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# The Bound of the Difference between Parameter Ideals and their Tight Closures

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## 1. Introduction.

Let $R$ be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and assume that $R$ possesses positive characteristic $p > 0$. For each $\mathfrak{m}$-primary ideal $I$ in $R$ let $\mathrm{e}_I(R)$ and $I^*$ denote the multiplicity of $R$ with respect to $I$ and the tight closure of $I$, respectively. Here let us briefly recall the definition of tight closures. For an ideal $\mathfrak{a}$ in $R$ let $\mathfrak{a}^{[q]} = (a^q \mid a \in \mathfrak{a})R$ where $q = p^e$ with $e \geq 0$. Let $R^0 = R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Min}\, R} \mathfrak{p}$. Let $\mathfrak{a}^*$ denote the set of elements $x \in R$ for which there exists $c \in R^0$ such that $cx^q \in \mathfrak{a}^{[q]}$ for all $q \gg 0$. Then the set $\mathfrak{a}^*$ forms an ideal in $R$ containing $\mathfrak{a}$, which we call the tight closure of $\mathfrak{a}$. The ideal $\mathfrak{a}$ is said to be tightly closed if $\mathfrak{a}^* = \mathfrak{a}$. A local ring $R$ is called $F$-rational if every parameter ideal in $R$ is tightly closed.

In this paper we investigate the behavior of $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ and $\sup_{\mathfrak{q}}\{\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\}$, where $\mathfrak{q}$ moves all parameter ideals in $R$. These two values are very closely related. In fact, they agree if $R$ is Cohen-Macaulay. By definition of $F$-rationality, it immediately follows that $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q}) = 0$ if and only if $R$ is $F$-rational. The second author studied $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ in [N2] and he gave some necessary conditins for $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ to be finite. On the other hand, Watanabe and Yoshida posed in [WY] the conjecture that the difference $\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)$ is not negative for every prameter ideal $\mathfrak{q}$ in $R$ if $R$ is unmixed. Moreover, they conjectured that $\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*) = 0$ for some parameter ideal $\mathfrak{q}$, then $R$ is Cohen-Macaulay and $F$-rational. This problem is affirmatively solved in [GN] under the condition that $R$ is a homomorphic image of a Cohen-Macaulay ring of chracteristic $p > 0$ and $\mathrm{Ass}\, R = \mathrm{Assh}\, R$. We further investigate the finiteness of the supremum of the difference $\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)$ in this paper. The first result gives a necessary and sufficient condition for $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ to be finite. Namely,

THEOREM 1.1. *Let $R$ be excellent and equidimensional. Then the following three conditions are equivalent.*

(1) $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q}) < \infty$.

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(2) *Local rings* $R_{\mathfrak{p}}$ *are* $F$*-rational for all* $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.

(3) $R$ *is FLC and* $(0)^*_{\mathrm{H}^d_{\mathfrak{m}}(R)}$ *is a module of finite length.*

*When this is the case, we have that* $\sup_{\mathfrak{q}}\{\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\} < \infty$.

Here $(0)^*_{\mathrm{H}^d_{\mathfrak{m}}(R)}$ stands for the tight closure of (0) in the local cohomology module $\mathrm{H}^d_{\mathfrak{m}}(R)$ ($d = \dim R$) (cf. [HH1, Chap 8]). We also refer e.g. [GY] for the definition of $FLC$ rings. The theorem concludes, for instance, that $\sup_{\mathfrak{q}} \ell_R(\mathfrak{q}^*/\mathfrak{q})$ is finite for excellent local domains $R$ having isolated singularity. The finiteness of $\sup_{\mathfrak{q}}\{\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\}$ yields the following.

THEOREM 1.2. *Let* $R$ *be a homomorphic image of a Cohen-Macaulay local ring of characteristic* $p > 0$ *and assume that* $\operatorname{Ass} R \subseteq \operatorname{Assh} R \cup \{\mathfrak{m}\}$. *If* $\sup_{\mathfrak{q}}\{e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\} < \infty$, *Then*

$$\{\mathfrak{p} \mid R_{\mathfrak{p}} \text{ is Cohen-Macaulay}\} \setminus \{\mathfrak{m}\} = \{\mathfrak{p} \mid R_{\mathfrak{p}} \text{ is } F\text{-rational}\} \setminus \{\mathfrak{m}\}.$$

We state how this paper is organized. Section 2 is devoted to preliminaries. We recall the concept of $d$-sequences and $USD$-sequences, which plays a key role throughout our argument. We also briefly explain the tight closure of modules and argue the relation between the tight closure of parameter ideals and the tight closure of (0) in the local cohomology module $\mathrm{H}^d_{\mathfrak{m}}(R)$ under the assumption that $R$ is excellent and equidimensional. In Section 3 we shall give the proof of our theorems.

Throughout this paper let $(R, \mathfrak{m})$ denote a Noetherian local ring of dimension $d$. For a finitely generated $R$-module $M$ we denote by $\ell_R(M)$ the length of $M$ and by $\mathrm{e}_I(M)$ the multiplicity of $M$ with respect to an $\mathfrak{m}$-primary ideal $I$. Let $\mathrm{H}^i_{\mathfrak{m}}(M)$ ($i \in \mathbf{Z}$) stand for the $i$-th local cohomology module of $M$ with respect to $\mathfrak{m}$.

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## 2. Preliminaries.

We first recall the definition of $d$-sequences and $USD$-sequences. A sequence $a_1, a_2, \cdots, a_s$ of elements in $R$ is called a $d$-sequence on $R$ (cf. [H1]) if the equality

$$(a_1, a_2, \cdots, a_{i-1}) : a_i a_j = (a_1, a_2, \cdots, a_{i-1}) : a_j$$

holds for all $1 \le i \le j \le s$, and moreover, it is called an unconditioned strong $d$-sequence (a $USD$-sequence for short) on $R$ if $a_1^{n_1}, a_2^{n_2}, \cdots, a_s^{n_s}$ is a $d$-sequence in any order and for all integers $n_1, n_2, \cdots, n_s > 0$.

For a system $a_1, a_2, \cdots, a_d$ of parameters of $R$, it is known that

$$\{R/(a_1^n, a_2^n, \cdots, a_d^n) \xrightarrow{a_1 \cdots a_d} R/(a_1^{n+1}, a_2^{n+1}, \cdots, a_d^{n+1})\}$$

forms a direct system whose direct limit is $\mathrm{H}^d_{\mathfrak{m}}(R)$. Let $K_n$ be the kernel of the canonical map from $R$, through $R/(a_1^n, a_2^n, \cdots, a_d^n)R$, to the direct limit $\mathrm{H}^d_{\mathfrak{m}}(R)$. Then the system $\{R/K_n \xrightarrow{a_1 \cdots a_d} R/K_{n+1}\}$ forms a direct system again whose maps are injective and whose direct limit is also $\mathrm{H}^d_{\mathfrak{m}}(R)$. Hence we can write that $\varinjlim R/K_n = \bigcup_{n>0} R/K_n = \mathrm{H}^d_{\mathfrak{m}}(R)$.

Furthermore, when $a_1, a_2, \cdots, a_d$ is a $USD$-sequence, by the monomial property of a $USD$-sequence (cf. Theorem 2.3 of [GY]), $K_n$ coincides with

$$\Sigma(a_1^n, a_2^n, \cdots, a_d^n) := \sum_{i=1}^{d}((a_1^n, \cdots, \widehat{a_i^n}, \cdots, a_d^n) : a_i^n) + (a_1^n, a_2^n, \cdots, a_d^n)\,.$$

We denote $\Sigma(\mathfrak{q}) = (a_1, a_2, \cdots, a_d)$ when $\mathfrak{q} = (a_1, a_2, \cdots, a_d)$. Also notice that $\Sigma(\mathfrak{q}) \subseteq \mathfrak{q}^*$, by colon capturing property (cf. [HH1, Theorem 7.9]) when $R$ is a homomorphic image of a Cohen-Macaulay local ring of characteristic $p > 0$ and equidimensional.

Next we briefly explain the tight closure of the zero-module of arbitrary $R$-modules. Suppose that $R$ has characterstic $p > 0$. The Frobenius endomorphism $F^e : R \to R$ is given by $F^e(x) = x^{p^e}$. When we consider $R$ as an $R$-algebra via $F^e$, we denote it by ${}^eR$. Let $M$ be an $R$-module and $x \in M$. We say that $x \in (0)^*_M$ if there exists $c \in R^0$ such that $c \otimes x = 0$ in ${}^eR \otimes_R M$ for all $e \gg 0$. The following properties readily comes from the definition.

(1) $(0)^*_{R/I} = I^*/I$ for any ideal $I$ of $R$.

(2) Let $f : M \to N$ be a homomorphism of $R$-modules. Then $f$ induces $(0)^*_M \to (0)^*_N$.

Kawasaki's result [K, Corollary 1.2] guarantees that an excellent local ring $R$ is a homomorphic image of a Cohen-Macaulay ring $A$. Furthermore, according to his theory, the Cohen-Macaulay ring $A$ is given as a Rees algebra over $R$. Thus, an excellent local ring $R$ of characteristic $p$ can be a homomorphic image of a Cohen-Macaulay local ring of characteristic $p$. We put $H = \mathrm{H}^d_{\mathfrak{m}}(R)$ for short.

PROPOSITION 2.1. *Let $R$ be an excellent and equidimensional local ring of characteristic $p$. Suppose that the system $a_1, a_2, \cdots, a_d$ of parameters of $R$ forms a $USD$-sequence. Then*

(1) $(0)^*_H = \varinjlim (a_1^n, a_2^n, \cdots, a_d^n)^*/(a_1^n, a_2^n, \cdots, a_d^n)$.

(2) $(0)^*_H = \bigcup_{n>0}(a_1^n, a_2^n, \cdots, a_d^n)^*/\Sigma(a_1^n, a_2^n, \cdots, a_d^n)$. *In particular, the length of $(0)^*_H$ is finite if and only if* $\sup_n\{\ell_R((a_1^n, a_2^n, \cdots, a_d^n)^*/\Sigma(a_1^n, a_2^n, \cdots, a_d^n))\} < \infty$.

PROOF. (1): We put $(\underline{a}^n) = (a_1^n, a_2^n, \cdots, a_d^n)$. The natural map $\alpha_n : R/(\underline{a}^n) \to H$ induces $\beta_n : (\underline{a}^n)^*/(\underline{a}^n) = (0)^*_{R/(\underline{a}^n)} \to (0)^*_H$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} \varinjlim (\underline{a}^n)^*/(\underline{a}^n) & \xrightarrow{\beta} & (0)^*_H \\ \downarrow & & \downarrow \\ \varinjlim R/(\underline{a}^n) & \xrightarrow{\alpha} & H\,, \end{array}$$

where vertical arrows are injective. Since $\alpha$ is isomorphic, the injectivity of $\beta$ follows. Next we shall prove that $\beta$ is surjective. Let $\xi \in (0)^*_H$. Then there exist $n > 0$ and $x \in R$ such that $\bar{x} \in R/(\underline{a}^n)$ and $\alpha_n(\bar{x}) = \xi$, while there exists $c \in R^0$ such that $c \otimes \xi = 0$ in ${}^eR \otimes_R H$ for all $e \gg 0$. Hence $c \otimes \bar{x}$ belongs to the kernel of ${}^eR \otimes \alpha_n : {}^eR \otimes_R R/(\underline{a}^n) \to {}^eR \otimes_R H$. On the other hand, passing to isomorphisms ${}^eR \otimes_R R/(\underline{a}^n) \cong R/(\underline{a}^{nq})$ and ${}^eR \otimes_R H \cong H$, where $q = p^e$, ${}^eR \otimes \alpha_n$ coincides with $\alpha_{nq}$, whence we have $\overline{cx^q} \in \mathrm{Ker}\,\alpha_{nq}$. Therefore,

$cx^q \in K_{nq} = \Sigma(a_1^{nq}, a_2^{nq}, \cdots, a_d^{nq}) \subseteq (a_1^{nq}, a_2^{nq}, \cdots, a_d^{nq})^*$ since $a_1, a_2, \cdots, a_d$ is a $USD$-sequence.

Let $R_{red} = R/\sqrt{(0)}$. For $y \in R$, let $y'$ stand for the image of $y$ in $R_{red}$. Now, we have $(cx^q)' \in ((a_1'^{nq}, a_2'^{nq}, \cdots, a_d'^{nq})R_{red})^*$. Besides, we can take $t \in R^0$ such that $t'$ is a test element of $R_{red}$ because $R_{red}$ is an excellent reduced local ring (cf. [HH1, Theorem 6.1]). Hence, it follows that

$$(tc)'x'^q \in (a_1'^n, a_2'^n, \cdots, a_d'^n)^{[q]}R_{red}$$

for all $q \gg 0$. Hence $x' \in ((a_1'^n, a_2'^n, \cdots, a_d'^n)R_{red})^*$, therefore we get $x \in (\underline{a}^n)^*$ by [HH1, Theorem 4.1]. This implies that the surjectivity of $\beta$.

(2): $\{(\underline{a}^n)^*/K_n\}$ is a direst system with injective maps. Besides, its direct limit coincides with that of $\{(\underline{a}^n)^*/(\underline{a}^n)\}$. The assertion follows. $\square$

### 3. Proof of the theorems.

Throughout this section, we assume that $R$ is a homomorphic image of a Cohen-Macaulay local ring of characteristic $p > 0$ and equidimensional. For an ideal $I$ in $R$ we put $\mathrm{U}(I) = \bigcup_{n>0} I : \mathfrak{m}^n$ and call it the unmixed component of $I$. A sequence $a_1, a_2, \cdots, a_s$ of elements of $R$ is called a filter regular sequence if $(a_1, a_2, \cdots, a_i) : a_{i+1} \subseteq \mathrm{U}((a_1, a_2, \cdots, a_i))$ for all $0 \leq i < s$. Note that any system $a_1, a_2, \cdots, a_d$ of parameters in $R$ forms a filter regular sequence if $R$ is $FLC$. For a parameter ideal $\mathfrak{q} = (a_1, a_2, \cdots, a_d)$, we put $\mathfrak{q}_i = (a_1, a_2, \cdots, a_i)$. We begin with the following Lemma.

LEMMA 3.1. *Let $\mathfrak{q} = (a_1, a_2, \cdots, a_d)$ be a parameter ideal of $R$. We put the function*

$$f(n) = \ell_R((\mathfrak{q}_{d-1} + (a_d^n))^*/(\mathrm{U}(\mathfrak{q}_{d-1}) + (a_d^n))$$

*for all $n > 0$. Then $f(n)$ is monotonous increasing function.*

PROOF. We first note that $\mathrm{U}(\mathfrak{q}_{d-1}) + (a_d^n) \subseteq (\mathfrak{q}_{d-1} + (a_d^n))^*$ by the colon capturing property (cf. [HH1, Theorem 7.9]). To get the lemma, it is enough to show that the following map is injective.

$$\Phi : (\mathfrak{q}_{d-1} + (a_d^n))^*/(\mathrm{U}(\mathfrak{q}_{d-1}) + (a_d^n)) \to (\mathfrak{q}_{d-1} + (a_d^{n+1}))^*/(\mathrm{U}(\mathfrak{q}_{d-1}) + (a_d^{n+1}))\,,$$

where $\Phi(\bar{x}) = \overline{a_d x}$. It readly follows that $\Phi$ is well-defined. Take $x \in R$ such that $a_d x \in \mathrm{U}(\mathfrak{q}_{d-1}) + (a_d^{n+1})$. Then there exists $y \in R$ such that $a_d(x - a_d^n y) \in \mathrm{U}(\mathfrak{q}_{d-1})$. Therefore $x - a_d^n y \in \mathrm{U}(\mathfrak{q}_{d-1})$ since $R/\mathrm{U}(\mathfrak{q}_{d-1})$ is a 1-dimensional Cohen-Macaulay ring. Thus $x \in \mathrm{U}(\mathfrak{q}_{d-1}) + (a_d^n)$ and lemma follows. $\square$

COROLLARY 3.2. *Let $R$ be an $FLC$ ring and $\mathfrak{q} = (a_1, a_2, \cdots, a_d)$ a parameter ideal in $R$. Then for any integer $n > 0$, we have*

$$\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}) \leq \mathrm{e}_{(a_1^n, a_2^n, \cdots, a_d^n)}(R) - \ell_R(R/(a_1^n, a_2^n, \cdots, a_d^n))\,.$$

PROOF. Note that $\mathrm{e}_{\mathfrak{q}}(R) = \ell_R(R/\mathrm{U}(\mathfrak{q}_{d-1}) + (a_d))$ because $a_1, a_2, \cdots, a_d$ forms a filter regular sequence (cf. [GN, Proof of Theorem 1.1]). We now have

$$\begin{aligned}
&\mathrm{e}_{(\mathfrak{q}_{d-1}+(a_d^n))}(R) - \ell_R(R/(\mathfrak{q}_{d-1} + (a_d^n))^*) \\
&\quad = \ell_R(R/\mathrm{U}(\mathfrak{q}_{d-1}) + (a_d^n)) - \ell_R(R/(\mathfrak{q}_{d-1} + (a_d^n))^*) \\
&\quad = \ell_R((\mathfrak{q}_{d-1} + (a_d^n))^*/(\mathrm{U}(\mathfrak{q}_{d-1}) + (a_d^n)) .
\end{aligned}$$

Hence by Lemma 3.1, we get

$$\mathrm{e}_{(\mathfrak{q}_{d-1}+(a_d^n))}(R) - \ell_R(R/(\mathfrak{q}_{d-1} + (a_d^n)) \le \mathrm{e}_{(\mathfrak{q}_{d-1}+(a_d^{n+1}))}(R) - \ell_R(R/(\mathfrak{q}_{d-1} + (a_d^{n+1})) .$$

Changing the order of the system of parameters and repeating the same procedure, the assertion follows. □

The following is an improvement of [N2, Proposition 4.4].

LEMMA 3.3. *Let $a_1, a_2, \cdots, a_{d-1}$ be a subsystem of parameters of $R$ of length $d-1$. Then $(a_1, a_2, \cdots, a_{d-1})^* R_{\mathfrak{p}} = ((a_1, a_2, \cdots, a_{d-1})R_{\mathfrak{p}})^*$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

PROOF. Let $I = (a_1, a_2, \cdots, a_{d-1})$. Notice that $\mathrm{ht}_R I = d-1$. We may assume that $\mathfrak{p} \supseteq I$ and $\mathfrak{p} \neq \mathfrak{m}$. Take $a_d \in R$ so that $a_1, \cdots, a_{d-1}, a_d$ forms a system of parameters. Let $S = R[1/a_d]$. Then it follows that $(IS)^* S_{\mathfrak{p}S} = (IR_{\mathfrak{p}})^*$ by [N1, Lemma 2.2]. To get the lemma, it is enough to show that $I^*S = (IS)^*$. The inclusion $I^*S \subseteq (IS)^*$ follows from [HH1, Lemma 4.11]. Conversely, take $x \in R$ with $x/1 \in (IS)^*$. Then there exists $c \in R^0$ such that $c/1 \cdot (x/1)^q \in I^{[q]}S$ for all $q \gg 0$. One can take $n$ so that

$$a_d^n c x^q \in (a_1^q, a_2^q, \cdots, a_{d-1}^q) \quad \text{for all } q \gg 0 . \tag{a}$$

(Here the integer $n$ may depend on $q$.)

We may put $R = A/J$, where $A$ is a Cohen-Macaulay local ring of characteristic $p$ and $J$ is an ideal of $A$ with $\mathrm{ht}_A J = 0$. Let $\tilde{x}, \tilde{c}, \widetilde{a_1}, \cdots, \widetilde{a_d} \in A$ be preimages of $x, c, a_1, \cdots, a_d \in R$, respectively. Here we can choose $\widetilde{a_1}, \cdots, \widetilde{a_d} \in A$ to be a system of parameters of $A$. (cf. e.g., [BH, Lemma 10.1.10]). Taking the preimage of the whole of (a) in $A$ yields

$$\tilde{a}_d^n \tilde{c} \tilde{x}^q \in (\tilde{a}_1^q, \tilde{a}_2^q, \cdots, \tilde{a}_{d-1}^q)A + J \quad \text{for all } q \gg 0 .$$

We take $d \in A \setminus \bigcup_{\mathfrak{p} \in \mathrm{Min}_A A/J} \mathfrak{p}$ and $q' = p^{e'}$ so that $d \cdot J^{[q']} = (0)$. Then we have $d\tilde{a}_d^{nq'} \tilde{c}^{q'} \tilde{x}^{qq'} \in (\tilde{a}_1^{qq'}, \tilde{a}_2^{qq'}, \cdots, \tilde{a}_{d-1}^{qq'})A$, whence $d\tilde{c}^{q'} \tilde{x}^{qq'} \in (\tilde{a}_1^{qq'}, \tilde{a}_2^{qq'}, \cdots, \tilde{a}_{d-1}^{qq'})A$ because $\tilde{a}_1, \cdots, \tilde{a}_d$ is a regular sequence on $A$. This implies that $d' x^q \in (a_1^q, \cdots, a_{d-1}^q)R$ for some $d' \in R^0$ and for all $q \gg 0$. Thus we get $x \in (a_1, a_2, \cdots, a_{d-1})^*$, whence $x/1 \in I^*S$. Thus the lemma follows. □

We come to the place to give the proof of Theorem 1.1.

PROOF. (1) $\Rightarrow$ (2): We first note that $R$ is $FLC$. In fact, for all parameter ideal $\mathfrak{q}$, we have

$$\ell_R(\mathfrak{q}^*/\mathfrak{q}) = (\ell_R(R/\mathfrak{q}) - \mathrm{e}_{\mathfrak{q}}(R)) + (\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)) , \tag{$*$}$$

while $e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)$ is non negative by [GN, Theorem 1.1]. Hence $\sup_{\mathfrak{q}}\{\ell_R(R/\mathfrak{q}) - e_{\mathfrak{q}}(R))\}$ is finite. This implies that $R$ is $FLC$ by [CTS].

Take $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \neq \mathfrak{m}$. Then $R_{\mathfrak{p}}$ is Cohen-Macaulay. We shall show that $R_{\mathfrak{p}}$ is $F$-rational. We may assume that $\mathrm{ht}_A \mathfrak{p} = d-1$. Let us take a parameter ideal $\mathfrak{q} = (a_1, a_2, \cdots, a_d)$ such that $\mathfrak{q}_{d-1} = (a_1, a_2, \cdots, a_{d-1}) \subseteq \mathfrak{p}$. Let $N = \sup_{\mathfrak{q}}\{\ell_R(\mathfrak{q}^*/\mathfrak{q})\}$. Then

$$\mathfrak{m}^N \mathfrak{q}_{d-1}^* \subseteq \mathfrak{m}^N(\mathfrak{q}_{d-1} + (a_d^n))^* \subseteq \mathfrak{q}_{d-1} + (a_d^n)$$

for all $n > 0$. The intersection theorem yields $\mathfrak{m}^N \mathfrak{q}_{d-1}^* \subseteq \mathfrak{q}_{d-1}$. Then it follows that $\mathfrak{q}_{d-1}^* R_{\mathfrak{p}} = \mathfrak{q}_{d-1} R_{\mathfrak{p}}$, while $\mathfrak{q}_{d-1}^* R_{\mathfrak{p}} = (\mathfrak{q}_{d-1} R_{\mathfrak{p}})^*$ by Lemma 3.3. Hence $R_{\mathfrak{p}}$ is $F$-rational since $\mathfrak{q}_{d-1} R_{\mathfrak{p}}$ is a parameter ideal of $R_{\mathfrak{p}}$ (cf. [FW, Proposition 2.2]).

(2) $\Rightarrow$ (3): We may assume that $\dim R \geq 1$. $R_{\mathfrak{p}}$ is Cohen-Macaulay for any $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$, since $R_{\mathfrak{p}}$ is excellent $F$-rational ring (cf. [H2, Theorem 4.2]). Thus $R$ is $FLC$ by [CTS] again. We shall show that the length of $(0)_H^*$ is finite. To do it, we may assume that $R$ is reduced. In fact, $(\sqrt{(0)})_{\mathfrak{p}} = (0)$ for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. So $R_{red} = R/\sqrt{(0)}$ satisfies the assertion (2). Besides, $\sqrt{(0)}$ is a module of finite length, whence we have that $R_{red}$ is $FLC$ and $\mathrm{H}_{\mathfrak{m}}^d(R) \cong \mathrm{H}_{\mathfrak{m}}^d(R_{red})$. (Notice also that $R_{red}^0 = \{\bar{c} \mid c \in R^0\}$.)

Now let $f_1, f_2, \cdots, f_d$ be a system of parameters in $R$. Then for each $i$, $f_i$ has a power $f_i^n$ which is a test element for parameter ideals in $R$ since $R[1/f_i]$ is $F$-rational (cf. [V, Theorem 3.9]). Put $\mathfrak{a} = (f_1^n, f_2^n, \cdots, f_d^n)R$. Then $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal for which $\mathfrak{a} I^* \subseteq I$ holds for all parameter ideals $I$. Let $(\underline{a}) = a_1, a_2, \cdots, a_d$ be a system of parameters which forms a $USD$-sequence. (This choice is possible because $R$ is $FLC$.) Since $(0)_H^* = \varinjlim (\underline{a})^*/(\underline{a})$ by Proposition 2.1, we have $\mathfrak{a}(0)_H^* = (0)$. Thus we have an embedding $(0)_H^* \hookrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, H)$ whose length are finite.

(3) $\Rightarrow$ (1): From the equation (*) it is enough to prove that $\sup_{\mathfrak{q}}\{e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\}$ is finite. To see this, by Corollary 3.2, we may assume that the generating system $a_1, a_2, \cdots, a_d$ of $\mathfrak{q}$ is a $USD$-sequence. (Taking a suitable power, $a_1^n, a_2^n, \cdots, a_d^n$ is a $USD$-sequence since $R$ is $FLC$.)

Now $\mathfrak{q} \subseteq \mathrm{U}(\mathfrak{q}_{d-1}) + (a_d) \subseteq \Sigma(\mathfrak{q}) \subseteq \mathfrak{q}^*$ (here we apply the colon capturing property), thus we have

$$\begin{aligned} e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*) &= \ell_R(\mathfrak{q}^*/\mathrm{U}(\mathfrak{q}_{d-1}) + (a_d)) \\ &= \ell_R(\mathfrak{q}^*/\Sigma(\mathfrak{q})) + \ell_R(\Sigma(\mathfrak{q})/\mathrm{U}(\mathfrak{q}_{d-1}) + (a_d)) \\ &\leq \ell_R(\mathfrak{q}^*/\Sigma(\mathfrak{q})) + \ell_R(\Sigma(\mathfrak{q})/\mathfrak{q}), \end{aligned}$$

where the first equality follows from the fact that $a_1, a_2, , \cdots, a_d$ is a filter regular sequence. Because $\ell_R(\Sigma(\mathfrak{q})/\mathfrak{q}) = \sum_{i=0}^{d-1}\binom{d}{i}\ell_R(\mathrm{H}_{\mathfrak{m}}^i(R))$ (cf. [G, Proposition 3.6]) and $\mathfrak{q}^*/\Sigma(\mathfrak{q}) \subseteq (0)_H^*$ by Proposition 2.1. Hence the finiteness of $\sup_{\mathfrak{q}}\{e_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\}$ follows. □

Before the proof of Theorem 1.2, we recall the following.

LEMMA 3.4 ([GN, Lemma 3.2]). *Suppose that* $\operatorname{Ass} R \subseteq \operatorname{Assh} R \cup \{\mathfrak{m}\}$. *Then*

$$\mathcal{F} := \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{ht}_R \mathfrak{p} > 1 = \operatorname{depth} R_{\mathfrak{p}}, \mathfrak{p} \neq \mathfrak{m}\}$$

*is a finite set.*

Let us recall the statement of Theorem 1.2.

THEOREM 1.2. *Suppose that* $\operatorname{Ass} R \subseteq \operatorname{Assh} R \cup \{\mathfrak{m}\}$. *If* $\sup_{\mathfrak{q}}\{\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\} < \infty$, *then*

$$\{\mathfrak{p} \mid R_{\mathfrak{p}} \text{ is } F\text{-rational}\} \setminus \{\mathfrak{m}\} = \{\mathfrak{p} \mid R_{\mathfrak{p}} \text{ is Cohen-Macaulay}\} \setminus \{\mathfrak{m}\}.$$

PROOF. The inclusion $\subseteq$ follows from [H2, Theorem 4.2]. Let us show the opposite inclusion. Let $\mathfrak{a} = \prod_{i<d} \operatorname{Ann}_R \mathrm{H}^i_{\mathfrak{m}}(R)$. By [K, Proposition 2.3], $V(\mathfrak{a})$ is the non Cohen-Macaulay locus of $R$. We can choose a generating system $f_1, f_2, \cdots, f_n$ of $\mathfrak{a}$ such that $f_i \in R^0$ for all $i$ because $\operatorname{ht}_R \mathfrak{a} > 0$. Let $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \neq \mathfrak{m}$ and assume that $R_{\mathfrak{p}}$ is Cohen-Macaulay. Then there exists $f_i$ with $f_i \notin \mathfrak{p}$. Let $S = R[1/f_i]$ and taken $\mathfrak{n} \in \operatorname{Max} S$ with $\mathfrak{n} \supseteq \mathfrak{p}S$. Let $P = \mathfrak{n} \cap R$. Notice that $R_P$ is Cohen-Macaulay. Besides, $S/\mathfrak{n} = S/PS = R/P[1/f_i]$, which is a field, whence $\dim R/P = 1$. Thus $\operatorname{ht}_R P = d - 1$.

CLAIM 1. *There exists* $a_1, a_2, \cdots, a_{d-1} \in P$ *which forms a filter regular sequence.*

PROOF. When $d = 1$, there is nothing to prove. Let $d \geq 2$. Let $\mathcal{F}$ be as in Lemma 3.4. Then $P \not\subseteq Q$ for any $Q \in \mathcal{F}$. We take

$$a_1 \in P \setminus \Big(\bigcup_{Q \in \mathcal{F}} Q\Big) \cup \Big(\bigcup_{Q \in \operatorname{Assh} R} Q\Big).$$

Then it follows that $\operatorname{Ass} R/(a_1) \subseteq \operatorname{Assh} R/(a_1) \cup \{\mathfrak{m}\}$. In fact, take $\mathfrak{p}_1 \in \operatorname{Ass}_R R/(a_1)$ with $\mathfrak{p}_1 \neq \mathfrak{m}$. Then depth $R_{\mathfrak{p}_1} = 1$ and it implies that $\operatorname{ht}_R \mathfrak{p}_1 = 1$. Thus, $\dim R/\mathfrak{p}_1 = \dim R/(a_1)$, whence $\mathfrak{p}_1 \in \operatorname{Assh} R/(a_1)$. From this choice it follows that $a_1$ is a filter regular sequence of length one. Repeating this procedure, the claim follows. □

Let $N = \sup_{\mathfrak{q}}\{\mathrm{e}_{\mathfrak{q}}(R) - \ell_R(R/\mathfrak{q}^*)\}$ and $a_1, a_2, \cdots, a_{d-1} \in P$ be as above. We take $a_d \in R$ so that $a_1, \cdots, a_{d-1}, a_d$ is a system of parameters in $R$. Notice that it forms a filter regular sequence. Let $I = (a_1, a_2, \cdots, a_{d-1})$ and $J = I + (a_d^n)$. Then

$$\begin{aligned} \ell_R(J^*/\mathrm{U}(I) + (a_d^n)) &= \ell_R(R/\mathrm{U}(I) + (a_d^n)) - \ell_R(R/J^*) \\ &= \mathrm{e}_J(R) - \ell_R(R/J^*) \\ &\leq N \end{aligned}$$

Hence, $\mathfrak{m}^N I^* \subseteq \mathfrak{m}^N J^* \subseteq \mathrm{U}(I) + (a_d^n)$ for all $n > 0$. The intersection theorem yields $\mathfrak{m}^N I^* \subseteq \mathrm{U}(I)$. Hence $I^* = \mathrm{U}(I)$ by colon capturing property. Now $I^* R_P = (I R_P)^*$ by Lemma 3.3, while $\mathrm{U}(I)_P = I R_P$. It implies that $I R_P$ is tightly closed, whence $R_P$ is $F$-rational Cohen-Macaulay ring (cf. [FW, Proposition 2.2]). Therefore, $R_{\mathfrak{p}}$ is $F$-rational. □

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