega_{\delta}, \end{cases}$$

has a unique solution $\mathbf{u}_{\delta} \in V_0^{\delta} = V_0(\Omega_{\delta})$ where we have denoted

$$\mathbb{C}_{\lambda}^{\delta}(\boldsymbol{\xi}) = 2\frac{\rho^{\delta}(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^2}\boldsymbol{\xi} \otimes \boldsymbol{\xi} + 2\lambda F_0^{\delta}(|\boldsymbol{\xi}|)\mathbb{I}$$

and $\Omega_{\delta} = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) \geq \delta\}$. Let us first discuss the convergence of operators. The following proposition states that the above sequence of nonlocal operators converge to a well known local operator.

Proposition 5.1. *For all $\mathbf{w} \in C_c^{\infty}(\Omega; \mathbb{R}^d)$, for all $\mathbf{x} \in \Omega$, and all $\lambda \in \mathbb{R}$ we have*

$$-(\mathcal{L}^{\delta} + \lambda\mathcal{F}^{\delta})\mathbf{w}(\mathbf{x}) \longrightarrow -\mathcal{L}_0\mathbf{w}(\mathbf{x}), \quad \text{as } \delta \rightarrow 0,$$

where $-\mathcal{L}_0$ is the (local) Navier operator (with Poisson ratio 1/4):

$$-\mathcal{L}_0\mathbf{w}(\mathbf{x}) = -\mu\Delta\mathbf{w}(\mathbf{x}) - 2\mu\nabla\text{div}\mathbf{u}(\mathbf{x}), \quad (\mu = \frac{\omega_d}{d+2}, \quad \omega_d = |B(0,1)|).$$

Moreover, there exists a constant $C = C(d, \mathbf{w})$ such that

$$(5.1) \quad \sup_{\delta > 0} \sup_{x \in \Omega} |(\mathcal{L}^{\delta} + \lambda\mathcal{F}^{\delta})\mathbf{w}(\mathbf{x})| \leq C.$$

Proof. We will first evaluate the limit

$$-\lim_{\delta \rightarrow 0} 2 \int_{B(\mathbf{x}, \delta)} \frac{\rho^{\delta}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} ((\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x}))(\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x}))d\mathbf{x}',$$

for all $\mathbf{x} \in \Omega$. Note, for sufficiently small δ , after change of variables the above expression is

$$-2 \int_{B(0, \delta)} \frac{\rho^{\delta}(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^2} (\boldsymbol{\xi} \otimes \boldsymbol{\xi})(\mathbf{w}(\mathbf{x} + \boldsymbol{\xi}) - \mathbf{w}(\mathbf{x}))d\boldsymbol{\xi}.$$

Since \mathbf{w} is smooth we can use Taylor expansion to write

$$\mathbf{w}(\mathbf{x} + \boldsymbol{\xi}) - \mathbf{w}(\mathbf{x}) = \nabla\mathbf{w}(\mathbf{x})\boldsymbol{\xi} + \frac{1}{2}D^2\mathbf{w}(\mathbf{x})\boldsymbol{\xi} \otimes \boldsymbol{\xi} + R(\mathbf{x}, |\boldsymbol{\xi}|)$$

where $R(\mathbf{x}, |\boldsymbol{\xi}|) \leq C|\boldsymbol{\xi}|^3$, and therefore,

$$\boldsymbol{\xi} \cdot (\mathbf{w}(\mathbf{x} + \boldsymbol{\xi}) - \mathbf{w}(\mathbf{x})) = (\nabla\mathbf{w}(\mathbf{x})\boldsymbol{\xi}, \boldsymbol{\xi}) + \frac{1}{2}(D^2\mathbf{w}(\mathbf{x})\boldsymbol{\xi} \otimes \boldsymbol{\xi}, \boldsymbol{\xi}) + \tilde{R}(\mathbf{x}, |\boldsymbol{\xi}|)$$

where $|\tilde{R}(\mathbf{x}, |\boldsymbol{\xi}|)| \leq C|\boldsymbol{\xi}|^4$, for all $\mathbf{x} \in \Omega$ and $|\boldsymbol{\xi}| \leq 1$. Using the above expansion and noting that

$$\int_{B(0, \delta)} \frac{\rho^{\delta}(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^2} (\nabla\mathbf{w}(\mathbf{x})\boldsymbol{\xi}, \boldsymbol{\xi})\boldsymbol{\xi}d\boldsymbol{\xi} = 0,$$

we have that

$$\begin{aligned} & -2 \int_{B(0, \delta)} \frac{\rho^{\delta}(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^2} (\boldsymbol{\xi} \otimes \boldsymbol{\xi})(\mathbf{w}(\mathbf{x} + \boldsymbol{\xi}) - \mathbf{w}(\mathbf{x}))d\boldsymbol{\xi} \\ &= - \int_{B(0, \delta)} \frac{\rho^{\delta}(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^2} (D^2\mathbf{w}(\mathbf{x})\boldsymbol{\xi} \otimes \boldsymbol{\xi}, \boldsymbol{\xi})\boldsymbol{\xi}d\boldsymbol{\xi} - 2 \int_{B(0, \delta)} \frac{\rho^{\delta}(|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|^2} \tilde{R}(\mathbf{x}, |\boldsymbol{\xi}|)\boldsymbol{\xi}d\boldsymbol{\xi} \end{aligned}$$

On the one hand, it is not difficult to show that

$$\lim_{\delta \rightarrow 0} -2 \int_{B(0,\delta)} \frac{\rho^\delta(|\xi|)}{|\xi|^2} \tilde{R}(\mathbf{x}, |\xi|) \xi d\xi = 0.$$

On the other hand,

$$\begin{aligned} & - \lim_{\delta \rightarrow 0} \int_{B(0,\delta)} \frac{\rho^\delta(|\xi|)}{|\xi|^2} (D^2 \mathbf{w}(\mathbf{x}) \xi \otimes \xi, \xi) \xi d\xi \\ &= - \lim_{\delta \rightarrow 0} \int_{B(0,\delta)} \hat{\rho}^\delta(|\xi|) \left(D^2 \mathbf{w}(\mathbf{x}) \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}, \frac{\xi}{|\xi|} \right) \frac{\xi}{|\xi|} d\xi \\ &= - \int_{\mathbb{S}^{d-1}} (D^2 \mathbf{w}(\mathbf{x}) \mathbf{s} \otimes \mathbf{s}, \mathbf{s}) s d\sigma(\mathbf{s}). \\ &= -\mu \Delta \mathbf{w}(x) - 2\mu \nabla \operatorname{div} \mathbf{w}(x), \end{aligned}$$

where $\mu = \frac{\omega_d}{d+2}$, and ω_d is the volume of the unit sphere. One can check the last equality with a mere evaluation of the surface integral, [20, See Appendix of].

Similar calculation as above imply that for all $\mathbf{x} \in \Omega$,

$$\begin{aligned} & 2 \int_{\Omega} F_0^\delta(|\mathbf{x}' - \mathbf{x}|) (\mathbf{w}(\mathbf{x}') - \mathbf{w}(\mathbf{x})) d\mathbf{x}' \\ &= \int_{B_\delta(0)} F_0^\delta(|\xi|) \nabla \mathbf{w}(\mathbf{x}) \xi d\xi + \int_{B_\delta(0)} F_0^\delta(|\xi|) r(\mathbf{x}, |\xi|) d\xi \end{aligned}$$

where $|r(x, \xi)| \leq C|\xi|^2$ for all $\mathbf{x} \in \Omega$. The first term on the right hand side is 0 since F_0 is a radial function. We show that the second term goes to zero as $\delta \rightarrow 0$. Indeed,

$$\left| \int_{B_\delta(0)} F_0^\delta(|\xi|) r(\mathbf{x}, |\xi|) d\xi \right| \leq \delta^2 \int_{B_\delta(0)} |F_0^\delta(|\xi|)| d\xi = \delta^2 \int_{B_1(0)} |F_0(|\xi|)| d\xi \rightarrow 0,$$

as $\delta \rightarrow 0$, since $F_0(|\xi|) \in L^1$.

To prove estimate (5.1), we note that when $2\delta < h = \operatorname{dist}(\operatorname{Supp}(\mathbf{w}); \partial\Omega)$, then for any $\mathbf{x} \in \Omega \setminus \Omega_{h/2}$, $(\mathcal{L}^\delta + \mathcal{F}^\delta) \mathbf{w}(\mathbf{x}) = 0$ while for $\mathbf{x} \in \Omega_{h/2}$, using similar arguments as in the proof of Part b) i) of Proposition 3.1, we have

$$|(\mathcal{L}^\delta + \mathcal{F}^\delta) \mathbf{w}(\mathbf{x})| \leq C \|D^2 \mathbf{w}\|_{L^\infty}.$$

□

Next we study the behavior of the sequence of solutions as $\delta \rightarrow 0$. Recall that for all $\mathbf{w} \in V_0^\delta$,

$$(5.2) \quad \langle -\mathcal{L}^\delta \mathbf{u}_\delta, \mathbf{w} \rangle + \lambda (\mathcal{F}^\delta \mathbf{u}_\delta, \mathbf{w}) = (\mathbf{b}, \mathbf{w}).$$

We begin by obtaining estimates that are uniform in δ . Our means is the nonlocal Poincaré-type inequality. We note, however, that the standard Poincaré-type inequality proved in Proposition 2.6 is not good enough to offer precise estimates that show the explicit dependence on δ as the constant depends on the subspace $V_0(\Omega_\delta)$, hence on δ . We thus need the following lemma which is a sharper version. Its proof is an adaptation to our setting of the argument used in [25] for a similar nonlocal Poincaré-type inequality for functions. Our proof of the sharper version uses the compactness result [20, Theorem 5.1] (see also [5, Theorem 4] or [25, Theorem 1.2] for functions) which we state here as a lemma.

Lemma 5.2. *Suppose that \mathbf{u}_n is a bounded sequence in $L^2(\Omega; \mathbb{R}^d)$ with compact support in Ω . Then if*

$$\sup_n \int_{\Omega} \int_{\Omega} \rho^{\delta_n}(|\mathbf{x}' - \mathbf{x}|) \left| \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \cdot (\mathbf{u}_n(\mathbf{x}') - \mathbf{u}_n(\mathbf{x})) \right|^2 d\mathbf{x}' d\mathbf{x} < \infty,$$

then \mathbf{u}_n is precompact in $L^2(\Omega; \mathbb{R}^d)$. Moreover, any limit point $\mathbf{u} \in W_0^{1,2}(\Omega; \mathbb{R}^d)$.

Our sharper Poincaré-type inequality is the following.

Proposition 5.3. *There exists δ_0 and $C(\delta_0)$ such that for all $\delta \in (0, \delta_0]$,*

$$\|\mathbf{u}\|_{L^2(\Omega)}^2 \leq C(\delta_0) \int_{\Omega} \int_{\Omega} \frac{\rho^{\delta}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} \quad \forall \mathbf{u} \in V_0(\Omega_{\delta}).$$

Proof. Let

$$\frac{1}{A} = \inf \left\{ \int_{\Omega} 2\mu |e(\nabla \mathbf{u}(\mathbf{x}))|^2 + \mu (\operatorname{div} \mathbf{u}(\mathbf{x}))^2 d\mathbf{x} : \mathbf{u} \in W_0^{1,2}(\Omega; \mathbb{R}^d), \|\mathbf{u}\|_{L^2} = 1 \right\}$$

where the constant $\mu = \frac{\omega_d}{d+2}$. By standard local Poincaré inequality, $\infty > A > 0$. We claim that given ϵ , there exists $\delta_0(\epsilon)$ such that for all $\delta < \delta_0$ the lemma holds with $C(\delta_0) = A + \epsilon$.

We prove the above statement by contradiction. Suppose there exists $C > A$, such that for all n , there exists, $0 < \delta_n < 1/n$, and \mathbf{u}_n with the property that

$$\begin{aligned} \mathbf{u}_{\delta_n} &\in V_0(\Omega_{\delta_n}), \quad \int_{\Omega} |\mathbf{u}_{\delta_n}(\mathbf{x})|^2 d\mathbf{x} = 1, \\ \text{and} \quad \int_{\Omega} \int_{\Omega} \frac{\rho^{\delta_n}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}_{\delta_n}(\mathbf{x}') - \mathbf{u}_{\delta_n}(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} &< \frac{1}{C}. \end{aligned}$$

By Lemma 5.2, \mathbf{u}_{δ_n} is precompact in $L^2(\Omega; \mathbb{R}^d)$. Moreover, any limit point \mathbf{u} will be in $W_0^{1,2}(\Omega; \mathbb{R}^d)$, and $\|\mathbf{u}\|_{L^2} = 1$. In addition, we claim that \mathbf{u} satisfies

$$\int_{\Omega} 2\mu |e(\nabla \mathbf{u}(\mathbf{x}))|^2 + \mu (\operatorname{div} \mathbf{u}(\mathbf{x}))^2 d\mathbf{x} \leq 1/C.$$

This gives the desired contradiction as A is the best Poincaré constant. Let us now prove the claim. Let ϕ_{ϵ} be a standard mollifier. Then it is not difficult to show that for each ϵ , $\phi_{\epsilon} * \mathbf{u}_{\delta_n}$ satisfies the bound

$$(5.3) \quad \int_{\Omega_{\epsilon}} \int_{\Omega_{\epsilon}} \frac{\rho^{\delta_n}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\phi_{\epsilon} * \mathbf{u}_{\delta_n}(\mathbf{x}') - \phi_{\epsilon} * \mathbf{u}_{\delta_n}(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} < \frac{1}{C}$$

Observe that for each fixed ϵ the mollified sequence $\phi_{\epsilon} * \mathbf{u}_{\delta_n} \rightarrow \phi_{\epsilon} * \mathbf{u}$ in $C^{\infty}(\overline{\Omega_{\epsilon}}; \mathbb{R}^d)$ as $n \rightarrow \infty$. Then we have

$$\begin{aligned} &\int_{\Omega_{\epsilon}} \int_{\Omega_{\epsilon}} \frac{\rho^{\delta_n}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\phi_{\epsilon} * \mathbf{u}(\mathbf{x}') - \phi_{\epsilon} * \mathbf{u}(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} \\ &\leq \int_{\Omega_{\epsilon}} \int_{\Omega_{\epsilon}} \frac{\rho^{\delta_n}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\phi_{\epsilon} * \mathbf{u}(\mathbf{x}') - \phi_{\epsilon} * \mathbf{u}_{\delta_n}(\mathbf{x}'))|^2 d\mathbf{x}' d\mathbf{x} \\ &\quad + \int_{\Omega_{\epsilon}} \int_{\Omega_{\epsilon}} \frac{\rho^{\delta_n}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\phi_{\epsilon} * \mathbf{u}_{\delta_n}(\mathbf{x}') - \phi_{\epsilon} * \mathbf{u}_{\delta_n}(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} \\ &\quad + \int_{\Omega_{\epsilon}} \int_{\Omega_{\epsilon}} \frac{\rho^{\delta_n}(|\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\phi_{\epsilon} * \mathbf{u}_{\delta_n}(\mathbf{x}) - \phi_{\epsilon} * \mathbf{u}(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} \end{aligned}$$

and letting $n \rightarrow \infty$ in the above and applying [20, Corollary 2.5], we obtain

$$2\mu \int_{\Omega_\epsilon} \left(|e(\nabla(\phi_\epsilon * \mathbf{u})(\mathbf{x}))|^2 + \frac{1}{2} |\operatorname{div}(\phi_\epsilon * \mathbf{u})(\mathbf{x})|^2 \right) dx \leq \frac{1}{C}$$

Now we take $\epsilon \rightarrow 0$ to obtain that

$$2\mu \int_{\Omega} \left(|e(\nabla \mathbf{u}(\mathbf{x}))|^2 + \frac{1}{2} |\operatorname{div} \mathbf{u}(\mathbf{x})|^2 \right) dx \leq \frac{1}{C}.$$

as asserted. \square

We now establish the convergence of the solutions \mathbf{u}_δ to \mathbf{u} of the Navier system.

Theorem 5.4. *Given $\hat{\rho}$ and F_0 as above, there exists $\lambda_0 > 0$ such that for all $|\lambda| \leq \lambda_0$, and $\mathbf{b} \in L^2(\Omega; \mathbb{R}^d)$ the sequence of solutions \mathbf{u}_δ converges strongly in $L^2(\Omega; \mathbb{R}^d)$ to $\mathbf{u} \in W_0^{1,2}(\Omega; \mathbb{R}^d)$, where \mathbf{u} solves the Navier system*

$$\begin{cases} -\mu \Delta \mathbf{u}(\mathbf{x}) - 2\mu \nabla \operatorname{div} \mathbf{u}(\mathbf{x}) = \mathbf{b}(\mathbf{x}) & \text{a.e. } \mathbf{x} \in \Omega \\ \mathbf{u}(\mathbf{x}) = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. We begin the proof by obtaining some uniform estimates. Plugging in \mathbf{u}_δ in place of \mathbf{w} in (5.2) we obtain that

$$(5.4) \quad \begin{aligned} & \int_{\Omega} \int_{\Omega} \frac{\rho^\delta(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}_\delta(\mathbf{x}') - \mathbf{u}_\delta(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} \\ & = -\lambda \int_{\Omega} \int_{\Omega} F_0^\delta(\mathbf{x}' - \mathbf{x}) |(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}_\delta(\mathbf{x}') - \mathbf{u}_\delta(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} + \int_{\Omega} \mathbf{b} \cdot \mathbf{u}_\delta d\mathbf{x} \end{aligned}$$

A simple calculation yields the following estimate

$$(5.5) \quad \int_{\Omega} \int_{\Omega} \frac{\rho^\delta(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}_\delta(\mathbf{x}') - \mathbf{u}_\delta(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} \leq 4|\lambda| \|F_0\|_{L^1} \|\mathbf{u}_\delta\|_{L^2}^2 + \|\mathbf{b}\|_{L^2} \|\mathbf{u}_\delta\|_{L^2}.$$

We next show that the left hand side of (5.5) is bounded uniformly in δ . Combining (5.5) and the estimate Proposition 5.3, we notice that for all $\delta \in I$, there exists C independent of δ such that

$$\|\mathbf{u}_\delta\|_{L^2}^2 \leq 4C|\lambda| \|F_0\|_{L^1} \|\mathbf{u}_\delta\|_{L^2}^2 + C\|\mathbf{b}\|_{L^2} \|\mathbf{u}_\delta\|_{L^2}.$$

Now if we choose λ_0 small enough such that for all $|\lambda| < \lambda_0$, the number $\nu = \frac{C}{1-4C|\lambda|\|F_0\|_{L^1}} > 0$, then

$$\|\mathbf{u}_\delta\|_{L^2} \leq \nu \|\mathbf{b}\|_{L^2}$$

where ν is independent of δ . Plugging this estimate in (5.5) we obtain that

$$\int_{\Omega} \int_{\Omega} \frac{\rho^\delta(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^2} |(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{u}_\delta(\mathbf{x}') - \mathbf{u}_\delta(\mathbf{x}))|^2 d\mathbf{x}' d\mathbf{x} \leq C$$

where the constant C is independent of δ .

With these uniform estimates at hand, we may apply Lemma 5.2 to conclude that the sequence $\{\mathbf{u}_\delta\}$ is precompact in $L^2(\Omega; \mathbb{R}^d)$, and any limit point $\mathbf{u} \in W_0^{1,2}(\Omega; \mathbb{R}^d)$. Let us show that any limit point will solve the Navier system and therefore unique and thus the entire sequence actually strongly converge to the unique solution \mathbf{u} .

Let $\mathbf{w} \in C_c^\infty(\Omega; \mathbb{R}^d)$. Then since the operator are self adjoint and $\mathcal{L}^\delta \mathbf{w} \in L^2(\Omega; \mathbb{R}^d)$ (see Proposition 3.1) we may rewrite (5.2) as

$$(5.6) \quad ((-\mathcal{L}^\delta + \lambda \mathcal{F}^\delta) \mathbf{w}, \mathbf{u}_\delta) = (\mathbf{b}, \mathbf{w}).$$

Using Proposition 5.1, the facts that $\mathbf{u}_\delta \rightarrow \mathbf{u}$ strongly in $L^2(\Omega; \mathbb{R}^d)$, $\mathbf{u}_\delta = 0$ outside of Ω_δ and $\Omega_\delta \subset \text{supp}(\mathbf{w})$ as $\delta \rightarrow 0$, we obtain from (5.6) that for all $\mathbf{w} \in C_c^\infty(\Omega; \mathbb{R}^d)$,

$$-\mu \int_{\Omega} (\Delta \mathbf{w}(x) + 2\nabla \text{div} \mathbf{w}(x)) \cdot \mathbf{u}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \mathbf{b}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x}.$$

as $\delta \rightarrow 0$, verifying that \mathbf{u} solves the Navier system. \square

Remark 5.1. *When $\lambda = 0$, that is in the absence of any perturbation, the nonlocal solutions \mathbf{u}_δ converge to \mathbf{u} solving the same Navier system. This implies that the 'large scale' Navier system does not see the effect of the addition of F_0 while the 'small scale' peridynamic system can detect the effect.*

6. CONCLUSION

In this work we have analyzed a linear peridynamic system modeling microelastic and isotropic materials. We have studied the equilibrium equations posing as a system of nonlocal constrained value problems with pure Dirichlet-type volumetric constraint. The necessity in considering a sign-changing kernel for practical applications has been highlighted in the original works of Silling [23], so that the contribution of the paper is proving the well posedness of the systems for more general class of PD models that are of close relevance to material modeling. We have used standard variational methods and Fredholm Alternative Theorem was used to prove well-posedness for the equilibrium system. We have presented conditions on potentially indefinite micromodulus tensors that give rise to equilibrium solutions minimizing a potential energy functional. These results are obtained as a consequence a careful study of the energy space and the nonlocal operator. The energy space is shown to be a separable Hilbert space with respect to a naturally defined inner product. We have managed to prove the validity of a nonlocal Poincaré-type inequality in the energy space. Conditions on the micromodulus tensor are provided for the space to be compactly embedded in $L^2(\Omega; \mathbb{R}^d)$. We have demonstrated the relationship between the nonlocal system with that the classical Navier system. Indeed, we have proved the convergence of solutions to the nonlocal problem to that of the local system when the extent of the nonlocal interaction is vanishing. Our studies provide much theoretical basis to further numerical and modeling works based on the peridynamic models. In the future, one may naturally consider the extension to more general systems such as the state-based peridynamic Navier systems that cover a larger class of elastic materials [24, 11]. Our results on nonlinear nonlocal variational problems also have close relevance to a number of interesting theoretical development such as the nonlocal interaction models originally proposed by Van der Vaals [26] and the nonlocal Cahn-Hilliard models [8, 9] of phase transitions which have received much interests in recent years. In addition, the issue of regularity of solutions relative to given data may also be investigated.

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