THE BOUNDARY CORRESPONDENCE UNDER QUASICONFORMAL MAPPINGS

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1. Introduction

1.1. The dilatation $D \geqq 1$ of a differentiable topological mapping f: $(x,y) \to (u,v)$ between plane regions is determined by

$$D + \frac{1}{D} = \frac{u_x^2 + u_y^2 + v_x^2 + v_y^2}{|u_x v_y - u_y v_x|}. \tag{1}$$

Geometrically, D represents the ratio between major and minor axis of the infinitesimal ellipse obtained by mapping an infinitesimal circle of center (x,y). From this interpretation it is evident that the dilatation is unaffected by conformal mappings of the planes. Furthermore, a mapping f and its inverse f^{-1} have the same dilatation at corresponding points.

A mapping is said to be *quasiconformal* if D is bounded. The least upper bound of D is referred to as the *maximal dilatation*.

1.2. It is known that a quasiconformal mapping of $x^2 + y^2 < 1$ onto $u^2 + v^2 < 1$ remains continuous on the boundary.[1] Hence it induces a topological correspondence between the two circumferences. We shall be concerned with the problem of characterizing this correspondence by simple necessary and sufficient conditions. More generally, we shall look for conditions which are compatible with a mapping whose maximal dilatation does not exceed a given number $K > 1$.

In view of the invariance with respect to conformal mappings we may replace the disk by the upper half-planes $y > 0$ and $v > 0$, and we may assume that the points at ∞

[1] L. Ahlfors, On quasiconformal mappings. *Journal d'Analyse Mathematique*, vol. 7.1 (1953/54).

correspond to each other under the mapping. In these circumstances the boundary correspondence is determined by a strictly monotone continuous function $\mu(x)$, in the sense that the point $(x,0)$ is mapped on $(\mu(x),0)$. It is sufficient to consider the case of an increasing $\mu(x)$.

1.3. The main theorem that will be proved in this paper is the following:

THEOREM 1. *There exists a quasiconformal mapping of the half-planes with the boundary correspondence $x \to \mu(x)$ if and only if*

$$\frac{1}{\varrho} \leqq \frac{\mu(x+t)-\mu(x)}{\mu(x)-\mu(x-t)} \leqq \varrho \tag{2}$$

for some constant $\varrho \geqq 1$ and for all real x and t.

More precisely, if (2) is fulfilled there exists a mapping whose dilatation does not exceed ϱ^2. On the other hand, every mapping with the boundary correspondence μ must have a maximal dilatation $\geqq 1+A \log \varrho$ where A is a certain numerical constant $(=.2284)$.

Condition (2), which will be referred to as the ϱ-condition, indicates that $\mu(x)$ must possess a degree of approximate symmetry when x approaches any value from the right and from the left. The ϱ-condition is not invariant with respect to arbitrary linear transformations, nor does the inverse mapping satisfy the same ϱ-condition. However, the very simple form that the condition takes when the points at ∞ are singled out is sufficient reason to give preference to the formulation that we have chosen.

1.4. In Section 2 we use Theorem 1 to derive a different characteristic of quasiconformal mappings. In Section 3 the problem is further analyzed, and in Section 4 we prove the easy part of Theorem 1. Section 5 deals with a class of explicit mappings and serves to exhibit limits beyond which the estimates in Theorem 1 cannot be improved. The complete proof of Theorem 1 follows in Section 6, and in a final section we give the answer to an open question concerning absolute continuity.

The investigation requires rather extensive computations. In order to facilitate the reading most of these computations are given in complete detail. A reader who is interested only in the qualitative aspects may omit the computations in Section 4 and all of Section 5.

2. Compact Families of Mappings

2.1. In this section we show that the ϱ-condition can also be interpreted as a compactness condition. This analysis can be carried out alternatively for transformations of the unit circle or for transformations of the line which leave the point at ∞ fixed. Since the transition is very easy we shall only treat the case of the line.

As before, $\mu(x)$ shall denote an increasing function which sets up a 1-1 correspondence of the real line with itself. The linear transformations $x \to ax + b$, $a > 0$ will be denoted by S, T. We say that a family M of transformations μ is closed under linear transformations if all composed mappings $S\mu T$ are contained in M together with μ.

We shall also say that a mapping is *normalized* if $\mu(0) = 0$, $\mu(1) = 1$.

2.2. Let us consider the following compactness condition:

(a) *Every infinite set of normalized mappings $\mu \in M$ contains a sequence $\{\mu_n\}_1^\infty$ which converges to a strictly increasing limit function.*

The following theorem holds:

THEOREM 2. *The mappings μ in a family M, which is closed under linear transformations, satisfy a ϱ-condition, the same for all μ, if and only if condition (a) is fulfilled.*

2.3. Suppose first that the ϱ-condition is satisfied. If μ is normalized we find

$$\mu(2^{-n+1}) \geqq \left(1 + \frac{1}{\varrho}\right) \mu(2^{-n})$$

for any n, and since $\mu(1) = 1$ it follows that

$$\mu(2^{-n}) \leqq \left(\frac{\varrho}{\varrho + 1}\right)^n \tag{3}$$

for positive and negative integers n.

For any a the function

$$\frac{\mu(a + x) - \mu(a)}{\mu(a + 1) - \mu(a)}$$

is normalized. By (3) we have consequently

$$\mu(a + x) - \mu(a) \leqq (\mu(a + 1) - \mu(a)) \left(\frac{\varrho}{\varrho + 1}\right)^n \tag{4}$$

as soon as $0 \leqq x \leqq 2^{-n}$. On the other hand, if a is restricted to a finite interval, inequality (3) with negative n yields a bound for $\mu(a + 1) - \mu(a)$. Hence (4) constitutes an equicontinuity condition on any compact set.

It follows that we can select a convergent sequence from every infinite set of normalized mappings. The limit function is normalized and satisfies the ϱ-condition. For this reason it cannot be constant on any interval. In other words, condition (a) is fulfilled.

2.4. Conversely, suppose that (a) is fulfilled. We set

$$\alpha = \inf \mu(-1), \qquad \beta = \sup \mu(-1),$$

where μ ranges over all normalized mappings in M. There exists a sequence of $\mu_n \in M$ such that $\mu_n(-1) \to \beta$. Since a subsequence converges to a strictly monotone function it follows that $\beta < 0$. The same reasoning yields $\alpha > -\infty$.

Consider any μ in M. The mapping

$$\nu(x) = \frac{\mu(y+tx) - \mu(y)}{\mu(y+t) - \mu(y)}$$

is normalized and in M. Hence

$$\alpha \leq \frac{\mu(y-t) - \mu(y)}{\mu(y+t) - \mu(y)} \leq \beta$$

or

$$-\frac{1}{\alpha} \leq \frac{\mu(y+t) - \mu(y)}{\mu(y) - \mu(y-t)} \leq -\frac{1}{\beta}.$$

Clearly, this implies the ϱ-condition for $\varrho = \max\left(-\alpha, -\frac{1}{\beta}\right)$.

2.5. Theorems 1 and 2 yield the following:

COROLLARY. *A boundary mapping μ can be extended to a quasiconformal mapping of the halfplanes if and only if the family of all mappings $S\mu T$ satisfies condition* (a).

Instead of relying on quantitative statements this criterion emphasizes a distinctive qualitative feature.

3. Quantities Related to Dilatation

3.1. If f is a differentiable mapping of the half-plane on itself we shall denote its maximal dilatation by K_f. For a given boundary correspondence μ we set

$$K(\mu) = \inf K_f,$$

where the infimum is with respect to all mappings f such that $f = \mu$ on the real axis.

The quantity $K(\mu)$ is in the foreground of our interest, but we shall also find it illuminating to introduce some other quantities of similar character.

3.2. A quasiconformal mapping with the maximal dilatation K has the following property: if the real-valued functions U_1 and U_2 are related by $U_1(z) = U_2(f(z))$, then the Dirichlet integrals of U_1 and U_2 over corresponding domains have a ratio which lies between $1/K$ and K.

In particular, we may confine the attention to Dirichlet integrals over the whole half-plane. Let u_1 and u_2 be defined on the real axis and related by $u_1(x) = u_2(\mu(x))$. We choose U_2 as the solution of Dirichlet's problem with boundary values u_2. Then the Dirichlet integral $\mathcal{D}(U_2)$ is given explicitly by $I(u_2)$ where

$$I(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{u(x) - u(y)}{x - y} \right)^2 dx\, dy$$

is the well-known Douglas functional.

By use of the Dirichlet principle we find at once that

$$I(u_1) \leqq \mathcal{D}(U_1) \leqq K \mathcal{D}(U_2) = K I(u_2).$$

Since the same reasoning can be applied to the inverse mapping we have also $I(u_2) \leqq K I(u_1)$.

This result leads us to introduce a quantity $K_1(\mu)$, defined as the least number K_1 such that

$$\frac{1}{K_1} \leqq \frac{I(u_2)}{I(u_1)} \leqq K_1 \tag{5}$$

for all pairs of corresponding functions u_1, u_2. We have just shown that

$$K_1(\mu) \leqq K(\mu).$$

The quantity $K_1(\mu)$ may be regarded as more explicit than $K(\mu)$ inasmuch as its definition does not involve two-dimensional mappings.

3.3. In the circular representation, let α_1, β_1 be any two disjoint arcs, and denote their images under the mapping μ by α_2, β_2. The extremal distance between α_1 and β_1, to be denoted by $d(\alpha_1, \beta_1)$, is a function-theoretic quantity which in this simple case can be computed explicitly. It is equal to $1/\mathcal{D}(U_1)$, where U_1 is the harmonic function with values 0,1 on α_1, β_1 whose normal derivative vanishes on the complementary arcs.

The function U_1 minimizes the Dirichlet integral for the prescribed boundary values. Therefore,

$$\frac{1}{d(\alpha_1, \beta_1)} = \min \mathcal{D}(U_1) = \min I(u_1)$$

for the class of all function u_1 which are 0 and 1 on α_1, β_1. But it follows from (5) that $\min I(u_1) \leqq K_1 \min I(u_2)$, and hence that $d(\alpha_2, \beta_2) \leqq K_1 d(\alpha_1, \beta_1)$.

In the present case there is symmetry between μ and its inverse mapping, for the complementary arcs of α_1, β_1 have the extremal distance $1/d(\alpha_1, \beta_1)$. Hence the definition

$$K_0(\mu) = \sup \frac{d(\alpha_2, \beta_2)}{d(\alpha_1, \beta_1)} \tag{6}$$

implies the double inequality

$$\frac{1}{K_0(\mu)} \leqq \frac{d(\alpha_2, \beta_2)}{d(\alpha_1, \beta_1)} \leqq K_0(\mu).$$

We have proved that

$$K_0(\mu) \leqq K_1(\mu) \leqq K(\mu).$$

Of these three quantities K_0 is the most explicit, for it is defined as the maximum of a function rather than a functional.

3.4. For a given mapping μ of the real axis we denote by $\varrho(\mu)$ the smallest value of ϱ such that the ϱ-condition (2) is fulfilled. Our efforts will be directed towards proving inequalities of the form

$$\Phi(\varrho(\mu)) \leqq K_0(\mu) \leqq K(\mu) \leqq \Psi(\varrho(\mu)).$$

Here we may let Φ and Ψ denote the best possible functions of their kind. This amounts to setting

$$\Phi(\varrho) = \inf K_0(\mu) \quad \text{for} \quad \varrho(\mu) \geqq \varrho$$
$$\Psi(\varrho) = \sup K(\mu) \quad \text{for} \quad \varrho(\mu) \leqq \varrho.$$

The necessity in Theorem 1 will be proved if we show that $\lim_{\varrho\to\infty} \Phi(\varrho) = \infty$, and the sufficiency follows upon showing that $\Psi(\varrho)$ is finite. It turns out that $\Phi(\varrho)$ can be determined explicitly, although in transcendental form, and the transcendental expression leads to the elementary minorant mentioned in the theorem. As for $\Psi(\varrho)$ we prove a slightly better inequality than $\Psi(\varrho) \leqq \varrho^2$, and for comparison we shall also derive a minorant of Ψ.

4. Proof of the Necessity

4.1. In order to determine a minorant of $K_0(\mu)$, defined by (6), we observe that the extremal distance $d(\alpha_1, \beta_1)$ is invariant under linear transformations, and hence a function of the cross-ratio of the end points of α_1 and β_1. If $\alpha_1 = (t_1, t_2)$, $\beta_1 = (t_3, t_4)$ we set

$$\lambda = \frac{t_3 - t_2}{t_2 - t_1} : \frac{t_4 - t_3}{t_4 - t_1}$$

and obtain $d(\alpha_1, \beta_1) = P(\lambda)$, where P is a known function.

The notation is such that $P(0) = 0$, $P(\infty) = \infty$. Also, since the complementary intervals (t_2, t_3), (t_4, t_1) lead to the reciprocal cross-ratio, $P(\lambda)P(1/\lambda) = 1$ and $P(1) = 1$.

4.2. A minorant for $K_0(\mu)$ is obtained by restricting the choice of α_1, β_1 to intervals of the form $(x - t, x)$ and $(x + t, \infty)$. In this case $\lambda = 1$, while the corresponding intervals α_2, β_2 determine

$$\lambda' = \frac{\mu(x+t) - \mu(x)}{\mu(x) - \mu(x-t)}.$$

Since $\sup \lambda' = \varrho(\mu)$ it follows by (6) that

$$K_0 \geqq P(\varrho).$$

4.3. On the other hand, let $\varrho > 1$ be given. The upper half-plane with vertices at $-1, 0, 1, \infty$ is conformally a square, and the half-plane with vertices at $-1, 0, \varrho, \infty$ is a rectangle whose sides have the ratio $P(\varrho)$. An affine mapping of the square on the rectangle determines an extremal quasiconformal mapping of the half-plane with constant dilatation $P(\varrho)$. If the induced mapping of the real axis is denoted by ν we have thus $K(\nu) = P(\varrho)$, while evidently $\varrho(\nu) \geqq \varrho$. The latter inequality implies $K_0(\nu) \geqq P(\varrho)$, and therefore $K_0(\nu) = K(\nu) = P(\varrho)$. This proves:

The best lower bound $\Phi(\varrho)$ of $K_0(\mu)$ is equal to $P(\varrho)$, the extremal distance between $(-1, 0)$ and (ϱ, ∞).

4.4. It is worthwhile to determine elementary estimates for $P(\varrho)$. The exact expression for $P(\varrho)$ reads

$$P(\varrho) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x+\varrho)}} : \int_0^1 \frac{dx}{\sqrt{x(1-x)(x+\varrho)}}.$$

On introducing the hypergeometric function

$$F(t) = F(\tfrac{1}{2}, \tfrac{1}{2}, 1, t) = \frac{1}{\pi} \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-xt)}}$$

it is possible to write

$$P(\varrho) = \frac{F\left(\frac{\varrho}{1+\varrho}\right)}{F\left(\frac{1}{1+\varrho}\right)}.$$

A classical computation (Carathéodory, *Funktionentheorie* II, p. 169) yields

$$\varrho P'(\varrho) = \frac{1}{\pi} F\left(\frac{1}{\varrho+1}\right)^{-2}.$$

Here $F[1/(\varrho+1)]$ varies between $F(0) = 1$ and $F(\tfrac{1}{2}) = 1.1803$. Consequently we can write

$$P(\varrho) - 1 = \theta(\varrho) \log \varrho,$$

where $\theta(\varrho)$ increases from $\theta(1) = .2284$ to $\theta(\infty) = 1/\pi = .3183$.

5. A Class of Explicit Mappings

5.1. In this section we study a class of explicit mappings for which $K(\mu)$ and $\varrho(\mu)$ can be computed. They are used to determine a minorant of $\Psi(\varrho)$ (see 3.4 for the definition of Ψ).

For any given $\alpha > 0$ we consider

$$\mu(x) = \operatorname{sgn} x \cdot |x|^{\alpha}. \tag{7}$$

A corresponding mapping is obtained by setting $|f(z)| = |z|^{\alpha}$, $\arg f(z) = \arg z$. In terms of $s = \log z$ and $\sigma = \log f(z)$ the mapping is affine,

$$\sigma = \alpha \operatorname{Re} s + i \operatorname{Im} s,$$

and we see at once that the dilatation is constantly equal to α if $\alpha \geqq 1$ and $1/\alpha$ if $\alpha \leqq 1$. It follows that $K(\mu) \leqq \max(\alpha, 1/\alpha)$.

5.2. In order to determine $K_0(\mu)$ we consider the intervals $(-r, -1/r)$ and $(1/r, r)$ for $r > 1$. Their extremal distance is given by $P(4r^2/(r^2-1)^2)$, and the image intervals have the extremal distance $P(4r^{2\alpha}/(r^{2\alpha}-1)^2)$. From the asymptotic development $P(t) \sim \pi/\log(1/t)$ for small t we obtain

$$P\left(\frac{4r^2}{(r^2-1)^2}\right) \sim \frac{\pi}{2\log r} \qquad r \to \infty$$

$$P\left(\frac{4r^{2\alpha}}{(r^{2\alpha}-1)^2}\right) \sim \frac{\pi}{2\alpha \log r} \qquad r \to \infty,$$

and hence the ratio tends to α. We conclude that $K_0(\mu) \geqq \max(\alpha, 1/\alpha)$. Together with $K_0(\mu) \leqq K(\mu) \leqq \max(\alpha, 1/\alpha)$ it follows that

$$K_0(\mu) = K(\mu) = \max(\alpha, 1/\alpha).$$

5.3. Because $\mu(kx) = \operatorname{sgn} k \cdot |k|^{\alpha} \mu(x)$, the ratio

$$\frac{\mu(x+t) - \mu(x)}{\mu(x) - \mu(x-t)}$$

takes the same values as

$$\frac{\mu(1+t) - 1}{1 - \mu(1-t)}. \tag{8}$$

We must therefore study the maximum of (8) in its dependence on the parameter α. The cases $\alpha \geqq 1$ and $\alpha \leqq 1$ will be treated separately.

5.4. Suppose first that $\alpha > 1$. For $0 < t < 1$ (8) takes the form

$$\frac{(1+t)^{\alpha}-1}{1-(1-t)^{\alpha}}. \tag{9}$$

A look at the graph of x^{α} shows that $[(1+t)^{\alpha}-1]/t$ increases and $[1-(1-t)^{\alpha}]/t$ decreases with increasing t. Hence (9) is increasing and attains its maximum for $t=1$, its minimum 1 for $t=0$.

For $t > 1$ (8) becomes

$$q(t) = \frac{(t+1)^{\alpha}-1}{(t-1)^{\alpha}+1}.$$

A simple calculation shows that $q'(t)$ has the same sign as

$$p(t) = (t+1)^{1-\alpha} + (t-1)^{1-\alpha} - 2.$$

But $p(t)$ decreases from ∞ for $t=1$ to -2 for $t=\infty$. Therefore $q(t)$ has a single maximum which is attained at the root t_{α} of the equation

$$(t+1)^{1-\alpha} + (t-1)^{1-\alpha} = 2.$$

The value of the maximum equals

$$q(t_{\alpha}) = \left(\frac{t_{\alpha}+1}{t_{\alpha}-1}\right)^{\alpha-1} = 2(t_{\alpha}+1)^{\alpha-1} - 1. \tag{10}$$

From what we have said it follows also that the values of (8) are > 1 for $t > 0$, and hence < 1 for $t < 0$. We conclude that $\varrho(\mu) = q(t_{\alpha})$.

5.5. As $\alpha \to 1$ it is clear that $t_{\alpha} \to \sqrt{2}$. To obtain an elementary bound for $q(t_{\alpha})$ we observe that $p(\sqrt{2}) > 0$ for the simple reason that $(\sqrt{2}+1)^{1-\alpha}$ and $(\sqrt{2}-1)^{1-\alpha}$ are reciprocals. This implies $t_{\alpha} > \sqrt{2}$, and by (10) we find that

$$\varrho(\mu) = q(t_{\alpha}) \leq (\sqrt{2}+1)^{2(\alpha-1)}.$$

We know that $K(\mu) = \alpha$ and are thus able to conclude that

$$\Psi(\varrho) \geqq 1 + \frac{\log \varrho}{2 \log (\sqrt{2}+1)}. \tag{11}$$

5.6. For large values of ϱ the estimate (11) can be replaced by a much better one which we find by choosing $\alpha = 1/K < 1$. The corresponding value of $\varrho(\mu)$ can be calculated in the same way as above, with the difference that (8) is now < 1 for positive t. Accordingly, we find that $\varrho(\mu) = 1/q(t_{\alpha})$ where $q(t)$ and t_{α} have the same meaning as before.

This time we are interested in small values of α, and it is easily seen that $t_\alpha \sim 1 + \alpha \log 2$ for $\alpha \to 0$. We shall show in a moment that $p(1 + \alpha \log 2) > 0$, provided that α is sufficiently small. Since $p(t)$ is now increasing this implies $t_\alpha < 1 + \alpha \log 2$.

By use of (10) we obtain

$$q(t_\alpha) \geqq 2(2 + \alpha \log 2)^{\alpha - 1} - 1 \geqq \frac{2^\alpha}{1 + \frac{\alpha}{2} \log 2} - 1 \geqq \frac{1 + \alpha \log 2}{1 + \frac{\alpha}{2} \log 2} - 1$$

$$= \frac{\frac{\alpha}{2} \log 2}{1 + \frac{\alpha}{2} \log 2}.$$

On substituting $q(t_\alpha) = 1/\varrho$ and $\alpha = 1/K$ we are led to the estimate

$$\Psi(\varrho) \geqq \frac{\log 2}{2}(\varrho - 1) \qquad (12)$$

which is valid for sufficiently large ϱ.

To prove our contention that $p(1 + \alpha \log 2) > 0$ we make use of the inequalities $e^x \geqq 1 + x$ and $(1 + x)^{-\alpha} \geqq 1 - \alpha x$. We obtain

$$(2 + \alpha \log 2)^{1-\alpha} = (2 + \alpha \log 2)\, 2^{-\alpha} \left(1 + \frac{\alpha}{2} \log 2\right)^{-\alpha}$$

$$\geqq (2 + \alpha \log 2)(1 - \alpha \log 2)\left(1 - \frac{\alpha^2}{2} \log 2\right)$$

$$\geqq 2 - \alpha \log 2 - \alpha^2 (1 + \log 2) \log 2$$

and further

$$(\alpha \log 2)^{1-\alpha} = \alpha \log 2\, (\alpha \log 2)^{-\alpha}$$

$$\geqq \alpha \log 2\, (1 - \alpha \log \alpha - \alpha \log \log 2).$$

When these inequalities are added we find that $p(1 + \alpha \log 2) \geqq 0$ as soon as

$$\log 1/\alpha \geqq 1 + \log 2 + \log \log 2,$$

i.e. if

$$\alpha \leqq \frac{1}{2e \log 2}.$$

Hence the estimate (12) is valid for $\varrho > 4e + 1$.

6. The Sufficiency Proof

6.1. It remains to find an upper bound for K when ϱ is given. To this end we must construct explicit quasiconformal mappings with a given boundary correspondence μ.

We define a mapping $f(x, y) = u(x, y) + iv(x, y)$ by

$$\left.\begin{aligned} u(x, y) &= \int_{-\infty}^{\infty} \frac{1}{y} K_1\left(\frac{x-t}{y}\right) \mu(t)\, dt = \int_{-\infty}^{\infty} K_1(t)\, \mu(x+yt)\, dt \\ v(x, y) &= r \int_{-\infty}^{\infty} \frac{1}{y} K_2\left(\frac{x-t}{y}\right) \mu(t)\, dt = r \int_{-\infty}^{\infty} K_2(t)\, \mu(x+yt)\, dt \end{aligned}\right\}, \tag{13}$$

where the kernels K_1, K_2 are given as follows

$$K_1(x) = \begin{cases} \frac{1}{2} & \text{for } -1 < x < 1 \\ 0 & \text{for } |x| \geqq 1 \end{cases}, \qquad K_2(x) = K_1(x) \operatorname{sign} x.$$

We observe that $f(x, y)$ is defined by a linear operation on the boundary values $\mu(x)$,

$$f(x, y) = \int_{-\infty}^{\infty} \frac{1}{y} K\left(\frac{x-t}{y}\right) \mu(t)\, dt, \qquad (K = K_1 + i r K_2).$$

The constant $r > 0$ is a parameter which we shall use later to make the dilatation as small as we can.

Without use of the notations K_1, K_2 the definition can be rephrased as

$$\left.\begin{aligned} u(x, y) &= \frac{1}{2} \int_0^1 [\mu(x+ty) + \mu(x-ty)]\, dt \\ v(x, y) &= \frac{r}{2} \int_0^1 [\mu(x+ty) - \mu(x-ty)]\, dt \end{aligned}\right\}. \tag{14}$$

6.2. The following properties are evident: u and v are defined and continuous in the closed half-plane $y \geqq 0, v > 0$ for $y > 0$, and $u(x,0) = \mu(x)$, $v(x,0) = 0$. Hence $f(z)$ has the right boundary values.

As to the behavior at ∞ we see that $u(x, y) \to +\infty$ for $x \to +\infty$ and $u(x, y) \to -\infty$ for $x \to -\infty$, uniformly in y. Similarly, $v(x, y) \to +\infty$ for $y \to +\infty$, and the convergence is uniform when x is restricted to a finite interval. We conclude that $f(z) \to \infty$ for $z \to \infty$.

It will be seen that the Jacobian of the mapping is always positive. In view of this fact, and by virtue of the boundary correspondence, it is a simple matter to show that the mapping is automatically topological, and that the image of the half-plane, $y > 0$ is the whole half-plane $v > 0$.

6.3. We shall have to determine the partial derivatives of u and v at a point (x_0, y_0), $y_0 > 0$. It is easy to prove that the derivatives of a convolution of the form used in (13) are given by

$$\frac{\partial}{\partial x}\int_{-\infty}^{\infty} K(t)\,\mu(x+yt)\,dt=\frac{1}{y}\int_{-\infty}^{\infty} K(t)\,d\mu(x+yt)$$

$$\frac{\partial}{\partial y}\int_{-\infty}^{\infty} K(t)\,\mu(x+yt)\,dt=\frac{1}{y}\int_{-\infty}^{\infty} t\,K(t)\,d\mu(x+yt).$$

Consequently, if we introduce the notations

$$\left.\begin{aligned}\alpha&=\frac{1}{y_0}\int_0^1 d\mu(x_0+y_0t), & \beta&=\frac{1}{y_0}\int_{-1}^0 d\mu(x_0+y_0t)\\ \alpha'&=\frac{1}{y_0}\int_0^1 t\,d\mu(x_0+y_0t), & \beta'&=-\frac{1}{y_0}\int_{-1}^0 t\,d\mu(x_0+y_0t)\end{aligned}\right\}, \tag{15}$$

we obtain

$$u_x=\alpha+\beta,\qquad u_y=\alpha'-\beta'$$
$$v_x=r(\alpha-\beta),\qquad v_y=r(\alpha'+\beta').$$

On substituting these values in (1) we find

$$D+\frac{1}{D}=\frac{(\alpha^2+\beta^2+\alpha'^2+\beta'^2)(1+r^2)+2(\alpha\beta-\alpha'\beta')(1-r^2)}{2r(\alpha\beta'+\alpha'\beta)}. \tag{16}$$

It is to be noted that α, β, α', β' are all positive. This substantiates our claim that $u_xv_y-u_yv_x=2r(\alpha\beta'+\alpha'\beta)>0$.

6.4. Since $\nu(t)=\mu(x_0+y_0t)$ satisfies the same ϱ-condition as $\mu(t)$ we lose no generality if we choose $x_0=0$, $y_0=1$ in (15). In other words, it amounts merely to a change of notation if we replace (15) by

$$\alpha=\int_0^1 d\mu\qquad \beta=\int_{-1}^0 d\mu$$

$$\alpha'=\int_0^1 t\,d\mu\qquad \beta'=-\int_{-1}^0 t\,d\mu.$$

By means of the ϱ-condition

$$\frac{1}{\varrho}\leqq\frac{\mu(x+t)-\mu(x)}{\mu(x)-\mu(x-t)}\leqq\varrho \tag{17}$$

we obtain at once

$$\alpha/\varrho\leqq\beta\leqq\varrho\,\alpha. \tag{18}$$

6.5. Corresponding estimates of α'/α and β'/β are a little less obvious. We prove in this respect:

LEMMA. *The ratios α'/α and β'/β lie between $1/(\varrho+1)$ and $\varrho/(\varrho+1)$.*

For the purpose of proving the Lemma we may assume that $\mu(0)=0, \mu(1)=1$. Then

$$\alpha' = 1 - \int_0^1 \mu\, dt,$$

and we have to prove that

$$\frac{1}{\varrho+1} \leqq \int_0^1 \mu\, dt \leqq \frac{\varrho}{\varrho+1}. \tag{19}$$

Let C_ϱ be the family of all μ which are normalized by $\mu(0)=0, \mu(1)=1$ and satisfy (17) for points $x-t$, x, $x+t$ contained in the interval $(0,1)$. We set

$$M(x) = \sup_{\mu \in C_\varrho} \mu(x).$$

It is clear that $M(\tfrac{1}{2}) \leqq \varrho/(\varrho+1)$. Furthermore, it follows from the definition of $M(x)$ that

$$M(x) \leqq M(\tfrac{1}{2}) M(2x), \qquad 0 \leqq x \leqq \tfrac{1}{2}$$

$$M(\tfrac{1}{2}+x) \leqq M(\tfrac{1}{2}) + (1 - M(\tfrac{1}{2})) M(2x), \qquad 0 \leqq x \leqq \tfrac{1}{2}.$$

Hence

$$M(x) + M(\tfrac{1}{2}+x) \leqq M(\tfrac{1}{2}) + M(2x)$$

and

$$\int_0^1 M(x)\, dx = \int_0^{1/2} [M(x) + M(\tfrac{1}{2}+x)]\, dx \leqq \tfrac{1}{2} M(\tfrac{1}{2}) + \tfrac{1}{2}\int_0^1 M(x)\, dx.$$

We conclude that

$$\int_0^1 \mu\, dt \leqq \int_0^1 M(x)\, dx \leqq M(\tfrac{1}{2}) \leqq \frac{\varrho}{\varrho+1}.$$

The left-hand inequality in (19) follows on replacing $\mu(t)$ by $1-\mu(1-t)$. The same bounds for β'/β are found when we replace $\mu(t)$ by $-\mu(-t)$.

6.6. We simplify (16) by writing $\alpha' = \xi\alpha$, $\beta' = \eta\beta$. Then

$$D + \frac{1}{D} = \frac{1}{2r(\xi+\eta)} \left\{ \left[\frac{\alpha}{\beta}(1+\xi^2) + \frac{\beta}{\alpha}(1+\eta^2) \right] (1+r^2) + 2(1-\xi\eta)(1-r^2) \right\},$$

and the point (ξ, η) is restricted to the square

$$\frac{1}{\varrho+1} \leqq \xi, \eta \leqq \frac{\varrho}{\varrho+1}.$$

Because of the symmetry we may suppose that $\xi \geqq \eta$. Under this condition our expression has its largest value when α/β attains its maximum ϱ. Hence we find that

$$D+\frac{1}{D} \leqq F(\xi, \eta) \tag{20}$$

where
$$F(\xi, \eta)=\frac{a(\xi, \eta)}{r}+b(\xi, \eta)\, r,$$

$$a(\xi, \eta)=\frac{(\varrho+1)^2+(\varrho\,\xi-\eta)^2}{2(\xi+\eta)}, \qquad b(\xi, \eta)=\frac{(\varrho-1)^2+(\varrho\,\xi+\eta)^2}{2(\xi+\eta)}.$$

The functions $a(\xi, \eta)$ and $b(\xi, \eta)$ are seen to be convex (from below) for $\xi+\eta>0$. Hence the maximum of $F(\xi, \eta)$ in the triangle under consideration can only be attained at one of the vertices

$$\left(\frac{1}{\varrho+1}, \frac{1}{\varrho+1}\right), \left(\frac{\varrho}{\varrho+1}, \frac{1}{\varrho+1}\right) \quad \text{or} \quad \left(\frac{\varrho}{\varrho+1}, \frac{\varrho}{\varrho+1}\right).$$

We denote the values at these points by F_1, F_2, F_3 respectively, and set $F_i=a_i/r+b_i r$.

6.7. We have already proved that the dilatation is bounded, for it follows from (20) that

$$D+\frac{1}{D} \leqq \max(F_1, F_2, F_3).$$

It remains to determine an explicit bound by suitable choice of r.

In order to treat this technical question we compute

$$a_1-a_2=\frac{(\varrho-1)(\varrho^2+3\varrho+4)}{4(\varrho+1)}$$

$$b_1-b_2=\frac{(\varrho-1)(\varrho^3-\varrho-4)}{4(\varrho+1)^2}$$

$$a_1-a_3=\frac{(\varrho-1)(\varrho^4+3\varrho^3+8\varrho^2+3\varrho+1)}{4\varrho^2(\varrho+1)}$$

$$b_1-b_3=\frac{(\varrho-1)(\varrho^5-\varrho^3+3\varrho+1)}{4\varrho^2(\varrho+1)^2}.$$

It is seen that a_1-a_3 and b_1-b_3 are both $\geqq 0$ for $\varrho \geqq 1$, and hence $F_1 \geqq F_3$ for all r. Consequently, we need only compare F_1 and F_2.

The inequality $F_1 \geqq F_2$ holds if

$$(a_1-a_2)+(b_1-b_2)\,r^2 \geqq 0.$$

The minimum of F_1 is attained for $r = \sqrt{a_1/b_1}$, and we conclude that this minimum is greater than the corresponding value of F_2 if

$$(a_1 - a_2)\, b_1 + (b_1 - b_2)\, a_1 \geqq 0.$$

After a lengthy computation one finds that this expression equals

$$\frac{(\varrho - 1)^2 (\varrho^5 + 5\varrho^4 + 11\varrho^3 + 9\varrho^2 - 6)}{8(\varrho + 1)^3},$$

and hence that it is indeed positive.

Accordingly, we have proved that

$$K + \frac{1}{K} \leqq \min F_1 = 2\sqrt{a_1 b_1},$$

and hence that

$$K \leqq \sqrt{a_1 b_1} + \sqrt{a_1 b_1 - 1}.$$

We prefer to replace this complicated result by the simpler inequality $K \leq \varrho^2$ announced in the theorem. To obtain it, it is sufficient to show that

$$4\, a_1 b_1 \leqq \left(\varrho^2 + \frac{1}{\varrho^2}\right)^2.$$

Explicitly, this inequality reads

$$(\varrho - 1)(3\varrho^7 + \varrho^6 + 8\varrho^5 + 12\varrho^4 - 4\varrho - 4) \geqq 0,$$

and it is obviously satisfied for all $\varrho \geqq 1$.

7. Absolute Continuity

7.1. It has been an open question whether the boundary correspondence induced by a quasiconformal mapping is always given by an absolutely continuous function μ. Our Theorem 1 reduces this question to one that deals only with monotone functions of a real variable. The answer is in the negative, and even the following stronger statement is true:

THEOREM 3. *There exists a quasiconformal mapping of the half-plane on itself whose boundary correspondence is given by a completely singular function μ with $\varrho(\mu)$ arbitrary close to* 1.[1]

[1] We wish to acknowledge that Mr. Errett Bishop has also constructed a function which satisfies a ϱ-condition without being absolutely continuous. The construction that we shall use is essentially different from his.

From a function-theoretic vantage point the most striking consequence of this theorem is that the distinction between sets of zero and positive harmonic measure is not preserved under quasiconformal mappings.

7.2. Let $\varrho > 1$ be given. We choose an increasing sequence of numbers ϱ_ν, $1 < \varrho_\nu < \varrho$, and a fixed number λ, $0 < \lambda < (\varrho_1 - 1)/(\varrho_1 + 1)$.

We are going to construct an infinite sequence of integers $0 < n_1 < n_2 < \cdots$ with the property that the functions $\mu_\nu(x)$, defined by

$$\mu_\nu(x) = \int_0^x \prod_{i=1}^{\nu} (1 + \lambda \cos n_i x)\, dx,$$

satisfy $\varrho(\mu_\nu) \leq \varrho_\nu$ and converge to a singular function $\mu(x)$ with $\varrho(\mu) \leq \varrho$.

To simplify the notation we shall use the same symbols μ_ν and μ for the corresponding interval and set functions. Since

$$\mu_{\nu+1}(x) = \int_0^x (1 + \lambda \cos n_{\nu+1} x)\, \mu_\nu'(x)\, dx$$

we have

$$1 - \lambda \leq \frac{\mu_{\nu+1}(\omega)}{\mu_\nu(\omega)} \leq 1 + \lambda$$

for any interval ω. For any pair of intervals ω, ω' we obtain

$$\frac{1-\lambda}{1+\lambda} \leq \frac{\mu_{\nu+1}(\omega)}{\mu_{\nu+1}(\omega')} : \frac{\mu_\nu(\omega)}{\mu_\nu(\omega')} \leq \frac{1+\lambda}{1-\lambda}. \tag{21}$$

7.3. We can choose an arbitrary $n_1 > 0$, for it is obvious that $\varrho(\mu_1) \leq (1 + \lambda)/(1 - \lambda) < \varrho_1$. Suppose that $n_1, \ldots, n_\nu$ have been determined. If

$$n_{\nu+1} > N_\nu = \sum_{i=1}^{\nu} n_i,$$

then

$$\int_0^{2\pi} \mu_\nu'(x) \cos n_{\nu+1} x\, dx = 0.$$

For this reason, and by Riemann–Lebesgue's Lemma,

$$\int_0^x \mu_\nu'(x) \cos n_{\nu+1} x\, dx \to 0$$

when $n_{\nu+1} \to \infty$, uniformly for all x. Hence $\mu_{\nu+1}$ tends uniformly to μ_ν.

Let ω and ω' denote two neighboring intervals $(x - t, x)$ and $(x, x + t)$ of length t. Since $\mu_\nu'(x)$ is uniformly continuous we can find $\eta_\nu > 0$ so that

$$\frac{1+\lambda}{1-\lambda}\frac{1}{\varrho_{\nu+1}} \leq \frac{\mu_\nu(\omega)}{\mu_\nu(\omega')} \leq \frac{1-\lambda}{1+\lambda}\varrho_{\nu+1}$$

whenever $t < \eta_\nu$. Together with (21) it follows that

$$\frac{1}{\varrho_{\nu+1}} \leq \frac{\mu_{\nu+1}(\omega)}{\mu_{\nu+1}(\omega')} \leq \varrho_{\nu+1} \tag{22}$$

under the same condition on t.

Because $\mu_{\nu+1}$ tends uniformly to μ_ν, and because $\varrho(\mu_\nu) \leq \varrho_\nu < \varrho_{\nu+1}$, we can choose $n_{\nu+1}$ so that (22) is also true for $t \geq \eta_\nu$. In other words, we can choose $n_{\nu+1}$ so that $\varrho(\mu_{\nu+1}) \leq \varrho_{\nu+1}$. In addition, we can and will make sure that

$$|\mu_{\nu+1}(x) - \mu_\nu(x)| < \frac{1}{\nu^2 N_\nu} \tag{23}$$

for all x.

7.4. For $\nu \to \infty$ the sequence $\{\mu_\nu\}$ converges uniformly to a non-decreasing limit function μ. It is clear that $\mu(0) = 0$, $\mu(x+2\pi) = \mu(x) + 2\pi$, and $\varrho(\mu) \leq \varrho$. We claim that μ is purely singular.

To see this we write

$$g(x) = \log(1 + \lambda \cos x) = \sum_{-\infty}^{\infty} \gamma_k e^{ikx},$$

where
$$\gamma_0 = \frac{1}{2\pi}\int_0^{2\pi} \log(1 + \lambda\cos x)\,dx = -a < 0$$

and $\sum_{-\infty}^{\infty} |\gamma_k| < \infty$. We determine q so that

$$\sum_{|k|>q} |\gamma_k| < \frac{a}{2}.$$

Then
$$g(x) < -\frac{a}{2} + \sum_{1 \leq |k| \leq q} \gamma_k e^{ikx}$$

and
$$\log \mu_\nu'(x) = \sum_{j=1}^{\nu} g(n_j x) < -\frac{\nu a}{2} + \sum_{1 \leq |k| \leq q} \left(\sum_{j=1}^{\nu} \gamma_k e^{ikn_j x}\right) \tag{24}$$
$$= -\frac{\nu a}{2} + \phi_\nu(x).$$

The Fourier expansion of ϕ_ν contains at most $2q\nu$ different powers, and in each term the coefficient is

$$< S = \sum_{-\infty}^{\infty} |\gamma_k|.$$

Hence
$$\frac{1}{2\pi}\int_0^{2\pi} \phi_\nu^2\, dx \leqq 2\, q\, \nu\, S^2.$$

As a result the subset of $(0, 2\pi)$ on which $\phi_\nu > a\nu/4$ is of measure $< c/\nu$, where c is a constant that depends only on λ. Because of (24) this set contains the set E_ν defined by the inequality

$$\mu_\nu'(x) > e^{-\frac{a\nu}{4}}.$$

Consequently the measure $m(E_\nu)$ of the latter set tends to 0, while on the other hand $\mu(E_\nu) \to 2\pi$.

We make use of the fact that $\mu_\nu'(x)$ is a trigonometric polynomial of degree $\leqq N_\nu = \sum_1^\nu n_i$. For this reason E_ν, considered on the unit circle, consists of at most N_ν arcs. From (23) we obtain $|\mu(x) - \mu_\nu(x)| < 1/(\nu - 1)N_\nu$, and we conclude that $|\mu(E_\nu) - \mu_\nu(E_\nu)| < 1/(\nu - 1)$. Hence $\mu(E_\nu) \to 2\pi$, and since $m(E_\nu) \to 0$ it follows that μ is purely singular on $(0, 2\pi)$, and consequently on the whole real axis.