THE BORDISM CLASS OF A BUNDLE SPACE

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Dedicated to R. L. Wilder on his seventieth birthday.

1. INTRODUCTION

In an earlier note [6], it was shown that if M^n is a closed manifold with even Euler characteristic, then the unoriented bordism class $[M^n]_2 \in \mathfrak{N}_n$ can be represented by a manifold fibred differentiably over the circle with structure group Z_2. Now we shall prove that if M^{4m} is a closed oriented manifold with index 0, then modulo an element of order 2 its oriented bordism class is represented by a manifold differentiably fibred over S^2 with structure group SO(2) and an oriented fibre. As a corollary of this, Burdick [2] uses his methods to show that $[M^{4m}]$ is represented, modulo torsion, by a manifold differentiably fibred over S^1 with structure group Z and an oriented fibre that bounds.

The proof of the above result appears in Section 6. We shall briefly outline the steps involved. It was shown by Milnor [9] that $\Omega/p\Omega$, where Ω is the oriented bordism ring of a point, is for any odd prime p a graded polynomial ring over Z_p with a generator in each dimension divisible by 4. A bordism class $[M^{4n}]$ is a generator of $\Omega/p\Omega$ if and only if modulo $p\Omega$ it cannot be expressed as a sum of products of lower-dimensional elements.

In Section 2 we discuss a construction of oriented (or weakly complex) bordism classes. This assigns to a complex (k + 1)-plane bundle $\xi \to M^{2n}$ over a closed oriented manifold the total space $CP(\xi)$ of the associated projective space bundle with fibre CP(k). In (4.1) we find the formula needed to determine whether $[CP(\xi)] \in \Omega_{2(n+k)}$ is a generator of $\Omega/p\Omega$. This is done by computing a numerical invariant in terms of the Chern classes of $\xi \to M^{2n}$.

In Section 4 we obtain a series of corollaries that represent the cases in which an effective computation is possible. In Section 6 a collection of manifolds fibred over S^2 is described. By means of the results established in Section 4 it is shown that some of these manifolds are generators of $\Omega/p\Omega$. The principal result will follow.

Section 5 shows, at least in part, just how effective the construction method of Section 2 is in finding generators of $\Omega/p\Omega$. Previously, several devices have been used in presenting generators, but as far as we know this is the first attempt at determining the efficiency of a construction.

The computations in Section 3 and for (4.1) appear to be tedious. We hope, however, that (4.1) sufficiently unifies this problem to eliminate this kind of work in the future.

Finally, in Sections 2 to 5 we use the weakly complex bordism ring $\mathfrak{U}$ of a point [10], [7, Chapter 1] rather than Ω. It is easier for us to work with, and as we point out in Section 6, there is no loss of generality.

The author expresses his admiration for Professor R. L. Wilder and his distinguished career in topology.

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2. GENERATORS OF $\mathfrak{U}/p\mathfrak{U}$

An element of the weakly complex bordism group $\mathfrak{U}_{2n}(BU(k+1))$ can be interpreted as the suitably defined bordism class of a smooth complex $(k+1)$-plane bundle $\pi: \xi \to M^{2n}$ over a closed weakly complex manifold. This interpretation stems immediately from the fact that the $(k+1)$-plane bundles over a finite complex X are in natural one-to-one correspondence with the homotopy classes of maps of X into the classifying space $BU(k+1)$.

We shall describe a natural homomorphism $\mathfrak{U}_{2n}(BU(k+1)) \to \mathfrak{U}_{2(n+k)}$. Let $p: CP(\xi) \to M^{2n}$ be the associated complex projective space bundle with fibre $CP(k)$; then the total space $CP(\xi)$ is itself a closed $2(n+k)$-manifold. We define a weakly complex structure on $CP(\xi)$ so that the homomorphism is

$$[\xi \to M^{2n}] \to [CP(\xi)] \in \mathfrak{U}_{2(n+k)}.$$

We denote by $\tau_1 \to M^{2n}$ the tangent bundle to the base, and by $\tau_2 \to CP(\xi)$ the complex k-plane bundle of vectors tangent to the fibres. According to [1], the tangent bundle of $CP(\xi)$ is $p^*(\tau_1) + \tau_2$; thus we assign to $CP(\xi)$ the weakly complex structure determined by that on τ_1 together with the complex structure on τ_2. The reader may verify that this homomorphism is well-defined. The following is one of our basic results.

(2.1) THEOREM. *If* p *is a fixed prime and* $n \leq k+1$, *then the image of*

$$\mathfrak{U}_{2n}(BU(k+1)) \to \mathfrak{U}_{2(n+k)}$$

contains an element that is indecomposable (*mod* $p\mathfrak{U}$), *except in the four cases*

(i) $n = 1$,

(ii) $n \equiv 1$ (*mod* p), $n < p^j$, $n+k = \alpha p^{j+1} + \beta p^j$ $(0 < \beta < p)$,

(iii) $n+k \neq p^j - 1$, $n < p^j$, $n+k = \alpha p^{j+1} + \beta p^j - 1$ $(0 < \beta < p)$,

(iv) $n+k = p^j - 1$ *and* $n < p^{j-1}$.

Thus we obtain a specific construction for generators of $\mathfrak{U}/p\mathfrak{U}$, and we know why this construction will fail in certain dimensions.

To demonstrate (2.1), we must introduce

$$\mathfrak{M}_{2m} = \sum_0^m \mathfrak{U}_{2n}(BU(m-n))$$

together with

$$\mathfrak{M} = \sum_0^\infty \mathfrak{M}_{2m}.$$

Obviously, $\mathfrak{M}$ is bigraded; that is, a homogeneous element is a bordism class of a complex vector bundle $[\xi \to M^{2n}]$ whose total degree is the dimension of the base plus the real dimension of the fibre. We shall agree that $\mathfrak{U}_{2m}(BU(0)) = \mathfrak{U}_{2m}$; thus $\mathfrak{U}_{2m} \subset \mathfrak{M}_{2m}$ as the 0-dimensional bundles over closed weakly complex manifolds.

A product is introduced into $\mathfrak{M}$ by setting

$$[\xi_1 \to M^{2n}][\xi_2 \to V^{2m}] = [\xi_1 \times \xi_2 \to M^{2n} \times V^{2m}].$$

In this way, $\mathfrak{M}$ becomes a graded commutative algebra over $\mathfrak{U}$ with unit. The unit is the 0-bundle over a point.

(2.2) LEMMA. *As an algebra over* $\mathfrak{U}$, $\mathfrak{M}$ *is a graded polynomial algebra whose generators are the Hopf line bundles* $\{[\eta \to CP(n)]\}_0^\infty$.

Actually, an alalogous result was proved for orthogonal bundles and unoriented bordism in [5, Section 28]. We shall merely sketch the proof, here. With $k+1$ fixed, consider all sequences $j_1 \leq \cdots \leq j_{k+1} = \omega$ of nonnegative integers, and with each associate the product

$$X_\omega = [\eta \to CP(j_1)] \cdots [\eta \to CP(j_{k+1})]$$

in $\mathfrak{U}_{2n}(BU(k+1))$, where $n = j_1 + \cdots + j_{k+1}$. Under the Thom homomorphism

$$\mu\colon \mathfrak{U}_*(BU(k+1)) \to H_*(BU(k+1); Z)$$

the elements $\mu(X_\omega)$ provide a homogeneous base for $H_*(BU(k+1); Z)$. We argue just as in [5, (18.1)] that $\mathfrak{U}_*(BU(k+1))$ is a free $\mathfrak{U}$-module generated by the X_ω. Since

$$\mathfrak{M} = \sum_{-1}^{\infty} \mathfrak{U}_*(BU(k+1)),$$

Lemma (2.2) will follow.

3. ALGEBRAIC PRELIMINARIES

If $\pi\colon \xi \to X$ is a complex $(k+1)$-plane bundle over a finite complex, then we shall denote by $p\colon CP(\xi) \to X$ the associated complex projective space bundle with fibre $CP(k)$. There is a canonical line bundle $\eta \to CP(\xi)$. A point in η is a pair consisting of a line in $\pi^{-1}(x)$ together with a vector in this line. We shall denote by $t \in H^2(CP(\xi); Z)$ the characteristic class of η. *Via* the induced homomorphism p^*, $H^*(CP(\xi); Z)$ is a free graded $H^*(X; Z)$-module with base $1, t, \cdots, t^k$ [8]. Borel and Hirzebruch in [1] completely determined the structure of the ring $H^*(CP(\xi); Z)$ by showing that

$$\sum_{0}^{k+1} (-1)^{k+1-j} p^*(v_j) t^{k+1-j} = 0,$$

where $v_0 = 1, v_1, \cdots, v_{k+1}$ are the Chern classes of ξ. We shall give a formula for the powers t^{n+k+1} analogous to that found for real bundles in [4, Section 2].

For $n \geq 0$, there are unique classes $V_{j,n} \in H^{2(j+n)}(X; Z)$ $(1 \leq j \leq k+1)$ for which

$$(-1)^{n+k+1} t^{n+k+1} + \sum_{1}^{k+1} (-1)^{k+1-j} p^*(V_{j,n}) t^{k+1-j} = 0.$$

If we multiply this defining equation by $-t$, we obtain by an elementary calculation the recursion formula $V_{j,n+1} = V_{j+1,n} - v_j V_{1,n}$.

(3.1) LEMMA. *In terms of the dual Chern classes,* $V_{1,n} = -\bar{v}_{n+1}$ *and*

$$V_{j,n+1} = \sum_{0}^{n+1} \bar{v}_i v_{n+1+j-i}.$$

If we show that $V_{1,n} = \sum_0^n \bar{v}_i v_{n+1-i}$, then $V_{1,n} = -\bar{v}_{n+1}$ since by definition $\bar{v}_{n+1} + \sum_0^n \bar{v}_i \bar{v}_{n+1-i} = 0$. The reader may verify the lemma for $n = 0$. Proceeding inductively, we use the recursion formula to write

$$V_{j,n+1} = \sum_{0}^{n} \bar{v}_i v_{n+j+1-i} + v_j \bar{v}_{n+1} = \sum_{0}^{n+1} \bar{v}_i v_{n+1+j-i},$$

which is the assertion of (3.1).

The total Chern class of $\xi \to X$ can be expressed in the factored form $\sum_1^{k+1} (1 + t_j)$, and for each $m > 0$ we introduce the symmetric functions $S_m(\xi) = \sum_1^{k+1} (t_j)^m$. Each $S_m(\xi)$ can be uniquely expressed as a homogeneous polynomial in the Chern classes (the elementary symmetric functions), hence $S_m(\xi) \in H^{2m}(X; Z)$. For a Whitney sum,

$$S_m(\xi_1 + \xi_2) = S_m(\xi_1) + S_m(\xi_2).$$

If M^{2n} is a weakly complex manifold, then in terms of the Chern classes of the weakly complex structure on the tangent bundle we obtain an $S_n(2(m - n)R + \tau)$ in $H^{2n}(M^{2n}; Z)$. If $\sigma_{2n} \in H_{2n}(M^{2n}; Z)$ is the fundamental class determined by the weakly complex structure, then we set

$$S_n(M^{2n}) = \langle S_n(2(m - n)R + \tau), \sigma_{2n} \rangle \in Z.$$

This is an invariant of $[M^{2n}] \in \mathfrak{U}_{2n}$. Milnor [9] has announced the following result.

(3.2) THEOREM. *If* p *is a fixed prime, then* $[M^{2n}]$ *is indecomposable* mod p *if and only if*

(i) $S_n(M^{2n}) \neq 0$ (mod p) *and* $n \neq p^j - 1$,

(ii) $S_n(M^{2n}) \neq 0$ (mod p^2) *and* $n = p^j - 1$.

The question of indecomposability must therefore be settled by the computation of the invariant $S_{n+k}(CP(\xi))$, and that will be carried out in Section 4.

We shall need an elementary result about the $S_m(\xi)$, and since we have no reference, we include an outline of the proof.

(3.3) LEMMA. *For any complex vector bundle and any* $m > 0$,

$$\sum_{1}^{m} (-1)^i S_i(\xi) \bar{v}_{m-i} = m \bar{v}_m.$$

The formula is trivial for a line bundle. Let ξ_1, ξ_2 be a pair of bundles over X for which the formula is valid; we shall show that (3.3) is then also valid for the sum

$\xi = \xi_1 + \xi_2$. Let $\bar{x}_i$ and $\bar{y}_i$ denote the dual Chern classes of ξ_1 and ξ_2, respectively. We begin with

$$\sum_1^m (-1)^i s_i(\xi)\bar{v}_{m-i} = \sum_1^m (-1)^i s_i(\xi_1)\bar{v}_{m-i} + \sum_1^m (-1)^i s_i(\xi_2)\bar{v}_{m-i}.$$

The first sum can be rewritten as

$$\begin{aligned}\sum_1^m (-1)^i s_i(\xi_1)\bar{v}_{m-i} &= \sum_1^m (-1)^i s_i(\xi_1)\left(\sum_i^{m-i} \bar{x}_{m-i-j}\bar{y}_j\right) \\ &= \sum_0^m \left(\sum_1^{m-j} (-1)^i s_i(\xi_1)\bar{x}_{m-i-j}\right)\bar{y}_j \\ &= \sum_0^m (m-j)\bar{x}_{m-j}\bar{y}_j.\end{aligned}$$

Similarly, $\sum_1^m (-1)^i s_i(\xi_2)\bar{v}_{m-i} = \sum_0^m j\bar{x}_{m-j}\bar{y}_j$, and therefore

$$\sum_1^m (-1)^i s_i(\xi)\bar{v}_{m-i} = m\bar{v}_m.$$

From this point the reader may complete the proof by a suitable splitting argument.

4. COMPUTATION OF S_{n+k}

The invariant $S_{n+k}(CP(\xi))$ can be evaluated as follows.

(4.1) THEOREM. *If $\xi \to M^{2n}$ is a smooth complex $(k+1)$-plane bundle over a closed weakly complex manifold, then*

$$S_{n+k}(CP(\xi)) = \pm\left\langle (k+1)\bar{v}_n + \sum_1^n (n+k,\, i)s_i(\xi)\bar{v}_{n-i},\ \sigma_{2n}\right\rangle.$$

Thus we see that the answer depends only on the characteristic classes of ξ. We recall that the weakly complex structure on $CP(\xi)$ was taken from

$$p^*(2(m-n)R + \tau_1) + \tau_2.$$

Now

$$S_{n+k}(p^*(2(m-n)R + \tau_1) + \tau_2) = p^*(S_{n+k}(2(m-n)R + \tau_1) + S_{n+k}(\tau_2)) = S_{n+k}(\tau_2),$$

since $k > 0$. Therefore $S_{n+k}(CP(\xi)) = \langle S_{n+k}(\tau_2), \sigma_{2(n+k)}\rangle \in Z$.

We can compute $S_{n+k}(\tau_2)$, since

$$\tau_2 + C = \bar{\eta} \otimes p^*(\xi) \quad \text{and} \quad S_{n+k}(\tau_2 + C) = S_{n+k}(\tau_2).$$

If $\pi_1^{k+1}(1+t_j)$ is the factored form of the total Chern class of ξ, then $\pi_1^{k+1}(1+p^*(t_j)-t)$ is the total Chern class of $\tau_2 + C$ and thus

$$S_{n+k}(\tau_2) = \sum_1^{k+1} (p^*(t_j) - t)^{n+k}.$$

We use the binomial expansion

$$S_{n+k}(\tau_2) = \sum_1^{k+1} \left(\sum_0^{n+k} (-1)^i (n+k, i) p^*(t_j^{n+k-i}) t^i \right)$$

$$= \sum_0^{n+k-1} (-1)^i (n+k, i) p^*(S_{n+k-i}(\xi)) t^i + (k+1)(-1)^{n+k} t^{n+k}.$$

Since $S_{n+k-i}(\tau_2) = 0$ if $i < k$, we can replace i by $n + k - i$ to obtain the relation

$$S_{n+k}(\tau_2) = \sum_1^n (-1)^{n+k-i} (n+k, i) p^*(S_i(\xi)) t^{n+k-i} + (k+1)(-1)^{n+k} t^{n+k}.$$

If $i < n$, we write $n + k - i = (k + 1) + (n - i - 1)$, so that

$$(-1)^{n+k-i} p^*(S_i(\xi)) t^{n+k-i} = \sum_1^{k+1} (-1)^{k-j} p^*(S_i(\xi) V_{j,n-i-1}) t^{k+1-j}.$$

Note, however, that $S_i(\xi) V_{j,n-i-1} \in H^{2(n-1+2)}(M^{2n}; Z)$, so that

$$(-1)^{n+k-i} p^*(S_i(\xi)) t^{n+k-i} = (-1)^k p^*(S_i(\xi) \bar{v}_{n-i}) t^k.$$

A similar argument shows that

$$(-1)^{n+k} t^{n+k} = (-1)^k p^*(\bar{v}_n) t^k,$$

and since $\bar{v}_0 = 1$, we may write

$$S_{n+k}(\tau_2) = \pm p^* \left(\sum_1^n (n+k, i) S_i(\xi) \bar{v}_{n-i} + (k+1) \bar{v}_n \right) t^k.$$

The fundamental cycle $\sigma_{2(n+k)}$ on $CP(\xi)$ is related to σ_{2n} by the rule $p_*(t^k \cap \sigma_{2(n+k)}) = \pm \sigma_{2n}$. From this it follows that

$$S_{n+k}(CP(\xi)) = \left\langle S_{n+k}(\tau_2), \sigma_{2(n+k)} \right\rangle = \pm \left\langle (k+1)\bar{v}_n + \sum_1^n (n+k, i) S_i(\xi) \bar{v}_{n-i}, \sigma_{2n} \right\rangle.$$

We let the reader observe that if $n = 1$, then $S_{1+k}(CP(\xi)) = 0$. To obtain some corollaries of (4.1), we need a few classical facts about binomial coefficients.

(4.2) LEMMA. *If* p *is a prime and*

$$n+k = a_0 + a_1 p + \cdots + a_s p^s, \qquad i = b_0 + b_1 p + \cdots + b_s p^s,$$

where $0 \le a_j, b_j \le p - 1$, *then*

$$(n+k, i) = (a_0, b_0)(a_1, b_1)\cdots(a_s, b_s) \pmod p.$$

If $n+k = p^{j+1} - 1$, $j \ge 0$, *then*

$$(p^{j+1} - 1, i) = (-1)^i \pmod p \qquad (0 \le i \le p^{j+1} - 1),$$

$$(p^{j+1} - 1, i) = (-1)^i \pmod{p^2} \qquad (0 \le i < p^j),$$

$$(p^{j+1} - 1, p^j) = (-1)^{p^j - 1}(p - 1) \pmod{p^2}.$$

The corollaries we obtain will concern the value of $S_{n+k}(CP(\xi))$ modulo p or p^2.

(4.3) COROLLARY. *Let* $\xi \to M^{2n}$ *be a smooth* $(k+1)$*-plane bundle over a closed weakly complex manifold.*

(i) *If* $n+k = \alpha p^{j+1} + \beta p^j$ $(\alpha \ge 0,\ 0 < \beta < p)$ *and* $n < p^j$, *then*

$$S_{n+k}(CP(\xi)) = \pm(n-1)\left\langle \bar{v}_n, \sigma_{2n} \right\rangle \ (mod\ p).$$

(ii) *If* $n+k = \alpha p^{j+1} + \beta p^j - 1$ $(\alpha \ge 0,\ 0 < \beta < p)$ *and* $n < p^j$, *then*

$$S_{n+k}(CP(\xi)) = 0 \pmod p.$$

(iii) *If* $n+k$ *has the form in* (ii) *and* $n = p^j < n+k$, *then*

$$S_{n+k}(CP(\xi)) = \pm\beta\left\langle v_1^n, \sigma_{2n} \right\rangle \ (mod\ p).$$

(iv) *If* $n+k = p^{j+1} - 1$ *and* $n < p^j$, *then*

$$S_{n+k}(CP(\xi)) = 0 \ (mod\ p^2).$$

(v) *If* $n+k = p^{j+1} - 1$ *and* $p^j = n$, *then*

$$S_{n+k}(CP(\xi)) = \pm p\left\langle v_1^n, \sigma_{2n} \right\rangle \ (mod\ p^2).$$

The first case follows immediately from (4.1), since

$$(n+k, i) = 0 \pmod p \qquad \text{for } 1 \le i \le n < p^j$$

and $n+k = 0 \pmod p$ implies that $k+1 = 1 - n \pmod p$. In case (ii) we use the fact that

$$(n+k, i) = (p^j - 1, i) = (-1)^i \pmod p \text{ for } 1 \le i \le n < p^j;$$

with the aid of (3.3) we see that

$$\sum_1^n (n+k, i) S_i(\xi)\bar{v}_{n-i} + (k+1)\bar{v}_n = (n+k+1)\bar{v}_n = 0 \pmod p.$$

In the third case, $(n+k,\ i) = (-1)^i \pmod p$ $(1 \leq i < n = p^j)$, but

$$(n+k,\ n) = \beta - 1 \pmod p.$$

Again referring to (3.3), we find that

$$\sum_{1}^{n-1} (-1)^i S_i(\xi)\bar{v}_{n-i} = n\bar{v}_n + S_n(\xi) \pmod p;$$

thus

$$\sum_{1}^{n} (n+k,\ i) S_i(\xi)\bar{v}_{n-i} + (k+1)\bar{v}_n = (n+k+1)\bar{v}_n + (\beta - 1) S_n(\xi) = \beta S_n(\xi) \pmod p.$$

Since $n = p^j$, however, $S_n(\xi) = v_1^n \pmod p$. Cases (iv) and (v) are similar to (ii) and (iii), but we should note that $pS_n(\xi) = pv_1^n \pmod{p^2}$ if $n = p^j$.

This corollary concerns only the values of $n+k$ that are congruent to 0 or $p-1 \pmod p$. In the description of a list of generators for $\mathfrak{U}/p\mathfrak{U}$, it is these two cases that present the special problems. To examine those values of $n+k$ that lie between 0 and $p-1 \pmod p$, we shall use complex bundles whose only nontrivial Chern class is v_1.

(4.4) LEMMA. *Let* $\xi \to M^{2n}$ *be a smooth* $(k+1)$*-plane bundle for which* $v_j = 0$ *if* $j > 1$; *then*

$$S_{n+k}(CP(\xi)) = \pm\left[\sum_{0}^{n} (-1)^i (n+k,\ i) + k\right]\langle v_1^n,\ \sigma_{2n}\rangle$$

$$= \pm\left[(-1)^{n+k+1}\sum_{0}^{k-1} (-1)^i (n+k,\ i) + k\right]\langle v_1^n,\ \sigma_{2n}\rangle.$$

An elementary computation using (3.3) shows that $S_i(\xi)\bar{v}_{n-i} = (-1)^{n+i} v_1^n$; therefore the first formula is a corollary of (4.1). The second formula follows from the identity

$$\sum_{0}^{n} (-1)^i (n+k,\ i) = \sum_{0}^{k-1} (-1)^{n+k+1-i}(n+k,\ i).$$

(4.5) COROLLARY. *Let* $\xi \to M^{2n}$ *be a smooth* $(k+1)$*-plane bundle with* $v_j = 0$ *if* $j > 1$. *If*

$$n+k = \alpha p^{j+1} + \beta p^j + r \qquad (\alpha \geq 0,\ 0 < \beta < p,\ 0 \leq r < p)$$

and either $r \leq n < p^j$ *or* $r \leq k-1 < p^j$, *then*

$$S_{n+k}(CP(\xi)) = \pm k\langle v_1^n,\ \sigma_{2n}\rangle \ (\textit{mod}\ p).$$

Under the hypothesis $r \leq n < p^j$, we have the relations

$(n+k, i) = (r, i) \pmod p \quad (0 \le i \le r)$ and $(n+k, i) = 0 \pmod p \quad (r < i \le n < p^j)$.

Thus

$$\sum_0^n (-1)^i (n+k, i) = \sum_0^r (-1)^i (r, i) = 0 \pmod p,$$

and we apply (4.4). A similar argument is used if $r \le k - 1 < p^j$. Note that if $n = p^j - 1$, then $k = r + 1 \pmod p$. One more corollary will be pointed out to cover the case $n + k \le p - 1$.

(4.6) COROLLARY. *Let* $\xi \to M^{2n}$ *be a smooth* $(k+1)$*-plane bundle with* $v_j = 0$ *if* $j > 1$.

(i) *If* $k = 1$, $S_{n+1}(CP(\xi)) = \pm[1 + (-1)^n] \langle v_1^n, \sigma_{2n} \rangle$.

(ii) *If* $k = 2$, $S_{n+2}(CP(\xi)) = [(-1)^n (n+1) + 2] \langle v_1^n, \sigma_{2n} \rangle$.

An important application of (ii) is to the case $n + 2 = p - 1$ (p odd), since $S_{n+2}(CP(\xi)) = \pm p \langle v_1^n, \sigma_{2n} \rangle$. For the case (i), the interesting application is to the value $n + 1 = 2^j - 1$, for here $S_{n+1}(CP(\xi)) = \pm 2 \langle v_1^n, \sigma_{2n} \rangle$.

We fix a prime p and briefly run through a list of generators for $\mathfrak{U}/p\mathfrak{U}$. We begin with $CP(1)$. As in Section 2, $\eta \to CP(n)$ is the Hopf line bundle. If

$$n + k = \alpha p^{j+1} + \beta p^j + r \qquad (0 < \beta < p,\ 0 \le r < p - 1),$$

we take $\xi = \eta + kC \to CP(p^j - 1)$; then $[CP(\xi)]$ is a generator, by (4.5). If

$$n + k = \alpha p^{j+1} + \beta p^j - 1 \qquad \text{and either } \alpha > 0 \text{ or } \beta > 1,$$

we take $\xi = \eta + kC \to CP(p^j)$ and conclude by (4.3) that $[CP(\xi)]$ is again a generator. For $1 < n + k \le p - 1$, we use $\xi = \eta + C \to CP(n)$ if $n + k$ is odd, and $\xi = \eta + 2C \to CP(n)$ if $n + k$ is even, and we apply (4.6). Finally, if $n + k = p^{j+1} - 1$ ($j > 0$), then $\xi = \eta + kC \to CP(p^j)$ supplies the generator. This is about the simplest set of choices for the generators of $\mathfrak{U}/p\mathfrak{U}$. The reader will have no trouble in finding other examples.

5. THE INVARIANTS $\langle S_i(\xi)\bar{v}_{n-i}, \sigma_{2n} \rangle$

In this section we shall prove (2.1). In (4.1), we saw how the invariants $\langle S_i(\xi)\bar{v}_{n-i}, \sigma_{2n} \rangle$ enter into the evaluation of $S_{n+k}(CP(\xi))$. The next lemma shows the degree of latitude we have in assigning values to the $\langle S_i(\xi)\bar{v}_{n-i}, \sigma_{2n} \rangle$.

(5.1) LEMMA. *If* $n \le k + 1$, *then for each sequence of integers* $\lambda_1, \cdots, \lambda_n$ *such that* $\sum_1^n (-1)^i \lambda_i = 0$ (*mod* n), *there exists some* $(k+1)$*-plane bundle* $\xi \to M^{2n}$ *over a closed weakly complex manifold for which* $\langle S_i(\xi)\bar{v}_{n-i}, \sigma_{2n} \rangle = \lambda_i$ *for* $1 \le i \le n$.

The requirement that $\sum_1^n (-1)^i \lambda_i = 0 \pmod n$ is necessary, according to (3.3). To demonstrate (5.1), we shall use the ring structure of $\mathfrak{M}$ found in (2.1).

For each integer $i > 0$, we define an additive homomorphism $S_i\colon \mathfrak{U}_{2n}(BU(m)) \to Z$ by

$$S_i([\xi \to M^{2n}]) = (-1)^i \langle S_i(\xi)\bar{v}_{n-i}, \sigma_{2n} \rangle .$$

Clearly, this only depends on the bordism class of the bundle $\xi \to M^{2n}$, and it is trivial for $i > n$. Similarly, we define $\overline{V}$: $\mathfrak{U}_{2n}(BU(m)) \to Z$ by

$$\overline{V}([\xi \to M^{2n}]) = \langle \bar{v}_{2n}, \sigma_{2n} \rangle .$$

In particular, $S_i(\mathfrak{U}_{2n}(BU(0))) = 0$, $\overline{V}(\mathfrak{U}_{2n}(BU(0))) = 0$ $(n > 0)$, and $\overline{V}(1) = 1$.

(5.2) LEMMA. *The homomorphism* $\overline{V}$: $\mathfrak{M} \to Z$ *is a ring homomorphism, and for each pair* $[\xi_1 \to M^{2n}]$, $[\xi_2 \to V^{2m}]$,

$$S_i([\xi_1 \to M^{2n}][\xi_2 \to V^{2m}])$$
$$= S_i([\xi_1 \to M^{2n}])(\overline{V}([\xi_2 \to V^{2m}])) + \overline{V}([\xi_1 \to M^{2n}])(S_i([\xi_2 \to V^{2m}])).$$

We shall denote by p_1 and p_2 the projections of $M^{2n} \times V^{2m}$ onto M^{2n} and V^{2m}, and $\bar{x}_i$ and $\bar{y}_j$ will denote the dual Chern classes of ξ_1 and ξ_2. Now

$$S_i(\xi_1 \times \xi_2) = S_i(p_1^*(\xi_1) + p_2^*(\xi_2)) = p_1^*(S_i(\xi_1)) + p_2^*(S_i(\xi_2)),$$

and furthermore $\bar{v}_{n+m-i} = \sum_0^{n+m-i} p_1^*(\bar{x}_j) p_2^*(\bar{y}_{n+m-i-j})$. By an elementary dimensional consideration,

$$p_1^*(S_i(\xi_1))\bar{v}_{n+m-i} = \sum_0^{n+m-i} p_1^*(S_i(\xi_1)\bar{x}_j) p_2^*(\bar{y}_{n+m-i-j}) = p_1^*(S_i(\xi_1)\bar{x}_{n-i}) p^*(\bar{y}_m)$$

and

$$p_2^*(S_i(\xi_2))\bar{v}_{n+m-i} = p_1^*(\bar{x}_n) p_2^*(S_i(\xi_2)\bar{y}_{m-i}).$$

The formula in (4.2) now follows immediately. That $\overline{V}$: $\mathfrak{M} \to Z$ is a ring homomorphism is trivial.

We showed that $\mathfrak{M}$ is a graded polynomial ring over $\mathfrak{U}$ whose generators are the Hopf line bundles $\{[\eta \to CP(n)]\}_0^\infty$. For $\eta \to CP(n)$, $S_i(\eta)\bar{v}_{n-i} = (-1)^{n-i} v_1^n$; thus

$$S_i([\eta \to CP(n)]) = (-1)^n \langle v_1^n, \sigma_{2n} \rangle = \overline{V}[\eta \to CP(n)]$$

for $1 \leq i \leq n$. With an appropriate choice of the orientation of CP(n), we may assume that

$$S_i([\eta \to CP(n)]) = \overline{V}[\eta \to CP(n)] = 1$$

for $1 \leq i \leq n$. Note that $\overline{V}[\eta \to CP(0)] = 1$.

With each sequence $\omega = (j_1 \leq \cdots \leq j_{k+1})$ of nonnegative integers we have associated the product

$$X_\omega = [\eta \to CP(j_1)] \cdots [\eta \to CP(j_{k+1})].$$

In view of (5.2),

$$S_i(X_\omega) = S_i[\eta \to CP(j_1)] + S_i[\eta \to CP(j_2)] + \cdots + S_i[\eta \to CP(j_{k+1})].$$

Thus $S_i(X_\omega)$ is the number of entries in ω that are at least equal to i.

We now prove (5.1) under the assumption $n = k + 1$. Each element in $\mathfrak{U}_{2n}(BU(k+1))$ can be uniquely expressed in the form $\sum X_\omega [M^{2m}]$, where the sum is taken over all ω with $j_1 + \cdots + j_{k+1} \leq n$ and

$$[M^{2m}] \in \mathfrak{U}_{2m} \qquad (m = n - (j_1 + \cdots + j_{k+1})).$$

Again by (5.2), $S_i(X_\omega [M^{2m}]) = 0$ if $m > 0$, so for our purposes we need only consider $a_\omega X_\omega$, where $j_1 + \cdots + j_{k+1} = n$ and a_ω is an integer.

Let $Y_1 = X_{(1,\cdots,1)}$ (1 repeated $k + 1 = n$ times), and let

$$Y_i = X_{(1,\cdots,1,i)} - X_{(1,\cdots,1,i-1)} \qquad \text{for } 2 \leq i \leq n.$$

Then $S_1(Y_1) = n$, $S_i(Y_1) = 0$ $(i > 1)$, while $S_1(Y_i) = -1$ $(2 \leq i \leq n)$. Furthermore, $S_j(Y_i) = 0$ $(i \neq j)$ and $S_i(Y_i) = 1$ $(2 \leq i \leq n)$. If $\lambda_1 = ns - (\lambda_2 + \cdots + \lambda_n)$, we form the linear combination $sY_1 + \lambda_2 Y_2 + \cdots + \lambda_n Y_n$; then $S_i = (-1)^i \langle S_i(\xi)\bar{v}_{n-i}, \sigma_{2n}\rangle$, evaluated on this combination, has the value λ_i $(1 \leq i \leq n)$. Thus (5.1) is proved for $n = k + 1$. The reader can obtain the case $n < k + 1$ as a corollary. A moment's reflection shows that (5.1) is not true if $n > k + 1$. We raise the question as to what a full set of relations among the invariants $\langle S_i(\xi)\bar{v}_{n-i}, \sigma_{2n}\rangle$ must be if $n > k + 1$.

We apply (5.1) to the proof of (2.1) as follows, under the assumption that $n \leq k + 1$. For each sequence $\lambda, \lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ there is a $(k+1)$-plane bundle $\xi \to M^{2n}$ such that

$$\langle S_i(\xi)\bar{v}_{n-i}, \sigma_{2n}\rangle = \lambda_i \qquad (1 \leq i \leq n - 1)$$

and

$$\langle S_n(\xi), \sigma_{2n}\rangle = (-1)^n n\lambda + \sum_{1}^{n-1} (-1)^{n+i+1}\lambda_i.$$

From (3.3) it follows that $n\langle \bar{v}_n, \sigma_{2n}\rangle = n\lambda$, hence $\langle \bar{v}_n, \sigma_{2n}\rangle = \lambda$, and

$$\sum_{1}^{n} (n+k, i)\langle S_i(\xi)\bar{v}_{n-i}, \sigma_{2n}\rangle + (k+1)\langle \bar{v}_n, \sigma_{2n}\rangle$$

$$= \sum_{1}^{n-1} [(n+k, i) + (-1)^{n+i+1}(n+k, n)]\lambda_i + [(-1)^n n(n+k, n) + k + 1]\lambda .$$

The image of $\mathfrak{U}_{2n}(BU(k+1)) \to \mathfrak{U}_{2(n+k)}$ fails to contain a generator of $\mathfrak{U}/p\mathfrak{U}$ if and only if the above expression is 0 (mod p), or (mod p^2) if $n + k = p^j - 1$, for every choice of $\lambda, \lambda_1, \cdots, \lambda_{n-1}$. This assertion is a corollary of (4.1).

(5.3) LEMMA. *If* $n \leq k + 1$ *and* $n + k = p^j - 1$ $(j > 0)$, *then the image of* $\mathfrak{U}_{2n}(BU(k+1)) \to \mathfrak{U}_{2(n+k)}$ *fails to contain an element indecomposable* (mod $p\mathfrak{U}$) *if and only if* $n = 1$ *or* $n < p^{j-1}$. *If* $n + k \neq p^j - 1$, *then the image fails to contain an indecomposable element if and only if*

(i) $(-1)^{i+1}(n+k) = (n+k, i) \pmod p \qquad (1 \leq i \leq n)$,

(ii) $-n(n+k) + k + 1 = 0 \pmod p$.

Consider first the case $n + k \neq p^j - 1$; here

$$(n+k,\ i) + (-1)^{n+i+1}(n+k,\ n) = 0 \pmod p \qquad (1 \le i \le n-1),$$

$$(-1)^n n(n+k,\ n) + k + 1 = 0 \pmod p.$$

Taking $i = 1$, we see that $(n+k,\ n) = (-1)^{n+1}(n+k) \pmod p$; thus

$$(n+k,\ i) = (-1)^{i+1}(n+k) \bmod p \qquad \text{for } 1 \le i \le n$$

and $-n(n+k) + k + 1 = 0 \pmod p$ also.

Next consider the case $n + k = p - 1$, with $1 \le n \le k+1$. Just as above,

$$(p-1,\ i) = (-1)^{i+1}(p-1) \pmod{p^2} \qquad (1 \le i \le n).$$

If $n \ge 2$, then $(p-1)(p-2)/2 = 1 - p + p^2$; in other words, $p^2 - 3p + 2 = 2 - 2p + 2p^2$, which means that $p = 0 \pmod{p^2}$, a contradiction.

The case $p^j - 1$ $(j > 1)$ follows, since by (4.2)

$$(p^j - 1,\ p^{j-1}) = (-1)^{p^{j-1}-1}(p-1) \ne (-1)^{p^{j-1}} \pmod{p^2}.$$

The details are omitted.

(5.4) LEMMA. *If* $1 < n \le k+1$ *and* $n + k \ne 0$ *or* $p - 1$ (mod p), *then the image of* $\mathfrak{U}_{2n}(BU(k+1)) \to \mathfrak{U}_{2(n+k)}$ *contains an element that is indecomposable* (mod p).

We can write $n + k = \alpha p^{j+1} + \beta p^j + r$, with $\alpha \ge 0$, $0 < \beta < p$, and $0 < r < p - 1$. Assume that the image fails to contain a generator, then, since $(n+k,\ i) = (r,\ i)$ (mod p) for $0 \le i < p^j$, it follows from (5.3) that

$$(-1)^{i+1} r = (r,\ i) \pmod p \qquad \text{for } 1 \le i \le n.$$

Since $1 < n$, $-r = (r,\ 2) = r(r-1)/2$ (mod p); but this means that $r^2 = -r$ (mod p) or $r = -1$ (mod p), contrary to our assumption that $0 < r < p - 1$.

We have narrowed down the cases in which the image does not contain a generator of $\mathfrak{U}/p\mathfrak{U}$. We now proceed to the proof of (2.1). The case $n + k = p^j - 1$ is covered by (5.3). We consider $n + k \ne p^j - 1$. In the proof of (4.3), we showed that $\mathfrak{U}_{2n}(BU(k+1)) \to \mathfrak{U}_{2(n+k)}$ fails to contain a generator of $\mathfrak{U}/p\mathfrak{U}$ in the first three cases listed under (2.1). We must now show that in all other cases a generator is present in the image.

If $n + k \ne p^j - 1$ and the image fails to contain a generator of $\mathfrak{U}/p\mathfrak{U}$, then $n + k = 0$ or $p - 1$ (mod p) by (5.4). If $n + k = \alpha p^{j+1} + \beta p^j$ and $n \ge p^j$, we see by (5.3) that $(n+k,\ p^j) = \beta = 0$ (mod p), which is a contradiction; thus $n < p^j$ and $n = 1$ (mod p), by (4.3). If $n + k = \alpha p^{j+1} + \beta p^j - 1$ and $n \ge p^j$, then

$$(n+k,\ p^j) = \beta - 1 = (-1)^{p^j} \pmod p,$$

which is again a contradiction. This completes the proof of (2.1).

The ring $\mathfrak{U}$ itself is a graded polynomial ring over Z with a generator in each even dimension. The generators are characterized by

$$S_n(M^{2n}) = \begin{cases} \pm 1 & \text{if } n \ne p^j - 1, \\ \pm p & \text{if } n = p^j - 1. \end{cases}$$

For $n \leq k + 1$, we may think of the correspondence

$$(\lambda, \lambda_1, \cdots, \lambda_{n-1}) \to ((-1)^n (n+k, n) + k + 1)\lambda + \sum_{1}^{n-1} [(n+k, i) + (-1)^{n+i+1}(n+k, n)]\lambda_i$$

as a homomorphism $Z^n \to Z$. If $n \leq k + 1$, then the image of

$$\mathfrak{U}_{2n}(BU(k+1)) \to \mathfrak{U}_{2(n+k)}$$

contains a generator of $\mathfrak{U}$ if and only if $Z^n \to Z$ is onto $(n + k \neq p^j - 1)$ or the image is pZ $(n = p^j - 1)$. From this viewpoint, we see that $\mathfrak{U}_{2n}(BU(k+1)) \to \mathfrak{U}_{2(n+k)}$ contains a generator of $\mathfrak{U}$ if for every prime p it contains a generator of $\mathfrak{U}/p\mathfrak{U}$.

(5.5) COROLLARY. *If* $n > 1$, *then the images of* $\mathfrak{U}_{2n}(BU(n))$ *in* $\mathfrak{U}_{4n-2}$ *and* $\mathfrak{U}_{2n}(BU(n+1))$ *in* $\mathfrak{U}_{4n}$ *both contain generators of* $\mathfrak{U}$.

For each prime p we can verify that none of the causes of failure listed in (2.1) can occur.

6. FIBRING OVER S^2

Let us first make precise the problem we wish to attack. In [5, Section 21] we introduced the unrestricted bordism group of all differentiable actions of $U(1)$ on closed oriented manifolds, and here we shall denote this by $I_n(U(1))$. An element of $I_n(U(1))$ is represented by a smooth action $(U(1), M^n)$ on a closed oriented n-manifold. This action bords if and only if there is an action $(U(1), B^{n+1})$ on a compact oriented manifold for which the induced action $(U(1), \partial B^{n+1})$ is equivariantly diffeomorphic to $(U(1), M^n)$. The pair $-(U(1), M^n)$ is represented by the same action of $U(1)$, but the orientation is reversed. Although we do not compute $I_n(U(1))$, its existence will be useful. The weak direct sum $I_*(U(1)) = \sum_0^\infty I_n(U(1))$ is given the structure of a graded Ω-module as follows. For any action $(U(1), M^n)$ on a closed oriented n-manifold, together with a closed oriented V^m, we set

$$[V^m][U(1), M^n] = [U(1), V^m \times M^n],$$

where the action on the cartesian product is given by $t(x, y) = (x, ty)$. We are interested in an Ω-module homomorphism

$$I_*(U(1)) \to \Omega$$

of degree $+2$, which we shall now define. Given $(U(1), M^n)$, we introduce $(U(1), M^n \times S^3)$ by $t(x, z, w) = (t^{-1}x, (tz, tw))$, and we consider the quotient manifold $(M^n \times S^3)/U(1)$ that is fibred over $S^2 = S^3/U(1)$ with fibre M^n and structure group $U(1)$. This quotient manifold is naturally oriented, and the reader may show that

$$[U(1), M^n] \to [(M^n \times S^3)/U(1)] \in \Omega_{n+2}$$

is a well-defined Ω-module homomorphism. Its image is a homogeneous ideal in Ω, and this is the ideal we would like to characterize. As stated in the introduction, we can do this only up to torsion.

So far, we have only dealt with the complex bordism ring $\mathfrak{U}$; but there is the natural homomorphism $\mathfrak{U} \to \Omega$ that ignores the weakly complex structure but preserves the orientation defined by this structure. Milnor noted that for any odd prime p, a generator for $\mathfrak{U}/p\mathfrak{U}$ in dimension 4n will also serve as a generator for $\Omega/p\Omega$. We now construct some manifolds, which will actually be complex analytic, and whose oriented bordism classes lie in the image of $I_*(U(1)) \to \Omega$.

For every pair of nonnegative integers n and k we define an action of the (n + 1)-dimensional toral group T^{n+1} on the cartesian product $S^{2k+1} \times (S^3)^n$ by

$$(t_1, \cdots, t_{n+1})[(\lambda_1, \cdots, \lambda_{k+1}), (z_2, w_2), \cdots, (z_{n+1}, w_{n+1})]$$
$$= [(t_1 t_2^{-1}\lambda_1, t_1\lambda_2, \cdots, t_1\lambda_{k+1}), (t_2 t_3^{-1} z_2, t_2, w_2), \cdots,$$
$$(t_n t_{n+1}^{-1} z_n, t_n w_n), (t_{n+1} z_{n+1}, t_{n+1} w_{n+1})].$$

This is a principal action, and we set $V(n, k) = (S^{2k+1} \times (S^3)^n)/T^{n+2}$ to obtain a closed 2(n + k)-manifold. We note that $V(n, 0) = V(n - 1, 1)$.

Let us refer to a point in $V(n, k)$ by its homogeneous coordinates $[(\lambda_1, \cdots, \lambda_{k+1}), (z_2, w_2), \cdots, (z_{n+1}, w_{n+1})]$; then we can define an action $(U(1), V(n, k))$ by

$$t[(\lambda_1, \cdots, \lambda_{k+1}), (z_2, w_2), \cdots, (z_{n+1}, w_{n+1})]$$
$$= [(\lambda_1, \cdots, \lambda_{k+1}), (z_2, w_2), \cdots, (z_n, w_n), (tz_{n+1}, w_{n+1})].$$

It follows immediately that $V(n + 1, k) = (V(n, k) \times S^3/U(1))$; thus

$$[V(n + 1, k)] \in \Omega_{2(n+k+1)}$$

lies in the image of $I_{2(n+k)}(U(1)) \to \Omega_{2(n+k+1)}$.

Let us describe $V(n, k)$ as fibred over $V(n, 0) = V(n - 1, 1)$ with fibre $CP(k)$. We consider $T^n \subset T^{n+1}$ as $(1, t_2, \cdots, t_{n+1})$ and let T^n act on $C \times (S^3)^n$ by

$$(t_2, \cdots, t_{n+1})(\lambda, (z_2, w_2), \cdots, (z_{n+1}, w_{n+1}))$$
$$= (t_2^{-1}\lambda, (t_2 t_3^{-1} z_2, t_2 w_2), \cdots, (t_{n+1} z_{n+1}, t_{n+1} w_{n+1})).$$

This produces a line bundle $\eta = (C \times (S^3)^n)/T^n \to V(n, 0)$. The reader will see that $V(n, k) = CP(\eta + kC)$. Thus we can apply the computations of Section 4; for if $v_1 \in H^2(V(n, 0); Z)$ is the Chern class of η, then, according to [4, (42.8)], $\langle v_1^n, \sigma_{2n}\rangle = \pm 1$.

We fix an odd prime p; if

$$n + k = \alpha p^{j+1} + \beta p^j + r \qquad (0 \leq r < p - 1),$$

then $[V(p^j - 1, p^{j+1} + (\beta - 1)p^j + r + 1)]$ is a generator of $\Omega/p\Omega$, by (4.5). On the other hand, if

$$n + k = \alpha p^{j+1} + \beta p^j - 1 \qquad (0 < \beta < p)$$

and either $\alpha > 0$ or $\beta > 1$, then by (4.3) $[V(p^j, p^{j+1} + (\beta - 1)p^j - 1)]$ is a generator of $\Omega/p\Omega$. Similarly, if $n + k = p^{j+1} - 1$ $(j > 0)$, we use $[V(p^j, p^j(p - 1) - 1)]$. If $n + k = 2$, we cannot get a generator, because n would have to be 1; but if

$2 < n + k \leq p - 1$, we take $k = 1$ if $n + k$ is odd and $k = 2$ if $n + k$ is even.

(6.1) LEMMA. *For each odd prime* p *and each* $n > 1$, *some bordism class* X^{4n} *in the image of* $I_{4n-2}(U(1)) \to \Omega_{4n}$ *is a generator of* $\Omega/p\Omega$.

We take $X^4 = [CP(2)]$ to complete the list of generators. An element in Ω_{4m} can be represented uniquely (mod $p\Omega_{4m}$) by an expression $\sum a_\omega X^{4n_1} \cdots X^{4n_m}$, where $\omega = (n_1 \leq \cdots \leq n_m)$ is a partition of m into nonnegative integers. The problem is the computation of $a_{(1,1,\cdots,1)}$, the coefficient of $(X^4)^m$. Since X^{4n} for $n > 1$ is represented by a manifold fibred over S^2, it follows from the Chern-Hirzebruch-Serre theorem [3] that index $(X^{4n}) = 0$. On the other hand, index $X^4 = 1$. Thus the index (mod p) of a bordism class is the coefficient (mod p) of $(X^4)^m$ in the above expansion. Recalling that the image of $I_*(U(1)) \to \Omega$ is an ideal, we have the following result.

(6.2) THEOREM. *If* p *is an odd prime and* M^{4m} *is a closed oriented manifold, then* $[M^{4m}]$ *lies in the image of* $I_{4m-2}(U(1)) \to \Omega_{4m}$ (mod $p\Omega_{4m}$) *if and only if* index $M^{4m} = 0$ (mod p).

The reader may show that if $n > 1$, then the image of $I_{4n-2}(U(1)) \to \Omega_{4n}$ contains a generator of Ω/Tor. From this we derive the following result.

(6.3) COROLLARY. *If* M^{4m} *is a closed oriented manifold of index* 0, *then modulo an element of order* 2, $[M^{4m}]$ *lies in the image of* $I_{4m-2}(U(1)) \to \Omega_{4m}$.

It seems doubtful that every element of order 2 lies in the image of $I_*(U(1)) \to \Omega$. In particular we are unable to show that the generator of $\Omega_5 \simeq Z_2$ is in this image.

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