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THE BOUNDARY LAYER NATURE OF SHOCK TRANSITION IN A REAL FLUID*

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Foreword. The usual way to deal with the so-called shoch phenomenon in compressible fluids is the following. On the one hand there is the fact that in many cases of observable flow there exist narrow zones across which pressure, density and velocity undergo rapid changes. On the other hand, it is well-known that the differential equations of inviscid perfect fluids fail to supply solutions satisfying certain boundary conditions that can be realized physically. One therefore makes the assumption that these differential equations are valid in regions of the (x, y, z, t) space which are separated from each other by discontinuity surfaces whose shape is a priori unknown. From physically plausible hypotheses one then derives *necessary* conditions for the values assumed by the physical variables on either side of the discontinuity surfaces. Such conditions were first given by Riemann, and later modified by Rankine and Hugoniot. It is finally assumed—and confirmed at least in special cases—that the differential equations combined with these transition conditions are sufficient to determine both the discontinuity surfaces themselves and the continuous flows in the regions between them (see [4], pp. 116-118, 134-138).

A different approach, as suggested by R. von Mises to the author, is followed in the present paper. The sole basis is formed by the system of partial differential equations (Navier-Stokes equations) which govern the motion of a viscous, heat-conducting, compressible fluid. No additional assumptions of any kind are introduced. It is *proved* that the integrals of these equations include a class of solutions of the boundary layer type; that is to say, solutions which asymptotically converge (with vanishing viscosity) towards flow patterns entirely different from those which are obtained when from the start viscosity is neglected. For a small viscosity coefficient μ, these flows have rapid changes of the physical variables across certain narrow regions, the widths of which converge to zero as $\mu \to 0$. In the limit the values of the variables on the two sides of the transition are subject to equations which are identical with the Rankine-Hugoniot conditions. These conditions, obtained here without any hypotheses, are thus proved to be not only necessary but also *sufficient* for the existence of shocks.

1. Introduction. In a recent paper [6], R. von Mises has discussed the occurrence of a shock in the one-dimensional steady flow of a real fluid. Here "real" will be used to denote "heat conducting, viscous and compressible." If we write his fundamental equa-

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tions (1) through (5) in a dimension-free form by referring ρ, u, x, p to standard values ρ_0, u_0, d, $\rho_0 u_0^2$, we obtain:

$$\rho u = m, \tag{1.1}$$

$$mu + p - \sigma = C_1 m, \tag{1.2}$$

$$m\left(\frac{u^2}{2} + \frac{T}{\kappa - 1}\right) + u(p - \sigma) - H = C_2 m, \tag{1.3}$$

$$p = \rho T, \tag{1.4}$$

with

$$\sigma = \frac{4}{3R}\frac{du}{dx}, \qquad H = \frac{\kappa}{\kappa - 1}\cdot\frac{1}{PR}\frac{dT}{dx}, \tag{1.5}$$

where T is referred to $u_0^2/\overline{R}$ ($\overline{R}$ is the gR of [6]), and we use the same symbols to denote the non-dimensional quantities, but change the $\dot{\gamma}$ of Reference 6 to κ for convenience. Here R is the Reynolds number,

$$R = \frac{\rho_0 u_0 d}{\mu}, \tag{1.6}$$

and P the Prandtl number,

$$P = \frac{\kappa}{\kappa - 1}\overline{R}\frac{\mu}{k}. \tag{1.7}$$

These equations are based on the assumption that the fluid is a perfect gas (1.4) with constant specific heats. Further, von Mises considers that P, μ, k may vary. For convenience we shall also assume μ, k and therefore P are constant, though the results are equally true without this restriction. Finally, if we write

$$x = \frac{s}{R}, \tag{1.8}$$

the equations (1.5) become

$$\sigma = \frac{4}{3}\frac{du}{ds}, \qquad H = \frac{\kappa}{\kappa - 1}\frac{1}{P}\frac{dT}{ds}, \tag{1.5$'$}$$

and the set of equations (1.1) through (1.4) and (1.5′) is now independent of R.

In order to discuss the boundary layer nature of shock transition, we recall the assumptions of the Prandtl boundary layer theory, according to which the Navier-Stokes equations are reduced to a new system of "boundary layer" equations (see [2]). If we denote by (v_1, v_2) the components of the velocity along the normals and parallels to a boundary line S in two-dimensional steady flow, and by $\partial/\partial\alpha$, $\partial/\partial\beta$ differentiations along these directions, then symbolically

$$\frac{\partial}{\partial\alpha} = O(R^{1/2}); \qquad \frac{\partial}{\partial\beta} = O(1), \tag{1.9}$$

$$v_1 = O(R^{-1/2}); \qquad v_2 = O(1), \tag{1.10}$$

and other physical quantities are $O(1)$. Thus we stretch the normal distance by a factor $R^{1/2}$, magnify v_1, the normal component of velocity, by $R^{1/2}$, and then consider that all new quantities and new derivatives have the same order of magnitude $O(1)$. Neglecting all but the terms of highest order in our equations, we obtain the reduced set of "boundary layer" equations. A solution of the latter equations constitutes an asymptotic integration of the Navier-Stokes equations. Now consider shock transition at S. Here, in contrast to (1.9), (1.10), we assume

$$\frac{\partial}{\partial\alpha} = O(R); \qquad \frac{\partial}{\partial\beta} = O(1), \tag{1.9$'$}$$

$$v_1 = O(1); \qquad v_2 = O(1), \tag{1.10$'$}$$

where these are suggested in the following manner. If we superpose a constant velocity $v_2 = O(1)$ on the motion discussed by von Mises, then we obtain a two-dimensional steady motion with S a straight line. For this motion, (1.9′) and (1.10′) clearly hold, in view of (1.5). Thus, in the general case, we stretch the normal distance by a factor R and then consider all new quantities and their new derivatives to have the same order of magnitude $O(1)$. Neglecting all but the terms of highest order in our equations, we obtain the reduced set of "shock transition" equations, a solution of which constitutes an asymptotic integration of the Navier-Stokes equations. These "shock transition" equations are found to be completely analogous to the system (1.1) through (1.4) and (1.5′). Hence the existence of "shock transition" regions in a real fluid can be inferred; further, the fact that as $R \to \infty$ these regions go over into discontinuity surfaces for which the Rankine-Hugoniot conditions of inviscid flow apply, can be shown.

Having discussed the points on which the boundary layer and the shock transition are analogous, we must next note two important differences. First, compressibility is an essential for the latter, but not for the former: shocks do not occur in an incompressible fluid. Second, time-variation of the motion produces essential changes in shock-transition, but not in the boundary layer. In the former, S has its own motion; in the latter, S is fixed. Hence the purpose of the paper is formulated precisely as follows: Given an arbitrary hypersurface S (which satisfies loose regularity conditions), a V^3 in (t, x, y, z) space it is possible to solve asymptotically the Navier-Stokes equations so that physical variables undergo rapid changes normal to S, and relatively slow changes parallel to S, in the neighborhood of S, provided that R is large enough. Moreover, in the limit $R \to \infty$, S becomes a discontinuity surface across which the Rankine-Hugoniot conditions are fulfilled. To make the discussion clearer, we shall work through two simpler cases first, one of steady motion and the other of unsteady.

Finally, we shall use the fact (inherent in von Mises' discussion) that du/ds, d^2u/ds^2, dT/ds, $d^2T/ds^2 \cdots$ are of the same order as u, $T \cdots$ in some fixed interval $[s]$, depending only on m, C_1, C_2, of s, and that the end points of this interval correspond very nearly to the Rankine-Hugoniot conditions which strictly hold for $s = \mp\infty$. This is equivalent to the result that the "thickness" of the shock is extremely small in von Mises' case, and that outside conditions are very nearly uniform.

2. Steady two-dimensional flow. The momentum equation in vector form may be written non-dimensionally:

$$\rho\left(\boldsymbol{\omega} \times \mathbf{V} + \operatorname{grad}\frac{1}{2}V^2\right) = -\operatorname{grad} p + \frac{4}{3R}\operatorname{grad}\vartheta - \frac{1}{R}\operatorname{curl}\boldsymbol{\omega} \tag{2.1}$$

where the vorticity $\boldsymbol{\omega}$ and the dilatation ϑ are given by

$$\boldsymbol{\omega} = \text{curl } \mathbf{V}, \qquad \vartheta = \text{div } \mathbf{V} \tag{2.2}$$

([1]—true also for compressible and heat-conducting fluid). In (2.1) R is the Reynold's number and the other quantities have their usual meaning. The equation of continuity may likewise be written:

$$\text{div } \rho\mathbf{V} = 0 \tag{2.3}$$

and the equation of state:

$$p = \rho T. \tag{2.4}$$

The energy equation is not commonly derived under such general conditions but a short computation yields (appendix A):

$$\rho\mathbf{V} \cdot \text{grad}\left(\frac{1}{2} V^2 + \frac{T}{\kappa - 1}\right) = \frac{\kappa - 1}{\kappa}\frac{1}{PR} \nabla^2 T - \text{div}\left[\left(p + \frac{2}{3R}\vartheta\right)\mathbf{V}\right] + \frac{1}{R}\nabla^2 V^2 - \frac{1}{R}\text{div }(\mathbf{V} \times \boldsymbol{\omega}), \tag{2.5}$$

where P is the Prandtl number (assumed constant) and the dilatation ϑ is given in (2.2) together with the vorticity $\boldsymbol{\omega}$.

These are the equations of motion and we wish to refer them to a particular set of orthogonal curvilinear coordinates having coordinate lines normal and parallel to the

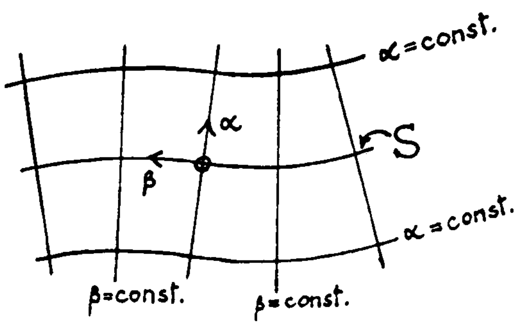

Fig. 1.

given line S, (Fig. 1). This system of coordinates was used by von Mises [2] to obtain the boundary layer equations for an arbitrary curved boundary line S. The square of the line element in these coordinates may be written

$$ds^2 = d\alpha^2 + \left(\frac{r + \alpha}{r}\right)^2 d\beta^2, \tag{2.6}$$

where β is the arc length on S measured from an arbitrary fixed point, $r = r(\beta)$ is the radius of curvature of S, and α is the normal distance from S. Hence if we write the components of velocity referred to these axes as (v_1, v_2) we have, instead of (2.1) and (2.2),

$$\rho\left\{v_1 \frac{\partial v_1}{\partial \alpha} + \frac{r}{r+\alpha} v_2 \frac{\partial v_1}{\partial \beta} - \frac{v_2^2}{r+\alpha}\right\} = -\frac{\partial p}{\partial \alpha} + \frac{4}{3R}\frac{\partial \vartheta}{\partial \alpha} - \frac{1}{R}\frac{r}{r+\alpha}\frac{\partial \omega}{\partial \beta}, \tag{2.7}$$

$$\rho\left\{v_1 \frac{\partial v_2}{\partial \alpha} + \frac{r}{r+\alpha} v_2 \frac{\partial v_2}{\partial \beta} + \frac{v_1 v_2}{r+\alpha}\right\} = -\frac{r}{r+\alpha}\frac{\partial p}{\partial \beta} + \frac{4}{3R}\frac{r}{r+\alpha}\frac{\partial \vartheta}{\partial \beta} + \frac{1}{R}\frac{\partial \omega}{\partial \alpha}, \tag{2.8}$$

$$\begin{aligned} \vartheta &= \frac{r}{r+\alpha}\left\{\frac{\partial}{\partial \alpha}\left(\frac{r+\alpha}{r} v_1\right) + \frac{\partial}{\partial \beta}(v_2)\right\}, \\ \omega &= \frac{r}{r+\alpha}\left\{\frac{\partial}{\partial \alpha}\left(\frac{r+\alpha}{r} v_2\right) - \frac{\partial}{\partial \beta}(v_1)\right\}, \end{aligned} \tag{2.9}$$

and instead of the continuity equation (2.3),

$$\frac{\partial}{\partial \alpha}\left(\frac{r+\alpha}{r}\rho v_1\right) + \frac{\partial}{\partial \beta}(\rho v_2) = 0 \tag{2.10}$$

The values of the various terms in the energy equation (2.5) may be obtained from

$$\mathbf{V}\cdot \operatorname{grad}\left(\frac{1}{2}V^2 + \frac{T}{\kappa-1}\right) = \left(v_1 \frac{\partial}{\partial \alpha} + v_2 \frac{r}{r+\alpha}\frac{\partial}{\partial \beta}\right)\left(\frac{1}{2}V^2 + \frac{T}{\kappa-1}\right) \tag{2.11a}$$

$$\nabla^2 = \frac{r}{r+\alpha}\left\{\frac{\partial}{\partial \alpha}\left(\frac{r+\alpha}{r}\frac{\partial}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{r}{r+\alpha}\frac{\partial}{\partial \beta}\right)\right\} \tag{2.11b}$$

$$\mathbf{V}\times \boldsymbol{\omega} = (v_2\omega, -v_1\omega) \tag{2.11c}$$

with use of (2.9), i.e. we remember, for instance, that the expression for ϑ gives us the divergence of a vector $\mathbf{V}$.

In agreement with the ideas of Sec. 1, we now replace α by s/R and consider in our new system of coordinates that all functions and their derivatives are of the same order of magnitude. Then considering only the highest order terms in $1/R$, we may write the previous equations (2.7) through (2.10) as:

$$\rho v_1 \frac{\partial v_1}{\partial s} = -\frac{\partial p}{\partial s} + \frac{4}{3}\frac{\partial}{\partial s}\left(\frac{\vartheta}{R}\right) \tag{2.7$'$}$$

$$\rho v_1 \frac{\partial v_2}{\partial s} = \frac{\partial}{\partial s}\left(\frac{\omega}{R}\right) \tag{2.8$'$}$$

$$\frac{\vartheta}{R} = \frac{\partial v_1}{\partial s}, \qquad \frac{\omega}{R} = \frac{\partial v_2}{\partial s} \tag{2.9$'$}$$

$$\frac{\partial}{\partial s}(\rho v_1) = 0 \tag{2.10$'$}$$

and the energy equation, from (2.11a) through (2.11c), as

$$\begin{aligned} \rho v_1 \frac{\partial}{\partial s}\left(\frac{1}{2}V^2 + \frac{T}{\kappa-1}\right) = \frac{\kappa-1}{\kappa}\cdot\frac{1}{P}\frac{\partial^2 T}{\partial s^2} - \frac{\partial}{\partial s}\left[\left(p + \frac{2\vartheta}{3R}\right)v_1\right] \\ + \frac{\partial^2}{\partial s^2}(V^2) - \frac{\partial}{\partial s}\left(v_2\frac{\omega}{R}\right) \end{aligned} \tag{2.11$'$}$$

It is clear immediately that (2.8′) with (2.9′) is satisfied by

$$v_2 = v_2(\beta) \tag{2.8''}$$

which is the mathematical expression of the fact that the component of velocity parallel to the line S remains unaltered along a direction prependicular to S. The equations (2.7′), (2.10′) and (2.11′), with use of (2.9′) may be immediately integrated to give

$$\rho v_1 = m(\beta), \tag{2.10''}$$

$$m v_1 + p - \frac{4}{3}\frac{\partial v_1}{\partial s} = c_1(\beta), \tag{2.7''}$$

$$m\left[\frac{1}{2}V^2 + \frac{T}{\kappa - 1}\right] + v_1\left(p - \frac{4}{3}\frac{\partial v_1}{\partial s}\right) - \frac{\kappa - 1}{\kappa}\frac{1}{P}\frac{\partial T}{\partial s} = c_2(\beta), \tag{2.11''}$$

where we have made use of (2.8″). The equations (2.4), (2.7″), (2.10″) and (2.11″) are equivalent to (1.1) through (1.4) and (1.5′), with v_1 for u and c_1 , c_2 for mC_1 , mC_2 .

These equations have been obtained under the sole assumptions that (i) R is large,

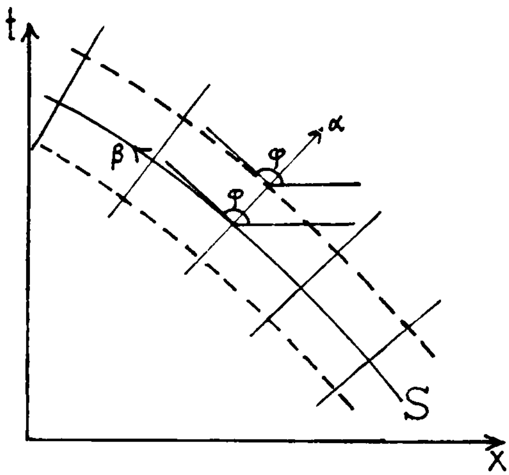

FIG. 2.

and (ii) changes along the shock are of the same order as $\partial u/\partial s$, $\partial^2 u/\partial s^2$, $\partial^2 T/\partial s^2$, etc. Thus they are accurate for the interval [s] of section 1. Hence the existence of a shock is demonstrated. Letting $R \to \infty$, the equations become *exact* without assumption (ii) and [s] may (in this case alone) be extended to

$$-\infty \leq s \leq +\infty \tag{2.12}$$

Thus S becomes a discontinuity and the Rankine-Hugoniot conditions hold.

3. Unsteady one-dimensional flow. The equations governing the motion are here, in non-dimensional form,

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p - \sigma) = 0, \tag{3.1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0, \tag{3.2}$$

$$\frac{\partial}{\partial t}\left\{\rho\left(\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right)\right\} + \frac{\partial}{\partial x}\left\{\rho u\left(\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right) + H + (\sigma - p)u\right\} = 0 \tag{3.3}$$

with

$$\sigma = \frac{4}{3R}\frac{\partial u}{\partial x}, \qquad H = \frac{\kappa - 1}{\kappa}\frac{1}{PR}\frac{\partial T}{\partial x} \tag{3.3a}$$

and the equation of state is

$$p = \rho T. \tag{3.4}$$

Our given surface S reduces to a line in the (x, t) plane and we set up the same system of coordinates as in the previous section (Fig. 2). Thus α = const. represents a motion fixed relative to the "shock" S, and β = const. represents motion directly across S. Now equations (3.1) through (3.3a) may be written as

$$\frac{\partial A_1^i}{\partial \alpha}\sin\varphi - \frac{\partial A_2^i}{\partial \alpha}\cos\varphi + \frac{r}{r + \alpha}\left(\frac{\partial A_1^i}{\partial \beta}\cos\varphi + \frac{\partial A_2^i}{\partial \beta}\sin\varphi\right) = 0, \qquad i = 1, 2, 3 \tag{3.5}$$

$$\begin{aligned} A_1^1 &= \rho u^2 + p - \sigma, & A_2^1 &= \rho u, \\ A_1^2 &= \rho u, & A_2^2 &= \rho, \\ A_1^3 &= \rho u\left(\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right) + H + (\sigma - p)u, & A_2^3 &= \rho\left(\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right), \end{aligned} \tag{3.5a}$$

and

$$\begin{aligned} \sigma &= \frac{4}{3R}\left\{\frac{\partial u}{\partial \alpha}\sin\varphi + \frac{r}{r + \alpha}\frac{\partial u}{\partial \beta}\cos\varphi\right\}, \\ H &= \frac{(\kappa - 1)}{\kappa}\frac{1}{PR}\left\{\frac{\partial T}{\partial \alpha}\sin\varphi + \frac{r}{r + \alpha}\frac{\partial T}{\partial \beta}\cos\varphi\right\}, \end{aligned} \tag{3.6}$$

where we have used the rules for differentiation under change of variables. We note that

$$\varphi = \varphi(\beta) \tag{3.7}$$

We now replace α by s/R and consider that in the new system of coordinates all functions and their derivatives are of the same order of magnitude. Then considering only the highest order terms in $1/R$, we may write (3.5) and (3.6) in the form:

$$\frac{\partial}{\partial s}(A_1^i - A_2^i \cot\varphi) = 0, \qquad i = 1, 2, 3. \tag{3.5$'$}$$

and

$$\sigma = \frac{4}{3}\sin\varphi\frac{\partial u}{\partial s}, \qquad H = \frac{(\kappa - 1)}{\kappa}\frac{1}{PR}\sin\varphi\frac{\partial T}{\partial s}\cdot \tag{3.6$'$}$$

If the velocity $\cot \varphi$ of the shock is written as c, the equations (3.5′) reduce to

$$mu + p - \sigma = c_1(\beta), \qquad \rho(u - c) = m(\beta),$$
$$m\left[\frac{1}{2}u^2 + \frac{T}{\kappa - 1}\right] - H + (p - \sigma)u = c_2(\beta). \tag{3.5''}$$

The equations (3.5″) and (3.6′) are equivalent to those of section 1 since (3.7) holds. Thus the discussion at the end of section 2 may be repeated.

4. Unsteady three-dimensional flow. In this, the most general case, the situation is a little more complicated. The differences between the present analysis and the preceding will be noted at each stage. First, the momentum equations in (t, x, y, z) space have no simple "vector" form which allows of simple transformation to new axes, as with (2.1) in (x, y) space. In fact, they may probably be dealt with best in the form:

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \operatorname{grad} \mathbf{V}\right) = -\operatorname{grad} p + \frac{1}{3R} \operatorname{grad} \vartheta + \frac{1}{R} \nabla^2 \mathbf{V} \tag{4.1}$$

with

$$\vartheta = \operatorname{div} \mathbf{V}, \qquad \mathbf{V} = (u, v, w). \tag{4.1a}$$

The equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{V} = 0 \tag{4.2}$$

likewise has the same drawback. The equation of state

$$p = \rho T, \tag{4.3}$$

and the energy equation (Appendix A)

$$\frac{\partial I}{\partial t} + \operatorname{div} (I\mathbf{V}) = \frac{\kappa - 1}{\kappa} \frac{1}{PR} \nabla^2 T - \operatorname{div} \left[\left(p + \frac{2}{3R} \vartheta\right)\mathbf{V}\right] + \frac{1}{2R} \nabla^2 \mathbf{V}^2 + \frac{1}{R} \operatorname{div} (\mathbf{V} \cdot \operatorname{grad} \mathbf{V}) \tag{4.4}$$

with

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{V}, \qquad I = \rho\left(\frac{1}{2} V^2 + \frac{T}{\kappa - 1}\right) \tag{4.5}$$

complete the system of equations governing the motion. The energy equation also is not of vector form.

Suppose that the Euclidean metric

$$ds^2 = dt^2 + dx^2 + dy^2 + dz^2 \tag{4.6}$$

is imposed on the (t, x, y, z) space. Then it is possible to set up a system of orthogonal curvilinear coordinates $(\alpha, \beta, \gamma, \delta)$ such that (4.6) reads:

$$ds^2 = d\alpha^2 + (h_1 + \lambda_1\alpha)^2 \, d\beta^2 + (h_2 + \lambda_2\alpha)^2 \, d\gamma^2 + (h_3 + \lambda_3\alpha)^2 \, d\delta^2 \tag{4.7}$$

where the lines $\beta, \gamma, \delta =$ const. are normal to the given manifold S (here a V^3) and the

3-spaces $\alpha =$ const. are parallel to S (see Appendix A). Moreover h_1, h_2, h_3, λ_1, λ_2, λ_3, are functions of β, γ, δ alone, and

$$ds^2 = h_1^2\, d\beta^2 + h_2^2\, d\gamma^2 + h_3^2\, d\delta^2 \tag{4.8}$$

is the metric of S. To illustrate the nature of this system, we note that in the two-dimensional case of unsteady flow the lines (β, γ const.) are normal to a surface S whose lines of curvature are $\alpha = 0$, $\beta =$ const. and $\alpha = 0$, $\gamma =$ const. Also in this case h_1/λ_1, h_2/λ_2 are the principal radii of curvature of S, and are functions of (β, γ).

We now wish to introduce a notation for vectors A_i in our 4-space. We shall denote by (A_1, A_2, A_3, A_4)—round brackets—the components of A_i referred to the (t, x, y, z) axes, and by $\{A_1', A_2', A_3', A_4'\}$—curled brackets—its components referred to the $(\alpha, \beta, \gamma, \delta)$ axes. Moreover we shall denote by div* the divergence i.e.

$$\text{div}^*\, A_i = \frac{\partial A_1}{\partial t} + \frac{\partial A_2}{\partial x} + \frac{\partial A_3}{\partial y} + \frac{\partial A_4}{\partial z} \tag{4.9}$$

and similarly the gradient by grad*. Finally we shall suppose that the relative orientation of the two sets of axes is given by the scheme

$$\begin{array}{c|cccc} & t & x & y & z \\ \hline \alpha & k_1 & l_2 & m_3 & n_4 \\ \beta & k_1 & l_2 & m_3 & n_4 \\ \gamma & k_1 & l_2 & m_3 & n_4 \\ \delta & k_1 & l_2 & m_3 & n_4 \end{array} \tag{4.10}$$

in which all elements are independent of α.

Consider now the first of the momentum equations (4.1). We have

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathbf{V} \cdot \text{grad}\, u &= (l, u, v, w) \cdot \text{grad}^*\, u \\ &= \{v_1, v_2, v_3, v_4\} \cdot \left\{\frac{\partial u}{\partial \alpha}, \frac{1}{h_1 + \lambda_1\alpha}\frac{\partial u}{\partial \beta}, \frac{1}{h_2 + \lambda_2\alpha}\frac{\partial u}{\partial \gamma}, \frac{1}{h_3 + \lambda_3\alpha}\frac{\partial u}{\partial \delta}\right\}, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} v_1 &= k_1 + l_1u + m_1v + n_1w, & v_2 &= k_2 + l_2u + m_2v + n_2w, \\ v_3 &= k_3 + l_3u + m_3v + n_3w, & v_4 &= k_4 + l_4u + m_4v + n_4w. \end{aligned} \tag{4.11a}$$

As in the previous sections we now replace α by s/R and (considering all functions and derivatives to be of the same order of magnitude in the new coordinates) retain only the lowest powers of $1/R$. Thus:

$$\frac{\partial u}{\partial t} + \mathbf{V} \cdot \text{grad}\, u = Rv_1 \frac{\partial u}{\partial s} \tag{4.12}$$

Also, to the same order,

$$\frac{\partial p}{\partial x} = Rl_1 \frac{\partial p}{\partial s}, \qquad \frac{\partial \vartheta}{\partial x} = Rl_1 \frac{\partial \vartheta}{\partial s}. \tag{4.13}$$

Further we have

$$\operatorname{div}^* \left(0, \frac{\partial u}{\partial x}, 0, 0\right) = \operatorname{div}^* \{l_1 U, l_2 U, l_3 U, l_4 U\}$$

with

$$U = l_1 \frac{\partial u}{\partial \alpha} + \frac{l_2}{h_1 + \lambda_1 \alpha} \frac{\partial u}{\partial \beta} + \frac{l_3}{h_2 + \lambda_2 \alpha} \frac{\partial u}{\partial \gamma} + \frac{l_4}{h_3 + \lambda_3 \alpha} \frac{\partial u}{\partial \delta},$$

so that

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{1}{(h_1 + \lambda_1 \alpha)(h_2 + \lambda_2 \alpha)(h_3 + \lambda_3 \alpha)} \left\{ \frac{\partial}{\partial \alpha} [(h_1 + \lambda_1 \alpha)(h_2 + \lambda_2 \alpha)(h_3 + \lambda_3 \alpha) l_1 U] \right. \\ &\qquad \left. + \frac{\partial}{\partial \beta} [(h_2 + \lambda_2 \alpha)(h_3 + \lambda_3 \alpha) l_2 U] + \cdots + \cdots \right\} \\ &= R^2 l_1^2 \frac{\partial^2 u}{\partial s^2}, \end{aligned} \tag{4.14}$$

to the highest order. We thus have from (4.14)

$$\nabla^2 u = R^2(1 - k_1^2) \frac{\partial^2 u}{\partial s^2} \tag{4.15}$$

To complete the discussion of (4.1) we have from (4.1a):

$$\begin{aligned} \vartheta &= \operatorname{div}^* (0, u, v, w) = \operatorname{div}^* \{v_1 - k_1, v_2 - k_2, v_3 - k_3, v_4 - k_4\} \\ &= R \frac{\partial}{\partial s} (v_1 - k_1) \end{aligned} \tag{4.16}$$

The x momentum equation, in $(\alpha, \beta, \gamma, \delta)$ coordinates, is thus,

$$\rho v_1 \frac{\partial u}{\partial s} = -l_1 \frac{\partial p}{\partial s} + \frac{1}{3} l_1 \frac{\partial^2}{\partial s^2} (v_1 - k_1) + (1 - k_1^2) \frac{\partial^2 u}{\partial s^2}. \tag{4.17a}$$

The y and z momentum equations are obtained in a similar manner: we merely change u to v and w, and l_1 to m_1 and n_1, respectively, to form two new equations (4.17b) and (4.17c). Unlike the case before, these equations are not component equations referred to the new system $(\alpha, \beta, \gamma, \delta)$ but are referred to the old system in new system terms.

The continuity equation (4.2) can be written as

$$0 = \operatorname{div}^* (\rho, \rho u, \rho v, \rho w) = \operatorname{div}^* \{\rho v_1, \rho v_2, \rho v_3, \rho v_4\}$$

and this becomes

$$\frac{\partial}{\partial s} (\rho v_1) = 0. \tag{4.18}$$

From the preceding discussion we may list immediately the following reductions of the terms of the energy equation (4.4)

$$\frac{\partial I}{\partial t} + \text{div }(I\mathbf{V}) = R\frac{\partial}{\partial s}(Iv_1), \tag{4.19a}$$

$$\nabla^2 T = R^2(1 - k_1^2)\frac{\partial^2 T}{\partial s^2}, \tag{4.19b}$$

$$\nabla^2(V^2) = R^2(1 - k_1^2)\frac{\partial^2 (V^2)}{\partial s^2}, \tag{4.19c}$$

$$\text{div}\left(p + \frac{2\vartheta}{3R}\right)\mathbf{V} = R\frac{\partial}{\partial s}\left[\left(p + \frac{2\vartheta}{3R}\right)(v_1 - k_1)\right]. \tag{4.19d}$$

Moreover similarly to (4.12)

$$\mathbf{V} \cdot \text{grad }\mathbf{V} = R(v_1 - k_1)\frac{\partial \mathbf{V}}{\partial s}$$

so that similarly to (4.16)

$$\begin{aligned}\text{div }(\mathbf{V} \cdot \text{grad }\mathbf{V}) &= R^2\frac{\partial}{\partial s}\left\{(v_1 - k_1)\frac{\partial}{\partial s}(v_1 - k_1)\right\} \\ &= R^2\frac{\partial^2}{\partial s^2}\left\{\frac{1}{2}(v_1 - k_1)^2\right\}.\end{aligned} \tag{4.19e}$$

Using (4.19a) through (4.19e) and (4.16), we may replace (4.4) by

$$\begin{aligned}\frac{\partial}{\partial s}\left[\rho v_1\left(\frac{1}{2}V^2 + \frac{T}{\kappa - 1}\right)\right] &= \frac{\kappa - 1}{\kappa}\frac{1}{P}(1 - k_1^2)\frac{\partial^2 T}{\partial s^2} - \frac{\partial}{\partial s}[p(v_1 - k_1)] \\ &\quad + \frac{1}{2}(1 - k_1^2)\frac{\partial^2(V^2)}{\partial s^2} + \frac{1}{6}\frac{\partial^2}{\partial s^2}(v_1 - k_1)^2.\end{aligned} \tag{4.20}$$

Before discussing these equations, we must introduce simplifications. If we write

$$D^2 = l_1^2 + m_1^2 + n_1^2 = 1 - k_1^2$$

it is clear that $\mathbf{n} = (l_1/D, m_1/D, n_1/D)$ is the unit normal in (x, y, z) space to the instantaneous position of the "shock." Moreover $-k_1/D$ is the normal velocity of the "shock" in this space. Denoting then the normal component of velocity by u_n and the normal velocity of the "shock" by U i.e.

$$l_1 u + m_1 v + n_1 w = Du_n, \qquad -k_1 = DU. \tag{4.21}$$

We may rewrite (4.17a), (4.17b), (4.17c), (4.18) and (4.20) in the more familiar form:

$$\rho(u_n - U)\frac{\partial u}{\partial s} = \frac{-l_1}{D}\frac{\partial p}{\partial s} + \frac{1}{3}l_1\frac{\partial^2 u_n}{\partial s^2} + D\frac{\partial^2 u}{\partial s^2}, \tag{4.17'a}$$

$$\rho(u_n - U)\frac{\partial v}{\partial s} = \frac{-m_1}{D}\frac{\partial p}{\partial s} + \frac{1}{3}m_1\frac{\partial^2 u_n}{\partial s^2} + D\frac{\partial^2 v}{\partial s^2}, \tag{4.17'b}$$

$$\rho(u_n - U)\frac{\partial w}{\partial s} = \frac{-n_1}{D}\frac{\partial p}{\partial s} + \frac{1}{3}n_1\frac{\partial u_n^2}{\partial s^2} + D\frac{\partial^2 w}{\partial s^2}, \tag{4.17'c}$$

$$\frac{\partial}{\partial s}[\rho(u_n - U)] = 0 \tag{4.18'}$$

$$\frac{\partial}{\partial s}\left[\rho(u_n - U)\left(\frac{1}{2}V^2 + \frac{T}{\kappa - 1}\right)\right] = \frac{\kappa - 1}{\kappa}\frac{D}{P}\frac{\partial^2 T}{\partial s^2} - \frac{\partial(pu_n)}{\partial s} + \frac{D}{2}\frac{\partial^2(V^2)}{\partial s^2} + \frac{D}{6}\frac{\partial^2}{\partial s^2}(u_n^2) \tag{4.20'}$$

If we multiply (4.17′a) by l_1, (4.17′b) by m_1, and (4.17′c) by n_1 and add we obtain

$$\rho(u_n - U)\frac{\partial u_n}{\partial s} = -\frac{\partial p}{\partial s} + \frac{4D}{3}\frac{\partial^2(u_n)}{\partial s^2} \tag{4.21}$$

which may be integrated by virtue of (4.18′), i.e.

$$\rho(u_n - U) = m(\beta, \gamma, \delta), \tag{4.22}$$

to give

$$mu_n + p - \sigma = c_1(\beta, \gamma, \delta) \tag{4.23}$$

with

$$\sigma = \frac{4}{3}D\frac{\partial(u_n)}{\partial s} \tag{4.23a}$$

Furthermore if we multiply (4.17′a) by m_1, (4.17′b) by l_1 and subtract we have

$$m\frac{\partial}{\partial s}(m_1 u - l_1 v) = D\frac{\partial^2}{\partial s^2}(m_1 u - l_1 v)$$

which is satisfied if

$$\frac{1}{D}(m_1 u - l_1 v) = f(\beta, \gamma, \delta). \tag{4.24}$$

Similarly the other components of $\mathbf{V} \times \mathbf{n}$ are independent of α. Hence the component of $\mathbf{V}$ tangential to the instantaneous position of the shock in the (x, y, z) is independent of α. Thus we may write

$$\frac{\partial}{\partial s}(V^2) = \frac{\partial}{\partial s}(u_n^2) \tag{4.25}$$

and the energy equation (4.20′) can be written in integrated form:

$$m\left(\frac{1}{2}u_n^2 + \frac{T}{\kappa - 1}\right) - H + (p - \sigma)u_n = c_2(\beta, \gamma, \delta) \tag{4.26}$$

with

$$H \doteq \frac{\kappa - 1}{\kappa} \frac{D}{P} \frac{\partial T}{\partial s} \tag{4.26a}$$

The equations (4.3), (4.22), (4.23), (4.23a), (4.26) and (4.26a) are equivalent to those of section 1 with u_n for u, and definition of $m(\beta, \gamma, \delta)$ slightly changed, since $D = D(\beta, \gamma, \delta)$ from (4.10). Hence discussion at end of section 2 applies equally well here.

Summary. Given an arbitrary hypersurface S in (t, x, y, z) space and an arbitrary distribution (within the loose restriction of the constant c of von Mises' paper) of $\mathbf{V}$, T, ρ and derivatives on it, we have the following:

(*i*) It is possible to construct a family of solutions of the Navier-Stokes equations having large normal derivatives on S and (comparatively) small derivatives parallel to S, provided μ is small enough. The solution is analogous to a solution of the steady one-dimensional equations of motion, in which the x-differentiation is replaced by

$$\frac{\partial}{\partial n} - U \frac{\partial}{\partial t},$$

where U is the normal velocity of the "instantaneous" section of S by (x, y, z) space and $\partial/\partial n$ denotes differentiation along the normal to this section. Moreover the Rankine-Hugoniot conditions are approximated very nearly, and the component of $\mathbf{V}$ in the tangent plane to the "instantaneous" section of S by (x, y, z) space is conserved along the normal direction of S.

(*ii*) As $\mu \to 0$ the surface S becomes a discontinuity surface and the Rankine-Hugoniot conditions hold exactly.

The purpose of this paper, as stated at the beginning, was to prove the existence of shocks, using as sole basis the Navier-Stokes equations for the motion of a perfect fluid. As a by-product of the analysis, we are given a method of investigating the structure of a shock in the most general motion of a perfect fluid. However, if such an application is made, it must be borne in mind that it will only be valid when the Navier-Stokes equations hold, that is when the fluid can be considered as a continuum, in other words, for weak shocks.

Appendix A

If we denote an element of the stress-tensor of a viscous fluid by σ_{ij} and an element of the rate of strain tensor by e_{ij}, the generalized concept of viscosity leads to [3]

$$\sigma_{ij} = -(p + 2/3\, \mu\vartheta)\delta_{ij} + \mu e_{ij} \tag{A.1}$$

with

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \qquad \vartheta = \frac{1}{2} e_{ii} = \operatorname{div} \mathbf{V}. \tag{A.2}$$

Here we use dimensional quantities, with μ the coefficient of viscosity and $(x, y, z) = (x_1, x_2, x_3)$, $(u, v, w) = (u_1, u_2, u_3)$. Since the total energy [4] is

$$I = \rho\left(\frac{1}{2} V^2 + \frac{\bar{R}T}{\kappa - 1}\right), \tag{A.3}$$

the energy equation is easily seen to be

$$\rho \frac{D}{Dt}\left(\frac{1}{2} V^2 + \frac{\bar{R}T}{\kappa - 1}\right) = k\nabla^2 T + \frac{\partial(u_i\sigma_{ij})}{\partial x_j} \tag{A.4}$$

From (A.1) and (A.2) it follows, however, that

$$u_i\sigma_{ij} = -\left(p + \frac{2}{3}\mu\vartheta\right)u_j + \frac{\partial}{\partial x_j}\left(\frac{1}{2} V^2\right) + u_i \frac{\partial u_j}{\partial x_i} \cdot \tag{A.5}$$

Inserting this into the right hand side of (A.4) and using the equation of continuity (4.2) on the left hand side, we obtain the dimensional form of (4.4).

For one-dimensional unsteady flow, we have $\mathbf{V} = (u, 0, 0)$ and

$$-\frac{2}{3}\mu \operatorname{div}(\vartheta\mathbf{V}) + \frac{\mu}{2}\nabla^2\mathbf{V}^2 + \mu \operatorname{div}(\mathbf{V}\cdot\operatorname{grad}\mathbf{V}) = \frac{4}{3}\mu\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right), \tag{A.6}$$

and (4.2) reduces to (3.3).

For steady two dimensional flow $\partial/\partial t = 0$ and the transformations of vector analysis yield

$$-\frac{1}{2}\mu\nabla^2 V^2 + \mu \operatorname{div}(\mathbf{V}\cdot\operatorname{grad}\mathbf{V}) = \mu \operatorname{div}(\boldsymbol{\omega}\times\mathbf{V}). \tag{A.7}$$

Hence (A.7) reduces (4.2) to the form (2.5).

Appendix B

Let $\mathbf{r} = \mathbf{r}(\nu_1, \nu_2, \nu_3)$ be the position vector of the manifold S (V^3) in the (t, x, y, z)-space, where (ν_1, ν_2, ν_3) form an orthogonal system of coordinates, i.e.

$$\mathbf{r}_i\cdot\mathbf{r}_j = |\mathbf{r}_i|\,|\mathbf{r}_j|\,\delta_{ij} \tag{B.1}$$

with

$$\mathbf{r}_i = \frac{\partial\mathbf{r}}{\partial\nu_i} \qquad (i = 1, 2, 3)$$

The unit normal $\mathbf{n}$ to S is uniquely determined by the equations:

$$\mathbf{n}\cdot\mathbf{r}_i = 0, \tag{B.2}$$

$$\mathbf{n}\cdot\mathbf{n} = 1. \tag{B.3}$$

From (B.3) we have by differentiation:

$$\mathbf{n}\cdot\mathbf{n}_i = 0, \tag{B.4}$$

and hence we may write

$$\mathbf{n}_i = \sum_{j=1}^{3} n_{ij}\frac{|\mathbf{r}_i|}{|\mathbf{r}_j|}\mathbf{r}_j \tag{B.5}$$

with suitable $n_{ij} = n_{ij}(\nu_1, \nu_2, \nu_3)$. Moreover from (B.2) we have

$$\mathbf{n}_j \cdot \mathbf{r}_i + \mathbf{n} \cdot \mathbf{r}_{ij} = 0,$$

whence

$$\mathbf{n}_j \cdot \mathbf{r}_i = \mathbf{n}_i \cdot \mathbf{r}_j \tag{B.6}$$

or substituting (B.5) and noting (B.1)

$$n_{ij} = n_{ji}\,. \tag{B.7}$$

Now suppose we have a unit vector in S defined by

$$\mathbf{s} = \sum_{i=1}^{3} s_i \frac{\mathbf{r}_i}{|\mathbf{r}_i|} \tag{B.8}$$

Then the derivative of $\mathbf{n}$ in the direction $\mathbf{s}$ is

$$\sum_{i=1}^{3} \mathbf{n}_i \frac{s_i}{|\mathbf{r}_i|} = \sum_{i=1}^{3} \sum_{j=1}^{3} n_{ij} s_i \frac{\mathbf{r}_j}{|\mathbf{r}_j|} \tag{B.9}$$

and this lies along $\mathbf{s}$ if

$$\sum_{i=1}^{3} n_{ij} s_i = k s_j \qquad (j = 1, 2, 3) \tag{B.10}$$

for some k. From the theory of matrices it is known [5] that since n_{ij} is symmetric (B.7) there exists at least one set of mutually orthogonal real unit eigenvectors $\mathbf{s}^{(\lambda)}(\lambda = 1, 2, 3)$ satisfying (B.10) with corresponding real eigenvalues $k = k^{(\lambda)}$. Taking these directions $\mathbf{s}^{(\lambda)}$ as the coordinate directions of a new set of orthogonal coordinates (β, γ, δ) in S we have

$$\mathbf{n}_\beta = \mu_1 \mathbf{r}_\beta\,, \qquad \mathbf{n}_\gamma = \mu_2 \mathbf{r}_\gamma\,, \qquad \mathbf{n}_\delta = \mu_3 \mathbf{r}_\delta \tag{B.11}$$

with μ_1, μ_2, μ_3 functions of (β, γ, δ).

Now let $\mathbf{n}$ complete a system of four orthogonal directions in 4-space and α be distance measured along it. Then if $\mathbf{R}$ is the position vector in this space

$$\mathbf{R} = \mathbf{r} + \alpha \mathbf{n}$$

with $\mathbf{r}$, $\mathbf{n}$ independent of α. Hence, using (B.11), we have

$$\begin{aligned} \mathbf{R}_\alpha &= \mathbf{n}, \qquad \mathbf{R}_\beta = \mathbf{r}_\beta + \alpha \mathbf{n}_\beta = \mathbf{r}_\beta (1 + \mu_1 \alpha), \\ \mathbf{R}_\gamma &= \mathbf{r}_\gamma (1 + \mu_2 \alpha), \qquad \mathbf{R}_\delta = \mathbf{r}_\delta (1 + \mu_3 \alpha). \end{aligned} \tag{B.12}$$

Hence $(\alpha, \beta, \gamma, \delta)$ form a system of orthogonal curvilinear coordinates with

$$ds^2 = d\alpha^2 + (h_1 + \lambda_1 \alpha)^2 \, d\beta^2 + (h_2 + \lambda_2 \alpha)^2 \, d\gamma^2 + (h_3 + \lambda_3 \alpha)^2 \, d\delta^2, \tag{B.13}$$

where $h_1^2 = \mathbf{r}_\beta^2$ etc., independent of α, and $\lambda_1 = h_1 \mu_1$ etc., independent of α.

We have thus demonstrated the validity of (4.7) and (4.8) as well as the statements about these equations. We have constructed V^3-spaces $\alpha = \text{const.}$ parallel to S (in the obvious sense of the term) and used them to form an orthogonal system of coordinates.

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