The boundary Riemann solver coming from the real vanishing viscosity approximation

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Abstract

We study the limit of the hyperbolic-parabolic approximation

$$\begin{cases} v^\varepsilon_t + \tilde A\big(v^\varepsilon, \, \varepsilon v^\varepsilon_x\big) v^\varepsilon_x = \varepsilon \tilde B(v^\varepsilon) v^\varepsilon_{xx} \qquad v^\varepsilon \in \mathbb{R}^N \\ \tilde{\beta}(v^\varepsilon(t, \, 0)) \equiv \bar g \\ v^\varepsilon(0, \, x) \equiv \bar v_0. \end{cases}$$

The function $\tilde{\beta}$ is defined in such a way to guarantee that the initial boundary value problem is well posed even if $\tilde B$ is not invertible. The data $\bar g$ and $\bar v_0$ are constant.

When $\tilde B$ is invertible, the previous problem takes the simpler form

$$\begin{cases} v^\varepsilon_t + \tilde A\big(v^\varepsilon, \, \varepsilon v^\varepsilon_x\big) v^\varepsilon_x = \varepsilon \tilde B(v^\varepsilon) v^\varepsilon_{xx} \qquad v^\varepsilon \in \mathbb{R}^N \\ v^\varepsilon(t, \, 0) \equiv \bar v_b \\ v^\varepsilon(0, \, x) \equiv \bar v_0. \end{cases}$$

Again, the data $\bar v_b$ and $\bar v_0$ are constant. The conservative case is included in the previous formulations.

It is assumed convergence of the v^ε, smallness of the total variation and other technical hypotheses and it is provided a complete characterization of the limit.

The most interesting points are the following two.

First, the boundary characteristic case is considered, i.e. one eigenvalue of $\tilde A$ can be 0.

Second, as pointed out before we take into account the possibility that $\tilde B$ is not invertible. To deal with this case, we take as hypotheses conditions that were introduced by Kawashima and Shizuta relying on physically meaningful examples. We also introduce a new condition of block linear degeneracy. We prove that, if it is not satisfied, then pathological behaviours may occur.

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1 Introduction

The aim of this work is to describe the limit of the parabolic approximation

$$\begin{cases} v^\varepsilon_t + \tilde A\big(v^\varepsilon, \, \varepsilon v^\varepsilon_x\big) v^\varepsilon_x = \varepsilon \tilde B(v^\varepsilon) v^\varepsilon_{xx} \qquad v^\varepsilon \in \mathbb{R}^N \\ \tilde{\beta}(v^\varepsilon(t, \, 0)) \equiv \bar g \\ v^\varepsilon(0, \, x) \equiv \bar v_0 \end{cases} \tag{1.1}$$

for $\varepsilon \to 0^+$. In the previous expression, the function $\tilde{\beta}$ is needed to have well posedness in the case the matrix $\tilde B$ is not invertible. The function $\tilde{\beta}$ is defined in Section 2. When the matrix $\tilde B$ is indeed invertible,

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system (1.1) takes the simpler form

$$\begin{cases} v^\varepsilon_t + \tilde{A}\big(v^\varepsilon, \, \varepsilon v^\varepsilon_x\big) v^\varepsilon_x = \varepsilon \tilde{B}(v^\varepsilon) v^\varepsilon_{xx} \qquad v^\varepsilon \in \mathbb{R}^N \\ v^\varepsilon(t, \, 0) \equiv \bar{v}_b \\ v^\varepsilon(0, \, x) \equiv \bar{v}_0. \end{cases} \tag{1.2}$$

Even if the equations in (1.1) and (1.2) are not necessarily in conservation form, nevertheless the conservative case

$$v^\varepsilon_t + f(v^\varepsilon)_x = \varepsilon \Big(\tilde{B}(v^\varepsilon) v^\varepsilon_x \Big)_x$$

is included in the previous formulation. Indeed, one can define

$$\tilde{A}(v^\varepsilon, \, \varepsilon v^\varepsilon_x) := Df(v^\varepsilon) - \varepsilon \Big(\tilde{B}(v^\varepsilon) \Big)_x$$

and obtain an equation like the one in (1.1) or (1.2).

In the present paper we assume that when $\varepsilon \to 0^+$ the solutions v^ε converge, in the sense that will be specified in the following, to a unique limit v. Since in both (1.1) and (1.2) the initial and boundary data are constant, then the limit v solves a so called boundary Riemann problem, i.e. an hyperbolic initial boundary value problem with constant data. Results in [2] show that the study of boundary Riemann problems is a key point to determine the semigroup of solutions for an hyperbolic initial boundary value problem. We will came back to this point at the end of the introduction.

The goal of the work is to determine the value of $v(t, \, x)$ for a.e. point $(t, \, x)$. In particular, we determine the value of the trace $\bar{v}$ of the limit on the axis $x = 0$. The reason why this is interesting is the following. Let us focus for simplicity on the case in which $\tilde{B}$ is invertible, i.e. on (1.2). It is known that, in general, the trace of the limit, which we denote by $\bar{v}$, is different from $\bar{v}_b$, the boundary datum imposed in (1.2). The relation between $\bar{v}_b$ and $\bar{v}$ is relevant for the study of hyperbolic initial boundary value problems and was first investigated (as far as we know) in [27]. Also, in [28] it was proved that the value of $\bar{v}$ in general depends on the choice of the matrx $\tilde{B}$. In other words, if in (1.2) one keeps fixed $\bar{v}_0$, $\bar{v}_b$ and the function $\tilde{A}$ and changes only the matrix $\tilde{B}$, then in general the value of $\bar{v}$ will change, even if the system is in conservation form.

The most interesting points in the work are the following two. First, we cover the characteristic case, which occurs when an eigenvalue of the matrix $\tilde{A}$ can attain the value 0. The non characteristic case occurs when none of the eigenvalues of $\tilde{A}$ can attain the value 0. The characteristic case is more complicated to handle than the non characteristic one. Loosely speaking, the reason is the following. Suppose that the k-th eigenvalue can assume the value 0. Then in the non linear case we do not know a priori if the waves of the k-th family are entering or are leaving the domain.

The second point is that we cover the case of a non invertible viscosity matrix $\tilde{B}$. To tackle this case we assume that the so called Kawashima Shizuta condition holds. We also introduce a new condition of block linear degeneracy. We provide a counterexample which shows that, if this condition is violated, then there may be pathological behaviours, in the sense that will be specified in the following.

The exposition is organized as follows. In Section 1.1 we give an overview of the paper, introducing the main ideas involved in the analysis. In Section 2 we discuss the hypotheses assumed in the work. In Section 3 we give a characterization of the limit of the parabolic approximation (1.2), i.e. when the viscosity matrix $\tilde{B}$ is invertible. Finally, in Section 4 we discuss the limit of (1.1) when the matrix $\tilde{B}$ is singular.

1.1 Overview of the paper

1.1.1 Section 2: hypotheses

Section 2 describes the hypotheses exploited in the work and it is divided into three parts.

Section 2.1 describes the hypotheses assumed in the case the matrix $\tilde{B}$ is invertible. These hypotheses were already considered in several previous works and they are automatically satisfied when the system admits a dissipative entropy.

In Section 2.2 we discuss the hypotheses assumed in the case the matrix $\tilde{B}$ is singular. These hypotheses can be divided into two groups.

The first group is composed by conditions that were already exploited in several previous works (e.g. in [34, 35, 36, 37, 47, 43, 44]). In particular one assumes that there exists a regular and invertible change of variables $u = u(v^\varepsilon)$ such that the following holds. If v satisfies

$$v^\varepsilon_t + \tilde{A}\big(v^\varepsilon, \, \varepsilon v^\varepsilon_x\big) v^\varepsilon_x = \varepsilon \tilde{B}(v^\varepsilon) v^\varepsilon_{xx},$$

then u satisfies

$$E(u) u_t + A(u, \, u_x) u_x = B(u) u_{xx}. \tag{1.3}$$

The matrix B has constant rank r and admits the block decomposition

$$B(u) = \begin{pmatrix} 0 & 0 \\ 0 & b(u) \end{pmatrix} \tag{1.4}$$

for a suitable $b(u) \in \mathbb{M}^{r \times r}$. Also, B and A satisfy suitable hypotheses that are reasonable from the physical point of view since they were introduced in [34, 35, 36, 37] relying on examples with a physical meaning. In particular, we assume that the so called Kawashima Shizuta condition is satisfied.

Apart from these hypotheses, we introduce a new condition of block linear degeneracy, which is the following. Let

$$A(u, \, u_x) = \begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u, \, u_x) \end{pmatrix} \qquad E(u) = \begin{pmatrix} E_{11}(u) & E_{12}(u) \\ E_{21}(u) & E_{22}(u) \end{pmatrix} \tag{1.5}$$

be the block decomposition of A corresponding to (1.4), namely A_{11} and E_{11} belong to $\mathbb{M}^{(N-r)\times(N-r)}$ and A_{22} and E_{22} belong to $\mathbb{M}^{r \times r}$. The condition of block linear degeneracy says that, for every given real number σ, the dimension of the kernel of $[A_{11}(u) - \sigma E_{11}(u)]$ is constant with respect to u. In other words, the dimension of the kernel may vary as σ varies, but it cannot change when u varies.

Block linear degeneracy is not just a technical condition. Indeed, in Section 2.2.2 we discuss counterexamples which show how, when the block linear degeneracy is violated, one can have pathological behaviors. More precisely, we exhibit examples in which block linear degeneracy does not hold and there is a solution of (1.3) which is not $\mathcal{C}^1$. These can be considered a pathological behaviour since one usually expects that the parabolic approximation has a regularizing effect.

On the other side, block linear degeneracy is not an optimal condition, in the following sense. It is possible to show that block linear degeneracy is satisfied by the Navier Stokes equation written using Lagrangian coordinates, but it is not satisfied by the Navier Stokes equation written in Eulerian coordinates. On the other side the two formulations of the Navier Stokes equation are equivalent, provided that the density of the fluid is strictly positive. This remark was first proposed by Frederic Rousset in [41] and it suggests that it is interesting to look for a condition strong enought to prevent pathological behaviours but at the same time sufficiently weak to be satisfied by the Navier Stokes equation written in Eulerian coordinates. This problem is tackled in the forthcoming paper [13].

In Section 2.2.1 it is defined the function ß which is used to define the boundary condition in (1.1). The point here is the following.

Consider an hyperbolic initial boundary value problem, and for simplicity let us focus on the conservative case:

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0, \, x) = \bar{u}_0 \\ u(t, \, 0) = \bar{u} \end{cases} \tag{1.6}$$

It is known that if one assigns a datum $\bar{u} \in \mathbb{R}^N$ (i.e. if one assigns N boundary conditions), then the initial boundary value problem (1.6) may be ill posed, in the following sense. In general, there is no function u which is a solution of

$$u_t + f(u)_x = 0$$

in the sense of distributions, which assumes the initial datum for $t = 0$ and satisfies

$$\lim_{x \to 0^+} u(t, \, x) = \bar{u}$$

for almost every t. Also, it is know that a necessary condition to obtain a well posed problem is to assign a number of conditions on the boundary which is in general smaller then N. Assume the boundary is non characteristic, i.e. assume that none of the eigenvalues of the jacobian $Df(u)$ can attain the value 0. In this case, one can impose on the boundary datum a number of conditions equal to the number of positive eigenvalues of $Df(u)$.

We can now came back to the parabolic equation

$$u_t + A(u, u_x)u_x = B(u)u_{xx}. \tag{1.7}$$

Let us write $u = (u_1, u_2)^T$, where $u_1 \in \mathbb{R}^{N-r}$, $u_2 \in \mathbb{R}^r$. Here r is the rank of B, as in (1.4). With this notations equation (1.7) can be rewritten as

$$\begin{cases} u_{1t} + A_{11}u_{1x} + A_{21}u_{2x} = 0 \\ u_{2t} + A_{12}u_{1x} + A_{22}u_{2x} = bu_{2xx} \end{cases} \tag{1.8}$$

Roughly speaking, the reason why one has to introduce the function ß is the following. Let n_{11} be the number of strictly negative eigenvalues of the block A_{11}, which itself is a $(N-r) \times (N-r)$ matrix. Also, let q denote the dimension of the kernel of A_{11}. The second line in (1.8) contains a second order derivative of u_2 and hence u_2 can be seen as a parabolic component. On the other side, only first derivatives of u_1 appear and hence u_1 can be seen as an hyperbolic component. Actually, there is an interaction between the two components (this is ensured by the Kawashima-Shizuta condition). On the other side, because of the hyperbolic component one is not completely free to assign the boundary condition in (1.1). As pointed out in previous works (e.g in [43]) the number of conditions one can impose on the boundary is $N - n_{11} - q$. Indeed, one can impose r conditions on u_2. On the other hand, one can impose on u_1 a number of conditions equal to the number of positive eigenvalues of A_{11}, i.e. to $N - r - n_{11} - q$. Summing up one obtains exactly $N - n_{11} - q$.

Thus, the function $\tilde{ß}$ in (1.1) takes values in $\mathbb{R}^{N-n_{11}-q}$ and $\bar{g}$ is a fixed vector in $\mathbb{R}^{N-n_{11}-q}$. The precise definition of $\tilde{ß}$ is given in section 2.2.1 and it is such that the initial boundary value problem (1.1) is well posed. In Section 4.3 it is given a more general definition for the function ß.

Section 2.2.2 discusses three examples. The first two show that, if the condition of block linear degeneracy is violated, then there may be solution of

$$u_t + A(u, u_x)u_x = B(u)u_{xx}$$

exhibiting pathological behaviors, in the sense explained before. More precisely, the first example deal with steady solutions

$$A(u, u_x)u_x = B(u)u_{xx}, \tag{1.9}$$

while the second one deals with travelling waves,

$$[A(u, u') - \sigma E(u)]u' = Bu''.$$

In the previous expression, σ represents the speed of the wave and it is a real parameter. Finally, the third example in Section 2.2.2 shows that if the rank of the matrix B is not constant, then there may be solutions of (1.9) exhibiting pathological behaviours of the same kind discussed before.

Section 2.3 discusses the hypotheses that are assumed in both cases, when the matrix $\tilde{B}$ in (1.1) is invertible and when it is not. It is assumed that the system is strictly hyperbolic (see Section 2.3 for a definition of strict hyperbolicty). Also, it is assumed that when $\varepsilon \to 0^+$ the solutions of v^ε of (1.1) converge to a unique limit. Also, it is assumed that the approximation is stable with respect to the initial and the boundary data and that the limit has finite propagation speed.

We refer to Section 2.3 for the exact statement of the hypotheses, here instead we underline another point. The proof of the convergence of v^ε in the case of a generic matrix $\tilde{B}$ is still an open problem. However, there are results that provide a justification of our hypotheses. In particular, in [28] it is proved the local in time convergence in the case $\tilde{B}$ is invertible, but in general different from the identity. Moreover, in [3] the authors proved the global in time convergence in the case of an artificial viscosity ($\tilde{B}(v^\varepsilon)$ is identically equal to I_N). The analysis in [3] exploits techniques that were introduced in [8, 9, 10, 11] to deal with the Cauchy problem. In [3] it is proved the same kind of convergence we assume in the present properties. Also, other properties we assume here (stability of the approximation, finite propagation speed of the limit) are as well proved in [3]. Analogous results were proved in [48] for a special class of problems with 2 boundaries.

Also, we point out that there are several works that study the stability of the approximation in the case of a very general viscosity matrix $\tilde{B}$. Actually, the literature concerning this topic is very wide and hence we will quote only works that concern specifically initial boundary value problems: [47, 42, 43, 44].

1.1.2 Section 3: the characterization of the hyperbolic limit in the case of an invertible viscosity matrix

Section 3 discusses the characterization of the limit of (1.2) when the matrix $\tilde{B}$ is invertible. Actually, because of the hypotheses we assume in Section 2.1, we study the equivalent (in the sense specified therein) problem

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t,\, 0) \equiv \bar{u}_b \\ u^\varepsilon(0,\, x) \equiv \bar{u}_0. \end{cases} \tag{1.10}$$

Also, Section 3 is divided into fours parts.

Section 3.1 collects preliminary results that are needed in the following.

Section 3.2 gives a quick review of some results concerning the characterization of the limit in the Riemann problem. These results were introduced in [7].

The Riemann problem is a Cauchy problem with a piecewise constant inial datum with a single jump. Let us focus for simplicity on the conservative case:

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0,\, x) = \begin{cases} u^- & x < 0 \\ u^+ & x \geq 0 \end{cases} \end{cases} \tag{1.11}$$

A solution of (1.11) was first described in [39] assuming some technical hypotheses (i.e. that all the fields are either genuinely non linear or linearly degenerate). Since we will need in the following, here we briefly review the ideas exploited in [39] to obtain a solution of (1.11).

Denote by

$$\lambda_1(u) < \dots < \lambda_N(u)$$

the eigenvalues of the jacobian $Df(u)$ and by $r_1(u) \dots r_N(u)$ the corresponding eigenvectors. For simplicity, we assume that all the fields are genuinely non linear. In this case, one can chose the orientation of r_i in such a way that $\nabla \lambda_i \cdot r_i > 0$. For every $i = 1 \dots N$ we denote by $\mathcal{S}^i_{s_i}(u^+)$ a curve in $\mathbb{R}^N$ which is parameterized by s_i and which enjoys the following property. For every value $\mathcal{S}^i_{s_i}(u^+)$ there exists a speed λ close to $\lambda_i(u^+)$ such that the Rankine Hugoniot condition is satisfied, i.e.

$$f(u^+) - f\Big(\mathcal{S}^i_{s_i}(u^+)\Big) = \lambda\Big[u^+ - \mathcal{S}^i_{s_i}(u^+)\Big]. \tag{1.12}$$

Also, let $\mathcal{R}^i_{s_i}(u^+)$ be the integral curve of $r_i(u)$ starting at u^+, in other words $\mathcal{R}^i_{s_i}(u^+)$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{ds}\mathcal{R}^i = r_i(\mathcal{R}^i_s(u^+)) \\ \\ \mathcal{R}^i_0(u^+) = u^+ \end{cases}$$

Finally, let $\mathcal{T}^i_{s_i}(u^+)$ be defined as follows:

$$T^i_{s_i}(u^+) = \begin{cases} \mathcal{R}^i_{s_i}(u^+) & s_i \leq 0 \\ \mathcal{S}^i_{s_i}(u^+) & s_i > 0 \end{cases}$$

In [39] it is proved that $T^i_{s_i}(u^+)$ is a $\mathcal{C}^2$ curve such that

$$\frac{dT^i_{s_i}(u^+)}{ds_i}\Bigg|_{s_i=0} = r_i(u^+).$$

In the following, we will say that $T^i_{s_i}(u^+)$ is the i-th curve of admissible states. Indeed, every state $T^i_{s_i}(u^+)$ can be connected to u^+ by either a rarefaction wave or a shock which is admissible in the sense of Liu. In other words, if $s_i \le 0$ then the solution of the Riemann problem

$$\begin{cases} u_t + f(u)_x = 0 \\ u(0,\, x) = \begin{cases} u^+ & x \ge 0 \\ \mathcal{T}^i_{s_i}(u^+) & x < 0 \end{cases} \end{cases} \tag{1.13}$$

is

$$u(t,\, x) = \begin{cases} u^+ & x \ge t\lambda_i(u^+) \\ \\ T^i_{s_i}(u^+) & x \le t\lambda_i\Big(T^i_{s_i}(u^+)\Big) \\ \\ T^i_s(u^+) & t\lambda_i\Big(T^i_{s_i}(u^+)\Big) < x < t\lambda_i(u^+), \quad x = t\lambda_i\Big(T^i_s(u^+)\Big) \end{cases} \tag{1.14}$$

The meaning of the third line is that $u(t,\, x)$ is equal to $T^i_s(u^+)$ (the value assumed by the curve at the point s) when x is exactly equal to λ_i evaluated at the point $T^i_s(u^+)$. The value of u is well defined because of the condition $\nabla\lambda_i \cdot r_i > 0$.

On the other hand, if $s_i > 0$ then the solution of the Riemann problem (1.13) is

$$u(t,\, x) = \begin{cases} u^+ & x \ge t\lambda \\ \\ T^i_{s_i}(u^+) & x \le t\lambda. \end{cases} \tag{1.15}$$

The speed λ satisfies the Rankine Hugoniot condition and it is close to $\lambda_i(u^+)$.

In this way one obtains N curve of admissible states $\mathcal{T}^1_{s_1}(u^+), \dots\ T^N_{s_N}(u^+)$. To define the solution of the Riemann problem (1.11) one can proceed as follows. Consider the function

$$\psi(s_1 \dots s_N,\, u^+) = T^1_{s_1} \circ \dots T^{N-1}_{s_{N-1}} \circ T^N_{s_N}(u^+). \tag{1.16}$$

With the notation $T^{N-1}_{s_{N-1}} \circ T^N_{s_N}(u^+)$ we mean that the starting point of $T^{N-1}_{s_{N-1}}$ is $T^N_{s_N}(u^+)$, i.e. that

$$T^{N-1}_0 \circ T^N_{s_N}(u^+) = T^N_{s_N}(u^+).$$

It is proved in [39] that the map ψ is locally invertible with respect to $s_1 \dots s_N$. In other words, the values of $s_1 \dots s_N$ are uniquely determined if one imposes

$$u^- = \psi(s_1 \dots s_N,\, u^+),$$

at least if u^- is a close enough to u^+. One takes u^- as in (1.11) and obtains the values $s_1 \dots s_N$. Indeed, we assume, here and in the following, $|u^+ - u^-| << 1$. Once $s_1 \dots s_N$ are known, one can obtain the limit gluing together pieces like (1.14) and (1.15).

The construction in [39] was extended in [40] to more general systems. Also, in [7] it was given a characterization of the limit of

$$\begin{cases} u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} & u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(0,\, x) = \begin{cases} u^+ & x \ge 0 \\ u^- & x < 0 \end{cases} \end{cases} \tag{1.17}$$

when $|u^+ - u^-|$ is sufficiently small (under hypotheses slightly different from the ones we consider here).

The construction works as follows. Consider a travelling profile for

$$u_t + A\big(u,\, u_x\big)u_x = B(u)u_{xx},$$

i.e. $u(x - \sigma t)$ such that

$$B(u)u'' = \Big(A(u,\, u') - \sigma E(u)\Big)u'.$$

In the previous, expression, the speed of the wave σ is a real parameter. Then u solves

$$\begin{cases} u' = v \\ B(u)v' = \Big(A(u,\, v) - \sigma E(u)\Big)v \\ \sigma' = 0. \end{cases} \tag{1.18}$$

The point $(u^+,\, \vec{0},\, \lambda_i(u^+))$ is an equilibrium for (1.18). Also, one can prove that a center manifold around $(u^+,\, \vec{0},\, \lambda_i(u^+))$ has dimension $N+2$.

Fix a center manifold $\mathcal{M}^c$, then $\mathcal{M}^c$ is invariant for (1.18). Also, if $(u,\, v,\, \sigma)$ is a solution to (1.18) laying on $\mathcal{M}^c$, then

$$\begin{cases} u' = v_i \tilde{r}_i(u,\, v_i,\, \sigma_i) \\ v_i' = \phi_i(u,\, v_i,\, \sigma_i)v_i \\ \sigma_i' = 0. \end{cases} \tag{1.19}$$

The functions $\tilde{r}_i$ and ϕ_i are defined in Section 3.2.1.

The construction of $T^i_{s_i}u^+$ works as follows. Fix $s_i > 0$ and consider the following fixed point problem, defined on a interval $[0,\, s_i]$:

$$\begin{cases} u(\tau) = u^+ + \displaystyle\int_0^\tau \tilde{r}_i(u(\xi),\, v_i(\xi),\, \sigma_i(\xi))d\xi \\ v_i(\tau) = f_i(\tau,\, u,\, v_i,\, \sigma_i) - \text{conc}f_i(\tau,\, u,\, v_i,\, \sigma_i) \\ \sigma_i(\tau) = \frac{1}{c_E(\bar{u}_0)}\dfrac{d}{d\tau}\text{conc}f_i(\tau,\, u,\, v_i,\, \sigma_i). \end{cases} \tag{1.20}$$

We have used the following notations:

$$f_i(\tau) = \int_0^\tau \tilde{\lambda}_i[u_i,\, v_i,\, \sigma_i](\xi)d\xi,$$

where

$$\tilde{\lambda}_i[u_i,\, v_i,\, \sigma_i](\xi) = \phi_i\Big(u_i(\xi),\, v_i(\xi),\, \sigma_i(\xi)\Big) + c_E(\bar{u}_0)\sigma.$$

Also, $\text{conc}f_i$ denotes the concave envelope of the function f_i:

$$\text{conc}f_i(\tau) = \inf\{h(s):\; h \text{ is concave},\; h(y) \geq f_i(y)\; \forall\, y \in [0,\, s_i]\}.$$

One can show that the fixed point problem (1.20) admits a unique solution.

The link between (1.20) and (1.18) is the following: let $(u_i,\, v_i,\, \sigma_i)$ satisfy (3.17). Assume that $v_i < 0$ on $]a,\, b[$ and that $v_k(a) = v_k(b) = 0$. Define $\alpha_i(\tau)$ as the solution of the Cauchy problem

$$\begin{cases} \dfrac{d\alpha_i}{d\tau} = -\dfrac{1}{v_i(\tau)} \\ \alpha_i\Big(\dfrac{a+b}{2}\Big) = 0 \end{cases}$$

then $(u_i \circ \alpha_i,\, v_i \circ \alpha_i,\, \sigma_i \circ \alpha_i)$ is a solution to (3.16) satisfying

$$\lim_{x \to -\infty} u_i \circ \alpha_i(x) = u_i(a) \qquad \lim_{x \to +\infty} u_i \circ \alpha_i(x) = u_i(b).$$

Thus, $u_i(a)$ and $u_i(b)$ are connected by a travelling wave profile.

On the other side, if $v_i \equiv 0$ on the interval $[c, d]$, then the following holds. Consider $\mathcal{R}^i_{s_i} u(c)$, the integral curve of $r_i(u)$ such that $\mathcal{R}^i_0 u(c) = u(c)$. Then $u(d)$ lays on $\mathcal{R}^i_{s_i} u(c)$, thus $u(c)$ and $u(d)$ are connected by a rarefaction or by a contact discontinuity.

If $s_i < 0$, one considers a fixed problem like (1.20), but instead of the concave envelope of f_i one takes the convex envelope:

$$\text{conv} f_i(\tau) = \sup\{h(s) : \; h \text{ is convex}, \; h(y) \leq f_i(y) \, \forall \, y \in [0, \, s_i]\}.$$

Again, one can prove the existence of a unique fixed point $(u_i, \, v_i, \, \sigma_i)$.

The curve $T^i_{s_i} u^+$ is defined setting

$$T^i_{s_i} u^+ := u(s_i).$$

This curve contains states that are connected to u^+ by rarefaction waves and shocks with speed close to $\lambda_i(u^+)$.

If $u^- = T^i_{s_i} u^+$, then the limit of the approximation (1.17) is

$$u(t, \, x) = \begin{cases} u^+ & x \geq \sigma_i(0)t \\ u_i(s) & x = \sigma_i(s)t \\ u_i(s_i) = T^i_{s_i} u^+ & x \leq \sigma_i(s_i)t \end{cases} \tag{1.21}$$

In the previous expression, σ_i is given by (1.20) and it is a monotone non increasing function.

It can be shown that in the case of conservative systems with only genuinely non linear or linearly degenerate fields the i-th curve of admissible states $T^i_{s_i} u^+$ defined in [7] coincides with the one described in [39]. Once $T^i_{s_i} u^+$ is known, then one defines ψ as in (1.16) and find the limit gluing together pieces like (1.21).

In Section 3.2.2 we give a characterization of the limit of

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon, \, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} & u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t, \, 0) \equiv \bar{u}_b \\ u^\varepsilon(0, \, x) \equiv \bar{u}_0. \end{cases} \tag{1.22}$$

when the boundary is not characteristic, i.e. when none of the eigenvalues of $E^{-1}A$ can attain the value 0. The idea is to costruct a locally invertible map ϕ which describes all the states that can be connected to $\bar{u}_0$, in the sense that is specified in the following. Loosely speaking, the map ϕ represents for the initial boundary value problem what the map ψ defined in (1.16) represents for the Cauchy problem. Once ϕ is defined, one takes $\bar{u}_b$ as in (1.22) and imposes

$$\bar{u}_b = \phi(\bar{u}_0, \, s_1 \ldots s_N).$$

If $\bar{u}_0$ and $\bar{u}_b$ are sufficiently close, then this relation uniquely determines the values of $s_1 \ldots s_N$. Once these values are known, then the limit $u(t, \, x)$ is uniquely determined. More precisely, one can define the value of $u(t, \, x)$ for a.e. $(t, \, x)$. We will come to this point later.

The construction of the map ϕ works as follows. Denote by $\lambda_1(u) \ldots \lambda_N(u)$ the eigenvalues of $E^{-1}(u)A(u, \, 0)$ and by $r_1(u) \ldots r_N(u)$ the associated eigenvectors. Also, assume that for every u,

$$\lambda_1(u) < \ldots \lambda_n(u) < -c < 0 < c < \lambda_{n+1} < \ldots \lambda_N(u)$$

In other words, n is the number of negative eigenvalues of $E^{-1}(u)A(u, \, 0)$. These eigenvalues are real because this is one of the hypotheses listed in Section 2 (Hypothesis 3).

For $i = n + 1 \ldots N$ consider the i-th curve of admissible states. Fix $N - n$ parameters $s_{n+1} \ldots s_N$ and define

$$\bar{u} = T^{n+1}_{s_{n+1}} \circ \ldots T^{N-1}_{s_{N-1}} \circ T^N_{s_N} \bar{u}_0.$$

As before, the notation $T^{N-1}_{s_{N-1}} \circ T^N_{s_N} \bar u_0$ means that the starting point of $T^{N-1}_{s_{N-1}}$ is $T^N_{s_N} \bar u_0$, $T^{N-1}_0 = T^N_{s_N} \bar u_0$. Thanks to the results in [7] we quoted before, $\bar u$ is connected to $\bar u_0$ by a sequence of rarefaction waves and shocks with stricly positive speed.

To complete the construction, one considers steady solutions of

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx}$$

i.e. couples $(U,\, p)$ such that

$$\begin{cases} U' = p \\ B(u)p' = A(u,\, p)p \end{cases} \tag{1.23}$$

The point $(\bar u,\, 0)$ is an equilibrium for (1.23). As shown in section 3.2.2, the stable manifold around $(\bar u,\, 0)$ has dimension n, i.e. has dimension equal to the number of strictly negative eigenvalues of $E^{-1}A$. Also, the following holds. Let ψ be a map that parameterizes the stable manifold, then ψ takes values into $\mathbb{R}^N \times \mathbb{R}^N$ and it is defined on a space of dimension n. To underline the dependence on $\bar u$ we will write $\psi(\bar u,\, s_1 \dots s_n)$. Denote by π_u the projection

$$\begin{aligned} \pi_u :& \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \\ & (u,\, p) \mapsto u \end{aligned}$$

Fix n parameters $s_1 \dots s_n$ and consider $\pi_u \circ \psi(\bar u,\, s_1 \dots s_n)$. Consider the problem

$$\begin{cases} A(u,\, u_x)u_x = B(u)u_{xx} \\ u(0) = \pi_u \circ \psi(\bar u,\, s_1 \dots s_n), \end{cases}$$

then there exists a unique solution of this problem such that

$$\lim_{x \to +\infty} u(x) = \bar u.$$

Setting $u^\varepsilon(x) := u(x/\varepsilon)$, one finds a solution of

$$A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = B(u^\varepsilon)u^\varepsilon_{xx}$$

such that $u^\varepsilon(0) = \pi_u \circ \psi(\bar u,\, s_1 \dots s_n)$ and for every $x > 0$,

$$\lim_{\varepsilon \to 0^+} u^\varepsilon(x) = \bar u.$$

In this sense, we say that there is a loss of boundary condition when passing to the hyperbolic limit, because the boundary condition $\pi_u \circ \psi(\bar u,\, s_1 \dots s_n)$ disappears in the limit. We point out that the idea of studying steady solutions to take into account the loss of boundary condition was already exploited in many previous work, e.g in [28].

To complete the characterization of the limit, we define ϕ as

$$\phi(\bar u_0,\, s_1 \dots s_N) = \pi_u \circ \psi\Big(T^{n+1}_{s_{n+1}} \circ \dots T^{N-1}_{s_{N-1}} \circ T^N_{s_N} \bar u_0,\, s_1 \dots s_n\Big).$$

In section 3.2.2 we prove that the map ϕ is locally invertible, i.e. the values of $s_1 \dots s_N$ are completely determined is one sets

$$\bar u_b = \phi(\bar u_0,\, s_1 \dots s_N).$$

We take the same $\bar u_b$ as in (1.22). As pointed out at the beginning of the paragraph, once $s_1 \dots s_N$ are known, then the value of the self similar solution $u(t,\, x)$ is determined for a.e. $(t,\, x)$. One can indeed glue together pieces like (1.21). In particular, it turns out that the trace of $u(t,\, x)$ on the axis $x = 0$ is

$$\bar u = T^{n+1}_{s_{n+1}} \circ \dots T^{N-1}_{s_{N-1}} \circ T^N_{s_N} \bar u_0.$$

As we have already underlined before, the relation between $\bar u_b$ (the boundary datum in (1.22)) and $\bar u$ is interesting for the study of hyperbolic initial boundary value problems.

In Section 3.2.3 we give a characterization of the limit of the parabolic approximation

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t,\, 0) \equiv \bar u_b \\ u^\varepsilon(0,\, x) \equiv \bar u_0. \end{cases} \tag{1.24}$$

when the boundary is characteristic, i.e. one eigenvalue of $E^{-1}A$ can attain the value 0. The characterization of the limit works as follows. We costruct a locally invertible map ϕ which describes all the states that can be connected to $\bar u_0$. Once ϕ is defined, one takes $\bar u_b$ as in (1.24) and imposes

$$\bar u_b = \phi(\bar u_0,\, s_1 \dots s_N).$$

If $\bar u_0$ and $\bar u_b$ are sufficiently close, then this relation uniquely determines the values of $s_1 \dots s_N$. Once these values are known, then the limit $u(t,\, x)$ is uniquely determined.

Formally, the idea is the same as in Section 3.2.2. However, the construction of the map ϕ is definitely more complicated in the boundary chararacteristic case.

Roughly speaking, the reason is the following. Let $\lambda_1(u) \dots \lambda_N(u)$ be the eigenvalues of $E^{-1}(u)A(u,\, 0)$. They are real by Hypothesis 3. Assume

$$\lambda_1(u) < \dots \lambda_{k-1}(u) < -c < \lambda_k(u) < c < \lambda_{k+1} < \dots \lambda_N(u),$$

where c is a suitable positive constant. In other words, there are at least $k-1$ strictly negative eigenvalues, $N-k$ strictly positive eigenvalues and one eigenvalue close to 0.

Define

$$\bar u_k = T^{k+1}_{s_{k+1}} \circ \dots T^{N-1}_{s_{N-1}} \circ T^N_{s_N} \bar u_0,$$

then $\bar u_k$ is connected to $\bar u_0$ by rarefaction waves and shocks with stricly positive speed. We now want to define the k-th curve of admissible states. To define $T^k_{s_k}\bar u_k$, we might try to consider the fixed point problem

$$\begin{cases} u(\tau) = \bar u_k + \displaystyle\int_0^\tau \tilde r_k(u(\xi),\, v_k(\xi),\, \sigma_k(\xi))d\xi \\ v_k(\tau) = f_k(\tau,\, u,\, v_k,\, \sigma_k) - \text{conc} f_k(\tau,\, u,\, v_k,\, \sigma_k) \\ \sigma_k(\tau) = \frac{1}{c_E(\bar u_0)} \dfrac{d}{d\tau}\text{conc} f_k(\tau,\, u,\, v_k,\, \sigma_k), \end{cases} \tag{1.25}$$

where $\tilde r_k$, f_k and c_E are the same as in (1.20). However, if we consider (1.25) we are not doing the right thing. Indeed, we might have that the speed σ_k is negative at a certain point τ. Since eventually we want to define the limit $u(t,\, x)$ as in (1.21), we want σ_k to be greater then 0.

Another problem is the following. Consider the system satisfied by steady solutions of

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx},$$

i.e. consider

$$\begin{cases} U' = p \\ B(u)p' = A(u,\, p)p \end{cases} \tag{1.26}$$

Also, consider the equilibrium point $(\bar u_k,\, 0)$. Let $\mathcal{M}^s$ be the stable manifold of (1.26) around $(\bar u_k,\, 0)$. For simplicity, assume $\lambda_k(\bar u_k) = 0$. Then, there might be a solution $(U,\, p)$ such that

$$\lim_{x \to +\infty} \Big(U(x),\, p(x)\Big) = \Big(\bar u_k,\, 0\Big)$$

but $(U,\, p)$ does *not* belong to the stable manifold. However, this kind of solution should be taken into account when we study the loss of boundary condition.

To tackle these difficulties one can proceed as follows. Instead of the fixed point problem (1.25), one considers

$$\begin{cases} u(\tau) = \bar u_k + \displaystyle\int_0^\tau \tilde r_k(u(\xi),\, v_k(\xi),\, \sigma_k(\xi))d\xi \\ v_k(\tau) = f_k(\tau,\, u,\, v_k,\, \sigma_k) - \text{mon} f_k(\tau,\, u,\, v_k,\, \sigma_k) \\ \sigma_k(\tau) = \frac{1}{c_E(\bar u_0)} \dfrac{d}{d\tau}\text{mon} f_k(\tau,\, u,\, v_k,\, \sigma_k). \end{cases} \tag{1.27}$$

In the previous expression, $\mathrm{mon}f_k$ denotes the monotone concave envelope of the function f_k,

$$\mathrm{mon}f_k(\tau) = \inf \Big\{ g(\tau): \ g(s) \geq f_k(s) \ \forall s, \ g \text{ concave monotone non decreasing in } [0, \, s_i] \ \Big\}.$$

Some properties of the monotone envelope are discussed in Section 3.1.2, here we stress that $\mathrm{mon}f_k$ is a concave and non decreasing function, thus the solution σ_k of (1.27) is always non negative.

Also, the following holds. Denote by $(u_k, \, v_k, \, \sigma_k)$ the solution of (1.27) (existence and uniqueness are proved in Section 3.2.3). Define

$$\bar{s} = \min\{s: \ \sigma_k(s) = 0\} \tag{1.28}$$

and

$$\underline{s} = \max\{s: \ \sigma_k(s) = 0, \ v_k(s) = 0\}.$$

Assume $0 < \bar{s} \leq \underline{s} < s_k$. Then $u_k(\bar{s})$ is connected to $\bar{u}_k$ by a sequence of rarefaction and shocks with positive speed. Also, one can show that there exists a steady solution U,

$$A(U, \, U_x) U_x = B(U) U_{xx}$$

such that $U(0) = u_k(s_k)$ and

$$\lim_{x \to +\infty} U(x) = u_k(\underline{s}).$$

However, in general this solution *does not* belong to the stable manifold of system (1.26). This means that considering system (1.27) we also manage to take into account the converging steady solutions we were missing considering just the stable manifold of (1.26).

Heuristically, to complete the construction one should consider the stable manifold of (1.26) and hence take into account the steady solution that, for x that goes to $+\infty$, converge to $u_k(\underline{s})$ with fast exponential decay, in the sense specified in Section 3.2.3. Actually, the situation is more complex. The reason, loosely speaking, is the following . There may be a solution U that converges to $u_k(\underline{s})$ and such that some of its components converge with fast exponential decay, but other components converge more slowly. This possibility is not covered if we consider only the solutions laying on the stable manifold and those given by (1.27). To take into account this possibility some technical tools are introduced. More precisely, one considers suitable manifolds: center stable manifold and uniformly stable manifold. The existence of these manifolds is a consequence of results in [33], but some of the most important properties are recalled in Section 3.2.3.

Eventually, one manages to define a locally invertible function ϕ. One then takes $\bar{u}_b$ as in (1.24) and imposes

$$\bar{u}_b = \phi(\bar{u}_0, \, s_1 \dots s_N).$$

If $\bar{u}_0$ and $\bar{u}_b$ are sufficiently close, then this relation uniquely determines the values of $s_1 \dots s_N$. Once these values are known, then the limit $u(t, \, x)$ is uniquely determined and can be obtained gluing together pieces like (1.21). In particular, it turns out that the trace of u on the x axis is $u_k(\bar{s})$, where u_k solves (1.27) and $\bar{s}$ is given by (1.28).

1.1.3 Section 4: the characterization of the hyperbolic limit in the case of a singular viscosity matrix

In Section 4 we discuss the characterization of the limit of (1.1) when the matrix $\tilde{B}$ is not invertible. Actually, because of the hypotheses introduced in Section 2, one studies the limit of

$$\begin{cases} u^\varepsilon_t + A\big(u^\varepsilon, \, \varepsilon u^\varepsilon_x\big) u^\varepsilon_x = \varepsilon B(u^\varepsilon) u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ ß(u^\varepsilon(t, \, 0)) \equiv \bar{g} \\ u^\varepsilon(0, \, x) \equiv \bar{u}_0 \end{cases} \tag{1.29}$$

This system is equivalent to (1.1) in the sense specified in Section 2.2. Also, $ß = \tilde{ß}\Big(u(v^\varepsilon)\Big)$: a precise definition is given Section 2. In particular, ß ensures that the initial boundary value problem (1.29) is well posed. Section 4 is divided into several parts.

In Section 4.1 we introduce some preliminary results. The point here is the following. In Section 3 we give a characterization of the hyperbolic limit when the viscosity matrix is invertible. A key point in the analysis is the study of travelling waves

$$[A(U, U') - \sigma E(U)]U' = B(U)U'' \tag{1.30}$$

and of steady solutions

$$A(U, U')U' = B(U)U''. \tag{1.31}$$

To give a characterization of the hyperbolic limit when the viscosity matrix is not invertible, we have to study again systems (1.30) and (1.31). However, being the viscosity matrix B singular, a technical difficulty arises. Let us focus, for simplicity, on the case of steady solutions. If B is invertible, we can write

$$\begin{cases} U' = p \\ p' = B(u)^{-1} A(u, p)p \end{cases} \tag{1.32}$$

In this way, we write system (1.31) in an explicit form. On the other side, if the matrix B is singular, additional work is required to reduce (1.31) in a form like (1.32). This is indeed done in Section 4.1. What we actually obtain is not a $2N$-dimensional first order system like (1.32), but a lower dimensional first order system. The exact dimension depends on the structure of the matrix A, in the sense specified in Section 4.1.

In Section 4.2.1 we review the characterization of the hyperbolic limit in the case of a Riemann problem, i.e. the limit of

$$\begin{cases} u^\varepsilon_t + A\big(u^\varepsilon, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} & u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(0, x) = \begin{cases} u^+ & x \geq 0 \\ u^- & x < 0 \end{cases} \end{cases}$$

As for the case of an invertible B (Section 3.2.1), the key point in the analysis is the description of the i-th curve of admissible states $T^i_{s_i} u^+$. However, there are technical difficulties due to the fact that B is not invertible. Actually, in Section 4.2.1 we only give a sketch of the construction, and we refer to [7] for the complete analysis.

In Section 4.2.2 we introduce a technical lemma. The problem is the following. Consider a steady solution (1.31) and assume that it is written in an explicit form like (1.32). This is possible thanks to the considerations carried on in Section 4.1. Given an equilibrium point for this new system, consider the stable manifold around that equilibrium point. For reasons explained in Section 4.2, we need to know the dimension of this stable manifold. Lemma 4.7 ensures that the dimension of the stable manifold is equal to $n - n_{11} - q$, where n is the number of strictly negative eigenvalues of A, n_{11} is the number of strictly negative eigenvalues of the block A_{11} and q is the dimension of the kernel of the block A_{11}. The block A_{11} is defined by (1.5). Lemma 4.7 gives an answer to a question left open in [44].

In Section 4.2 we discuss the characterization of the hyperbolic limit of

$$\begin{cases} u^\varepsilon_t + A\big(u^\varepsilon, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} & u^\varepsilon \in \mathbb{R}^N \\ ß(u^\varepsilon(t, 0)) \equiv \bar{g} \\ u^\varepsilon(0, x) \equiv \bar{u}_0 \end{cases} \tag{1.33}$$

when the matrix B is singular, but the boundary is not characteristic, i.e. none of the eigenvalues of $E^{-1}(u)A(u, 0)$ can attain the value 0.

To provide a characterization, we construct a map $\phi(\bar{u}_0, s_{n_{11}+q+1} \dots s_N)$ which enjoys the following properties. It takes values in $\mathbb{R}^N$ and when $s_{n_{11}+q+1} \dots s_N$ vary it describes all that states that can be connected to $\bar{u}_0$, in the sense specified in the following. Note that similar functions are constructed in Section 3.2.2 and Section 3.2.3 and they are used to describe the limit of the parabolic approximation when the viscosity is *invertible*. However, in those cases the functions depend on N variables, while in the case of a singular matrix B the function ϕ depends only on $(N - n_{11} - q)$ variables.

Also, in the case of a singular viscosity we have to compose ϕ with the function ß, which is used to assign the boundary datum in (1.33). It is possible to show that the map $ß \circ \phi$ is locally invertible, in other words the values of $s_{n_{11}+q+1} \dots s_N$ are uniquely determined if one imposes

$$ß \circ \phi(s_{n_{11}+q+1} \dots s_N) = \bar{g}, \tag{1.34}$$

provided that $|ß(\bar{u}_0) - \bar{g}|$ is small enough. We will plug the same $\bar{g}$ as in (1.33). Once the values of $s_{n_{11}+q+1} \dots s_N$ are known, the limit $u(t, x)$ can be determined a.e. (t, x) in the same way as in Sections 3.2.2 and 3.2.3. In particular, one can determine exactly the value of the trace of the limit u on the axis $x = 0$. As pointed out before, this is important for the study of hyperbolic initial boundary value problems.

The construction of the map ϕ works as follows. Denote as before by n the number of the eigenvalues of $E^{-1}(u)A(u, 0)$ that are strictly negative, since the boundary is not characteristic the number of strictly positive eigenvalues is $N - n$. For $i = n + 1 \dots N$, let $T^i_{s_i}$ be the i-th curve of admissible states, whose construction is reviewed in Section 4.2.1. Fix $s_{n+1} \dots s_N$ and define

$$\bar{u} = T^{n+1}_{s_{n+1}} \circ \dots T^{N-1}_{s_{N-1}} \circ T^N_{s_N} \bar{u}_0. \tag{1.35}$$

As before, with the notation $T^{N-1}_{s_{N-1}} \circ T^N_{s_N} \bar{u}_0$ we mean that the starting point of $T^{N-1}_{s_{N-1}}$ is $T^N_{s_N} \bar{u}_0$, i.e.

$$T^{N-1}_0 \circ T^N_{s_N} \bar{u}_0 = T^N_{s_N} \bar{u}_0.$$

The value $\bar{u}$ is connected to $\bar{u}_0$ by a sequence of rarefactions and shocks with strictly positive speed.

To complete the construction, we consider steady solutions of

$$E(u)u_t + A(u, u_x)u_x = B(u)u_{xx},$$

i.e. we consider solutions of

$$A(u, u_x)u_x = B(u)u_{xx} \tag{1.36}$$

In Section 4.1.2 we discuss this system and we explain how to write it as first order O.D.E.. Also, in Section 4.2.2 we study the stable manifold of (1.36) around an equilibrium point such that $u = \bar{u}$ and $u_x = 0$. In particular, we prove that this manifold has dimension $n - n_{11} - q$. Let $\psi(\bar{u}, s_{n_{11}+q} \dots s_n)$ a map that parameterizes the stable manifold (we are also putting in evidence the dependence on $\bar{u}$). For every $s_{n_{11}+q} \dots s_n$ there exists a solution u of (1.36) such that $u(0) = \psi(\bar{u}, s_{n_{11}+q} \dots s_n)$ and

$$\lim_{x \to +\infty} u(x) = \bar{u}.$$

setting $u^\varepsilon(x) := u(x/\varepsilon)$ one obtains a steady solution of

$$E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon, \varepsilon u^\varepsilon_x)u^\varepsilon_x = B(u^\varepsilon)u^\varepsilon_{xx},$$

such that for every $x > 0$

$$\lim_{\varepsilon \to 0^+} u^\varepsilon(x) = \bar{u}.$$

In other words, we experience again a possibile loss of boundary condition from the parabolic approximation to the hyperbolic limit.

The map ϕ is then defined as follows:

$$\phi(\bar{u}_0, s_{n_{11}+q+1} \dots s_N) := \psi\Big(T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_N} \bar{u}_0, s_{n_{11}+q+1} \dots s_n\Big).$$

In Section 4.3 we prove that $ß \circ \phi$ is locally invertible with respect to $s_{n_{11}+q+1} \dots s_N$.

Here, instead, we point out the following. The fact that the dimension of the stable manifold of 1.36 has dimension $n - n_{11} - q$ is proved in Section 4.2.2 and it is not, a priori, obvious. However, $n - n_{11} - q$ is exactly the number that makes things works if one wants the function $ß \circ \phi$ to be locally invertible, in the sense that $ß \circ \phi$ takes values in $\mathbb{R}^{N - n_{11} - q}$ and hence to be locally invertible it should depend on $N - n_{11} - q$ variables. Since we have to take into account $N - n$ curve of admissible states, we are left with $n - n_{11} - q$ variables for the stable manifold.

In Section 3.2.2 we provide a characterization of the limit of

$$\begin{cases} u^\varepsilon_t + A\big(u^\varepsilon, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ ß(u^\varepsilon(t, 0)) \equiv \bar{g} \\ u^\varepsilon(0, x) \equiv \bar{u}_0 \end{cases} \tag{1.37}$$

when the matrix B is singular and the boundary is characteristic, i.e one of the eigenvalues of A can attain the value 0. Actually this case requires no new ideas, in the sense that one can combine the techniques that are described in Section 3.2.3 (to deal with the fact that the boundary is characteristic) and in Section 4.2.3 (to deal with the singularity of the matrix B).

Finally, in Section 4.3 we introduce a technical lemma which guarantees that the function $ß \circ \phi$ in (1.34) is locally invertible. We also provide a more general definition for the function ß. Such a definition still ensures that the initial boundary value problem (1.37) is well posed and that the map $ß \circ \phi$ is locally invertible.

1.2 Unquoted references

Existence results for hyperbolic initial value problems were obtained in [31] and [45] relying on an adaptation of the Glimm scheme of approximation introduced in [29].

These results were later improved by mean of wave front tracking techniques. These techniques were used in a series of papers ([14, 15, 18, 19, 20, 21, 22, 23]) to establish the well posedness of the Cauchy problem. A comprehensive account of the stability and uniqueness results for the Cauchy problem for a system of conservation laws can be found in [16]. There are many references for a general introduction to system of conservation laws, for example [25] and to [46]. As concerns initial boundary value problems, in [1] the existence results in [31] and [45] were substantially improved, while well posedness results were obtained in [26].

In [2] it is studied the limit of the wave front tracking approximation. In particular, the authors extended to initial boundary value problems the definition of Standard Riemann Semigroup. Such a notion was introduced in [15] for the Cauchy problem. Roughly speaking, the analysis in [2] guarantees that to identify the semigroup of solutions it is enough to consider the behaviour of the semigroup in the case the initial and the boundary data are constant. It hence one of the most important motivations for the study of the parabolic approximation (1.1).

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2 Hypotheses

This section introduces the hypotheses exploited in the work.

The exposition is organized as follows. In Section 2.1 we introduce the hypotheses we impose on system (1.2) when the viscosity matrix $\tilde{B}$ is invertible. In Section 2.2 we discuss the hypotheses exploited when the matrix $\tilde{B}$ in (1.1) is singular. Finally, in Section 2.3 we introduce the hypotheses needed in both the cases, when the matrix $\tilde{B}$ in (1.1) is invertible and when it is not.

In Section 2.2.1 we also discuss three examples which show that, if some of the conditions introduced are not satisfied, then there may be pathological behaviors, in the sense specified there. Also, we define the function $\tilde{ß}$ used to assign the boundary condition in (1.1).

In [13] it will be discussed a way to extend to a more general setting the condition of block linear degeneracy, which is the third condition in Hypothesis 2.

2.1 The hypotheses assumed when the viscosity matrix is invertible

In this section it is considered the system

$$v_t + \tilde{A}(v,\, v_x)v_x = \tilde{B}(v)v_{xx} \tag{2.1}$$

in the case the viscosity matrix $\tilde{B}(v)$ is invertible. We assume the following.

Hypothesis 1. *There exists an (invertible) smooth change of variables $v = v(u)$ such that* (2.1) *is equivalent to system*

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx}, \tag{2.2}$$

where

1. *for any u, the matrix $E(u)$ is real, symmetric and positive definite: there exists a constant $c_E(u)$ such that*
$$\forall\, \xi \in \mathbb{R}^N,\ \langle E(u)\xi,\, \xi\rangle \geq c_E(u)|\xi|^2.$$
2. *for any u, the matrix $A(u,\, 0)$ is symmetric.*
3. *for any u, the viscosity matrix $B(u)$ is real and there exists a constant $c_B(u) > 0$ such that*
$$\forall\, \xi \in \mathbb{R}^N,\ \langle B(u)\xi,\, \xi\rangle \geq c_B(u)|\xi|^2.$$

In particular, the matrix $E(u)$ may be the differential $Dv(u)$ of the change of coordinates.

The initial boundary value problem (1.2) is equivalent to

$$\begin{cases} u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t,\, 0) \equiv \bar{u}_b \\ u^\varepsilon(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{2.3}$$

where $v(\bar{u}_0) = \bar{v}_0$ and $v(\bar{u}_b) = \bar{v}_b$. Indeed, one can verify that, thanks to the invertibility of the viscosity matrix, it is possible to assign a full boundary condition $\bar{u}_b \in \mathbb{R}^N$.

In the case system (2.1) is in conservation form

$$v_t + f(v)_v = \Big(\tilde{B}(v)v_x\Big)_x, \tag{2.4}$$

Hypothesis 1 is guaranteed by the presence of a dissipative and convex entropy.

For completeness we recall that the entropy η is dissipative for (2.4) if for any v there exists a constant $c_D(v)$ such that

$$\forall\, \xi \in \mathbb{R}^N,\ \langle D^2\eta(v)\xi,\, \tilde{B}(v)\xi\rangle \geq c_D(v)|\xi|^2 \tag{2.5}$$

It is known (see for example [30]) that if system (2.1) admits a convex entropy η, then there exists a symmetrizing and invertible change of variable $v = v(u)$. More precisely, if the inverse function $u(v)$ is defined by $u = \nabla\eta(v)$, then v satisfies system (2.2), with $A(u,\, 0)$ symmetric and given by

$$A(u,\, 0) = Df\big(v(u)\big)\Big(D^2\eta\big(v(u)\big)\Big)^{-1} - \frac{\partial}{\partial x}\bigg(B\big(v(u)\big)\Big(D^2\eta\big(v(u)\big)\Big)^{-1}\bigg)\bigg|_{u_x=0} = Df\big(v(u)\big)\Big(D^2\eta\big(v(u)\big)\Big)^{-1}$$

Moreover,

$$E(u) = \Big(D^2\eta\big(v(u)\big)\Big)^{-1}$$

is symmetric (being the inverse of an hessian) and positive definite by convexity, while the dissipation condition (2.5) guarantees that

$$B(u) = \tilde{B}\big(v(u)\big)\Big(D^2\eta\big(v(u)\big)\Big)^{-1}$$

is positive definite. Hence Hypothesis 1 is satisfied.

2.2 The hypotheses assumed when the viscosity matrix is not invertible

The aim of this section is to introduce the hypotheses that will be needed in Section 4 to study system

$$v_t + \tilde{A}(v,\, v_x)v_x = \tilde{B}(v)v_{xx} \tag{2.6}$$

in the case the viscosity matrix is singular.

Hypothesis 2. *There exists an (invertible) smooth change of coordinates $v = v(u)$ such that (2.6) is equivalent to system*

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx}, \tag{2.7}$$

where

1. *the matrix $B(u)$ has constant rank $r < N$ and admits the block decomposition*

$$B(u) = \begin{pmatrix} 0 & 0 \\ 0 & b(u) \end{pmatrix} \tag{2.8}$$

   with $b(u) \in \mathbb{M}^{r \times r}$. Moreover, for every u there exists a constant $c_b(u) > 0$ such that

$$\forall\, \xi \in \mathbb{R}^r,\ \langle b(u)\xi,\, \xi \rangle \geq c_b(u)|\xi|^2.$$

2. *for any u the matrix $A(u,\, 0)$ is symmetric. Moreover, the block decomposition of $A(u,\, u_x)$ corresponding to (2.8) takes the form*

$$\begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u,\, u_x) \end{pmatrix}, \tag{2.9}$$

   with $A_{11} \in \mathbb{M}^{(N-r)\times(N-r)}$, $A_{12} \in \mathbb{M}^{(N-r)\times r}$, $A_{21} \in \mathbb{M}^{r \times (N-r)}$ and $A_{22} \in \mathbb{M}^{r \times r}$. Namely, only in the block A_{22} depends on u_x.

3. *a condition of block linear degeneracy holds. More precisely, for any $\sigma \in \mathbb{R}$ the dimension of the kernel ker$[A_{11}(u) - \sigma E_{11}(u)]$ does not depend on u, but only on σ.*

   This holds in particular for the limit cases in which the dimension of ker$(A_{11} - \sigma E_{11})$ is equal to $N - r$ or to 0.

4. *the so called Kawashima condition holds: for any u,*

$$\mathit{ker}\big(B(u)\big) \cap \{\text{eigenvectors of } E^{-1}(u)A(u,\, 0)\} = \emptyset$$

5. *for any u, the matrix $E(u)$ is real, symmetric and positive definite: there exists a constant $c_E(u)$ such that*

$$\forall\, \xi \in \mathbb{R}^N,\ \langle E(u)\xi,\, \xi \rangle \geq c_E(u)|\xi|^2.$$

   In the following, we will denote by

$$E(u) = \begin{pmatrix} E_{11}(u) & E_{12}(u) \\ E_{21}(u) & E_{22}(u) \end{pmatrix} \tag{2.10}$$

   the block decomposition of $E(u)$ corresponding to (2.8).

The change of variable $v = v(u)$ guarantees that the initial-boundary value problem (1.1) is equivalent to

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} & u^\varepsilon \in \mathbb{R}^N \\ \mathbb{B}(u^\varepsilon(t,\, 0)) \equiv \bar{g} \\ u^\varepsilon(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{2.11}$$

where $v(\bar{u}_0) = \bar{v}_0$, $v(\bar{u}_b) = \bar{v}_b$, $\mathbb{B}(u^\varepsilon) = \tilde{\mathbb{B}}(v(u^\varepsilon))$.

Relying on the block decomposition of the viscosity matrix described in Hypothesis 2, in Section 2.2.1 the explicit definition of the function ß is introduced. Moreover, in Section 4.3 such a definition is extended to a more general formulation.

In order to justify the assumptions summarized in Hypothesis 2 it will be made reference to the works of Kawashima and Shizuta, in particular to [34] and to [37].

In particular, in [37] it is assumed that the system in conservation form

$$v_t + f(v)_x = \Big(\tilde{B}(v)v_x\Big)_x \tag{2.12}$$

admits a convex and dissipative entropy η which moreover satisfies

$$\Big(D^2\eta(v)\Big)^{-1}\tilde{B}(v)^T = \tilde{B}(v)\Big(D^2\eta(v)\Big)^{-1}.$$

If one performs the change of variables defined by $w = \nabla\eta(v)$, finds that system (2.6) is equivalent to

$$\hat{E}(w)w_t + \hat{A}(w)w_x = \Big(\hat{B}(w)w_x\Big)_x,$$

with $\hat{A}$ and $\hat{B}$ symmetric.

It is then introduced the assumption

Condition N: the kernel of $\hat{B}(w)$ does not depend on w.

and it is proved that *Condition N* holds if and only if there exists a change of variable $w = w(u)$ which ensures that system (2.12) is equivalent to

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx}$$

for some $E(u)$ that satisfies condition 5 in Hypothesis 2 and some $A(u,\, u_x)$ and $B(u)$ as in condition 2 and 1 respectively.

Moreover, it is shown that *Condition N* is verified in the case of several systems with physical meaning.

On the other side, the fourth assumption in Hypothesis 2 is the so called Kawashima condition and was introduced in [34]. Roughly speaking, its meaning is to ensure that there exists an interaction between the parabolic and the hyperbolic component of system (2.6) and hence to guarantee the regularity of a solution (2.6).

Examples 2.1 and 2.2 in Section 2.2.2 show that, if the condition of block linear degeneracy is violated, then one encounters pathological behaviours, in the following sense. One may find a solution of (2.6) that are not continuously differentiable. This is a pathological behaviour in the sense that, when one introduces a parabolic approximation, one expects a regularizing effect. In Section 1.1.1 we explain why it is interesting to look for an extension of the condition of block linear degeneracy to a more general setting. This problem will be tackled in [13].

Finally, Example 2.3 show that if the first condition in Hypothesis 2 is violated, then one can have pathological behaviours like the one described before. In other words, if the rank of B can vary, then one may find a solution of (2.7) which is not continuously differentiable.

2.2.1 An explicit definition of boundary datum for the parabolic problem

Thanks to Hypothesis 2 system (1.2) is equivalent to

$$\begin{cases} E(u)u_t + A\big(u,\, u_x\big)u_x = B(u)u_{xx} \qquad u \in \mathbb{R}^N \\ ß(u)(t,\, 0)) \equiv \bar{g} \\ u(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{2.13}$$

where $ß(u^\varepsilon) = \tilde{ß}[v^\varepsilon(u)]$. In this section we define the function ß. Once $ß(u)$ is known, one can obtain $\tilde{ß}(v)$ exploiting the fact that the map $v(u)$ is invertible. In Section 4.3 we extend this definition to a more general setting.

Let r be, as in the statement of Hypothesis 2, the rank of the matrix B. Decompose u as $u = (u_1, u_2)^T$, where $u_1 \in \mathbb{R}^{N-r}$ and $u_2 \in \mathbb{R}^r$. then the equation

$$E(u)u_t + A\big(u, u_x\big)u_x = B(u)u_{xx}$$

can be written as

$$\begin{cases} E_{11}u_{1t} + E_{12}u_{2t} + A_{11}u_{1x} + A_{12}u_{2x} = 0 \\ E_{21}u_{1t} + E_{22}u_{2t} + A_{21}u_{1x} + A_{22}u_{2x} = bu_{2xx} \end{cases} \tag{2.14}$$

The blocks E_{11}, A_{11} and so on are as in (2.9) and (2.10).

In the first line of (2.14) only first order derivatives appear, while in the second line there is a second order derivative u_{2xx}. In this sense, u_1 can be regarded as the *hyperbolic* component of (2.14), while u_2 is the *parabolic* component. As explained in section 1.1.1, one can impose r boundary conditions on u_2, while one can impose on u_1 a number of boundary conditions equal to the number of eigenvalues of $E_{11}^{-1}A_{11}$ with strictly positive real part.

We recall that we denote by n_{11} the number of strictly negative eigenvalues of A_{11}, by q the dimension of $\ker A_{11}$ and by n the number of strictly negative eigenvalues of A. One can prove that the number of eigenvalues of $E_{11}^{-1}A_{11}$ with strictly negative real part is equal to n_{11} (see Lemma (3.1) in Section 3.1.1, which was actually introduced in [6]). Also, the dimension of the kernel of $E_{11}^{-1}A_{11}$ is q.

Let $\vec{\zeta}_i(u, 0) \in \mathbb{R}^{N-r}$ be an eigenvector of $E_{11}^{-1}A_{11}(u)$ associated to an eigenvalue η_i with non positive real part. Let $Z_i(u, 0) \in \mathbb{R}^N$ be defined by

$$Z_i := \begin{pmatrix} \vec{\zeta}_i \\ 0 \end{pmatrix} \tag{2.15}$$

and finally let

$$\mathcal{Z}(u) := \text{span}\langle Z_1(u, 0), \dots, Z_{n_{11}+q}(u, 0)\rangle.$$

Finally, let

$$\mathcal{W}(u) := \text{span}\Big\langle \begin{pmatrix} 0 \\ \vec{e}_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vec{e}_r \end{pmatrix}, \begin{pmatrix} \vec{w}_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \vec{w}_{N-r-n_{11}-q} \\ 0 \end{pmatrix} \Big\rangle,$$

where $\vec{e}_i \in \mathbb{R}^r$ are the vectors of a basis in $\mathbb{R}^r$ and $\vec{w}_j \in \mathbb{R}^{N-r}$ are the eigenvectors of $E_{11}^{-1}A_{11}$ associated to eigenvalues with strictly positive real part.

Since $\mathcal{W}(u) \oplus \mathcal{V}(u) = \mathbb{R}^N$, every $u \in \mathbb{R}^N$ can be written as

$$u = u_w + u_z, \quad u_w \in \mathcal{W}(u), u_z \in \mathcal{Z}(u). \tag{2.16}$$

in a unique way.

We define ß as follows.

Definition 2.1. The function ß which gives the boundary condition in (2.11) is

$$\begin{aligned} ß : \mathbb{R}^N &\to \mathbb{R}^{N-n_{11}-q} \\ u &\mapsto u_z, \end{aligned} \tag{2.17}$$

where u_z is the component of u along $\mathcal{W}(u)$, according to the decomposition (2.16).

We refer to Section 4.3 for an extension of this definition.

2.2.2 Examples

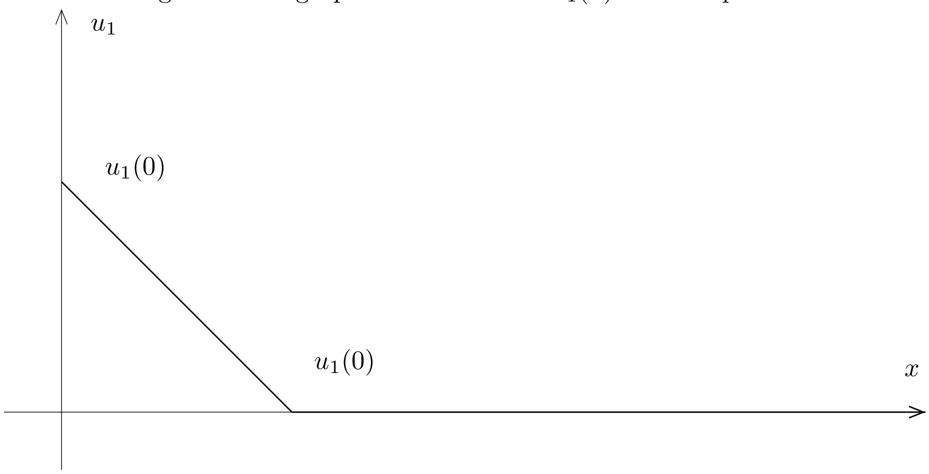
Example 2.1. This example deals with the equation

$$A(u, u_x)u_x = B(u)u_{xx}, \tag{2.18}$$

in the case the condition of block linear degeneracy (the third in Hypothesis 2) does not hold.

System (2.18) is satisfied by the steady solutions of (2.7). One therefore expects the solution of (2.18) to have good regularity properties: the example which is going to be discussed show that if the condition of

Figure 1: the graph of the function $u_1(x)$ in Example 2.1



block linear degeneracy does not hold a function u satisfying (2.18) may have the graph illustrated by Figure 1 and hence be not $\mathcal{C}^1$. Moreover, the figure suggests that a pathological behavior typical of the solution of the porous-media equation may occur, namely it may happen that a solution is different from zero on a interval and then vanishes identically. Let $u = (u_1,\, u_2)^T$ and let

$$B(u) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

and

$$A(u,\, u_x) = \begin{pmatrix} u_1 & 1 \\ 1 & 0, \end{pmatrix}.$$

Then the assumption of block linear degeneracy is not satisfied in a neighborhood of $u_1 = 0$: we therefore impose the limit conditions

$$\lim_{x \to +\infty} u_1(x) = 0 \qquad \lim_{x \to +\infty} u_2(x) = 0.$$

In this case equation (2.18) writes

$$\begin{cases} u_1\big(u_{1x} + 1\big) = 0 \\ u_1 = u_{2x} \end{cases} \tag{2.19}$$

Since the equation satisfied by u_1 admits more solutions in a neighborhood of $u_1 = 0$, to introduce a selection principle we proceed as follows.

We consider a solution $u^\nu = (u_1^\nu,\, u_2^\nu)^T$ of (2.18) such that

$$\lim_{x \to +\infty} u_1^\nu(x) = \nu.$$

The parameter ν is positive and we study the limit $\nu \to 0^+$. Fix the initial datum $u_{10} > 0$. The component u_1^ν satisfies the Cauchy problem

$$\begin{cases} u_{1x}^\nu = \dfrac{\nu - u_1^\nu}{u_1^\nu} \\ u_1^\nu(0) = u_{10}^\nu \end{cases} \tag{2.20}$$

If ν is sufficiently small then $u_{10} > \nu$ and hence u_{1x} is always negative. In other words, u_1^ν is a monotone non increasing function which satisfies $\nu < u_1^\nu(x) \leq u_{10}$ for every x. Also, note that if $\nu_1 < \nu_2$ then, for every x,

$$u_1^\nu(x) \leq u_2^\nu(x). \tag{2.21}$$

This can be deduced by a comparison argument applied to the Cauchy problem (2.20). Indeed, if $u_1^\nu > 0$ then

$$\frac{\nu - u_1^{\nu_1}}{u_1^{\nu_1}} < \frac{\nu - u_1^{\nu_2}}{u_1^{\nu_2}}.$$

In particular, from (2.21) we deduce that for every x $u_1^\nu(x)$ is monotone decreasing with respect to ν and hence it admits limit $\nu \to 0^+$.

Denote by u_1 the pointwise limit of u_1^ν for $\nu \to 0^+$: we claim that u_1 satisfies

$$u_1 \big(u_{1x} + 1 \big) = 0.$$

Indeed, let

$$x_0 := \min\{x : \ u^1(x) = 0\}.$$

If $x < x_0$, then by monotoniticity for every ν and for every $y < x$ $u_{10} \geq u_1^\nu(y) \geq u_1(x) > 0$. Also, if ν is sufficiently small then $u_1^\nu(x) > 0$ and by monotonicity $u_1^\nu(y) \geq u_1^\nu(x) > 0$. Consider the relation

$$u_1^\nu(x) = u_{10} + \int_0^x \frac{\nu - u_1^\nu(y)}{u_1^\nu(y)} dy.$$

We take the limit $\nu \to 0^+$ and, applying Lebesgue's dominated convergence theorem, we get

$$u_1(x) = u_{10} - x.$$

On the other side, if $x \geq x_0$ then by monotonicity $u_1(x) \leq 0$. On the other side, $u_1(x) \geq 0$ for every x. We conclude that $u_1(x) = 0$ if $x \geq x_0$. In other words, $x_0 = u_{10}$ and

$$u_1(x) = \begin{cases} u_{10} - x & x \leq u_{10} \\ 0 & x \geq u_{10}. \end{cases}$$

This function ha the graph illustrated in Figure 1 and it is not continuously differentiable. In this sense, we encounter a pathological behaviour.

Remark 2.1. An alternative interpretation of the previous considerations is the following.

Consider the family of systems

$$\begin{cases} u_1 u_{1x} + u_{2x} = 0 \\ u_{1x} = u_{2xx} \\ u_1(0) = \upsilon \qquad \lim_{x \to +\infty} u_1(x) = y \qquad \lim_{x \to +\infty} u_2(x) = 0 \end{cases} \tag{2.22}$$

parametrized by the limit value $y \in \mathbb{R}$. Let $\mathcal{U}_1(y)$ be the set of the values υ such that (2.22) admits a solution. The ODE satisfied by u_1 is

$$u_{1x} = \frac{y - u_1}{u_1}$$

when $u_1 \neq 0$. From a standard analysis it turns out that when $y > 0$, then $\mathcal{U}_1(y) =]0, +\infty[$. When $y = 0$, $\mathcal{U}_1(0) = [0, +\infty[$, while when $y < 0$, $\mathcal{U}_1(y) = \{y\}$.

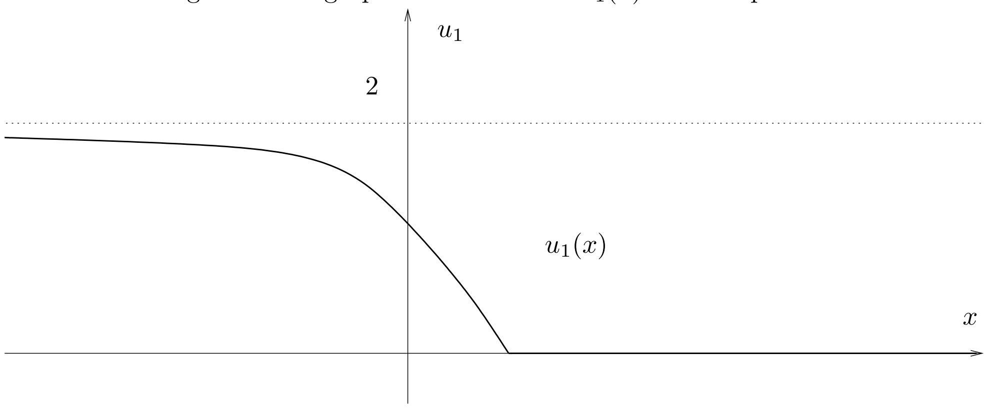
In other words, $\mathcal{U}_1$ is a manifold of dimension 1 when $y > 0$, a manifold with boundary and with dimension 1 when $y = 0$, while when $y < 0$ the dimension of the manifold drops to zero, i.e. the manifold reduces to a point.

Example 2.2. Example 2.1 shows that, if the condition of blcok linear degeneracy (the third in Hypothesis 2) is not satisfied, then one can find a steady solution of

$$E(u)u_t + A(u, \, u_x)u_x = B(u)u_{xx} \tag{2.23}$$

which is not continuously differentiable.

Figure 2: the graph of the function $u_1(x)$ in Example 2.2



As we see in the following sections, steady solutions are important when studying the limit of the parabolic approximation (2.13). Other solutions that play an important role are travelling wave solutions, i.e U such that

$$\Big(A(U,\, U') - \sigma E(U)\Big)U' = B(U)U''. \tag{2.24}$$

In the previous expression, σ is a real parameter and it is the speed of the travelling wave. More precisely, in the following sections we study travelling waves such that σ is close to an eigenvalue of the matrix $E^{-1}A$.

Let

$$A(u,\, u_x) := \begin{pmatrix} u_1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B(u) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.25}$$

which are obtained from a system in conservation form

$$u_t + f(u)_x = \Big(B(u)u_x\Big)_x$$

where

$$f(u) = \Big(u_1^2/2 + u_2,\, u_1 + u_2,\, 0\Big)^T$$

and $B(u)$ is the constant matrix defined by (2.25).

The matrix A defined by (2.25) has an eigenvalue identically equal to zero: we will focus on the travelling waves with speed $\sigma = 0$.

System (2.24) can then be rewritten as

$$\begin{cases} u_1 u_{1x} + u_{2x} = 0 \\ u_{1x} + u_{2x} = u_{2xx} \\ 0 = u_{3xx}. \end{cases} \tag{2.26}$$

As in the case considered in Example 2.1, the condition of block linear degeneracy is not satisfied in a neighborhood of $u_1 = 0$ and therefore we study travelling wave solutions such that

$$\lim_{x \to +\infty} u_1(x) = 0 \qquad \lim_{x \to +\infty} u_2(x) = 0 \qquad \lim_{x \to +\infty} u_3(x) = 0.$$

From system (2.26) and from the previous condition one obtains

$$\begin{cases} u_1^2/2 + u_2 = 0 \\ u_1 + u_2 = u_{2x} \\ u_3 \equiv 0. \end{cases}$$

Hence in the following we ignore the third component of the solution and we study only the first two lines of the system, which can be rewritten as

$$\begin{cases} u_2 = -u_1^2/2 \\ u_1 u_{1x} = u_1^2/2 - u_1. \end{cases} \tag{2.27}$$

If $u_1 \neq 0$, the second line is equivalent to

$$u_{1x} = u_1/2 - 1.$$

Fix u_{10} such that $0 < u_{10} < 2$. We impose $u_1(0) = u_{10}$ and we obtain

$$u_1(x) = 2 + (u_{10} - 2)e^{x/2},$$

This solution can be extended as far as $u_1 > 0$: when u_1 reaches the value zero, (2.27) admits more that one solution.

In order introduce a selection principle, we proceed as in Example 2.1 and we introduce a family of solutions $u^\nu = (u_1^\nu,\, u_2^\nu,\, u_3^\nu)$ of (2.24) such that

$$\lim_{x \to +\infty} u_1^\nu(x) = \nu \qquad \lim_{x \to +\infty} u_2^\nu(x) = 0 \qquad \lim_{x \to +\infty} u_3^\nu(x) = 0. \tag{2.28}$$

We also impose $u_1^\nu(0) = u_{10}$. In the previous expression, ν is a small and positive parameter and we study limit $\nu \to 0^+$. One can repeat the same considerations as in the previous example and conlude that when $\nu \to 0^+$ the solution u_1^ν converges pointwise to

$$u_1(x) = \begin{cases} u_1(x) = 2 + (u_1(0) - 2)e^{x/2} & x \leq 2\log\Big(2/\big(2 - u_{10}\big)\Big) \\ 0 & x > 2\log\Big(2/\big(2 - u_{10}\big)\Big) \end{cases} \tag{2.29}$$

This function has the graph illustrated in Figure 2 and it is not continuously differentiable.

Example 2.3. The aim of this example is to show that if in a system of the form (2.7) the rank of the matrix B is not constant, thus contradicting the first assumption of Hypothesis 2, then pathological behaviors of the same kind described before may appear. More precisely, we find a steady solution which is not continuously differentiable.

We consider the system in conservation form

$$u_t + f(u)_x = \Big(B(u)u_x\Big)_x$$

with

$$f(u_1,\, u_2) = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} \qquad B(u_1,\, u_2) = \begin{pmatrix} \gamma u_1^{\gamma - 1} & 0 \\ 0 & 1 \end{pmatrix}$$

for some $\gamma > 3$. In this case, the rank of $B(u)$ drops from 2 to 1 when u_1 reaches 0: to find out pathological behaviors it seems therefore natural to study the solution in a neighborhood of $u_1 = 0$. More precisely, the attention is focused on steady solutions

$$f(u)_x = \Big(B(u)u_x\Big)_x \tag{2.30}$$

such that

$$\lim_{x \to +\infty} u_1(x) = 0 \qquad u_1(x) \geq 0 \qquad \lim_{x \to +\infty} u_2(x) = 0 \tag{2.31}$$

and it will be shown that the graph of the first component u_1 has the shape illustrated in Figure 3. Hence u_1 is not continuously differentiable and presents a behaviour like the one typical of the solutions of the porous-media equation: it is different from zero on a interval and then vanishes identically.

In this case equation (2.30) writes

$$\begin{cases} u_{2x} = \Big(u_1^\gamma\Big)_{xx} \\ u_1 = u_{2x} \end{cases} \tag{2.32}$$

and hence after some computations one obtains

$$\begin{cases} u_{1x} = -\sqrt{\dfrac{2}{\gamma(\gamma+1)} u_1^{(3-\gamma)}} \\ \\ u_2 = \displaystyle\int_{+\infty}^{x} u_1(y)dy. \end{cases} \tag{2.33}$$

The equation satisfied by u_1 admits more than one solution in a neighborhood of $u_1 = 0$.

To introduce a selection principle we consider the matrix

$$B^\nu(u^\nu) = \begin{pmatrix} \nu + \gamma(u_1^\nu)^{\gamma-1} & 0 \\ 0 & 1 \end{pmatrix},$$

and the equation

$$u_{xx}^\nu = \Big(B^\nu(u^\nu)\Big)^{-1} f(u^\nu)_x. \tag{2.34}$$

In the previous expression, ν is a positive parameter and we study the limit $\nu \to 0^+$. We keep the limit conditions (2.31) fixed. In other words, this time we do not perturb the limit condition (as in the previous examples), but the equation itself. Note that now B^ν is invertible. The solution of (2.34) with limit conditions (2.31) satisfies

$$(u_2^\nu)^2 = \nu(u_1^\nu)^2 + \frac{2\gamma}{\gamma+1}(u_1^\nu)^{\gamma+1}. \tag{2.35}$$

Indeed, from

$$\begin{cases} u_{2x}^\nu = \Big(\nu u_{1x}^\nu + \gamma(u_1^\nu)^{\gamma-1} u_{1x}^\nu\Big)_x \\ u_{1x}^\nu = u_{2xx}^\nu \end{cases}$$

one obtains integrating

$$\begin{cases} u_2^\nu = \nu u_{1x}^\nu + \gamma(u_1^\nu)^{\gamma-1} u_{1x}^\nu \\ u_1^\nu = u_{2x}^\nu. \end{cases}$$

Multiplying the first line by $u_1 = u_{2x}$ and then integrating again one obtains (2.35). From (2.35) we get

$$u_{2x}^\nu = -\frac{(2\nu + \gamma(u_1^\nu)^\gamma) u_{1x}^\nu}{\sqrt{\nu u_1^\nu + \dfrac{2\gamma}{\gamma+1}(u_1^\nu)^{\gamma+1}}}.$$

Taking $u_{2x}^\nu = u_1^\nu$ one eventually gets

$$u_{1x}^\nu = -\frac{\sqrt{\nu u_1^\nu + \frac{2\gamma}{\gamma+1}(u_1^\nu)^{\gamma+1}}}{2\nu + \gamma(u_1^\nu)^\gamma}.$$

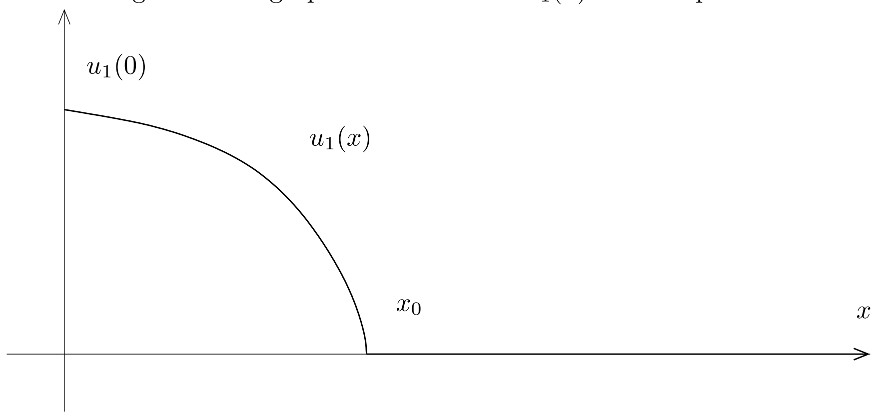
Fix $u_{10} > 0$ and consider the Cauchy problem

$$\begin{cases} u_{1x}^\nu = -\dfrac{\sqrt{\nu u_1^\nu + \frac{2\gamma}{\gamma+1}(u_1^\nu)^{\gamma+1}}}{2\nu + \gamma(u_1^\nu)^\gamma} \\ u_1^\nu(0) = u_{10}^\nu \end{cases}$$

One can then exploit the same considerations as in the previous examples: in particular, the fact that for every x $u_1^\nu(x)$ is monotone decreasing with respect to ν follows from

$$\frac{\partial}{\partial \nu}\left(-\frac{\sqrt{\nu u_1^\nu + \frac{2\gamma}{\gamma+1}(u_1^\nu)^{\gamma+1}}}{2\nu + \gamma(u_1^\nu)^\gamma} \right)\Bigg|_{\nu=0,\ u_1^\nu>0} < 0$$

Figure 3: the graph of the function $u_1(x)$ in Example 2.3



if $\gamma < 3$. One can the prove (proceeding as in the previous examples) that when $\nu \to 0^+$, u_1^ν converges pointwise to a function u_1 satisfying

$$u_1(x) = \begin{cases} \sqrt[\gamma-1]{\dfrac{(\gamma-1)^2}{2\gamma(\gamma+1)}(x-x_0)^2} & x \le x_0 \\ 0 & x > x_0, \end{cases}$$

where

$$x_0 = \sqrt{\frac{2\gamma(\gamma+1)}{(\gamma-1)^2} u_{10}^{\gamma-1}}.$$

This function has the graph illustrated in Figure 3 and it is not continuously differentiable.

2.3 General hypotheses

This section introduces the general hypotheses required in both cases, i.e. when the viscosity matrix $\tilde{B}$ in (1.1) is invertible and when it is singular. In the statements of the hypotheses we actually make reference to the formulation (2.1) and (2.13). Thus, in particular, we consider the same value $\bar{u}_0$ as in (2.1) and (2.13).

First of all, we assume strict hyperbolicity:

Hypothesis 3. *There exists $\delta > 0$ such that, if u belongs to a neighbourhood of $\bar{u}_0$ of size δ then all the eigenvalues of the matrix $E^{-1}(u)A(u, 0)$ are real. Moreover, there exists a constant $c > 0$ such that*

$$\inf_u \left\{ \left| \lambda_i(u, 0) - \lambda_j(u, 0) \right| \right\} \ge c > 0 \quad \forall\, i \ne j.$$

We also introduce an hypothesis of convergence:

Hypothesis 4. *Let*

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \\ \beta(u^\varepsilon(t, 0)) = g(t) \\ u^\varepsilon(0, x) = \bar{u}_0(x) \end{cases}$$

be a parabolic initial boundary value problem such that $g,\, \bar{u}_0 \in L^1_{loc}$ and

$$\left| g(0) - \beta\big(\bar{u}_0(0)\big) \right|,\ \mathrm{TotVar}\big\{g\big\},\ \mathrm{TotVar}\big\{\bar{u}_0\big\} \le \delta \tag{2.36}$$

For a suitable constant $\delta << 1$. Then we assume that there exists a time T_0, which depends only on the bound δ and on the matrices E, A and B, such that

$$\mathrm{TotVar}\big\{u^\varepsilon(t,\,\cdot)\big\} \leq C\delta \quad \forall\,\varepsilon,\ t \in [0,\,T_0].$$

We also assume directly a result of convergence and of uniqueness of the limit:

$$\forall\, t \in [0,\,T_0] \quad u^\varepsilon(t,\,\cdot) \to u(t)\ L^1_{loc} \text{ when } \varepsilon \to 0^+ \qquad \mathrm{TotVar}\big\{u(t)\big\} \leq C\delta.$$

We point out that the uniqueness of the limit is actually implied by the next hypotheses and the uniqueness results for the Standard Riemann Semigroup with boundary: it is made reference to [2] for the extension of the definition of SRS to initial boundary value problems, while an application of this notion to prove the uniqueness of the limit of the vanishing viscosity solutions was introduced in [11] in the case of the Cauchy problem.

It is also assumed:

Hypothesis 5. *It holds*

$$|\beta(\bar{u}_0) - \bar{g}| < \delta$$

for the same constant $\delta << 1$ that appears in Hypothesis 4.

An hypothesis of stability with respect to L^1 perturbations in the initial and boundary data is also introduced:

Hypothesis 6. *There exists a constant $L > 0$ such that the following holds.*

Two families of parabolic initial boundary value problems are fixed:

$$\begin{cases} u^{1\,\varepsilon}_t + A(u^{1\,\varepsilon},\,\varepsilon u^{1\,\varepsilon}_x)u^{1\,\varepsilon}_x = \varepsilon B(u^{1\,\varepsilon})u^{1\,\varepsilon}_{xx} \\ \beta(u^{1\,\varepsilon}(t,\,0)) = \bar{g}^1(t). \\ u^{1\,\varepsilon}(0,\,x) = \bar{u}^1_0(x) \end{cases} \qquad \begin{cases} u^{2\,\varepsilon}_t + A(u^{2\,\varepsilon},\,\varepsilon u^{2\,\varepsilon}_x)u^{2\,\varepsilon}_x = \varepsilon B(u^{2\,\varepsilon})u^{2\,\varepsilon}_{xx} \\ \beta(u^{2\,\varepsilon}(t,\,0)) = \bar{g}^2(t). \\ u^{2\,\varepsilon}(0,\,x) = \bar{u}^2_0(x) \end{cases} \tag{2.37}$$

with $\bar{u}^1_0$, $\bar{g}^1$ and $\bar{u}^2_0$, $\bar{g}^2$ in L^1_{loc} and satisfying the assumption (2.36).

Then for all $t \in [0,\,T_0]$ and for all $\varepsilon > 0$ it holds

$$\|u^{1\,\varepsilon}(t) - u^{2\,\varepsilon}(t)\|_{L^1} \leq L\Big(\|\bar{u}^1_0 - \bar{u}^2_0\|_{L^1} + \|\bar{g}^1 - \bar{g}^2\|_{L^1}\Big).$$

From the stability of the approximating solutions and from the L^1_{loc} convergence one can deduce the stability of the limit. More precisely, let u^1 and u^2 the limits of the two approximations defined above, then

$$\|u^1(t) - u^2(t)\|_{L^1} \leq L\Big(\|\bar{u}^1_0 - \bar{u}^2_0\|_{L^1} + \|\bar{g}^1 - \bar{g}^2\|_{L^1}\Big).$$

Finally, it is assumed that in the hyperbolic limit there is a *finite propagation speed* of the disturbances:

Hypothesis 7. *There exist constants $\beta, c > 0$ such that the following holds.*

Let $\bar{u}^1_0$, $\bar{g}^1$ and $\bar{u}^2_0$, $\bar{g}^2$ in L^1_{loc} and bounded, let $u^{\varepsilon\,1}$ and $u^{\varepsilon\,2}$ be the solutions of (2.37) *and u^1 and u^2 the corresponding limits. If*

$$\bar{g}^1(t) = \bar{g}^2(t) \quad \forall\, t \leq t_0 \qquad \bar{u}^1_0(x) = \bar{u}^2_0(x) \quad \forall\, x \leq b$$

then

$$|u^{\varepsilon 1}(x,\,t) - u^{\varepsilon 2}(x,\,t)| \leq \mathcal{O}\Big(e^{-c\min\big\{|x-\max\{0,\,\beta(t-t_0)\}|,|x-b+\beta t|\big\}/\epsilon}\Big) \quad \forall x \in [\max\{0,\,\beta(t-t_0)\},\,b-\beta t].$$

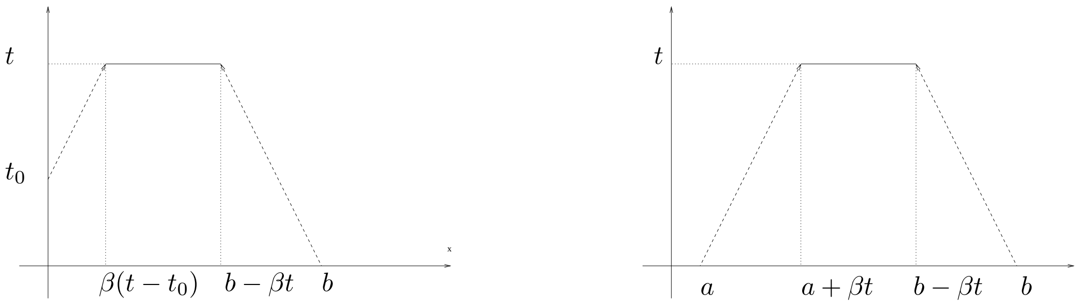
Analogously, if

$$\bar{u}^1_0(x) = \bar{u}^2_0(x) \quad \forall\, x \in [a,\,b]$$

then

$$|u^{\varepsilon 1}(x,\,t) - u^{\varepsilon 2}(x,\,t)| \leq \mathcal{O}\Big(e^{-c\min\big\{|x-a-\beta t|,|x-b+\beta t|\big\}/\epsilon}\Big) \quad \forall\, x \in [a+\beta t,\,b-\beta t].$$

Figure 4: the finite propagation speed in the hyperbolic limit



Remark 2.2. Hypothesis 7 may appear a bit technical. It implies the finite propagation speed of disturbances in the hyperbolic limit: an heuristic representation of this phenomenon is illustrated in Figure 4. Loosely speaking, the reason why we need finite propagation speed is the following. We have to be sure that the limit of (2.13) in the case of a generic couple of data $(\bar{u}_0, \bar{g})$ can be obtained gluing together the limit one obtains in the case of *cooked up* data, namely data connected by travelling wave profiles and boundary layer profiles.

3 Characterization of the hyperbolic limit in the case of an invertible viscosity matrix

The aim of this section is to provide a characterization of the limit of the parabolic approximation (1.1) when the viscosity matrix $\tilde{B}$ is invertible. The precise hypotheses that are assumed are listed in Section 2.1 and 2.3: in particular, these hypotheses guarantee that it is sufficient to study system

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \qquad u^\varepsilon \in \mathbb{R}^N \\ u^\varepsilon(t,\, 0) \equiv \bar{u}_b \\ u^\varepsilon(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{3.1}$$

where the matrices E, A and B satisfy Hypotheses 1 and 3.

The exposition is organized as follows. In Section 3.1 we discuss some preliminary results. More precisely, in Section 3.1.1 we recall some transversality results discussed in [6]. In Section 3.1.2 we recall the definition of monotone envelope of a function and we introduce some related results. In Section 3.2 we review some results in [7]. Namely, we give a characterization of the limit of parabolic approximation

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A\big(u^\varepsilon,\, \varepsilon u^\varepsilon_x\big)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \\ u^\varepsilon(0,\, x) = \begin{cases} u^- & x \le 0 \\ \bar{u}_0 & x > 0. \end{cases} \end{cases}$$

In Section 4.2.3 it is given a characterization of the limit of the parabolic approximation (3.1) in the case of a non characteristic boundary, i.e. when none of the eigenvalues of $E^{-1}A$ can attain the value 0. The case of a characteristic boundary occurs when one of the eigenvalues of $E^{-1}A$ can attain the value 0 and it is definitely more complicated than the previous one. The characterization of the limit (3.1) in the case of a characteristic boundary is discussed in Section 3.2.3.

3.1 Preliminary results

3.1.1 Transversality results

The following lemma is discussed in [28]. However, for completeness we will repeat the proof.

Lemma 3.1. *Assume that Hypotheses 1 and 3 hold. Then for any given u and u_x*

1. $B^{-1}(u)A(u,\, u_x)\vec{\xi} = 0 \iff A(u,\, u_x)\vec{\xi} = 0 \iff E^{-1}(u)A(u,\, u_x)\vec{\xi} = 0 \qquad \vec{\xi} \in \mathbb{R}^N$
2. *the number of eigenvalues of the matrix $B^{-1}(u)A(u,\, 0)$ with negative (respectively positive) real part is equal to the number of eigenvalues of $E^{-1}(u)A(u,\, 0)$ with negative (respectively positive) real part.*

Proof. The first point is actually an immediate observation.

To simplify the notation, in the proof of the second point, we will write A, B and E instead of $A(u,\, 0)$, $B(u)$ and $E(u)$ respectively. One can define the continuous path

$$\begin{aligned} F : [0,\, 1] &\to \mathbb{M}^{N \times N} \\ s &\mapsto (1-s)B + sE, \end{aligned}$$

which satisfies the following condition: for every $s \in [0,\, 1]$, $F(s)$ is positively definite and hence invertible. Indeed,

$$\forall \vec{\xi} \in \mathbb{R}^N \quad \langle F(s)\vec{\xi},\, \vec{\xi} \rangle \geq \min\{c_B(u),\, c_E(u)\}|\vec{\xi}|^2. \tag{3.2}$$

Moreover, by classical results (see for example [38]) from the continuity of the path $F(s)^{-1}A$ it follows the continuity of the corresponding paths of eigenvalues $\lambda_1(s) \dots \lambda_N(s)$.

Let $k-1$ be the number of eigenvalues of $F(0)^{-1}A = B^{-1}A$ with negative real part:

$$Re(\lambda_1(0)) < \dots Re(\lambda_{k-1}(0)) < 0 = Re(\lambda_k(0)) < Re(\lambda_{k+1}(0)) < \dots Re(\lambda_N(0)).$$

Because of the continuity of $\lambda_1(s) \dots \lambda_i(s)$, to prove that the number of negative eigenvalues of $F(1)^{-1}A = E^{-1}A$ is $(k-1)$ it is sufficient to prove that for any $i = 1 \dots (k-1)$, $\lambda_i(s)$ cannot cross the imaginary axis. Moreover, the case $\lambda_i(s) = 0$ is excluded because otherwise the corresponding eigenvector $\vec{\xi}$ should satisfy $F(s)^{-1}A\vec{\xi}(s) \equiv 0$ for any s and hence $\lambda_i(s) \equiv 0$. It remains therefore to consider the possibility that $\lambda_i(s)$ is purely imaginary.

By contradiction, assume that there exist $\bar{s} \in [0,\, 1]$, $\lambda \in \mathbb{R} \setminus \{0\}$, $r \in \mathbb{C}^N$ such that

$$F(\bar{s})^{-1}Ar = i\lambda r : \tag{3.3}$$

just to fix the ideas, it will be supposed that $\lambda > 0$. Moreover, let

$$r = r_1 + ir_2, \quad r_1,\ r_2 \in \mathbb{R}^N.$$

Then from (3.3) one gets

$$\begin{aligned} Ar_1 &= -\lambda F(\bar{s})r_2 \\ Ar_2 &= \lambda F(\bar{s})r_1 \end{aligned}$$

and hence from (3.2) and from the symmetry of A it follows

$$\begin{aligned} &\langle Ar_1,\ r_2 \rangle \leq -\lambda \min\{c_B(v),\, c_E(v)\}|r_2|^2 \\ &\langle Ar_1,\ r_2 \rangle = \langle Ar_2,\ r_1 \rangle \geq \lambda \min\{c_B(v),\, c_E(v)\}|r_1|^2, \end{aligned}$$

which is a contradiction since $\lambda > 0$. This ends the proof of Lemma 3.1. $\square$

In the following, given a matrix C we denote by $V^s(C)$ the direct sum of all eigenspaces associated with eigenvectors with strictly negative real part, by $V^c(C)$ the direct sum of the eigenspace associated with the eigenvector with zero real part, and $V^u(C)$ the the direct sum of all eigenspaces associated with eigenvectors with strictly positive real part. Lemma 3.1 ensures that, if Hypothesis 1 holds, then for all u, u_x

$$\begin{aligned} &V^c\big(E^{-1}(u)A(u,\, 0)\big) = V^c\big(A(u,\, 0)\big) = V^c\big(B^{-1}(u)A(u,\, 0)\big) \\ &dimV^u\big(E^{-1}(u)A(u,\, 0)\big) = dimV^u\big(B^{-1}(u)A(u,\, 0)\big) \\ &dimV^s\big(E^{-1}(u)A(u,\, u_x)\big) = dimV^s\big(B^{-1}(u)A(u,\, u_x)\big) \end{aligned}$$

Lemma 3.2. *Assume that Hypotheses 1 and 3 hold, then*

$$V^u\Big(E^{-1}(u)A(v,\,0)\Big)\cap V^s\Big(B^{-1}(u)A(v,\,0)\Big)=\{0\}.$$

Thanks to the previous lemma, which is a direct application of Lemma 7.1 in [6], one can conclude that actually

$$V^u\Big(E^{-1}(u)A(u,\,0)\Big)\oplus V^c\Big(E^{-1}(u)A(u,\,0)\Big)\oplus V^s\Big(B^{-1}(u)A(u,\,0)\Big)=\mathbb{R}^N. \tag{3.4}$$

3.1.2 Some results about the monotone envelope of a function

The aim of this section is to collect some results that will be needed in Section 3.2.3.

In the following, $\text{conc}_{[0,\,s]}f$ will denote the concave envelope of the function f on the interval $[0,\,s]$, namely

$$\Big(\text{conc}_{[0,\,s]}f\Big)(\tau):=\inf\Big\{h(\tau):\;h(t)\geq f(t)\quad\forall\,t\in[0,\,s],\;h\text{ is concave}\Big\}. \tag{3.5}$$

We will consider only the case of a function $f\in\mathcal{C}^{1,\,1}_k([0,\,s])$, i.e. $f\in\mathcal{C}^{1,\,1}$ and f' is Lipschitz continuous with Lipschitz constant smaller or equal to k.

The symbol $\text{mon}_{[0,\,s]}f$ will denote the monotone envelope of the function f on the interval $[0,\,s]$, i.e.

$$\Big(\text{mon}_{[0,\,s]}f\Big)(\tau):=\sup\Big\{h(\tau):\;h(t)\geq f(t)\quad\forall\,t\in[0,\,s],\;h\text{ is concave and nondecreasing}\Big\}. \tag{3.6}$$

.

Proposition 3.1 is very similar to more general results discussed for example in [5] and [32]. The case considered here is actually slightly different: an alternative proof of Proposition 3.1 can be thus found in the online version of the present paper, available at www.math.ntnu.no/conservation/2006/021.htm.

Proposition 3.1. *If* $f\in\mathcal{C}^{1,\,1}_k([0,\,s])$ *, then* $\text{conc}_{[0,\,s]}f\in\mathcal{C}^{1,\,1}_k([0,\,s])$*.*

The following result describes the relation between the concave and the monotone envelope.

Lemma 3.3. *Let* $f\in\mathcal{C}^{1,\,1}_k([0,\,s])$*, then*

$$\text{mon}_{[0,\,s]}f(\tau)=\begin{cases}\text{conc}_{[0,\,s]}f(\tau) & \text{if } \tau\leq\tau_0\\ \text{conc}_{[0,\,s]}f(\tau_0) & \text{if } \tau>\tau_0.\end{cases}$$

The value τ_0 *is given by*

$$\tau_0:=max\Big\{t\in[0,\,s]:\;\big(\text{conc}_{[0,\,s]}f\big)'(t)\geq 0\Big\}. \tag{3.7}$$

If $\big(\text{conc}_{[0,\,s]}f\big)'(t)$ *is negative for every* t*, then we set* $\tau_0=0$*.*

Proof. Let

$$h(\tau)=\begin{cases}\text{conc}_{[0,\,s]}f(\tau) & \text{if } \tau\leq\tau_0\\ \text{conc}_{[0,\,s]}f(\tau_0) & \text{if } \tau>\tau_0.\end{cases}$$

The function h is concave, monotone non decreasing and satisfies $h(\tau)\geq f(\tau)$ for every τ. Thus,

$$\text{mon}_{[0,\,s]}f(\tau)\leq h(\tau).$$

Assume by contradiction that there exists $\bar\tau$ such that

$$\text{mon}_{[0,\,s]}f(\bar\tau)<h(\bar\tau). \tag{3.8}$$

For every $\tau\in[0,\,s]$

$$\text{conc}_{[0,\,s]}f(\tau)\leq\text{mon}_{[0,\,s]}f(\tau),$$

thus $\bar\tau>\tau_0$ and moreover

$$\text{conc}_{[0,\,s]}f(\tau)=\text{mon}_{[0,\,s]}f(\tau)$$

for every $\tau\leq\tau_0$. Since $\text{mon}_{[0,\,s]}f$ is non decreasing, then

$$\text{mon}_{[0,\,s]}f(\tau)\geq\text{mon}_{[0,\,s]}f(\tau_0)=h(\tau)$$

for every $\tau\geq\tau_0$. This contradicts (3.8) and concludes the proof of the lemma. □

Combining Lemma 3.3 and Proposition 3.1 one deduces that if f belongs to $\mathcal{C}_k^{1,\,1}([0,\,s])$ then

$$\mathrm{mon}_{[0,\,s]} f \in \mathcal{C}_k^{1,\,1}([0,\,s_1]).$$

The following propostion collects some estimates that will be exploited in Section 3.2.3. Analogous results are discussed in [4] (Lemmata 2 and 3). The proof will be therefore omitted here, but can be found in the online version of the present paper, which is available at www.math.ntnu.no/conservation/2006/021.htm.

Proposition 3.2. *Let $f,\, g \in \mathcal{C}_k^{1,\,1}([0,\,s])$. Then*

1.

$$\|\mathrm{conc}_{[0,\,s]} f\|_{\mathcal{C}^0} \leq \|f\|_{\mathcal{C}^0} \qquad \|\big(\mathrm{conc}_{[0,\,s]} f\big)'\|_{\mathcal{C}^0} \leq \|f'\|_{\mathcal{C}^0}.$$

2.

$$\|\mathrm{conc}_{[0,\,s]} f - \mathrm{conc}_{[0,\,s]} g\|_{\mathcal{C}^0} \leq \|f-g\|_{\mathcal{C}^0} \qquad \|\big(\mathrm{conc}_{[0,\,s]} f - \mathrm{conc}_{[0,\,s]} g\big)'\|_{\mathcal{C}^0} \leq \|f'-g'\|_{\mathcal{C}^0} \tag{3.9}$$

The following result concerns the dependence of the concave envelope from the interval:

Proposition 3.3. *Let $s_1 \leq s_2$, $f \in \mathcal{C}_k^{1,\,1}([0,\,s_2])$ and assume that $\|f'\|_{\mathcal{C}^0([0,\,s_2])} \leq C_1\delta_0$. Then there are constants C_2 and C_3 such that*

$$\|\mathrm{conc}_{[0,\,s_1]} f - \mathrm{conc}_{[0,\,s_2]} f\|_{\mathcal{C}^0([0,\,s_1])} \leq C_2\delta_0(s_2-s_1) \tag{3.10}$$

and

$$\|(\mathrm{conc}_{[0,\,s_1]} f - \mathrm{conc}_{[0,\,s_2]} f)'\|_{\mathcal{C}^0([0,\,s_1])} \leq C_3\delta_0(s_2-s_1). \tag{3.11}$$

Combining Lemma 3.3 and Proposition 3.4 one finally gets

Proposition 3.4. *Let $s_1 \leq s_2$, $f \in \mathcal{C}_k^{1,\,1}([0,\,s_1])$ and assume that $\|f'\|_{\mathcal{C}^0([0,\,s_2])} \leq C_1\delta_0$. Then there are constants C_2 and C_3 such that*

$$\|\mathrm{mon}_{[0,\,s_1]} f - \mathrm{mon}_{[0,\,s_2]} f\|_{\mathcal{C}^0([0,\,s_1])} \leq C_2\delta_0(s_2-s_1)$$

and

$$\|(\mathrm{mon}_{[0,\,s_1]} f - \mathrm{mon}_{[0,\,s_2]} f)'\|_{\mathcal{C}^0([0,\,s_1])} \leq C_3\delta_0(s_2-s_1).$$

3.2 The hyperbolic limit in the case of an invertible viscosity matrix

3.2.1 The hyperbolic limit in the case of a Cauchy problem

In this section for completeness we will give a quick review of the construction of the Riemann solver for the Cauchy problem. We refer to [7] for the complete analysis. The goal is to characterize the limit of

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \\ u^\varepsilon(0,\, ,x) = \begin{cases} u^- & x<0 \\ \bar{u}_0 & x \geq 0 \end{cases} \end{cases} \tag{3.12}$$

under Hypotheses 1, 3, 5 and 6.

The construction works as follows. Consider a travelling profile

$$B(U)U'' = \Big(A(U,\, U') - \sigma E(U)\Big)U',$$

then U solves

$$\begin{cases} u' = v \\ B(u)v' = \Big(A(u,\, v) - \sigma E(u)\Big)v \\ \sigma' = 0. \end{cases} \tag{3.13}$$

Let $\lambda_i(\bar{u}_0)$ be the i-th eigenvalue of $E^{-1}(\bar{u}_0)A(\bar{u}_0,\, 0)$ and let $r_i(\bar{u}_0)$ be the corresponding eigenvalue. If one linearizes the previous system around the equilibrium point $(\bar{u}_0,\, \vec{0},\, \lambda_i(\bar{u}_0))$ one finds

$$\begin{pmatrix} 0 & I_N & 0 \\ 0 & A(\bar{u}_0,\, 0) - \lambda_i(\bar{u}_0)E(\bar{u}_0) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The generalized eigenspace corresponding to the 0 eigenvector is

$$V^c = \Big\{ (u,\, x r_i(\bar{u}_0),\, \sigma) : u \in \mathbb{R}^N,\, x,\, \sigma \in \mathbb{R} \Big\}.$$

It is then possible to define a *center* manifold which is parameterized by V^c: we refer to [33] for an extensive analysis, here we will just recall some of the fundamental properties of a center manifold.

Every center manifold is locally invariant with respect to (3.13), is defined in a neighbourhood of the equilibrium point $(\bar{u}_i,\, \vec{0},\, \lambda(\bar{u}_i)$ and it is tangent to V^c at $(\bar{u}_i,\, 0,\, \lambda(\bar{u}_i))$.

Also, the following holds. Define

$$V^{su} := \Big\{ (\bar{u}_0,\, \sum_{j \neq i} x_j r_j(\bar{u}_0),\, \sigma) : u \in \mathbb{R}^N,\, x_j,\, \sigma \in \mathbb{R} \Big\},$$

where $r_1(\bar{u}_0) \dots r_{i-1}(\bar{u}_0),\, r_{i+1}(\bar{u}_0) \dots r_N(\bar{u}_0)$ are eigenvectors of $E^{-1}(\bar{u}_0)A(\bar{u}_0,\, 0)$ different than $r_i(\bar{u}_0)$. If we write $\mathbb{R}^{2N+1} = V^c \oplus V^{su}$, then the map

$$\Phi_c : V^c \to V^c \oplus V^{us}$$

that parameterizes $\mathcal{M}^c$ can be chosen in such a way that if π_c is the projection from $V^c \oplus V^{us}$ onto V^c, then $\pi_c \circ \Phi_c$ is the identity.

Fix a center manifold $\mathcal{M}^c$. Putting all the previous considerations together, one gets that a point $(u,\, v,\, \sigma)$ belongs $\mathcal{M}^c$ if and only if

$$v = v_i r_i(\bar{u}_0) + \sum_{j \neq i} \psi_j(u,\, v_i,\, \sigma_i) r_j(\bar{u}_0). \tag{3.14}$$

Since all the equilibrium points $(u,\, 0,\, \sigma_i)$ lie on the center manifold, it turns out that when $v_i = 0$, then

$$\psi_j(u,\, 0,\, \sigma_i) = 0 \quad \forall\, j,\ u,\ \sigma_i.$$

Hence for all j, $\psi_j(u,\, v_i,\, \sigma_i) = v_i \phi_j(u,\, v_i,\, \sigma_i)$ for a suitable regular function ϕ_j. The relation (3.14) can be therefore rewritten as

$$v = v_i \Big(r_i(\bar{u}_0) + \sum_{j \neq i} \phi_j(u,\, v_i,\, \sigma_i) r_j(\bar{u}_0) \Big) := v_i \tilde{r}_i(u,\, v_i,\, \sigma_i).$$

Because of the tangency condition of the center manifold to the center space, it holds

$$\tilde{r}_i \Big(\bar{u}_0,\, 0,\, \lambda_i(\bar{u}_0 \Big) = r_i(\bar{u}_0).$$

Inserting the expression found for v into (3.13), one gets

$$v_{ix} B(u) \tilde{r}_i + v_i^2 B(u) D\tilde{r}_i \tilde{r}_i + v_i v_{ix} B(u) \tilde{r}_{iv} = v_i \Big(A(u,\, v_i \tilde{r}_i) - \sigma_i E(u) \Big) \tilde{r}_i.$$

Considering the scalar product of the previous expression with $\tilde{r}_i$ one obtains

$$v_{ix} \Big(\langle \tilde{r}_i,\, B(u)\tilde{r}_i \rangle + v_i \langle \tilde{r}_i,\, B(u)\tilde{r}_{iv} \rangle \Big) = v_i \Big(\langle \tilde{r}_i,\, \Big(A(u,\, v_i\tilde{r}_i) - \sigma_i E(u) \Big) \tilde{r}_i \rangle - v_i \langle \tilde{r}_i,\, B(u) D\tilde{r}_i \tilde{r}_i \rangle \Big).$$

Hence setting

$$\begin{aligned} c_i(u,\, v_i,\, \sigma_i) &:= \langle \tilde{r}_i,\, B(u)\tilde{r}_i \rangle + v_i \langle \tilde{r}_i,\, B(u)\tilde{r}_{iv} \rangle \\ a_i(u,\, v_i,\, \sigma_i) &:= \langle \tilde{r}_i,\, \Big(A(u,\, v_i\tilde{r}_i) - \sigma_i E(u) \Big) \tilde{r}_i \rangle - v_i \langle \tilde{r}_i,\, B(u) D\tilde{r}_i \tilde{r}_i \rangle \end{aligned}$$

one can define

$$\phi_i(u,\, v_i,\, \sigma_i) := \frac{a_i(u,\, v_i,\, \sigma_i)}{c_i(u,\, v_i,\, \sigma_i)}.$$

The fraction is well defined since $c_i(\bar u_0,\, 0,\, \lambda_i(\bar u_0)) \geq c_B(\bar u_0) > 0$ and hence c_i is strictly positive in a small neighborhood. The constant c_B is as in the third condition in Hypothesis 1.

Thus,

$$\frac{\partial \phi_i}{\partial \sigma_i}\Big|_{\big(\bar u_0,\, 0,\, \lambda_i(\bar u_0)\big)} = -\frac{a_i}{c_i^2}\frac{\partial c_i}{\partial \sigma_i} + \frac{1}{c_i}\frac{\partial a_i}{\partial \sigma_i}.$$

Since

$$a_i(\bar u_0,\, 0,\, \lambda_i(\bar u_0)) \geq c_B(\bar u_0) = 0$$

then

$$\begin{aligned}\frac{\partial \phi_i}{\partial \sigma_i}\Big|_{\big(\bar u_0,\, 0,\, \lambda_i(\bar u_0)\big)} &= \frac{1}{c_i}\Big(\langle \tilde r_{i\sigma},\, \Big(A(\bar u_0,\, 0) - \lambda_i E(\bar u_0)\Big) r_i(\bar u_0)\rangle - \langle r_i(\bar u_0),\, E(\bar u_0) r_i(\bar u_0)\rangle\Big)\\ &\quad + \frac{1}{c_i}\Big(\langle r_i(\bar u_0),\, \Big(A(\bar u_0,\, 0) - \lambda_i E(\bar u_0)\Big)\tilde r_{i\sigma}\rangle\Big)\\ &= -c_E(\bar u_0) < 0.\end{aligned} \tag{3.15}$$

In the previous computations, we have exploited the symmetry of $A(\bar u_0,\, 0)$ and $E(\bar u_0)$ and hence the fact that

$$\Big\langle r_i(\bar u_0,\, 0),\, \Big(A(\bar u_0,\, 0) - \lambda_i E(\bar u_0)\Big)\tilde r_{i\sigma}\Big\rangle = \Big\langle \Big(A(\bar u_0,\, 0) - \lambda_i E(\bar u_0)\Big) r_i(\bar u_0),\, \tilde r_{i\sigma}\Big\rangle = 0.$$

Also, the constant c_E in (3.15) is the same as in Hypothesis 1.

In conclusion, system (3.13) restricted to $\mathcal{M}^c$ can be rewritten as

$$\begin{cases} u' = v_i \tilde r_i(u,\, v_i,\, \sigma_i)\\ v_i' = \phi_i(u,\, v_i,\, \sigma_i) v_i\\ \sigma_i' = 0.\end{cases} \tag{3.16}$$

One actually studies the following fixed point problem, defined on a interval $[0,\, s_i]$:

$$\begin{cases} u(\tau) = \bar u_0 + \int_0^\tau \tilde r_i(u(\xi),\, v_i(\xi),\, \sigma_i(\xi)) d\xi\\ v_i(\tau) = f_i(\tau,\, u,\, v_i,\, \sigma_i) - \mathrm{conc} f_i(\tau,\, u,\, v_i,\, \sigma_i)\\ \sigma_i(\tau) = \frac{1}{c_E(\bar u_0)}\frac{d}{d\tau}\mathrm{conc} f_i(\tau,\, u,\, v_i,\, \sigma_i).\end{cases} \tag{3.17}$$

We have used the following notations:

$$f_i(\tau) = \int_0^\tau \tilde\lambda_i[u_i,\, v_i,\, \sigma_i](\xi) d\xi,$$

where

$$\tilde\lambda_i[u_i,\, v_i,\, \sigma_i](\xi) = \phi_i\Big(u_i(\xi),\, v_i(\xi),\, \sigma_i(\xi)\Big) + c_E(\bar u_0)\sigma.$$

Also, $\mathrm{conc} f_i$ denotes the concave envelope of the function f_i:

$$\mathrm{conc} f_i(\tau) = \inf\{h(s):\ \text{h is concave},\ h(y) \geq f_i(y)\ \forall\, y \in [0,\, s_i]\}.$$

The link between (3.17) and (3.13) is the following: let $(u_i,\, v_i,\, \sigma_i$ satisfy (3.17). Assume that $v_i < 0$ on $]a,\, b[$ and that $v_k(a) = v_k(b) = 0$. Define $\alpha_i(\tau)$ as the solution of the Cauchy problem

$$\begin{cases} \dfrac{d\alpha_i}{d\tau} = -\dfrac{1}{v_i(\tau)}\\ \alpha\Big(\dfrac{a+b}{2}\Big) = 0\end{cases}$$

then $(u_i \circ \alpha_i,\, v_i \circ \alpha_i,\, \sigma_i \circ \alpha_i)$ is a solution to (3.16) satisfying

$$\lim_{x \to -\infty} u_i \circ \alpha_i(x) = u_i(a) \qquad \lim_{x \to +\infty} u_i \circ \alpha_i(x) = u_i(b).$$

Thus, $u_i(a)$ and $u_i(b)$ are connected by a travelling wave profile.

As shown in [7], (3.17) admits a unique continuous solution $(u_i,\, v_i,\, \sigma_i)$. Also, one can show that $u_k(0)$ and $u_k(s_i)$ are connected by a sequence of rarefaction and travelling waves with speed close to $\lambda_i(\bar{u}_0)$. If $u(t,\, x)$ is the limit of

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \\ u^\varepsilon(0,\, ,x) = \begin{cases} \bar{u}_0 & x < 0 \\ u_i(s_i) & x \geq 0 \end{cases} \end{cases}$$

then as $\varepsilon \to 0^+$

$$u(t,\, x) = \begin{cases} \bar{u}_0 & x \leq \sigma_i(0)t \\ u_i(s) & x = \sigma_i(s)t \\ u_i(s_i) & x \geq \sigma_i(s_i)t \end{cases} \tag{3.18}$$

The i-th *curve of admissible states* is defined setting

$$T^i_{s_i}\bar{u}_0 := u_i(s_i).$$

If $s_i < 0$, one considers a fixed problem like (3.17), but instead of the concave envelope of f_i one takes the convex envelope:

$$\text{conv} f_i(\tau) = \sup\{h(s) :\ \text{h is convex},\ h(y) \leq f_i(y)\ \forall\, y\}.$$

Again, one can prove the existence of a unique fixed point $(u_i,\, v_i,\, \sigma_i)$.

Also, in [7] it is proved that the curve $T^i_{s_i}$ is Lipschitz continuos with respect to s_i and it is differentiable at $s_i = 0$ with derivative given by $\vec{r}_i(\bar{u}_0)$. Moreover, the function $T^i_{s_i}$ is Lipschitz continuos with respect to $\bar{u}_0$.

Consider the composite function

$$\psi(\bar{u}_0,\, s_1 \dots s_N) = T^1_{s_1} \circ \dots T^N_{s_N}\bar{u}_0$$

With the previous expression we mean that the starting point for T^{N-1}_s is not $\bar{u}_0$ but $T^N_{s_N}\bar{u}_0$. Thanks to the previous steps, the map ψ is Lipschitz continous with respect to $s_1 \dots s_N$ and differentiable at $s_1 = 0$, $\dots s_N = 0$. The column of the jacobian are $r_1(\bar{u}_0) \dots r_N(\bar{u}_0)$. Thus the jacobian is invertible and hence, exploiting the extension of the implicit function theorem discussed in [24] (page 253), the map $\phi(\bar{u}_0, \cdot)$ is invertible in a neighbourhood of $(s_1 \dots s_N) = (0 \dots 0)$. In other words, if u^- is fixed and is sufficiently close to $\bar{u}_0$, then the values of $s_1 \dots s_N$ are uniquely determined by the equation

$$u^- = \psi(\bar{u}_0,\, s_1 \dots s_N).$$

Taking the same u^- as in (3.12), one obtains the parameters $(s_1 \dots s_N)$ which can be used to reconstruct the hyperbolic limit u of (3.12). Indeed, once $(s_1 \dots s_N)$ are known then u can be obtained gluing together solutions like (3.18).

3.2.2 The hyperbolic limit in the non characteristic case

The goal of this section is to provide a characterization of the limit of the parabolic approximation (3.1) in the case of a non characteristic boundary, i.e. when none of the eigenvalues of $E^{-1}(u)A(u,\, u_x)$ can attain the value 0. More precisely, we assume the following.

Hypothesis 8. *Let $\lambda_1(u) \dots \lambda_N(u)$ be the eigenvalues of the matrix $E^{-1}(u)A(u,\, 0)$. Then there exists a constant c such that for every u*

$$\lambda_1(u) < \dots < \lambda_n(u) \leq -\frac{c}{2} < 0 < \frac{c}{2} \leq \lambda_{n+1}(u) < \dots < \lambda_N(u). \tag{3.19}$$

Thus, in the following n will denote the number of eigenvalues with strictly negative negative real part and $N-p$ the number of eigenvalues with strictly positive real part.

To give a characterization of the limit of (3.1) we will proceed as follows. We wil construct a map $\phi(\bar{u}_0, s_1 \dots s_N)$ which is a *boundary Riemann solver* in the sense that as $(s_1 \dots s_N)$ vary, describes states that can be connected to $\bar{u}_0$. We will the show that the map ϕ is locally invertible. Hence, given $\bar{u}_0$ and $\bar{u}_b$ sufficiently close, the values of $(s_1 \dots s_N)$ are uniquely determined by the equation

$$\bar{u}_b = \phi(\bar{u}_0,\, s_1 \dots s_N).$$

Once $(s_1 \dots s_N)$ are known the limit of (3.1) is completely characterized. The construction of the map ϕ is divided in some steps:

1. *Waves with positive speed*

   Consider the Cauchy datum $\bar{u}_0$, fix $(N-k)$ parameters $(s_{N-k} \dots s_N)$ and consider the value

   $$\bar{u} = T^{N-k}_{s_{N-k}} \circ \dots T^{N}_{s_N} \bar{u}_0.$$

   The curves $T^{N-k}_{s_{N-k}} \dots T^{N}_{s_N}$ are, as in Section 3.2.1, the *curves of admissible states* introduced in [7]. The state $\bar{u}_0$ is then connected to $\bar{u}$ by a sequence of rarefaction and travelling waves with positive speed.

2. *Boundary layers*

   We have now to characterize the set of values u such that the following problem admits a solution:

   $$\begin{cases} A(U, U_x) U_x = B(U) U_{xx} \\ U(0) = u \\ \lim_{x \to +\infty} U(x) = \bar{u}. \end{cases}$$

   We have thus to study system

   $$\begin{cases} U_x = p \\ p_x = B(U)^{-1} A(U,\, p) p \end{cases} \tag{3.20}$$

   Consider the equilibrium point $(\bar{u},\, 0)$, linearize at that point and denote by V^s the stable space, i.e. the eigenspace associated to the eigenvalues with strictly negative real part. Thanks to Lemma 3.1, the dimension of V^s is equal to the number of negative eigenvalues of $E^{-1}(\bar{u}) A(\bar{u},\, 0)$, i.e. to n. Also, V^s is given by

   $$V^s = \Big\{ \big(\bar{u} + \sum_{i=1}^{n} \frac{x_i}{\mu_i(\bar{u})} \vec{\chi}_i(\bar{u}),\; \sum_{i=1}^{n} x_i \vec{\chi}_i(\bar{u}) \big),\; x_1 \dots x_n \in \mathbb{R} \Big\},$$

   where $\mu_1(\bar{u}) \dots \mu_n(\bar{u})$ are the eigenvalues of $B^{-1}(\bar{u}) A((\bar{u}),\, 0)$ with negative real part and $\vec{\chi}_1(\bar{u}) \dots \vec{\chi}_n(\bar{u})$ are the corresponding eigenvectors.

   Denote by $\mathcal{M}^s$ the stable manifold, which is parameterized by V^s. Also, denote by ϕ_s a parameterization of $\mathcal{M}^s$:

   $$\phi_s : V^s \to \mathbb{R}^N.$$

   Let π_u be the projection

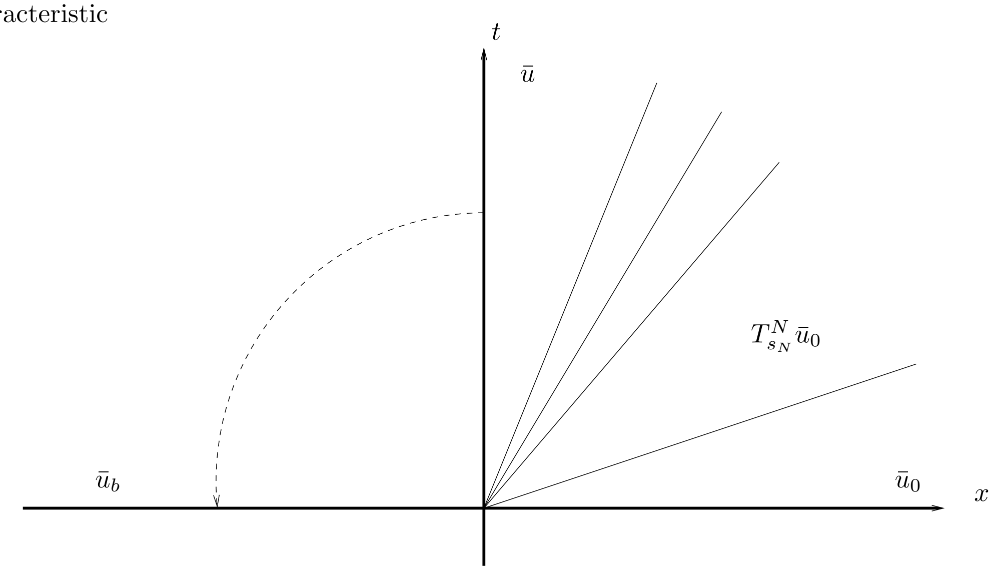
   $$\begin{aligned} \pi_u : &\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \\ &(u,\, p) \mapsto u \end{aligned}$$

   If $u \in \pi_u\big(\phi_s(s_1 \dots s_n)\big)$ for some $s_1 \dots s_n$, then system (3.20) admits a solution. Note that thanks to classical results about the stable manifold (see eg [33]) the map ϕ_s is differentiable and hence also $\pi_u \circ \phi_s$ is differentiable. In particular, the stable manifold is tangent at $s_1 = 0 \dots s_n = 0$ to V^s and hence the columns of the jacobian of $\pi_u \circ \phi_s$ computed at $s_1 = 0 \dots s_n = 0$ are $\vec{\chi}_1 \dots \vec{\chi}_n(\bar{u})$.

   Note that the map $\pi_u \circ \phi_s$ actually depends also on the point $\bar{u}$ and it does in a Lipschitz continuos way:

   $$|\pi_u \circ \phi_s(\bar{u}_1,\, s_1 \dots s_n) - \pi_u \circ \phi_s(\bar{u}_2,\, s_1 \dots s_n)| \le L |\bar{u}_1 - \bar{u}_2|$$

Figure 5: the solution of the boundary Riemann problem when the viscosity is invertible and the boundary is not characteristic



3. *Conclusion*

   Define the map ϕ as follows:

$$\phi(\bar{u}_0,\, s_1 \dots s_n) = \pi_u \circ \phi_s \Big(T^{N-k}_{s_{N-k}} \circ \dots T^N_{s_N} \bar{u}_0,\, s_1 \dots s_n \Big) \tag{3.21}$$

   From the previous steps it follows that ϕ is Lipschitz continuos and that it is differentiable at $s_1 = 0 \dots s_N = 0$. Also, the columns of the jacobian are $\vec{\chi}_1(\bar{u}_0) \dots \vec{\chi}_n(\bar{u}_0)$, $\vec{r}_{n+1}(\bar{u}_0) \dots \vec{r}_N$. Thus, thanks to Lemma 3.1, the jacobian is invertible. One can thus exploit the extension of the implicit function theorem discussed in [24] (page 253) and conclude that the map $\phi(\bar{u}_0, \cdot)$ is invertible in a neighbourhood of $(s_1 \dots s_N) = (0 \dots 0)$. In particular, if one takes $\bar{u}_b$ as in (3.1) and assumes that $|\bar{u}_0 - \bar{u}_b|$ is sufficiently small, then the values of $s_1 \dots s_N$ are uniquely determined by the equation

$$\bar{u}_b = \phi(\bar{u}_0,\, s_1 \dots s_n) \tag{3.22}$$

   Once the values of $s_1 \dots s_N$ are known, then the limit $u(t, x)$ can be reconstructed. In particular, the trace of u on the axis $x = 0$ is given by

$$\bar{u} := T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_n} \bar{u}_0. \tag{3.23}$$

   The self similar function u is represented in Figure 3.2.2 and can be obtained gluing together pieces like (3.18) .

Theorem 3.1. *Let Hypotheses 1, 3, 4, 5, 6 , 7 and 8 hold. Then there exists $\delta > 0$ small enough such that the following holds. If $|\bar{u}_0 - \bar{u}_b| << \delta$, then the limit of the parabolic approximation* (3.1) *satisfies*

$$\bar{u}_b = \phi(\bar{u}_0,\, s_1 \dots s_N)$$

for a suitable vector $(s_1 \dots s_N)$. The map ϕ is defined by (3.21)*. Given $\bar{u}_0$ and $\bar{u}_b$, one can invert ϕ and determine uniquely $(s_1 \dots s_N)$. Once $(s_1 \dots s_N)$ are known one can determine a.e. (t, x) the value $u(t, x)$ assumed by the limit function. In particular, the trace $\bar{u}$ of the hyperbolic limit in the axis $x = 0$ is given by* (3.23)*.*

3.2.3 The hyperbolic limit in the boundary characteristic case

The aim of this section is to provide a characterization of the limit of the parabolic approximation (3.1) when the matrix B is invertible, but the boundary is characteristic, i.e. one eigenvalue of $A(u, u_x)$ can attain the value 0.

Let δ be the bound on the total variation introduced in Hypothesis 4. Moreover, as in the previous section we will denote by $\lambda_1(u) \dots \lambda_N(u)$ the eigenvalues of the matrix $E^{-1}(u)A(u, 0)$. The eigenvalue $\lambda_k(u)$ is Lipschitz continuous with respect to u and hence there exists a suitable constant M such that, if $|\lambda_k(\bar{u}_0)| > M\delta$, then $|\lambda_k(u)| > 0$ for every u in a neighbourhood of $\bar{u}_0$ of size δ, $|u - \bar{u}_0| \leq \delta$. Thus, in particular, if $\lambda_k(\bar{u}_0) > M\delta$ then Hypothesis 8 is satisfied for some c and hence the boundary is non characteristic. In this section we will therefore make the following assumption:

Hypothesis 9. *Let δ the same constant as in Hypothesis 4 and let M the constant introduced before. Then*

$$|\lambda_k(\bar{u}_k)| \leq M\delta. \tag{3.24}$$

Note that, because of strict hyperbolicity (Hypothesis 3) the other eigenvalues of $E^{-1}(u)A(u, 0)$ are well separated from zero, in the following sense: there exist a constant $c > 0$ such that for all u satisfying $|\bar{u}_0 - u| \leq \delta$, it holds

$$\lambda_1(u) < \dots < \lambda_{k-1}(u) \leq -c < 0 < c \leq \lambda_{k+1}(u) < \dots < \lambda_N(u). \tag{3.25}$$

The notation used in this section is therefore the following: $(k-1)$ denotes the number of strictly negative eigenvalues of $E^{-1}A(u, 0)$, while $(N-k)$ is the number of strictly positive eigenvalues.

The study of the limit is more complicated in the boundary characteristic case than in the case considered in the previous section. The main ideas exploited in the analysis are described in the introduction, here instead we will give the technical details of the construction. As in the previous section, the characterization of the limit will be as follows: we will construct a map $\phi(\bar{u}_0, s_1 \dots s_N)$ which describes all the states that can be connected to $\bar{u}_0$. We will then show that the map is locally invertible: thus, given $\bar{u}_0$ and $\bar{u}_b$ sufficiently close, the values of $(s_1 \dots s_N)$ are uniquely determined by the equation

$$\bar{u}_b = \phi(\bar{u}_0, s_1 \dots s_N).$$

Once $(s_1 \dots s_N)$ are known one can proceed as in the previous section and determine a.e. (t, x) the value $u(t, x)$ assumed by the limit function. The construction of the map ϕ will be given in several steps:

1. *Study of the waves with uniformly positive speed*

   Consider the Cauchy datum $\bar{u}_0$, fix $(N-k)$ parameters $(s_{N-k} \dots s_N)$ and consider the value

   $$\bar{u}_k = T^{N-k}_{s_{N-k}} \circ \dots T^N_{s_N} \bar{u}_0.$$

   The curves $T^{N-k}_{s_{N-k}} \dots T^N_{s_N}$ are, as in Section 3.2.1, the *curves of admissible states* introduced in [7]. The state $\bar{u}_0$ is then connected to $\bar{u}_k$ by a sequence of rarefaction and travelling waves with uniformly positive speed. The speed is uniformly positive in the sense of (3.25).

2. *Reduction on the center stable manifold*

   Consider system

   $$\begin{cases} u_x = p \\ p_x = B(u)^{-1}\Big(A(u, p) - \sigma E(u)\Big)p \\ \sigma_x = 0 \end{cases} \tag{3.26}$$

   and the equilibrium point $\big(\bar{u}_k, 0, \lambda(\bar{u}_k)\big)$. Linearizing the system around $\big(\bar{u}_k, 0, \lambda(\bar{u}_k)\big)$ one obtains

   $$\begin{pmatrix} 0 & I^0_N & \\ 0 & B^{-1}(\bar{u}_k)\Big(A(\bar{u}_k, 0) - \lambda_k(\bar{u}_k E(\bar{u}_k)\Big) & \\ 0 & 0 & `0 \end{pmatrix} \tag{3.27}$$

Thanks to (3.25), the matrix $E(\bar{u}_k) - \lambda_k(\bar{u}_k) A(\bar{u}_k, 0)$ has 1 null eigenvalue and $(N-k)$ strictly negative eigenvalues. Because of Lemma 3.1 one can then conclude that the matrix (3.27) has $N+1$ null eigenvalues and $(N-k)$ eigenvalues with strictly negative real part.

Let V^c be the kernel of (3.27), V^s the eigenvspace associated eigenvalues with strictly negative real part. and V^u the eigenspace associated to eigenvalues with strictly positive real part. There exists a so called *center stable* manifold $\mathcal{M}^{cs}$ which is locally invariant for (3.26) and is parameterized by $V^s \oplus V^c$: we refer to [33] for an extensive analysis, here we will just recall some of the fundamental properties of a center stable manifold. If we write $\mathbb{R}^{2N+1} = V^c \oplus V^s \oplus V^u$, then the map

$$\Phi_{cs} : V^s \oplus V^c \to V^c \oplus V^s \oplus V^u$$

that parameterizes $\mathcal{M}^{cs}$ can be chosen in such a way that if π_{cs} is the projection from $V^c \oplus V^s \oplus V^u$ onto $V^c \oplus V^s$, then $\pi_{cs} \circ \Phi_{cs}$ is the identity. A center stable manifold $\mathcal{M}^{cs}$ is defined in a neighbourhood of the equilibrium point $(\bar{u}_k, 0, \lambda(\bar{u}_k)$ and it is tangent to $V^c \oplus V^s$ at $\big(\bar{u}_k, 0, \lambda(\bar{u}_k)\big)$.

In the following, we will fix a center stable manifold $\mathcal{M}^{cs}$ and we will focus on the solutions of (3.26) that lay on $\mathcal{M}^{cs}$. The center and the stable space of (3.27) are given respectively by

$$V^c = \Big\{ (u, v^k_{cs} \vec{r}_k(\bar{u}_k), \sigma) : \ u \in \mathbb{R}^N, \ v^k_{cs}, \ \sigma \in \mathbb{R} \Big\}$$

and

$$V^s = \Big\{ (\bar{u}^k + \sum_{i<k} \frac{v^i_{cs}}{\mu_i} \vec{\chi}_i(\bar{u}_k), \sum_{i<k} v^i_{cs} \vec{\chi}_i(\bar{u}_k), \ \lambda_k(\bar{u}_k)) \Big\}.$$

In the previous expression, $r_k(\bar{u}_k)$ is a unit vector in the kernel of $\Big(A(\bar{u}_k, 0) - \lambda_k(\bar{u}_k)\Big)$, while $\vec{\chi}_1 \dots \vec{\chi}_{k-1}$ are the eigenvectors of $B^{-1}(\bar{u}_k)\Big(A(\bar{u}_k, 0) - \lambda_k(\bar{u}_k)\Big)$ associated to the eigenvalues $\mu_1 \dots \mu_{k-1}$ with strictly negative real part. Since $\Phi_{cs} \circ \pi_{cs}$ is the identity, then a point (u, p, σ) belongs to the manifold $\mathcal{M}^{cs}$ if and only if

$$p = \sum_{i<k} v^i_{cs} \vec{\chi}_i(\bar{u}_k) + v^k_{cs} \vec{r}_k + \sum_{i>k} \phi^i_{cs}(u, \ v^1_{cs} \dots v^k_{cs}, \ \sigma) \vec{\chi}_i(\bar{u}_k),$$

where $\vec{\chi}_{k+1}(\bar{u}_k) \dots \vec{\chi}_N(\bar{u}_k)$ are the eigenvectors of $B^{-1}(\bar{u}_k)\Big(A(\bar{u}_k, 0) - \lambda_k(\bar{u}_k)\Big)$ associated to eigenvalues with strictly positive real part.. Since all the equilibrium points $(u, 0, \sigma)$ belong to the center stable manifold, then we must have $p = 0$ when $v^1_{cs} \dots v^k_{cs}$ are all 0. Thus, since $\vec{\chi}_{k+1}(\bar{u}_k) \dots \vec{\chi}_N(\bar{u}_k)$ are linearly independent, $\phi^i_{cs}(u, \vec{0}, \sigma) = 0$ for all $i = k+1 \dots N$ and for every u and σ. This means, in particular, that for every $i = k+1 \dots N$ there exists a k-dimensional vector $\tilde{\phi}^i_{cs}(u, \ v^1_{cs} \dots p^k_{cs}, \ \sigma)$ such that

$$\phi^i_{cs}(u, \ v^1_{cs} \dots v^k_{cs}) = \langle \tilde{\phi}^i_{cs}(u, \ v^1_{cs} \dots v^k_{cs}, \ \sigma), \ v^1_{cs} \dots v^k_{cs} \rangle.$$

Define

$$R_{cs}(u, \ v^1_{cs} \dots v^k_{cs}, \ \sigma) := \Big(\vec{\chi}_1 | \dots | \vec{\chi}_{k-1} | \vec{r}_k \Big) + \Big(\vec{\chi}_{k+1} | \dots | \vec{\chi}_N \Big) \left(\begin{array}{c} \tilde{\phi}^{k+1}_{cs}(u, \ v^1_{cs} \dots v^k_{cs}, \ \sigma) \\ \hline \dots \\ \hline \tilde{\phi}^N_{cs}(u, \ v^1_{cs} \dots v^k_{cs}, \ \sigma) \end{array} \right),$$

then a point (u, p, σ) belongs to $\mathcal{M}^{cs}$ if and only if

$$p = R_{cs}(u, \ V_{cs}, \ \sigma) V_{cs}, \tag{3.28}$$

where V_{cs} denotes $(v^1_{cs} \dots v^k_{cs})^T$. Since $\mathcal{M}^{cs}$ is tangent to $V^c \oplus V^s$ at $(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k))$, then

$$R_{cs}(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k)) = \Big(\vec{\chi}_1 | \dots | \vec{\chi}_{k-1} | \vec{r}_k \Big).$$

Plugging $p = R_{cs}(u, V_{cs}, \sigma)V_{cs}$ into system (3.49) one gets

$$\Big((D_u R_{cs})R_{cs}V_{cs} + (D_v R_{cs})V_{csx}\Big)V_{cs} + R_{cs}V_x^{cs} = B^{-1}\Big(A(u, R_{cs}V_{cs}) - \sigma E(u)\Big)R_{cs}V_{cs}. \tag{3.29}$$

Moreover, the matrix

$$D(u, V_{cs}) = (d_{ih}) \qquad d_{ih} = \sum_{j=1}^{k} \frac{\partial R_{csij}}{\partial V_h} V_j$$

satisfies

$$D(u, 0) = 0 \qquad \Big((D_v R_{cs})V_{csx}\Big)V_{cs} = D(u, V^{cs})V_x^{cs}.$$

Hence (3.29) can be rewritten as

$$\Big(R_{cs} + D(u, V_{cs})\Big)V_{csx} = \Big(B^{-1}\Big(A - \sigma E\Big) - (D_u R_{cs})R_{cs}V_{cs}\Big)R_{cs}V_{cs}. \tag{3.30}$$

Since

$$\Big(R_{cs} + D(u, V_{cs})\Big)(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k)) = \Big(\vec{\chi}_1|\dots|\vec{\chi}_{k-1}|\vec{r}_k\Big),$$

then the columns of $R_{cs} + D(u, V_{cs})$ are linearly independent in a small enough neighbourhood. Thus one can find a matrix $L(u, V_{cs}, \sigma)$ such that

$$L(u, V_{cs}, \sigma)\Big(R_{cs} + D(u, V_{cs})\Big) = I_k.$$

Multiplying (3.30) by $L(u, V_{cs}, \sigma)$ one finds

$$V_x^{cs} = \Big(LB^{-1}(A - \sigma E)R_{cs} - L(D_u R_{cs})R_{cs}V_{cs}\Big)V_{cs}. \tag{3.31}$$

Define

$$\Lambda_{cs}(u, V_{cs}, \sigma) := LB^{-1}(A - \sigma E)R_{cs} - L(D_u R_{cs})R_{cs}V_{cs}, \tag{3.32}$$

then by construction $\Lambda_{cs}(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k))$ is a $(k \times k)$ diagonal matrix with one eigenvalue equal to 0 and $N - k$ eigenvalues with strictly negative real part. Also, if $\vec{\ell}_k = (0 \dots 0, 1)$ and $\vec{e}_k = \vec{\ell}_k^T$, then

$$\vec{\ell}_k \left(\frac{\partial \Lambda_{cs}}{\partial \sigma} \bigg|_{u=\bar{u}_k,\, V_{cs}=\vec{0},\, \sigma=\lambda_k(\bar{u}_k)} \right) \vec{e}_k = -\bar{k} < 0 \tag{3.33}$$

for a suitable strictly positive constant $\bar{k}$. This is a consequence of conditions 1 and 3 in Hypothesis 1. In conclusion, the solutions of system (3.26) laying on $\mathcal{M}^{cs}$ satisfy

$$\begin{cases} u_x = R_{cs}(u, V_{cs}, \sigma)V_{cs} \\ V_{csx} = \Lambda_{cs}(u, V_{cs}, \sigma)V_{cs} \\ \sigma_x = 0 \end{cases} \tag{3.34}$$

where $\Lambda_{cs}(u, V_{cs}, \sigma)$ is defined by (3.32).

3. *Analysis of the uniformly stable component of* (3.34)

Linearizing system (3.34) around the equilibrium point $(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k))$ one obtains the matrix

$$\begin{pmatrix} 0 & R_{cs}(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k)) & 0 \\ 0 & \Lambda_{cs}(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k)) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.35}$$

which has $k-1$ distinct eigenvalues with strictly negative real part and the eigenvalue 0 with multiplicity $N+1$. Also, the manifold

$$E := \Big\{ (u,\, \vec{0},\, \sigma),\ u \in \mathbb{R}^N,\ \sigma \in \mathbb{R} \Big\}$$

is entirely constituted by equilibria.

Then there exists a *uniformly stable manifold* $\mathcal{M}^{us}_E$ which is invariant for (3.34) and is characterized by the following property: given $(u^0,\, V^0_{cs},\, \sigma^0)$ belonging to $\mathcal{M}^{us}_E$, denote by $(u(x),\, V_{cs}(x),\, \sigma(x))$ the solution of (3.34) starting at $(u^0,\, V^0_{cs},\, \sigma^0)$. Then there exists a point $(u^\infty,\, \vec{0},\, \sigma^\infty)$ belonging to E such that

$$\lim_{x \to +\infty} \Big(|u(x) - u^\infty| + |V_{cs}(x)| + |\sigma(x) - \sigma^\infty| \Big) e^{cx/2} = 0.$$

We remark that the uniformly stable manifold $\mathcal{M}^{cs}_E$ is contained but in general does not coincide with a given center stable manifold. Indeed, $\mathcal{M}^{us}_E$ does not contain, for example, the trajectories that converge to a point in E with speed slower than e^{-cx}.

The existence of the uniformly stable manifold is implied by Hadamard-Perron theorem, which is discussed for example in [33]. The application of the uniformly stable manifold to the study of hyperbolic initial boundary values problems is derived from [3]. In our case, the existence of the uniformly stable manifold is loosely speaking guaranteed by the strict hyperbolicity of A (Hypothesis 3), namely the fact that the eigenvalues of A are uniformly separated one from each other. In the following, we will just recall some of the properties of the uniformly stable manifold.

Let

$$\tilde{V}^s = \Big\{ \Big(\bar{u}_k + \sum_{i<k} v^i_s \vec{\chi}_i(\bar{u}_k),\, \sum_{i=1}^{k} v^i_s \vec{e}_i,\, \lambda_k(\bar{u}_k) \Big),\ v^1_s \dots v^{k-1}_s \in \mathbb{R} \Big\},$$

where $\vec{\chi}_1$ are as before the eigenvectors of $B^{-1}(A-E)$ and $\vec{e}_i \in \mathbb{R}^k$ are the vectors of the canonical basis of $\mathbb{R}^k$, $\vec{e}_1 = (1,\, 0 \dots 0)$ and so on. The uniformly stable manifold M^{cs}_E is parameterized by $E \oplus \tilde{V}^s$ and hence has dimension $N+k-1$. The parameterization

$$\Phi_{us} : \tilde{V}^s \oplus E \to \mathbb{R}^{k+2}$$

can be chosen in such a way that the following property is satisfied. Let

$$\tilde{V}^c = \Big\{ \Big(\bar{u}_k + v^k \vec{r}_k(\bar{u}_k),\, v^k \vec{e}_k,\, \lambda_k(\bar{u}_k) \Big),\ v^k \in \mathbb{R} \Big\}.$$

If we write $\mathbb{R}^{N+k+1}$ as $\tilde{V}^c \oplus \tilde{V}^s \oplus E$ and we denote by π_{us} the projection onto $\tilde{V}^s \oplus E$, then we can have $\pi_{us} \circ \Phi_{us}$ equal to the identity.

By considerations analogous, but easier, to those performed to get (3.28) one deduces that a point $(u,\, V_{cs},\, \sigma)$ belongs to $\mathcal{M}^{us}_E$ if and only if

$$V_{cs} = R_s(u,\, V_s,\, \sigma) V_s,$$

where $V_s = (v^1_{us} \dots v^{k-1}_{us})$ are the components of V_{cs} along $\vec{e}_1 \dots \vec{e}_{k-1}$. The matrix R_s belongs to $\mathbb{M}^{k \times (k-1)}$ and

$$R_s\Big(\bar{u}_k,\, \vec{0},\, \lambda_k(\bar{u}_k)\Big) = \Big(\vec{e}_1 | \dots | \vec{e}_{k-1}\Big).$$

We want now to compute $\Lambda_{cs} R_s V_s$. Plugging the relation $V_{sx} = \Lambda_{cs} V_s$ into (3.34) one gets

$$\Big[R_s + D_s \Big] V_{sx} = \Big[\Lambda_{cs} R_s - D_u R_s R_{cs} R_s V_s \Big] V_s, \tag{3.36}$$

where D_s is a matrix such that $D_s(u,\, \vec{0},\, \sigma)$ for every u and σ. Thus,

$$V_{sx} = \tilde{\Lambda}_s V_s$$

for a suitable matrix $\tilde{\Lambda}_s$ such that $\tilde{\Lambda}_s\Big(\bar{u}_k,\, \vec{0},\, \lambda_k(\bar{u}_k\Big)$ is a diagonal matrix with all the eigenvalues with strctly negative real part. Plugging back into (3.36) one gets

$$\Lambda_{cs} R_s V_s = \Big\{ \Big[R_s + D_s \Big] \tilde{\Lambda}_s + D_u R_s R_{cs} R_s V_s \Big\} V_s. \tag{3.37}$$

4. *Analysis of the center component of* (3.34)

   Linearizing system (3.34) around the equilibrium point $(\bar{u}_k,\, \vec{0},\, \lambda_k(\bar{u}_k))$ one obtains (3.35) and hence the center space is given by

   $$\tilde{V}^c = \Big\{ (u,\, v_k \vec{e}_k,\, \sigma) : \; u \in \mathbb{R}^N, \; v_k,\, \sigma \in \mathbb{R} \Big\}.$$

   Consider the center manifold $\mathcal{M}^c$, parameterized by $\tilde{V}^c$: thanks to considerations similar, but easier, than those performed in the previous steps one gets that a point $(u,\, V_{cs},\, \sigma)$ belongs to $\mathcal{M}^c$ if and only if

   $$V_{cs} = \check{r}_k(u,\, v_k,\, \sigma) v_k,$$

   where $\check{r}_k \in \mathbb{R}^k$ and $\check{r}_k(\bar{u}_k,\, \vec{0},\, \lambda_k(\bar{u}_k)) = \vec{e}_k$.

   Again, with considerations analogous, but easier than those performed at the previous step one gets

   $$\Lambda_{cs} \check{r}_k v_k = \Big[(\check{r}_k + \check{r}_{kv} v_k) \tilde{\phi}_k + D_u \check{r}_k R_{cs} \check{r}_k v_k \Big] v_k \tag{3.38}$$

   for a suitable function $\tilde{\phi}_k$ satisfying $\tilde{\phi}_k\Big(\bar{u}_k,\, \vec{0},\, \lambda_k(\bar{u}_k)\Big) = \vec{0}$. Also, thanks to (3.33),

   $$\frac{\partial \tilde{\phi}_k}{\partial \sigma}\Bigg|_{u=\bar{u}_k,\, V_{cs}=\vec{0},\, \sigma=\lambda_k(\bar{u}_k} = -\bar{k} < 0. \tag{3.39}$$

5. *Decomposition of* (3.34) *in center and uniformly stable component*

   In this step, we will fix a trajectory $(u,\, V_{cs},\, \sigma)$ of (3.34) and we will decompose it in a *uniformly stable* and in a *center* component, in the following sense. We exploits the manifolds $\mathcal{M}^c$ and $\mathcal{M}^{us}_E$ introduced in the previous steps and we decompose

   $$V_{cs} = R_s(u,\, V_s,\, \sigma) V_s + \check{r}_k(u,\, v_k,\, \sigma) v_k. \tag{3.40}$$

   Plugging this expression into the first line of (3.34) one gets

   $$u_x = R_{cs} R_s V_s + R_{cs} \check{r}_k v_k.$$

   From the second line of (3.34) one gets

   $$\begin{aligned} V_{csx} &= R_s V_{sx} + \Big[D_u R_s (R_{cs} R_s V_s + R_{cs} \check{r}_k v_k) + D_V R_s V_{sx} \Big] V_s \\ &\quad + \check{r}_k v_{kx} + \Big[D_u \check{r}_k (R_{cs} R_s V_s + R_{cs} \check{r}_k v_k) + \check{r}_{kv} v_{kx} \Big] v_k \\ &= \Lambda_{cs} R_s V_s + \Lambda_{cs} \check{r}_k v_k. \end{aligned}$$

   One can prove that

   $$D_V R_s V_{sx} V_s = D_s(u,\, V_s,\, \sigma) V_{sx}$$

for a suitable matrix D_s which satisfies $D(u, \vec{0}, \sigma) = 0$ for every u and σ (the same as in the previous step). Thus, exploiting (3.37) and (3.38), we get

$$\begin{aligned}
\Big[R_s + D_s\Big]V_{sx} + \Big[\check{r}_k + \check{r}_{kv}v_k\Big]v_{kx} &= \Lambda_{cs}R_sV_s + \Lambda_{cs}\check{r}_kv_k \\
&\quad - D_uR_s(R_{cs}R_sV_s + R_{cs}\check{r}_kv_k)V_s - D_u\check{r}_k(R_{cs}R_sV_s + R_{cs}\check{r}_kv_k)v_k \\
&= \Big\{\Big[R_s + D_s\Big]\tilde{\Lambda}_s + D_uR_sR_{cs}R_sV_s\Big\}V_s \\
&\quad + \Big[(\check{r}_k + \check{r}_{kv}v_k)\tilde{\phi}_k + D_u\check{r}_kR_{cs}\check{r}_kv_k\Big]v_k \\
&\quad - D_uR_s(R_{cs}R_sV_s + R_{cs}\check{r}_kv_k)V_s - D_u\check{r}_k(R_{cs}R_sV_s + R_{cs}\check{r}_kv_k)v_k \\
&= \Big[R_s + D_s\Big]\tilde{\Lambda}_sV_s + (\check{r}_k + \check{r}_{kv}v_k)\tilde{\phi}_kv_k \\
&\quad - \Big[D_uR_sR_{cs}\check{r}_k + D_u\check{r}_kR_{cs}R_s\Big]V_sv_k
\end{aligned} \tag{3.41}$$

The functions $R_s + D_s$ and $\check{r}_k + \check{r}_kv_k$ satisfy

$$(R_s + D_s)\Big(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k)\Big) = \Big(\vec{e}_1|\dots|\vec{e}_{k-1}\Big) \qquad (\check{r}_k + \check{r}_kv_k)\Big(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k)\Big) = \vec{e}_k$$

and hence in a neighbourhood of $(\bar{u}_k, \vec{0}, \lambda_k(\bar{u}_k))$ the columns of $R_s + D_s$ and $\check{r}_k + \check{r}_kv_k$ are all linearly independent. Denote by $\ell_1 \dots \ell_k$ the vectors of the dual basis and define

$$L_s(u, V_s, v_k, \sigma) := \begin{pmatrix} \ell_1 \\ \dots \\ \ell_{k-1} \end{pmatrix}$$

Then multiplying (3.41) on the left by L_s one gets

$$V_{sx} = \Lambda_s(u, v_k, V_s, \sigma)V_s,$$

where $\Lambda_s \in \mathbb{M}^{k-1\times k-1}$ is given by

$$\Lambda_s(u, v_k, V_s, \sigma) = L_s\Big[R_s + D_s\Big]\tilde{\Lambda}_s - L_s\Big[D_uR_sR_{cs}\check{r}_k + D_u\check{r}_kR_{cs}R_s\Big]v_k.$$

The matrix $\Lambda_s(\bar{u}_k, 0, \vec{0}, \lambda_k(\bar{u}_k))$ is diagonal and all the eigenvalues have strictly negative real part.
Also, multiplying (3.41) on the left by ℓ_k one gets

$$v_{kx} = \phi_kv_k,$$

where the real valued function $\phi_k(u, v_k, V_s, \sigma)$ is given by

$$\phi_k(u, v_k, V_s, \sigma) = \ell_k(\check{r}_k + \check{r}_{kv}v_k)\tilde{\phi}_k - \ell_k\Big[D_uR_sR_{cs}\check{r}_k + D_u\check{r}_kR_{cs}R_s\Big]V_s.$$

Also, $\phi_k(\bar{u}_k, 0, \vec{0}, \lambda_k(\bar{u}_k)) = 0$.
In conclusion, system (3.34) can be decomposed as

$$\begin{cases} u_x = R_{cs}(u, v_k\check{r}_k + R_sV_s, \sigma)\check{r}_kv_k + R_{cs}(u, v_k\check{r}_k + R_sV_s, \sigma)R_sV_s \\ v_{kx} = \phi_k(u, v_k, V_s, \sigma)v_k \\ V_{sx} = \Lambda_s(u, v_k, V_s, \sigma)V_s \\ \sigma_x = 0 \end{cases} \tag{3.42}$$

Note that, thanks to (3.39),

$$\frac{\partial\phi_k(u_k, v_k, V_s, \sigma_k)}{\partial\sigma}\bigg|_{u=\bar{u}_k,\, v_k=0,\, V_s=\vec{0},\, \sigma_k=\lambda_k(\bar{u}_k)} = -\bar{k} < 0. \tag{3.43}$$

To study system (3.42) we will first consider the solution of (3.42) when $u_s \equiv 0$ and $V_s \equiv 0$. Fix $s_k > 0$: we will actually study the following fixed point problem, which is defined on the interval $[0,\, s_k]$:

$$\begin{cases} u_k(\tau) = \bar{u}_k + \displaystyle\int_0^\tau \check{r}_k(u_k(\xi),\, v_k(\xi),\, 0,\, \sigma_k(\xi))d\xi \\ v_k(\tau) = f_k[u_k,\, v_k,\, \sigma_k](\tau) - \mathrm{mon}_{[0,\, s_k]} f_k[u_k,\, v_k,\, \sigma_k](\tau) \\ \sigma_k(\tau) = \dfrac{1}{\bar{c}}\dfrac{d}{d\tau}\mathrm{mon}_{[0,\, s_k]} f_k(\tau,\, u_k,\, v_k,\, \sigma_k). \end{cases} \tag{3.44}$$

In the previous expression,

$$f_k[u_k,\, v_k,\, \sigma_k](\tau) = \int_0^\tau \tilde{\lambda}_k[u_k,\, v_k,\, \sigma_k](\xi)d\xi \tag{3.45}$$

where

$$\tilde{\lambda}_k[u_k,\, v_k,\, \sigma_k](\xi) = \phi_k\Big(u_k(\xi),\, v_k(\xi),\, 0,\, \sigma_k(\xi)\Big) + \bar{k}\sigma_k(\xi). \tag{3.46}$$

The constant $\bar{k}$ is defined by (3.43).

If $s_k < 0$, one considers a fixed problem like (3.44) but instead of the monotone concave envelope of f_i one takes the monotone convex envelope:

$$\mathrm{monconv} f_i(\tau) = \sup\{h(s) :\ \text{h is convex and monotone non decreasing},\ h(y) \le f_i(y)\ \forall\, y \in [0,\, s_k]\}.$$

In the following we will consider only the case $s_k > 0$, the case $s_k < 0$ being entirely similar.

The link between system (3.44) and system (3.42) in the case $u_s \equiv 0$, $V_s \equiv \vec{0}$ is the following. Let $(u_k,\, v_k,\, \sigma_k)$ solve (3.44), assume that $v_k(s_k) < 0$ and define

$$\alpha(\tau) = -\int_\tau^{s_k} \frac{1}{v_k(s)}ds. \tag{3.47}$$

Let

$$\underline{s} := \max\{\tau \in [0,\, s_k] :\ v_k(\tau) = 0\},$$

in the following we will prove that $\alpha(\tau) < +\infty$ if and only if $\tau > \underline{s}$.

The function $(u_k \circ \alpha,\, v_k \circ \alpha,\, 0)$ solves (3.42) in the case $u_s \equiv 0$, $V_s \equiv \vec{0}$. If $v_k(s_k) = 0$ then $(u_k(s_k),\, 0,\, 0)$ is also a trivial solution of (3.42) in the case $u_s \equiv 0$, $V_s \equiv \vec{0}$ and the following properties are satisfied. Note that the function $u_k \circ \alpha$ is a steady solution of the original parabolic equation since $\sigma \equiv 0$:

$$A(u_k,\, u_{kx})u_{kx} = B(u_k)u_{kxx}.$$

Moreover,

$$\lim_{x \to \infty}(u_k \circ \alpha)(x) = u_k(\underline{s}). \tag{3.48}$$

From system (3.44) we also get that $\bar{u}_k$ is connected to $u_k(\underline{s})$ by a sequence of rarefaction waves and shocks with non negative speed.

After finding a solution of (3.44) in the case $u_s \equiv 0$, $V_s \equiv 0$ we will also find a solution of (3.44) in the case $u_k \equiv \bar{u}_k$, $v_k \equiv 0$ and $\sigma \equiv 0$. We will also impose

$$\lim_{x \to \infty} u_s(x) = 0$$

and hence we will study the fixed point problem

$$\begin{cases} u_s(x) = -\displaystyle\int_x^{+\infty} \check{R}^s\Big(\bar{u}_k + u_s(y),\, 0,\, V^s(y)\Big)V^s(y)dy \\ V_s(x) = e^{\bar{\Lambda}x}V^s(0) + \displaystyle\int_0^x e^{\bar{\Lambda}(x-y)}\Big[\check{\Lambda}^s\Big(u_s(y) + \bar{u}_k,\, 0,\, V^s(y)\Big) - \bar{\Lambda}\Big]V^s(y)dy, \end{cases} \tag{3.49}$$

where

$$\bar{\Lambda} = \check{\Lambda}^s\Big(\bar{u}_k,\, 0,\, 0\Big).$$

Note that again, being $\sigma = 0$, u_s provides a steady solution of the origin parabolic equation,

$$A(u_s,\, u_{sx})u_{sx} = B(u_s)u_{sxx}$$

Finally, we will consider a component of perturbation, due to interaction between the purely center component, defined by (3.44) and the purely stable component, which satisfies (3.49). More precisely, we will define $(U,\, q,\, p)$ in such a way that $u = U + u_k \circ \alpha + u_s$, $v_k + q$ and $V_s + p$ is a solution of (3.42).

6. **Analysis of the purely center component**

The purely center component is the solution of system (3.44).

Lemma 3.4. *Fix $\delta > 0$ such that $s_k \leq \delta << 1$. Then system* (3.44) *admits a unique solution $(u_k,\, v_k,\, \sigma_k)$ satisfying*

(a) $u_k \in \mathcal{C}^0([0,\, s_k])$, $\|u_k - \bar{u}_k\|_{\mathcal{C}^0} \leq c_u\delta$, u_k is Lipschitz with constant $Lip(u_k) \leq 2$.

(b) $v_k \in \mathcal{C}^0([0,\, s_k])$, $\|v_k\|_{\mathcal{C}^0} \leq c_v\delta$, v is Lipschitz with constant $Lip(v_k) \leq \tilde{c}_v\delta$.

(c) $\sigma_k \in \mathcal{C}^0([0,\, s_k])$, $\|\sigma_k - \lambda_k(\bar{u}_k)\|_{\mathcal{C}^0} \leq c_\sigma\delta/\eta_k$, σ_k is Lipschitz with constant $Lip(\sigma_k) \leq \tilde{c}_\sigma$.

The constants c_u, c_v, $\tilde{c}_v$, c_σ, η_k and $\tilde{c}_\sigma$ do not depend on δ. The function f defined by (3.45) *belongs to $\mathcal{C}^{1,\,1}_k([0,\, s_k])$ for a suitable constant k which again does not depend on δ.*

The symbol $\mathcal{C}^{1,\,1}_k([0,\, s_k])$ denotes the space of functions $f \in \mathcal{C}^1([0,\, s_k])$ such that f' is Lipschitz continuous and has Lipschitz constant $Lip(f') \leq k$.

Proof. The proof of the lemma relies on a fixed point argument. Only the fundamental steps are sketched here.

Define

$$\begin{aligned}
&X_{uk} := \big\{u_k \in \mathcal{C}^0([0,\, s_k],\, \mathbb{R}^N):\ \|u_k - \bar{u}_k\|_{\mathcal{C}^0} \leq c_u\delta,\ u_k \text{ is Lipschtiz},\ Lip(u_k) \leq 2\big\},\\
&X_{vk} := \big\{u_k \in \mathcal{C}^0([0,\, s_k],\, \mathbb{R}):\ \|v_k\|_{\mathcal{C}^0} \leq c_v\delta,\ v_k \text{ is Lipschtiz},\ Lip(v_k) \leq \tilde{c}_v\delta\big\},\\
&X_{\sigma k} := \big\{\sigma_k \in \mathcal{C}^0([0,\, s_k],\, \mathbb{R}):\ \|\sigma_k - \lambda_k(\bar{u}_k)\|_\sigma \leq c_\sigma\delta^2,\ \sigma_k \text{ is Lipschtiz},\ Lip(\sigma_k) \leq \tilde{c}_\sigma\big\}.
\end{aligned} \tag{3.50}$$

In the previous expression, $\|\cdot\|_\sigma$ is defined as follows:

$$\|\cdot\|_\sigma := \eta_k\delta\|\cdot\|_{\mathcal{C}^0},$$

for a suitable constant η_k whose exact value does not depend on δ and will be determined in the following. Also the constants c_u, c_v, $\tilde{c}_v$, c_σ and $\tilde{c}_\sigma$ do not depend on δ and their exact value will be estimated in the following.

If $(u_k,\, v_k,\, \sigma_k) \in X_{uk} \times X_{vk} \times X_{\sigma k}$ then the function f defined by (3.45) satisfies $f \in \mathcal{C}^{1,\,1}_k$ for a large enough constant k. Moreover, exploiting (3.24) and (3.46), one gets

$$\begin{aligned}
\|f\|_{\mathcal{C}^0} &\leq \delta\|\check{\lambda}_k\|_{\mathcal{C}^0}\\
&\leq \delta\Bigg(\Big|\phi_k\Big(\bar{u}_k,\, 0,\, \lambda_k(\bar{u}_k)\Big)\Big| + \Big\|\phi_k\Big(u_k,\, v_k,\, \sigma_k\Big) - \phi_k\Big(\bar{u}_k,\, 0,\, \lambda_k(\bar{u}_k)\Big)\Big\|_{\mathcal{C}^0} + \Big|\lambda_k(\bar{u}_k)\Big|\\
&\qquad + \|\lambda_k(\bar{u}_k) - \sigma_k\|_{\mathcal{C}^0}\Bigg)\\
&\leq \delta\Bigg(0 + \mathcal{O}(1)(c_u\delta + c_v\delta + c_\sigma\delta) + M\delta + \frac{c_\sigma}{\eta_k}\delta\Bigg)\\
&\leq \frac{1}{2}c_v\delta
\end{aligned} \tag{3.51}$$

if δ is sufficiently small and c_v sufficiently large.

The same computations ensure that

$$\|f'\|_{\mathcal{C}^0} \le \frac{1}{2}\tilde{c}_v \delta \tag{3.52}$$

for a large enough $\tilde{c}_v$. Finally,

$$\begin{aligned} |f'(x) - f'(y)| &= |\check{\lambda}_k\Big(u_k(x),\, v_k(x),\, \sigma_k(x)\Big) - \check{\lambda}_k\Big(u_k(y),\, v_k(y),\, \sigma_k(y)\Big)| \\ &\le \mathcal{O}(1)(2 + \tilde{c}_v\delta + \tilde{c}_\sigma\delta)|x-y| \le k|x-y| \end{aligned} \tag{3.53}$$

for a sufficiently large constant k. In the previous estimate, we have exploited the following observation: since by (3.43)

$$\frac{\partial \check{\lambda}}{\partial \sigma}\Bigg|_{\Big(u=\bar{u}_k,\, v_k=0,\, \sigma_k=\lambda_k(\bar{u}_k)\Big)} = 0,$$

then

$$\left\|\frac{\partial \check{\lambda}}{\partial \sigma}\right\|_{\mathcal{C}^0} \le \mathcal{O}(1)\delta. \tag{3.54}$$

Because of (3.51), (3.52) and (3.54), $f \in \mathcal{C}^{1\,1}_k$ and hence by Proposition 3.2 $\text{mon}_{[0,\, s_k]} f \in \mathcal{C}^{1\,1}_k$. Moreover,

$$\|\text{mon}_{[0,\, s_k]} f\|_{\mathcal{C}^0} \le \tilde{c}_v\delta \qquad \|(\text{mon}_{[0,\, s_k]} f)'\|_{\mathcal{C}^0} \le \tilde{c}_v\delta.$$

One can then conclude that the map T defined by the right hand side of (3.44) maps $X_{uk} \times X_{vk} \times X_{\sigma k}$ into itself.

Choosing η_k large enough (but independent from δ) it is possible to prove the contraction property. One has to exploit all the previous estimates and also properties (3.9) and (3.54), which are needed to handle the second and the third component of T.

It turns out that the Lipschitz constant of T (i.e., the constant in the contraction) is uniformly bounded with respect to δ.

□

In the following it will be useful to know how the solution of (3.44) depends on the length s_k of the interval of definition. More precisely, in order to underline the dependence from s_k, we will denote the first component of the solution of (3.44) as $u_k^{s_k}$. Let

$$F^k(\bar{u}_k,\, s_k) := u_k^{s_k}(s_k),$$

where the dependence from the initial point $\bar{u}_k$ is also made explicit.

To study how F^k depends on s_k we will exploit the following result:

Lemma 3.5. *Fix $\delta << 1$ and let $s_k^1,\, s_k^2$ such that $s_k^1 \le s_k^2 \le \delta$. Let $\big(u_k^1,\, v_k^1,\, \sigma_k^1\big)$ and $\big(u_s^2,\, v_k^2,\, \sigma_k^2\big)$ be the corresponding solutions of* (3.44)*. Then,*

$$\|u_k^1 - u_k^2\|_{\mathcal{C}^0([0,\, s_k^1])} + \|v_k^1 - v_k^2\|_{\mathcal{C}^0([0,\, s_k^1])} + \eta_k\delta\|\sigma_k^1 - \sigma_k^2\|_{\mathcal{C}^0([0,\, s_k^1])} \le L_k\delta|s_k^1 - s_k^2|, \tag{3.55}$$

where L_k is a suitable constant which does not depend on δ.

Proof. In the following we will denote by $T^{s_k^1}$ and $T^{s_k^2}$ the maps defined by the right hand side of (3.44) when $s_k = s_k^1$ and $s_k = s_k^2$ respectively. Moreover, to simplify notations, we will denote by $\big(u_s^2,\, v_k^2,\, \sigma_k^2\big)$ the restriction of the fixed point of $T^{s_k^2}$ to the interval $[0,\, s_k^1]$.

Since $\big(u_k^1,\, v_k^1,\, \sigma_k^1\big)$ is the fixed point of $T^{s_k^1}$, one then has

$$\big\|\big(u_k^1,\, v_k^1,\, \sigma_k^1\big)-\big(u_s^2,\, v_k^2,\, \sigma_k^2\big)\big\|_{X_{uk}\times X_{vk}\times X_{\sigma k}}\le \frac{1}{1-K}\big\|\big(u_k^2,\, v_k^2,\, \sigma_k^2\big)-T^{s_k^1}\big(u_s^2,\, v_k^2,\, \sigma_k^2\big)\big\|_{X_{uk}\times X_{vk}\times X_{\sigma k}}.$$

In the previous formula, K denotes the Lipschitz constant of $T^{s_k^1}$, which turns out to be uniformly bounded with respect to K as underlined in the proof of Lemma 3.4. Moreover, X_{uk}, X_{vk} and $X_{\sigma k}$ denote the spaces defined by (3.50) when $s_k = s_k^1$.

Thus, to prove the lemma one actually reduces to prove

$$\begin{aligned}\big\|\big(u_k^2,\, v_k^2,\, \sigma_k^2\big)-T^{s_k^1}\big(u_s^2,\, v_k^2,\, \sigma_k^2\big)\big\|_{X_{uk}\times X_{vk}\times X_{\sigma k}} =&\|u_k^2-T_u^{s_k^1}\big(u_s^2,\, v_k^2,\, \sigma_k^2\big)\|_{\mathcal{C}^0([0,\, s_k^1])}\\ &+\|v_k^2-T_v^{s_k^1}\big(u_s^2,\, v_k^2,\, \sigma_k^2\big)\|_{\mathcal{C}^0([0,\, s_k^1])}\\ &+\eta_k\|\sigma_k^2-T_\sigma^{s_k^1}\big(u_s^2,\, v_k^2,\, \sigma_k^2\big)\|_{\mathcal{C}^0([0,\, s_k^1])}\\ &\le \mathcal{O}(1)\delta|s_k^1-s_k^2|.\end{aligned}$$

In the previous formula, $T_u^{s_k^1}$, $T_v^{s_k^1}$ and $T_\sigma^{s_k^1}$ denote respectively the first, the second and the third component of $T^{s_k^1}$.

Since $\big(u_k^2,\, v_k^2,\, \sigma_k^2\big)$ is the fixed point of $T^{s_k^2}$, then

$$\begin{cases} u_k^2(\tau)=\bar u_k+\displaystyle\int_0^\tau \check r_k(u_k^2(\xi),\, v_k^2(\xi),\, 0,\, \sigma_k^2(\xi))d\xi\\ v_k^2(\tau)=f_k[u_k^2,\, v_k^2,\, \sigma_k^2](\tau)-\mathrm{mon}_{[0,\, s_k^2]}f_k[u_k^2,\, v_k^2,\, \sigma_k^2](\tau)\\ \sigma_k^2(\tau)=\dfrac{1}{\bar c}\dfrac{d}{d\tau}\mathrm{mon}_{[0,\, s_k^2]}f_k(\tau,\, u_k^2,\, v_k^2,\, \sigma_k^2).\end{cases}$$

Hence, to prove the lemma it is sufficient to show that

$$\|\mathrm{mon}_{[0,\, s_k^2]}f_k[u_k^2,\, v_k^2,\, \sigma_k^2]-\mathrm{mon}_{[0,\, s_k^1]}f_k[u_k^2,\, v_k^2,\, \sigma_k^2]\|_{\mathcal{C}^0[0,\, s_k^1]}\le \mathcal{O}(1)\delta(s_k^2-s_k^1)$$

and that

$$\|(\mathrm{mon}_{[0,\, s_k^2]}f_k[u_k^2,\, v_k^2,\, \sigma_k^2])'-(\mathrm{mon}_{[0,\, s_k^1]}f_k[u_k^2,\, v_k^2,\, \sigma_k^2])'\|_{\mathcal{C}^0[0,\, s_k^1]}\le \mathcal{O}(1)\delta(s_k^2-s_k^1).$$

This follows directly from Proposition 3.4. □

In particular, the previous Lemma implies that

$$|F(\bar u_k,\, s_k^1)-F(\bar u_k,\, ,s_k^2)|\le 3|s_k^1-s_k^2|. \tag{3.56}$$

Indeed, assuming for example that $s_k^1\le s_k^2$, one can write

$$\begin{aligned}|u_k^1(s_k^1)-u_k^2(s_k^2)|&\le |u_k^1(s_k^1)-u_k^2(s_k^1)|+|u_k^2(s_k^1)-u_k^2(s_k^2)|\\ &\le L_k\delta(s_k^2-s_k^1)+\Big|\int_{s_k^1}^{s_k^2}\check r_k\big(u_k^2(\tau),\, v_k^2(\tau),\, \sigma_k^2(\tau)\big)d\tau\Big|\\ &\le (L_k\delta+2)(s_k^2-s_k^1).\end{aligned}$$

Take $s_k^1=0$ in the previous estimate: since

$$\begin{aligned}|F^k(\bar u_k,\, s_k^1)-F^k(\bar u_k,\, s_k^2)|&=\int_0^{s_k^2}\check r_k\big(u_k^2(\tau),\, v_k^2(\tau),\, \sigma_k^2(\tau)\big)d\tau\\ &=\int_0^{s_k^2}\check r_k\big(\bar u_k,\, 0,\, 0\big)d\tau\\ &\quad+\int_0^{s_k^2}\big[\check r_k\big(u_k^2(\tau),\, v_k^2(\tau),\, \sigma_k^2(\tau)\big)-\check r_k\big(\bar u_k,\, 0,\, 0\big)\big]d\tau\\ &=\check r_k\big(\bar u_k,\, 0,\, 0\big)s_k^2+\mathcal{O}(1)\delta s_k^2,\end{aligned}\tag{3.57}$$

one gets

$$F^k(\bar{u}_k,\, s_k) = u^2_k(s_k) = \bar{u}_k + \check{r}_k\big(\bar{u}_k,\, 0,\, 0\big)s_k + \mathcal{O}(1)\delta^2. \tag{3.58}$$

Thus, the function F^k defined before is differentiable at $s_k = 0$ and the gradient is the eigenvector $r_k(\bar{u}_k) = \check{r}_k\big(\bar{u}_k,\, 0,\, 0\big)$.

The function F^k depends Lipschitz continuously on $\bar{u}_k$ too:

Lemma 3.6. *Fix δ such that $0 < \delta << 1$. Take $\bar{u}^1_k\ \bar{u}^2_k \in \mathbb{R}^N$ such that $|\bar{u}^1_k - \bar{u}^2_k| \le a\delta$ for a certain constant a. If $|s_k| \le \delta$, then*

$$|F(\bar{u}^1_k,\, s_k) - F(\bar{u}^2_k,\, s_k)| \le \tilde{L}_k |u^1_k - u^2_k| \tag{3.59}$$

for a suitably large constant $\tilde{L}_k$.

Proof. The value $F(\bar{u}^1_k,\, s_k)$ is the fixed point of the application T defined by (3.44). To underline the dependence of T on $\bar{u}_k$ we will write

$$T : \mathbb{R}^N \times X_{uk} \times X_{vk} \times X_{\sigma k} \to X_{uk} \times X_{vk} \times X_{\sigma k}$$

The map T depends Lipschitz continuously on $\bar{u}_k$, but one cannot apply directly the classical results (see e.g. [17]) on the dependence of the fixed point of a contraction from a parameter because of a technical difficulty. Indeed, the domain $X_{uk} \times X_{vk} \times X_{\sigma k}$ of the contraction depends on $\bar{u}_k$.

To overcome this problem, one can proceed as follows. Define X^1_{uk} and X^2_{uk} the spaces defined as in (3.50) with $\bar{u} = \bar{u}^1_k$ and $\bar{u}_k = \bar{u}^2_k$ respectively. If $\big(u^1_k,\, v^1_k,\, \sigma^1_k\big)$ is the fixed point of the application $T(\bar{u}^1_k,\, \cdot)$ defined on $X^1_{uk} \times X_{vk} \times X_{\sigma k}$, then

$$\|u^1_k - \bar{u}_k\|_{\mathcal{C}^0} \le \|u^1_k - \bar{u}^1_k\|_{\mathcal{C}^0} + |u^1_k - u^2_k| \le 2s_k + a\delta \le (2+a)\delta.$$

Choosing $c_u \ge (2+a)$, one gets that $\big(u^1_k,\, v^1_k,\, \sigma^1_k\big)$ belongs to $X^2_{uk} \times X_{vk} \times X_{\sigma k}$. Thus,

$$\|\big(u^1_k,\, v^1_k,\, \sigma^1_k\big) - \big(u^2_k,\, v^2_k,\, \sigma^2_k\big)\|_{X^2_{uk} \times X_{vk} \times X_{\sigma k}} \le \frac{1}{1-K} \|\big(u^1_k,\, v^1_k,\, \sigma^1_k\big) - T\big(\bar{u}^2_k,\, u^1_k,\, v^1_k,\, \sigma^1_k\big)\|_{X^2_{uk} \times X_{vk} \times X_{\sigma k}}.$$

In the previous expression, $\big(u^2_k,\, v^2_k,\, \sigma^2_k\big)$ is the fixed point of the application $T(\bar{u}^2_k,\, \cdot)$. One can then proceed as in [17] and conclude that

$$\|\big(u^1_k,\, v^1_k,\, \sigma^1_k\big) - \big(u^2_k,\, v^2_k,\, \sigma^2_k\big)\|_{X^2_{uk} \times X_{vk} \times X_{\sigma k}} \le \tilde{L}_k |\bar{u}^1_k - \bar{u}^2_k| \tag{3.60}$$

□

7. **Analysis of the purely stable component**

The purely center component satisfies (3.49).

Lemma 3.7. *Fix δ such that $|V^s(0)| \le \delta << 1$. Then system (3.49) defines a contraction on the space $X^s_u \times X^s_v$, where*

$$X^s_u := \Big\{u^s \in \mathcal{C}^0([0,\, +\infty),\, \mathbb{R}^N),\ \|u^s\|_{us} \le m_u\delta\Big\} \qquad X^s_v := \Big\{V^s \in \mathcal{C}^0([0,\, +\infty),\, \mathbb{R}^{k-1}),\ \|V^s\|_{vs} \le m_v\delta\Big\}.$$

The constants m_u and m_v do not depend on δ the norms $\|\ \cdot\ \|_{us}$ and $\|\ \cdot\ \|_{vs}$ are defined by

$$\|u^s\|_{us} := \eta_s \sup \big\{e^{cx/2}|u^s(x)|\big\} \qquad \|V^s\|_{vs} := \sup \big\{e^{cx/2}|V^s(x)|\big\}, \tag{3.61}$$

where η is a small enough constant which does not depend on δ and c is defined by (3.25).

Proof. We know that $\bar\Lambda$ is has eigenvalues $\lambda_1(\bar u_k)\dots\lambda_{k-1}(\bar u_k)$. Relying on (3.19), one obtains

$$|e^{\bar\Lambda x}V^s(0)| \le e^{-cx/2}|V^s(0)| \le e^{-cx/2}\delta.$$

Moreover, if $u^s \in X^s_u$ and $V^s \in X^s_v$, then

$$|\Lambda(\bar u_k + u^s,\, 0,\, V^s) - \bar\Lambda| \le \mathcal{O}(1)\delta,$$

where $\mathcal{O}(1)$ denotes (here and in the following) some constant that depends neither on η nor on δ.

The columns of $\check R^s(\bar u_k,\, 0,\, 0)$ are the eigenvectors $r_1(\bar u_k)\dots r_{k-1}(\bar u_k)$, which are all unit vectors. Thus, if $u \in X^s_u$ and $V^s \in X^s_v$, then

$$|\check R^s(\bar u_k + u^s,\, 0,\, V^s)| \le \mathcal{O}(1).$$

From the previous observations one gets that the application defined by (3.49) maps $X^s_u \times X^s_v$ into itself.

To prove that the application is a contraction, take $u^{1s},\ u^{2s} \in X^s_u$ and $V^{1s},\ V^{2s} \in X^s_v$. Then

$$\begin{aligned}
&\eta_s\Big|\int_x^{+\infty} \check R^s\Big(\bar u_k + u^{1s}(y),\, 0,\, V^{1s}(y)\Big)V^{1s}(y)dy - \int_x^{+\infty} \check R^s\Big(\bar u_k + u^{2s}(y),\, 0,\, V^{2s}(y)\Big)V^{2s}(y)dy\Big| \\
&\le \mathcal{O}(1)\delta\|u^{1s} - u^{2s}\|_{us} + \mathcal{O}(1)\delta\eta_s\|V^{1s} - V^{2s}\|_{vs} + \mathcal{O}(1)\eta_s\|V^{1s} - V^{2s}\|_{vs}.
\end{aligned}$$

Choosing η_s sufficiently small, one can suppose that $\mathcal{O}(1)\eta_s < 1/4$ in the previous expression. Moreover, one can suppose that δ is small enough to have that $\mathcal{O}(1)\delta < 1/2$, $\mathcal{O}(1)\delta\eta_s < 1/4$.

Finally,

$$\begin{aligned}
e^{cx/2}\Big|&\int_0^x e^{\bar\Lambda(x-y)}\Big[\check\Lambda^s\Big(u_{1s}(y) + \bar u_k,\, 0,\, V^{1s}(y)\Big) - \bar\Lambda\Big]V^{1s}(y)dy \\
&- \int_0^x e^{\bar\Lambda(x-y)}\Big[\check\Lambda^s\Big(u_{2s}(y) + \bar u_k,\, 0,\, V^{2s}(y)\Big) - \bar\Lambda\Big]V^{2s}(y)dy\Big| \\
\le \mathcal{O}(1)&\frac{\delta}{\eta_s}\|u^{1s} - u^{2s}\|_{us} + \mathcal{O}(1)\delta\|V^{1s} - V^{2s}\|_{vs}.
\end{aligned}$$

Assuming that δ is small enough, $\mathcal{O}(1)\delta/\eta_s < 1/2$ and $\mathcal{O}(1)\delta < 1/2$. Thus the map is contraction and the constant of the contraction is less or equal to $1/2$ uniformly for $\delta \to 0^+$. □

The solution of (3.49) depends on the parameter $V^s(0)$. The regularity of $\Big(u_s,\, V_s\Big)$ with respect to $V^s(0)$ is discussed in the following lemma.

Lemma 3.8. *Fix $\delta << 1$ and let $V^1_s(0)$, $V^2_s(0)$ two initial data such that $|V^1_s(0)|$, $|V^2_s(0)| \le \delta$. Let $(u^1_s,\, V^1_s)$ and $(u^2_s,\, V^2_s)$ the corresponding solutions of* (3.49)*. Then,*

$$\|u^1_s - u^2_s\|_{us} + \|V^{1s} - V^{2s}\|_{vs} \le L_s|V^1_s(0) - V^2_s(0)|, \tag{3.62}$$

where L_s is a suitable constant which does not depend on δ. Moreover, if $V^2_s(0) = \vec 0$, then $u^2_s \equiv 0$. Also,

$$u^1_s(0) = \check R^s(\bar u_k,\, 0,\, 0)\bar\Lambda^{-1}V^1_s(0) + \mathcal{O}(1)\delta^2. \tag{3.63}$$

Equation (3.63) guarantees, in particular, that the application

$$F^s(\bar u,\, V_s(0)) := u^s(0).$$

is differentiable with respect to $V_s(0)$ when $V_s(0) = 0$ and that the jacobian is the matrix $\check R^s(\bar u_k,\, 0,\, 0)\bar\Lambda^{-1}$, whose columns are the eigenvectors $r_1(\bar u_k)/\lambda_1(\bar u_k)\dots r_{k-1}(\bar u_k)/\lambda_{k-1}(\bar u_k)$.

Proof. Let T the application defined by the right hand side of (3.49). To underline the dependence on the parameter $V^s(0)$, we write

$$T : X_{us} \times X_{vs} \times \mathbb{R}^{k-1} \to X_{us} \times X_{vs}.$$

and denote by T_u and T_v respectively the first and the second component of T. For every $(u_s, V_s) \in X_{us} \times X_{vs}$,

$$\left\| T_u \Big(u_s,\, V_s,\, V_s^1(0) - T_u \Big(u_s,\, V_s,\, V_s^2(0) \Big) \right\|_{us} = 0$$

and

$$\left\| T_v \Big(u_s,\, V_s,\, V_s^1(0) - T_v \Big(u_s,\, V_s,\, V_s^2(0) \Big) \right\|_{vs} \le |V_s^1 - V_s^2|.$$

Hence, (3.62) holds with $L_s := 1/(1-K)$, where K is the Lipschitz constant of T and hence it is smaller than 1 because T is a contraction. Moreover, from the proof of Lemma 3.7 it follows that k is bounded away from 0 uniformly with respect to δ. This concludes the proof of the first part of the lemma.

To prove the second part, we observe that

$$V_s^1(x) = e^{\bar{\Lambda} x} V_s^1(0) + \mathcal{O}(1)\delta^2 e^{-cx/2}$$

and that

$$\left| \check{R}^s(\bar{u}_k + u_s^1,\, 0,\, V_s^1) - \check{R}^s(\bar{u}_k,\, 0,\, 0) \right| \le \mathcal{O}(1)\delta.$$

Hence,

$$\begin{aligned} u_s^1(0) &= \int_0^{+\infty} \check{R}^s(\bar{u}_k,\, 0,\, 0) V_s^1(y) dy + \int_0^{+\infty} \Big[\check{R}^s(\bar{u}_k + u_s^1,\, 0,\, V_s^1) - \check{R}^s(\bar{u}_k + u_s,\, 0,\, V_s) \Big] V_s^1(y) dy \\ &= \int_0^{+\infty} \check{R}^s(\bar{u}_k,\, 0,\, 0) e^{\bar{\Lambda} y} V_s^1(0) dy + \mathcal{O}(1)\delta^2 \int_{+\infty}^0 e^{-cy/2} dy + \mathcal{O}(1)\delta^2 \int_{+\infty}^0 e^{-cy/2} dy \\ &= \check{R}^s(\bar{u}_k,\, 0,\, 0) \bar{\Lambda}^{-1} V_s^1(0) + \mathcal{O}(1)\delta^2. \end{aligned}$$

This completes the proof of the lemma. □

To study the dependence of $F^s\big(\bar{u}_k,\, V_s(0),\, s_k\big)$ on $\bar{u}_k$ one can proceed as follow. Fix $\bar{u}_k^1,\, \bar{u}_k^2 \in \mathbb{R}^N$ close enough and denote by $\big(u_s^1,\, V_s^1\big)$ and $\big(u_s^2,\, V_s^2\big)$ the solution of (3.49) corresponding to $\bar{u}_k = \bar{u}_k^1$ and $\bar{u}_k = \bar{u}_k^2$ respectively. Then classical results (see e.g. [17]) on the dependence of the fixed point of a contraction on a parameter ensure that

$$\|u_k^1 - u_k^2\|_{us} + \|V^{1s} - V^{2s}\|_{vs} \le \tilde{L}_s |\bar{u}_k^1 - \bar{u}_k^2| \tag{3.64}$$

for a suitable constant $\tilde{L}^s$. In particular,

$$|F^s\big(\bar{u}_k^1,\, V^s(0)\big) - F^s\big(\bar{u}_k^2,\, V^s(0)\big)| \le \tilde{L}_s |\bar{u}_k^1 - \bar{u}_k^2|. \tag{3.65}$$

8. **The component of perturbation**

Assume that $v_k(s_k) < 0$ and define α as in (3.47):

$$\alpha(\tau) = -\int_\tau^{s_k} \frac{1}{v_k(s)} ds.$$

Let

$$\underline{s} := \max\{\tau \in [0,\, s_k] :\, v_k(\tau) = 0\},$$

we can now prove that $\alpha(\tau) < +\infty$ if and only if $\tau > \underline{s}$.

Indeed, for $s \geq \underline{s}$ it holds

$$-v_k(s) = -v_k(s) + v_k(\underline{s}) \leq \tilde{c}_v \delta (s - \underline{s})$$

and hence

$$\alpha(\tau) = \int_{\tau}^{s_k} -\frac{1}{v_k(s)} ds \geq \int_{\tau}^{s_k} \frac{1}{\tilde{c}_v \delta (s - \underline{s})} ds = +\infty.$$

Let

$$\beta : [0, +\infty[\to]\underline{s}, s_k]$$

the inverse function of α, $\beta \circ \alpha(\tau) = \tau$. If $v_k(s_k) = 0$, we set $\beta(x) = s_k$ for every x.

The component of perturbation (U, q, p) is the fixed point of the transformation

$$\left\{ \begin{aligned} U(x) = & \int_{+\infty}^{x} \Big[\check{R}^s \Big(u_k \circ \beta(y) + u_s(y) + U(y),\, v_k \circ \beta(y) + q(y),\, V_s(y) + p(y),\, 0 \Big) \\ & \qquad - \check{R}^s \Big(\bar{u}_k + u_s(y),\, 0,\, V_s(y),\, 0 \Big) \Big] V_s(y) dy \\ & + \int_{+\infty}^{x} \check{R}^s \Big(u_k \circ \beta(y) + u_s(y) + U(y),\, v_k \circ \beta(y) + q(y),\, V_s(y) + p(y),\, 0 \Big) p(y) dy \\ & +. \int_{+\infty}^{x} \Big[\check{r}^k \Big(u_k \circ \beta(y) + u_s(y) + U(y),\, v_k \circ \beta(y) + q(y),\, V_s(y) + p(y),\, 0 \Big) \\ & \qquad - \check{r}^k \Big(u_k \circ \beta(y),\, v_k \circ \beta(y),\, 0,\, 0 \Big) \Big] v_k \circ \beta(y) dy \\ & + \int_{+\infty}^{x} \check{r}^k \Big(u_k \circ \beta(y) + u_s(y) + U(y),\, v_k \circ \beta(y) + q(y),\, V_s(y) + p(y),\, 0 \Big) q(y) dy \\ q(x) = & \int_{+\infty}^{x} \Big[\phi_k \Big(u_k \circ \beta(y) + u_s(y) + U(y),\, v_k \circ \beta(y) + q(y),\, V_s(y) + p(y),\, \sigma \circ \beta \Big) \\ & \qquad - \phi_k \Big(u_k \circ \beta(y),\, v_k \circ \beta(y),\, 0,\, \sigma \circ \beta \Big) \Big] v_k \circ \beta(y) dy \\ & + \int_{+\infty}^{x} \phi_k \Big(u_k \circ \beta(y) + u_s(y) + U(y),\, v_k \circ \beta(y) + q(y),\, V_s(y) + p(y),\, \sigma_k \circ \beta \Big) q(y) dy \\ p(x) = & \int_{0}^{x} e^{\bar{\Lambda}(x-y)} \Big[\Lambda \Big(u_k \circ \beta(y) + u_s(y) + U(y),\, v_k \circ \beta(y) + q(y),\, V_s(y) + p(y) \Big) - \bar{\Lambda} \Big] p(y) dy \\ & + \int_{0}^{x} e^{\bar{\Lambda}(x-y)} \Big[\Lambda \Big(u_k \circ \beta(y) + u_s(y) + U(y),\, v_k \circ \beta(y) + q(y),\, V_s(y) + p(y) \Big) \\ & \qquad - \Lambda \Big(u_s(y), 0),\, V_s(y) \Big) \Big] V^s(y) dy \quad . \end{aligned} \right. \tag{3.66}$$

In the equation for p we used the same notation as in (3.49): $\bar{\Lambda} = \check{\Lambda}^s(\bar{u}_k, 0, 0)$.

Lemma 3.9. *Let (u_k, v_k, σ_k) and (u_s, V^s) be as in Lemma 3.4 and 3.7 respectively. Then there exists a unique solution (U, q, p) of (3.66) belonging to $X_U \times X_q \times X_p$, where the spaces X_U, X_q and X_p are defined as follows:*

$$X_U := \Big\{ U \in \mathcal{C}^0([0, +\infty[, \mathbb{R}^N) : \ |U(x)| \leq k_U \delta^2 e^{-cx/2} \Big\},$$

$$X_p := \Big\{ p \in \mathcal{C}^0([0, +\infty[, \mathbb{R}^{k-1}) : \ |p(x)| \leq k_p \delta^2 e^{-cx/2} \Big\},$$

and

$$X_q := \Big\{ q \in \mathcal{C}^0([0, +\infty[, \mathbb{R}) : \ |q(x)|_q \leq k_q \delta^2 e^{-cx/2}. \Big\}$$

In the previous expressions, the constants k_U, k_q and k_p do not depend on δ.

The space X_U is endowed with the norm

$$\|U\|_U = \eta_U \sup_x \Big\{ e^{cx/4} |U(x)| \Big\}$$

where η_U is a suitable constant which does not depend on δ. The spaces X_p and X_q are endowed respectively with the norms

$$\|p\|_p = \sup_x \Big\{ e^{cx/4} |p(x)| \Big\}$$

and

$$\|q\|_q = \sup_x \Big\{ e^{cx/4} |q(x)| \Big\}.$$

Proof. It is enough to prove that the right hand side of (3.66) defines a contraction on $X_U \times X_q \times X_p$. As in the proof of Lemma 3.7, one observes that $\bar{\Lambda}$ is a diagonal matrix with all the eigenvalues less or equal to $-c$. Thus,

$$|e^{\bar{\Lambda}x} \vec{\xi}| \leq e^{-cx} |\vec{\xi}| \quad \forall, \ \vec{\xi} \in \mathbb{R}^{k-1}.$$

Moreover, from Lemma 3.7 one gets that for every x,

$$|V_s(x)| \leq m_v \delta e^{-cx/2} \qquad |u_s(x)| \leq m_u \delta e^{-cx/2}.$$

Another useful observation is the following:

$$\int_{+\infty}^{0} v_k \circ \beta(y) dy \leq s_k \leq \delta. \tag{3.67}$$

Indeed, if $v_k(s_k) < 0$, then one can exploit the change of variable $\tau = \beta(y)$. Recalling that β is the inverse function of α, which is defined by (3.47), one has

$$\int_{+\infty}^{0} v_k \circ \beta(y) dy \leq s_k = \int_{\underline{s}}^{s_k} v_k(\tau) \alpha'(\tau) d\tau = \int_{\underline{s}}^{s_k} d\tau = (s_k - \underline{s}) \leq s_k.$$

If $v_k(s_k) = 0$, then $v_k \circ \beta \equiv 0$ and hence (3.67) is trivial.

One can also assume

$$|\phi_k \big(\bar{u}_k, 0, \, 0, \, \lambda_k(\bar{u}_k) \big)| \leq K\delta \tag{3.68}$$

and hence

$$|\phi_k \big(u_k \circ \beta(y) + u_s(y) + U(y), \, v_k \circ \beta(y) + q(y), \, V_s(y) + p(y), \, \sigma_k \circ \beta \big)| \leq \mathcal{O}(1)\delta.$$

Moreover, it holds

$$\Big| \Lambda \Big(u_k \circ \beta(y) + u_s(y) + U(y), \, v_k \circ \beta(y) + q(y), \, V_s(y) + p(y) \Big) - \bar{\Lambda} \Big| \leq \mathcal{O}(1)\delta$$

Since $\bar{\Lambda} = \check{\Lambda}^s(\bar{u}_k, \, 0, \, 0)$, this follows from the regularity of Λ and from the estimates

$$|u_k - \bar{u}_k|, \ |u_s|, \ |v_k|, \ |q|, \ |V_s|, \ |p| \leq \mathcal{O}(1)\delta.$$

Exploiting the previous observations, by direct check one gets that the right hand side of (3.66) belongs to $X_U \times X_q \times X_p$.

To prove that the map is actually a contraction, one has to exploit again the previous remarks, combined with the following: for every x,

$$|U^1 - U^2|(x) \leq \frac{1}{\eta_U} \|U^1 - U^2\|_U e^{-cx/4}, \quad |q^1 - q^2|(x) \leq \|q^1 - q^2\|_q e^{-cx/4}$$

and

$$|p^1 - p^2|(x) \leq \|p^1 - p^2\|_p e^{-cx/4}.$$

One can then check that the map is a contraction, provided that η_U is sufficiently small. □

Since $(u_k,\, v_k,\, \sigma_k)$ and $(u_s,\, V_s)$ depend on the parameters s_k and $V_s(0)$ respectively, then also $(U,\, q,\, p)$ does. Let

$$F^p\big(\bar{u}_k,\, V_s(0),\, s_k\big) := U(0).$$

The following lemma concerns the regularity of F^p with respect to $\big(V_s(0),\, s_k\big)$.

Lemma 3.10. *The function F^p is differentiable with respect to $\big(V_s(0),\, s_k\big)$ at $\big(V_s(0),\, s_k\big) = \vec{0}$ and the jacobian is the null matrix.*
Also, let $\big(V_s^1(0),\, s_k^1\big)$ and $\big(V_s^2(0),\, s_k^2\big)$ such that $\big|\big(V_s^1(0),\, s_k^1\big)\big|,\ \big|\big(V_s^2(0),\, s_k^2\big)\big| \le \delta$. Then,

$$\big|F^p\big(\bar{u}_k^1,\, V_s^1(0),\, s_k^1\big) - F^p\big(\bar{u}_k^2\, V_s^2(0),\, s_k^2\big)\big| \le L_p\Big(|s_k^1 - s_k^2| + |V_s^1(0) - V_s^2(0)|\Big). \tag{3.69}$$

Proof. By construction,

$$|U(0)| \le \mathcal{O}(1)\delta^2. \tag{3.70}$$

This implies the differentiability at $\big(V_s(0),\, s_k\big) = \vec{0}$.

To prove the second part of the lemma, we will focus only on the dependence from $V_s(0)$ and s_k. The proof of the Lipschitz continuous dependence from $\bar{u}_k$ is completely analogous.

We can observe that $(U,\, q,\, p)$ is the fixed point of a contraction T, defined by the right hand side of (3.66). To underline the dependence from $\big(V_s(0),\, s_k\big)$ we write

$$T : X_u \times X_q \times X_p \times \mathbb{R}^{k-1} \times \mathbb{R} \to X_u \times X_q \times X_p$$

and denote by T_U, T_q and T_p respectively the first, the second and the third component of T. Since the Lispchitz constant of T is not only smaller than 1 but also bounded away from 0 uniformly with respect to δ, then to prove (3.69) it is enough to prove that, for every $(U,\, q,\, p)$,

$$\begin{aligned}
&\big\|T_U\big(V_s^1(0),\, s_k^1,\, U,\, p,\, q\big) - T_U\big(V_s^2(0),\, s_k^2,\, U,\, p,\, q\big)\big\|_U \le \mathcal{O}(1)\Big(|s_k^1 - s_k^2| + |V_s^1(0) - V_s^2(0)|\Big)\\
&\big\|T_q\big(V_s^1(0),\, s_k^1,\, U,\, p,\, q\big) - T_q\big(V_s^2(0),\, s_k^2,\, U,\, p,\, q\big)\big\|_q \le \mathcal{O}(1)\Big(|s_k^1 - s_k^2| + |V_s^1(0) - V_s^2(0)|\Big)\\
&\big\|T_p\big(V_s^1(0),\, s_k^1,\, U,\, p,\, q\big) - T_p\big(V_s^2(0),\, s_k^2,\, U,\, p,\, q\big)\big\|_p \le \mathcal{O}(1)\Big(|s_k^1 - s_k^2| + |V_s^1(0) - V_s^2(0)|\Big).
\end{aligned} \tag{3.71}$$

Let

$$\alpha_1(\tau) = -\int_\tau^{s_k^1} \frac{1}{v_k^1(s)} ds \qquad \alpha_2(\tau) = -\int_\tau^{s_k^2} \frac{1}{v_k^2(s)} ds$$

and β^1 and β^2 the inverse functions. Just to fix the ideas, let us assume that $s_k^1 < s_k^2$, then $\beta_1(0) = s_k^1 < \beta_k^2(0) = s_k^2$. Exploiting Lemma 3.5 one gets that, as far as $\beta_1(x) \le \beta_2(x)$, it holds

$$\begin{aligned}
\frac{d(\beta_2 - \beta_1)}{dx} &= v_k^2 \circ \beta^2 - v_k^1 \circ \beta^1 = v_k^2 \circ \beta^2 - v_k^2 \circ \beta^1 + v_k^2 \circ \beta^1 - v_k^1 \circ \beta^1\\
&\le \|v_k^2 - v_k^1\|_{\mathcal{C}^0} + \tilde{c}_v \delta(\beta^2 - \beta^1)\\
&\le L_k \delta(s_k^2 - s_k^1) + \tilde{c}_v \delta(\beta^2 - \beta^1)
\end{aligned}$$

and hence

$$|\beta^1(x) - \beta^2(x)| \le \Big[\frac{L_k}{\tilde{c}_v}(s_k^2 - s_k^1) + (s_k^2 - s_k^1)\Big] e^{\tilde{c}_v \delta x} - \frac{L_k}{\tilde{c}_v}(s_k^2 - s_k^1) \le \mathcal{O}(1)(s_k^2 - s_k^1) e^{\tilde{c}_v \delta x} \tag{3.72}$$

If $\beta^1 > \beta^2$, then

$$\beta^1(x) - \beta^2(x) \le \frac{L_k}{\tilde{c}_v}\Big[e^{\tilde{c}_v \delta(x - \bar{x})} - 1\Big](s_k^2 - s_k^1),$$

where

$$\bar{x} = \max\{y < x : \ \beta^1(y) = \beta^2(y)\}.$$

Thus, estimate (3.72) still holds.

Exploiting (3.72), one gets

$$\begin{aligned} |u_k^2 \circ \beta^2(x) - u_k^1 \circ \beta^1(x)| &\leq |u_k^2 \circ \beta^2 - u_k^2 \circ \beta^1| + |u_k^2 \circ \beta^1(x) - u_k^1 \circ \beta^1(x)| \\ &\leq \|u_k^2 - u_k^1\|_{\mathcal{C}^0} + 2|\beta^2 - \beta^1| \leq L_k \delta (s_k^2 - s_k^1) + \mathcal{O}(1)(s_k^2 - s_k^1) e^{\tilde{c}_v \delta x} \\ &\leq \mathcal{O}(1)(s_k^2 - s_k^1) e^{\tilde{c}_v \delta x} \end{aligned} \tag{3.73}$$

and, by analogous considerations,

$$|v_k^2 \circ \beta^2(x) - v_k^1 \circ \beta^1(x)|, \ |\sigma_k^2 \circ \beta^2(x) - \sigma_k^1 \circ \beta^1(x)| \leq \mathcal{O}(1)(s_k^2 - s_k^1) e^{\tilde{c}_v \delta x}. \tag{3.74}$$

To prove (3.71), the most complicated terms to handle are in the form

$$\begin{aligned} &\Big| \int_{+\infty}^{x} \Big[\check{r}^k \Big(u_k^1 \circ \beta^1(y) + u_s^1(y) + U(y), \, v_k^1 \circ \beta^1(y) + q(y), \, V_s^1(y) + p(y), \, 0 \Big) \\ &\qquad - \check{r}^k \Big(u_k^1 \circ \beta^1(y), \, v_k^1 \circ \beta^1(y), \, 0, \, 0 \Big) \Big] v_k^1 \circ \beta^1(y) dy \\ &\quad - \int_{+\infty}^{x} \Big[\check{r}^k \Big(u_k^2 \circ \beta^2(y) + u_s^2(y) + U(y), \, v_k^2 \circ \beta^2(y) + q(y), \, V_s^2(y) + p(y), \, 0 \Big) \\ &\qquad - \check{r}^k \Big(u_k^2 \circ \beta^2(y), \, v_k^2 \circ \beta^2(y), \, 0, \, 0 \Big) \Big] v_k^2 \circ \beta^2(y) dy \Big| \\ &\leq \Big| \int_{+\infty}^{x} \Big[\check{r}^k \Big(u_k^1 \circ \beta^1(y) + u_s^1(y) + U(y), \, v_k^1 \circ \beta^1(y) + q(y), \, V_s^1(y) + p(y), \, 0 \Big) \\ &\qquad - \check{r}^k \Big(u_k^1 \circ \beta^1(y), \, v_k^1 \circ \beta^1(y), \, 0, \, 0 \Big) \Big] \Big(v_k^1 \circ \beta^1(y) - v_k^2 \circ \beta^2(y) \Big) dy \Big| \\ &+ \Big| \int_{+\infty}^{x} \Big\{ \Big[\check{r}^k \Big(u_k^1 \circ \beta^1(y) + u_s^1(y) + U(y), \, v_k^1 \circ \beta^1(y) + q(y), \, V_s^1(y) + p(y), \, 0 \Big) \\ &\qquad - \check{r}^k \Big(u_k^1 \circ \beta^1(y), \, v_k^1 \circ \beta^1(y), \, 0, \, 0 \Big) \Big] \\ &\quad - \Big[\check{r}^k \Big(u_k^2 \circ \beta^2(y) + u_s^2(y) + U(y), \, v_k^2 \circ \beta^2(y) + q(y), \, V_s^2(y) + p(y), \, 0 \Big) \\ &\qquad - \check{r}^k \Big(u_k^2 \circ \beta^2(y), \, v_k^2 \circ \beta^2(y), \, 0, \, 0 \Big) \Big] \Big\} v_k^2 \circ \beta^2(y) dy \Big| \end{aligned} \tag{3.75}$$

Exploiting (3.74) the first term in the previous sum can be estimated as follows:

$$\begin{aligned} &\Big| \int_{+\infty}^{x} \Big[\check{r}^k \Big(u_k^1 \circ \beta^1(y) + u_s^1(y) + U(y), \, v_k^1 \circ \beta^1(y) + q(y), \, V_s^1(y) + p(y), \, 0 \Big) \\ &\qquad - \check{r}^k \Big(u_k^1 \circ \beta^1(y), \, v_k^1 \circ \beta^1(y), \, 0, \, 0 \Big) \Big] \Big(v_k^1 \circ \beta^1(y) - v_k^2 \circ \beta^2(y) \Big) dy \Big| \\ &\leq \int_{+\infty}^{x} e^{-cy/2} \Big[m_u \delta + k_u \delta^2 + k_q \delta^2 + m_v \delta + k + p \delta^2 \Big] \mathcal{O}(1)(s_k^2 - s_k^1) e^{\tilde{c}_v \delta y} dy \\ &\leq \mathcal{O}(1)(s_k^2 - s_k^1) \delta e^{-cx/4}, \end{aligned}$$

where we have supposed that δ is small enough to obtain $\tilde{c}_v \delta \leq c/4$.

To handle with the second term in (3.75) one can observe that

$$\Big[\check{r}^k\Big(u_k^1\circ\beta^1(y)+u_s^1(y)+U(y),\,v_k^1\circ\beta^1(y)+q(y),\,V_s^1(y)+p(y),\,0\Big)$$
$$-\check{r}^k\Big(u_k^1\circ\beta^1(y),\,v_k^1\circ\beta^1(y),\,0,\,0\Big)\Big]$$
$$=\int_0^1 D_u\check{r}^k\Big(u_k^1\circ\beta^1(y)+tu_s^1(y)+tU(y),\,v_k^1\circ\beta^1(y)+tq(y),\,tV_s^1(y)+tp(y),\,0\Big)\Big[u_s^1(y)+U(y)\Big]dt$$
$$+\int_0^1 D_v\check{r}^k\Big(u_k^1\circ\beta^1(y)+tu_s^1(y)+tU(y),\,v_k^1\circ\beta^1(y)+tq(y),\,tV_s^1(y)+tp(y),\,0\Big)\Big[q(y)\Big]dt$$
$$+\int_0^1 D_V\check{r}^k\Big(u_k^1\circ\beta^1(y)+tu_s^1(y)+tU(y),\,v_k^1\circ\beta^1(y)+tq(y),\,tV_s^1(y)+tp(y),\,0\Big)\Big[V_s^1(y)+p(y)\Big]dt.$$

Moreover,

$$\Bigg|\int_{+\infty}^x\Bigg\{\int_0^1 D_u\check{r}^k\Big(u_k^1\circ\beta^1(y)+tu_s^1(y)+tU(y),\,v_k^1\circ\beta^1(y)+tq(y),\,tV_s^1(y)+tp(y),\,0\Big)\Big[u_s^1(y)+U(y)\Big]dt$$
$$-D_u\check{r}^k\Big(u_k^2\circ\beta^2(y)+tu_s^2(y)+tU(y),\,v_k^2\circ\beta^2(y)$$
$$+tq(y),\,tV_s^2(y)+tp(y),\,0\Big)\Big[u_s^2(y)+U(y)\Big]dt\Bigg\}v_k^2\circ\beta^2(y)dy\Bigg|$$
$$\le\Bigg|\int_{+\infty}^x\Bigg\{\int_0^1 D_u\check{r}^k\Big(u_k^1\circ\beta^1(y)+tu_s^1(y)+tU(y),\,v_k^1\circ\beta^1(y)$$
$$+tq(y),\,tV_s^1(y)+tp(y),\,0\Big)\Big[u_s^1(y)-u_s^2(y)\Big]dt\Bigg\}v_k^2\circ\beta^2(y)dy\Bigg|$$
$$+\Bigg|\int_{+\infty}^x\Bigg\{\int_0^1\Big[D_u\check{r}^k\Big(u_k^1\circ\beta^1(y)+tu_s^1(y)+tU(y),\,v_k^1\circ\beta^1(y)+tq(y),\,tV_s^1(y)+tp(y),\,0\Big)$$
$$-D_u\check{r}^k\Big(u_k^2\circ\beta^2(y)+tu_s^2(y)+tU(y),\,v_k^2\circ\beta^2(y)$$
$$+tq(y),\,tV_s^2(y)+tp(y),\,0\Big)\Big][u_s^2(y)+U(y)]dt\Bigg\}v_k^2\circ\beta^2(y)dy\Bigg|$$
$$\le\int_{+\infty}^x\Bigg\{\int_0^1\mathcal{O}(1)L_s|V_s^1(0)-V_s^2(0)|e^{-cy/2}dt\Bigg\}c_v\delta dy$$
$$+\int_{+\infty}^x\Bigg\{\int_0^1\Big[\mathcal{O}(1)(s_k^2-s_k^1)e^{\tilde{c}_v\delta y}+L_s|V_s^1(0)-V_s^2(0)|e^{-cy/2}\Big]\delta e^{-cy/2}dt\Bigg\}\delta$$
$$\le\mathcal{O}(1)\delta\Big(|V_s^1(0)-V_s^2(0)|+(s_k^2-s_k^1)\Big)e^{-cx/4}.$$

By analogous considerations, one can conclude that the second term in (3.75) can be bounded by $\mathcal{O}(1)\delta\Big(|V_s^1(0)-V_s^2(0)|+(s_k^2-s_k^1)\Big)e^{-cx/4}$.

One can also perform similar estimates on all the other terms that appear in

$$\big\|T_U\big(V_s^1(0),\,s_k^1,\,U,\,p,\,q\big)-T_U\big(V_s^2(0),\,s_k^2,\,U,\,p,\,q\big)\big\|_U,$$
$$\big\|T_q\big(V_s^1(0),\,s_k^1,\,U,\,p,\,q\big)-T_q\big(V_s^2(0),\,s_k^2,\,U,\,p,\,q\big)\big\|_q$$

and

$$\big\|T_p\big(V_s^1(0),\,s_k^1,\,U,\,p,\,q\big)-T_p\big(V_s^2(0),\,s_k^2,\,U,\,p,\,q\big)\big\|_p$$

and conclude that (3.71) holds. □

Set $V_s(0) = (s_1 \dots s_{k-1})$ and define

$$F(\bar{u}_k, \, s_1, \dots s_k) := F^k(\bar{u}_k, \, s_k) + F^s(\bar{u}_k, \, s_1, \dots s_{k-1}) + F^p(\bar{u}_k, \, s_1, \dots s_k). \tag{3.76}$$

Combining (3.55), (3.62) and (3.69) one gets that F is Lipschitz continuous with respect to $(s_1 \dots s_k)$:

$$|F(\bar{u}_k, \, s_1, \dots s_k) - F(\bar{u}_k, \, \tilde{s}_1, \dots \tilde{s}_k)| \le L\big(|s_1 - \tilde{s}_1| + \dots |s_k - \tilde{s}_k|\big) \tag{3.77}$$

for a suitable constant L.

Moreover, F is differentiable at $(s_1 \dots s_k) = \vec{0}$ because of (3.58) , (3.63) and (3.70). The colums of the jacobian matrix are the eigenvectors $r_1(\bar{u}_k) \dots r_k(\bar{u}_k)$.

Finally, F is also Lipschitz continuous with respect to $\bar{u}_k$, namely

$$|F(\bar{u}_k^1, \, s_1 \dots s_k) - F(\bar{u}_k^2, \, s_1 \dots s_k)| \le \tilde{L}|\bar{u}_k^1 - \bar{u}_k^2|. \tag{3.78}$$

Define

$$\phi(\bar{u}_0, \, s_1 \dots s_N) := F\Big(T^{k+1}_{s_{k+1}} \circ \dots \circ T^N_{s_N}(\bar{u}_0), \, s_1 \dots s_k\Big). \tag{3.79}$$

We recall that the composite map $T^{k+1}_{s_{k+1}} \circ \dots \circ T^N_{s_N}(\bar{u}_0)$ is Lipschitz continuous and differentiable at $(s_{k+1} \dots s_N) = (0 \dots 0)$, with the columns of the jacobian given by the eigenvectors $r_{k+1}(\bar{u}_0) \dots r_N(\bar{u}_0)$. This was proved in [7].

Combining this with (3.78) and (3.79) one gets that ϕ is Lipschitz continuous. Moreover, it is differentiable at $(s_1 \dots s_N)$ and the columns of the jacobian are the eigenvectors $r_1(\bar{u}_0) \dots r_N(\bar{u}_0)$. Thus, thanks to (3.4), the jacobian is invertible and hence, exploiting the extension of the implicit function theorem discussed in [24] (page 253), the map $\phi(\bar{u}_0, \cdot)$ is invertible in a neighbourhood of $(s_1 \dots s_N) = (0 \dots 0)$.

Thus, if $\bar{u}_b$ is fixed and is sufficiently close to $\bar{u}_0$, then the values of $s_1 \dots s_N$ are uniquely determined by the equation

$$\bar{u}_b = \phi(\bar{u}_0, \, s_1 \dots s_N). \tag{3.80}$$

We will take $\bar{u}_b$ equal to the boundary datum imposed on the parabolic approximation (3.1). Once $(s_1 \dots s_N)$ are determined one can determine the values of the hyperbolic limit $u(t, \, x)$ a.e. $(t, \, x)$. This can be done glueing together pieces like (3.18). Here we will just make a remark concerning the trace $\bar{u}$ of the hyperbolic limit on the axis $x = 0$. Let s_k determined by equation (3.80) and let $(u_k, \, v_k, \, \sigma_k)$ solve the fixed point problem (3.44). Define

$$s := \min\{s \in \, [0, \, s_k] : \; \sigma_k(s) = 0\},$$

then the trace $\bar{u}$ of the hyperbolic limit on $x = 0$ is

$$\bar{u} = u_k(\bar{s}) \tag{3.81}$$

The following theorem collects the results discussed in this section.

Theorem 3.2. *Let Hypotheses 1, 3, 4, 5, 6 , 7 and 9 hold. Then there exists $\delta > 0$ small enough such that the following holds. If $|\bar{u}_0 - \bar{u}_b| << \delta$, then the limit of the parabolic approximation* (3.1) *satisfies*

$$\bar{u}_b = \phi(\bar{u}_0, \, s_1 \dots s_N)$$

for a suitable vector $(s_1 \dots s_N)$. The map ϕ is given by

$$\phi(\bar{u}_0, \, s_1 \dots s_N) := F\Big(T^{k+1}_{s_{k+1}} \circ \dots \circ T^N_{s_N}(\bar{u}_0), \, s_1 \dots s_k\Big),$$

where $T^{k+1}_{s_{k+1}} \dots T^N_{s_N}$ are the curves of admissible states defined in [7], while the function F is defined by (3.76). *The map ϕ is locally invertible: given $\bar{u}_0$ and $\bar{u}_b$, one can determine uniquely $(s_1 \dots s_N)$. Once $(s_1 \dots s_N)$ are known, the value $u(t, \, x)$ assumed by the limit is determined for a.e. $(t, \, x)$. In particular, the trace $\bar{u}$ of the hyperbolic limit in the axis $x = 0$ is given by* (3.81).

4 The characterization of the hyperbolic limit in the case of a singular viscosity matrix

The aim of this section is to describe the limit of the parabolic approximation (1.1) when the viscosity matrix $\tilde{B}$ is not invertible. The precise hypotheses assumed in this case are introduced in Sections 2.3 and 2.2. In particular, they guarantee that it is sufficient to study the family of systems

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} & u^\varepsilon \in \mathbb{R}^N \\ ß(u^\varepsilon(t,\, 0)) \equiv \bar{g} \qquad u^\varepsilon(0,\, x) \equiv \bar{u}_0, \end{cases} \tag{4.1}$$

with E, A and B satisfying Hypothesis 2, ß given by Definition 2.1.

The exposition is organized as follows. In Section 4.1 we introduce some preliminary results, while in Section 4.2 we give a characterization of the limit of (4.1). More precisely, in Section 4.2.1 we briefly recall the characterization of the limit in the case of the Cauchy problem:

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, \varepsilon u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \\ u^\varepsilon(0,\, x) = \begin{cases} \bar{u}_0 & x \geq 0 \\ u^- & x < 0 \end{cases} \end{cases}$$

We refer to [7] for the complete analysis. In Section 4.2.2 we introduce a lemma on the dimension of the stable manifold of the equation

$$A(u,\, u_x)u_x = B(u)u_{xx},$$

which is the equation satisfied by the steady solutions of (4.1). Such a lemma gives an answer to question which was left open in [44]. In Sections 4.2.3 we give a characterization of the limit in the non characteristic case, i.e. when none of the eigenvalues of $E^{-1}A$ can attain the value zero. The boundary characteristic case occurs when an eigenvalue of $E^{-1}A$ can attain the value 0 and it is discussed in Section 4.2.4. Finally, in Section 4.3 we discuss a transversality lemma, a technical result which is needed to complete the analysis in Sections 4.2.3 and 4.2.4.

4.1 Preliminary results

This section introduces the preliminary results needed in Section 4.2. The exposition is organized as follows: Section 4.1.1 gives some results about the structure of the matrix $A(u,\, u_x)$, all implied by Hypothesis 2 and in particular by Kawashima condition. In Section 4.1.2 we rewrite equations

$$E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx} \tag{4.2}$$

and

$$A(u,\, u_x)u_x = B(u)u_{xx} \tag{4.3}$$

in a form easier to handle.

Some notations have to be introduced: given a real parameter σ, let

$$A(u,\, u_x,\, \sigma) := A(u,\, u_x) - \sigma E(u). \tag{4.4}$$

Consequently, both systems (4.2) and (4.3) may be written in the form

$$A(u,\, u_x,\, \sigma)u_x = B(u)u_{xx}. \tag{4.5}$$

Finally, let

$$\begin{pmatrix} A_{11}(u,\, \sigma) & A_{12}(u,\, \sigma) \\ A_{21}(u,\, \sigma) & A_{22}(u,\, u_x,\, \sigma) \end{pmatrix} = \begin{pmatrix} A_{11}(u) - \sigma E_{11}(u) & A_{12}(u) - \sigma E_{12}(u) \\ A_{21}(u) - \sigma E_{21}(u) & A_{22}(u,\, u_x) - \sigma E_{22}(u) \end{pmatrix} \tag{4.6}$$

and

$$\Xi = (w,\, z)^T, \quad w \in \mathbb{R}^{N-r},\ z \in \mathbb{R}^r \tag{4.7}$$

be the block decompositions of $A(u,\, u_x,\, \sigma)$ and of $\Xi \in \mathbb{R}^N$ corresponding to (2.8).

4.1.1 Some results about the structure of $A(u, u_x, \sigma)$

Even if it is not explicitly specified in the statement, all the results of this section presuppose that Hypothesis 2 holds.

The first observation is immediate, but useful:

Lemma 4.1. *Let $A_{ij}(u, u_x, \sigma)$ be the blocks of $A(u, u_x, \sigma)$ defined by (4.6). Then*

$$A_{12}(u, \sigma) = A_{21}^T(u, \sigma) \qquad A_{11}^T(u, \sigma) = A_{11}(u, \sigma)$$

Some further notations are required: let $P_0(u, \sigma)$ denote the projection of $\mathbb{R}^{N-r}$ on $\ker A_{11}(u, \sigma)$. Thanks to the third condition in Hypothesis 2, the dimension of this subspace is constant in u: we will denote it by q. The operator P_0 can be identified with a matrix $P_0(u, \sigma) \in \mathbb{M}^{q\times(N-r)}$. Similarly, the projection on the subspace orthogonal to $\ker A_{11}(u, \sigma)$ is identified with a matrix $P_\perp(u, \sigma) \in \mathbb{M}^{(N-r-q)\times(N-r)}$. The decomposition (4.7) is hence refined setting

$$\bar{w} = P_0 w \qquad \tilde{w} = P_\perp w \qquad \bar{w} \in \mathbb{R}^q, \;\; \tilde{w} \in \mathbb{R}^{N-r-q}. \tag{4.8}$$

Using the previous notations and recalling Lemma 4.1, the product $A(u, u_x, \sigma)\vec{\Xi}$ writes

$$\begin{pmatrix} A_{11}(u, \sigma) & A_{21}(u, \sigma)^T \\ A_{21}(u, \sigma) & A_{22}(u, u_x, \sigma) \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & \big(A_{21}^I\big)^T(u, \sigma) \\ 0 & \tilde{A}_{11}(u, \sigma) & \big(A_{21}^{II}\big)^T(u, \sigma) \\ A_{21}^I(u) & A_{21}^{II}(u, \sigma) & A_{22}(u, \sigma, u_x) \end{pmatrix} \begin{pmatrix} \bar{w} \\ \tilde{w} \\ z \end{pmatrix}, \tag{4.9}$$

where

$$A_{21}^I(u, \sigma) = A_{21}(u, \sigma) P_0^T(u, \sigma) \in \mathbb{M}^{r\times q} \qquad\qquad A_{21}^{II}(u, \sigma) = A_{21}(u, \sigma) P_\perp^T(u, \sigma) \in \mathbb{M}^{r\times(N-r-q)}$$

and

$$\tilde{A}_{11}(u, \sigma) = P_\perp(u, \sigma) A_{11}(u, \sigma) P_\perp^T(u, \sigma) \in \mathbb{M}^{(N-r-q)\times(N-r-q)}$$

It is now possible to state the following result, a consequence of Kawashima condition:

Lemma 4.2. *The matrix $A_{21}^I(u, \sigma) \in \mathbb{M}^{r\times q}$ has rank q. Thus, in particular, $q \le r$.*

Proof. First of all, one observes that

$$E^{-1}A(u, 0)\Xi = \sigma\Xi \iff A(u, 0, \sigma)\Xi = 0$$

and hence Kawashima condition writes

$$\ker\big(A(u, 0, \sigma)\big) \cap \ker\big(B(u)\big) = \big\{0\big\} \qquad \forall\, u, \sigma. \tag{4.10}$$

We claim that this implies

$$A_{21}^I \bar{w} = 0 \Rightarrow \bar{w} = 0. \tag{4.11}$$

Indeed, suppose by contradiction that there exists $\bar{w} \neq 0$ such that $A_{21}^I \bar{w} = 0$. Then

$$\begin{pmatrix} 0 & 0 & (A_{21}^I)^T \\ 0 & \tilde{A}_{11} & (A^I I_{21})^T \\ A_{21}^I & A^I I_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \bar{w} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix} \begin{pmatrix} \bar{w} \\ 0 \\ 0 \end{pmatrix},$$

which contradicts (4.10). Hence (4.11) holds.

From (4.11) one then deduces that A_{21}^I has q independent columns and hence it admits a submatrix of rank q. □

In the following we will isolate an invertible submatrix of $A^I_{21}(u,\,\sigma)$ and we will denoted it by $a_{11}(u,\,\sigma)$. More precisely, a_{11} is obtained from $A^I_{21}(u,\,\sigma)$ selecting q independent rows. By continuity, one can suppose that the rows selected are the same for every u.

With considerations analogous to the ones that lead to (4.9), one can write

$$z = \bar{P}^T(\sigma)\bar{z} + \tilde{P}^T(\sigma)\tilde{z},$$

where $\bar{P} \in \mathbb{M}^{q\times r}$ is the projection such that $a_{11} = A^I_{21}\bar{P}^T$ and $\tilde{P} \in \mathbb{M}^{(r-q)\times r}$ is the projection on the subspace orthogonal to the image of $\mathbb{R}^r$ through $\bar{P}$.

Thanks to the previous considerations, one obtains that $A(u,\,u_x,\,\sigma)\Xi$ writes

$$\begin{pmatrix} 0 & 0 & a^T_{11}(u,\,\sigma) & a^T_{21}(u,\,\sigma) \\ 0 & \tilde{A}_{11}(u,\,\sigma) & a^T_{12}(u,\,\sigma) & a^T_{22}(u,\,\sigma) \\ a_{11}(u,\,\sigma) & a_{12}(u,\,\sigma) & \alpha_{11}(u,\,u_x,\,\sigma) & \alpha^T_{21}(u,\,u_x,\,\sigma) \\ a_{21}(u) & a_{22}(u,\,\sigma) & \alpha_{21}(u,\,u_x,\,\sigma) & \alpha_{22}(u,\,u_x,\,\sigma) \end{pmatrix} \begin{pmatrix} \bar{w} \\ \tilde{w} \\ \bar{z} \\ \tilde{z}, \end{pmatrix} \tag{4.12}$$

with

$$a_{12} \in \mathbb{M}^{q\times(N-r-q)}, \quad a_{22} \in \mathbb{M}^{(r-q)\times(N-r-q)}$$
$$\alpha_{11} \in \mathbb{M}^{q\times q}, \quad \alpha_{21} \in \mathbb{M}^{(r-q)\times q}, \quad \alpha_{22} \in \mathbb{M}^{(r-q)\times(r-q)}$$
$$\bar{z} \in \mathbb{R}^q \quad \tilde{z} \in \mathbb{R}^{r-q}$$

The corresponding decomposition of $B(u)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_{11}(u) & b_{12}(u) \\ 0 & 0 & b_{21}(u) & b_{22}(u) \end{pmatrix} \tag{4.13}$$

with

$$b_{11} \in \mathbb{M}^{q\times q}, \quad b_{12} \in \mathbb{M}^{q\times(r-q)}, \quad b_{21} \in \mathbb{M}^{(r-q)\times q}, \quad b_{22} \in \mathbb{M}^{(r-q)\times(r-q)}.$$

It is worth underling explicitly that formulations (4.12) and (4.13) include in particular the limit cases $q = 0$ and $q = N - r$: indeed, it is enough to assume $w = \tilde{w}$, $z = \tilde{z}$ and $w = \bar{w}$, $z = \bar{z}$ respectively.

Lemma 4.3. *The matrix*

$$a_{21}a^{-1}_{11}\Big(b_{11}(a^T_{11})^{-1}a^T_{21} - b_{12}\Big) - b_{21}(a^T_{11})^{-1}a^T_{21} + b_{22}$$

is invertible.

Proof. By contradiction, assume that there exists an $(r-q)$-dimensional vector $\vec{\xi} \neq 0$ such that

$$\Big(a_{21}a^{-1}_{11}\Big(b_{11}(a^T_{11})^{-1}a^T_{21} - b_{12}\Big) - b_{21}(a^T_{11})^{-1}a^T_{21} + b_{22}\Big)\vec{\xi} = 0.$$

It follows that

$$(a_{21}a^{-1}_{11},\, -I_{r-q}) \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} \left(a^T_{11}\right)^{-1} a^T_{21} \\ -I_{r-q} \end{pmatrix} \vec{\xi} = 0 \tag{4.14}$$

Define

$$D = \begin{pmatrix} (a^{-1}_{11})^T a^T_{21} \\ I_{r-q} \end{pmatrix},$$

then (4.14) implies

$$0 = \langle D\vec{\xi},\, bD\vec{\xi}\rangle.$$

Since $D\,\xi \neq \vec{0}$, this is a contradiction because of the second condition in Hypothesis 2. □

4.1.2 The explicit form of system (4.5)

The following result guarantees that system (4.5), and hence systems (4.2) and (4.3) can be reduced to an explicit form like (4.15)

Lemma 4.4. *There exists a sufficiently small constant $\delta > 0$ such that if $|u - \bar{u}_0| < \delta$ and $|u_x| < \delta$, then the following holds. System (4.5) can be rewritten as*

$$\begin{cases} u_x = \Big(\bar{w}(u,\, \tilde{z},\, \sigma),\ \tilde{w}(u,\, \tilde{z},\, \sigma),\ \bar{z}(u,\, \tilde{z},\, \sigma),\ \tilde{z}\Big)^T \\ \tilde{z}_x = f(u,\, \tilde{z},\, \sigma) \end{cases} \tag{4.15}$$

for suitable functions

$$\bar{w} : \mathbb{R}^N \times \mathbb{R}^{r-q} \times \mathbb{R} \to \mathbb{R}^q \qquad \tilde{w} : \mathbb{R}^N \times \mathbb{R}^{r-q} \times \mathbb{R} \to \mathbb{R}^{N-r-q} \qquad \bar{z} : \mathbb{R}^N \times \mathbb{R}^{r-q} \times \mathbb{R} \to \mathbb{R}^q$$
$$f : \mathbb{R}^N \times \mathbb{R}^{r-q} \times \mathbb{R} \to \mathbb{R}^{r-q}.$$

For every σ and for every u the following holds.

$$\bar{w}(u,\, \vec{0},\, \sigma) = \vec{0} \qquad \tilde{w}(u,\, \vec{0},\, \sigma) = \vec{0} \qquad \bar{z}(u,\, \vec{0},\, \sigma) = \vec{0} \qquad f(u,\, \vec{0},\, \sigma) = \vec{0} \tag{4.16}$$

Also, these functions satisfy

$$D_u \bar{z}\big|_{u=\bar{u}_0,\, \tilde{z}=0,\, \sigma} = 0 \qquad D_\sigma \bar{z}\big|_{u=\bar{u}_0,\, \tilde{z}=0,\, \sigma} = 0 \qquad D_{\tilde{z}} \bar{z}\big|_{u=\bar{u}_0,\, \tilde{z}=0} = -(a_{11}^T)^{-1} a_{21}^T \tag{4.17}$$

and

$$\begin{gathered} D_u \tilde{w}\big|_{u=\bar{u}_0,\, \tilde{z}=0,\, \sigma} = 0 \qquad D_\sigma \tilde{w}\big|_{u=\bar{u}_0,\, \tilde{z}=0,\, \sigma} = 0 \\ D_{\tilde{z}} \tilde{w}\big|_{u=\bar{u}_0,\, \tilde{z}=0,\, \sigma} = -\tilde{A}_{11}^{-1}\big(a_{12}^T D_{\tilde{z}} \bar{z} + a_{22}^T\big) = -\tilde{A}_{11}^{-1}\big(-a_{12}^T (a_{11}^T)^{-1} a_{21}^T + a_{22}^T\big). \end{gathered} \tag{4.18}$$

Moreover,

$$\begin{aligned} & D_u \bar{w}\big|_{u=\bar{u}_0,\, \tilde{z}=0,\, \sigma} = 0 \qquad D_\sigma \bar{w}\big|_{u=\bar{u}_0,\, \tilde{z}=0,\, \sigma} = 0 \\ D_{\tilde{z}} \bar{w}\big|_{u=\bar{u}_0,\, \tilde{z}=0} &= -a_{11}^{-1}\Big(a_{12} D_{\tilde{z}} \tilde{w} + \alpha_{11} D_{\tilde{z}} \bar{z} + \alpha_{12}^T\Big) + a_{11}^{-1} b_{11} D_{\tilde{z}} \bar{z} D_{\tilde{z}} f + a_{11}^{-1} b_{12} D_{\tilde{z}} f \\ &= a_{11}^{-1}\Big(a_{12} \tilde{A}_{11}^{-1}\big(a_{22}^T - a_{12}^T (a_{11}^T)^{-1} a_{21}^T\big) + \alpha_{11} (a_{11}^T)^{-1} a_{21}^T - \alpha_{12}\Big) + a_{11}^{-1}\Big(b_{12} \\ &\quad - b_{11} (a_{11}^T)^{-1} a_{21}^T\Big)\Big(a_{21} a_{11}^{-1}\Big(b_{11} (a_{11}^T)^{-1} a_{21}^T - b_{12}\Big) - b_{21} (a_{11}^T)^{-1} a_{21}^T \\ &\quad + b_{22}\Big)^{-1}\Big(a_{21} a_{11}^{-1}\Big(a_{12} \tilde{A}_{11}^{-1}\big(a_{22}^T - a_{12}^T (a_{11}^T)^{-1} a_{21}^T\big) + \alpha_{11} (a_{11}^T)^{-1} a_{21}^T - \alpha_{12}\Big) \\ &\quad + a_{22} \tilde{A}_{11}^{-1}\big(a_{12}^T (a_{11}^T)^{-1} a_{21}^T - a_{22}^T - \alpha_{21} (a_{11}^T)^{-1} a_{21}^T + \alpha_{22}\big)\Big) \end{aligned} \tag{4.19}$$

and

$$
\begin{aligned}
&D_u f\big|_{u=\bar u_0,\,\tilde z=0,\,\sigma} = 0 \qquad D_\sigma f\big|_{u=\bar u_0,\,\tilde z=0,\,\sigma} = 0 \\
&\quad D_{\tilde z} f\big|_{u=\bar u_0,\,\tilde z=0} = \Big(b_{21} D_{\tilde z}\bar z + b_{22} - a_{21}\Big(a_{11}^{-1} b_{11} D_{\tilde z}\bar z + a_{11}^{-1} b_{12}\Big)\Big)^{-1}\Big(-a_{21}a_{11}^{-1}\Big(a_{12}D_{\tilde z}\tilde w + \alpha_{11} D_{\tilde z}\bar z \\
&\qquad\qquad + \alpha_{12}^T\Big) + a_{22} D_{\tilde z}\tilde w + \alpha_{21} D_{\tilde z}\bar w + \alpha_{22}\Big) \\
&\qquad = \Big(a_{21}a_{11}^{-1}\Big(b_{11}(a_{11}^T)^{-1}a_{21}^T - b_{12}\Big) - b_{21}(a_{11}^T)^{-1}a_{21}^T + b_{22}\Big)^{-1}\Big(a_{21}a_{11}^{-1}\Big(a_{12}A_{11}^{-1}\big(a_{22}^T \\
&\qquad\qquad - a_{12}^T(a_{11}^T)^{-1}a_{21}^T\big) + \alpha_{11}(a_{11}^T)^{-1}a_{21}^T - \alpha_{21}^T\Big) + a_{22}\tilde A_{11}^{-1}\Big(a_{12}^T(a_{11}^T)^{-1}a_{21}^T \\
&\qquad\qquad - a_{22}^T\Big) - \alpha_{21}(a_{11}^T)^{-1}a_{21}^T + \alpha_{22}\Big).
\end{aligned} \tag{4.20}
$$

Proof. Using notation (4.12) and (4.13), equation rewrites as

$$
\begin{pmatrix}
0 & 0 & a_{11}^T(u,\sigma) & a_{21}^T(u,\sigma) \\
0 & \tilde A_{11}(u,\sigma) & a_{12}^T(u,\sigma) & a_{22}^T(u\,\sigma) \\
a_{11}(u,\sigma) & a_{12}(u,\sigma) & \alpha_{11}(u,\bar w,\tilde w,\bar z,\tilde z,\sigma) & \alpha_{21}^T(u,\bar w,\tilde w,\bar z,\tilde z,\sigma) \\
a_{21}(u,\sigma) & a_{22}(u,\sigma) & \alpha_{21}(u,\bar w,\tilde w,\bar z,\tilde z,\sigma) & \alpha_{22}(u,\bar w,\tilde w,\bar z,\tilde z\,\sigma)
\end{pmatrix}
\begin{pmatrix} \bar w \\ \tilde w \\ \bar z \\ \tilde z \end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & b_{11}(u) & b_{12}(u) \\
0 & 0 & b_{21}(u) & b_{22}(u)
\end{pmatrix}
\begin{pmatrix} \bar w_x \\ \tilde w_x \\ \bar z_x \\ \tilde z_x \end{pmatrix}
$$

Hence from the first line it follows

$$a_{11}^T(u,\sigma)\bar z + a_{21}^T(u,\sigma)\tilde z = 0$$

and therefore

$$\bar z(u,\tilde z) = -\big(a_{11}^T(u,\sigma)\big)^{-1} a_{21}^T(u,\sigma)\tilde z.$$

The second line then reads

$$\tilde A_{11}(u,\sigma)\tilde w - a_{12}^T(u)\big(a_{11}^T(u,\sigma)\big)^{-1} a_{21}^T(u,\sigma)\tilde z + a_{22}^T(u,\sigma)\tilde z = 0$$

and hence thanks to the invertibility of $\tilde A_{11}(u)$

$$\tilde w(u,\tilde z) = \tilde A_{11}^{-1}(u,\sigma)\Big(a_{12}^T(u)\big(a_{11}^T(u,\sigma)\big)^{-1} a_{21}^T(u,\sigma) + a_{22}^T(u,\sigma)\Big)\tilde z.$$

Moreover, one has

$$\big(\bar z(u,\tilde z)\big)_x = -\big(a_{11}^T(u,\sigma)\big)^{-1} a_{21}^T(u,\sigma)\tilde z_x - D_u\Big(\big(a_{11}^T(u,\sigma)\big)^{-1} a_{21}^T(u,\sigma)\Big)u_x = \bar z_x(u,\bar w,\tilde z,\tilde z_x)$$

and hence the third line reads

$$a_{11}(u)\bar w + a_{12}(u)\tilde w(u,\tilde z) + \alpha_{11}(u,\bar w,\tilde z)\bar z(u,\tilde z) + \alpha_{21}^T(u,\bar w,\tilde z)\tilde z = b_{11}(u)\bar z_x(u,\bar w,\tilde z,\tilde z_x) + b_{12}(u)\tilde z_x$$

and therefore, thanks to the invertibility of a_{11}, in a small enough neighborhood of $(u = \bar u_0,\, \bar w = 0,\, \tilde z = 0,\, \tilde z_x = 0)$ it is implicitly defined a map $\bar w = \bar w(u,\tilde z,\tilde z_x)$ such that

$$
\begin{aligned}
&D_u \bar w\big|_{u=\bar u_0,\,\tilde z=0,\,\tilde z_x=0} = 0 \qquad D_{\tilde z_x}\bar w\big|_{u=\bar u_0,\,\tilde z=0,\,\tilde z_x=0} = a_{11}^{-1}\Big(b_{12} - b_{11}(a_{11}^T)^{-1}a_{21}^T\Big) \\
&D_{\tilde z}\bar w\big|_{u=\bar u_0,\,\tilde z=0,\,\tilde z_x=0} = a_{11}^{-1}\Big(a_{12}A_{11}^{-1}\big(a_{22}^T - a_{12}^T(a_{11}^T)^{-1}a_{21}^T\big) + \alpha_{11}(a_{11}^T)^{-1}a_{21}^T - \alpha_{21}^T\Big).
\end{aligned}
$$

Then the fourth line reads

$$a_{21}(u)\bar{w}(u,\,\tilde{z},\,\tilde{z}_x)+a_{22}(u)\tilde{w}(u,\,\tilde{z})+\alpha_{21}(u,\,\tilde{z},\,\tilde{z}_x)\bar{z}_x(u,\,\tilde{z},\,\tilde{z}_x)+\alpha_{22}(u,\,\tilde{z},\,\tilde{z}_x)\tilde{z}=b_{21}(u)\bar{z}_x(u,\,\tilde{z},\,\tilde{z}_x)+b_{22}(u)\tilde{z}_x$$

Hence thanks to Lemma 4.3, it is implicitly defined a map $\tilde{z}_x = f(u,\,\tilde{z})$, which satisfies the hypotheses described in the statement of the lemma. □

In order to simplify the notations, we set

$$\begin{aligned}\underline{a}(u,\,\tilde{z},\,\sigma) &:= a_{21}a_{11}^{-1}\Big(a_{12}\tilde{A}_{11}^{-1}\big(a_{22}^T - a_{12}^T(a_{11}^T)^{-1}a_{21}^T\big) + \alpha_{11}(a_{11}^T)^{-1}a_{21}^T - \alpha_{21}^T\Big) + a_{22}\tilde{A}_{11}^{-1}\big(a_{12}^T(a_{11}^T)^{-1}a_{21}^T \\ &\quad - a_{22}^T - \alpha_{21}(a_{11}^T)^{-1}a_{21}^T + \alpha_{22}\big) \in \mathbb{M}^{(r-q)\times(r-q)} \\ \underline{b}(u,\,\sigma) &:= a_{21}a_{11}^{-1}\Big(b_{11}(a_{11}^T)^{-1}a_{21}^T - b_{12}\Big) - b_{21}(a_{11}^T)^{-1}a_{21}^T + b_{22} \in \mathbb{M}^{(r-q)\times(r-q)}\end{aligned} \tag{4.21}$$

With this notations,

$$D_{\tilde{z}}f\Big|_{u=\bar{u}_0,\,\tilde{z}=0,\,\sigma} = \underline{b}^{-1}(\bar{u}_0,\,0,\,\sigma)\underline{a}(\bar{u}_0,\,0,\,\sigma).$$

Also, if we consider the jacobian

$$D_{\tilde{z}}\Big(\bar{w}(u,\,\tilde{z},\,\sigma),\,\tilde{w}(u,\,\tilde{z},\,\sigma),\,\bar{z}(u,\,\tilde{z},\,\sigma),\,\tilde{z}(u,\,\tilde{z},\,\sigma)\Big)$$

and we compute it at the point $\Big(u = \bar{u}_0,\,\tilde{z}=0,\,\sigma\big)$, we get

$$\begin{pmatrix} a_{11}^{-1}a_{12}\tilde{A}_{11}^{-1}\Big(a_{22}^T - a_{12}^T(a_{11}^T)^{-1}a_{21}^T\Big)\vec{\xi} + a_{11}^{-1}\alpha_{11}(a_{11}^T)^{-1}a_{21}^T\vec{\xi} - a_{11}^{-1}\alpha_{21}^T\vec{\xi} + \Big(a_{11}^{-1}b_{12} - a_{11}^{-1}b_{11}(a_{11}^{-1})^T a_{21}^T\Big)\underline{b}^{-1}\underline{a} \\ \tilde{A}_{11}^{-1}\Big(a_{12}^T(a_{11}^T)^{-1}a_{21}^T - a_{22}^T\Big) \\ -(a_{11}^T)^{-1}a_{21}^T \\ I_{r-q}\end{pmatrix} \tag{4.22}$$

Finally, the proof of Lemma 4.4 implies the following result:

Lemma 4.5. *Given $\vec{\xi} \in \mathbb{R}^{r-q}$, the condition $\underline{a}(u,\,0,\,\sigma)\vec{\xi} = \lambda\underline{b}(u,\sigma)\vec{\xi}$ is equivalent to $A(u,\,0,\,\sigma)\;\Xi = \lambda B(u)\;\Xi$, where $\Xi \in \mathbb{R}^N$ is given by*

$$\Xi = \begin{pmatrix} a_{11}^{-1}a_{12}\tilde{A}_{11}^{-1}\Big(a_{22}^T - a_{12}^T(a_{11}^T)^{-1}a_{21}^T\Big)\vec{\xi} + a_{11}^{-1}\alpha_{11}(a_{11}^T)^{-1}a_{21}^T\vec{\xi} - a_{11}^{-1}\alpha_{21}^T\vec{\xi} + \Big(a_{11}^{-1}b_{12} - a_{11}^{-1}b_{11}(a_{11}^{-1})^T a_{21}^T\Big)\underline{b}^{-1}\underline{a}\vec{\xi} \\ \tilde{A}_{11}^{-1}\Big(a_{12}^T(a_{11}^T)^{-1}a_{21}^T - a_{22}^T\Big)\vec{\xi} \\ -(a_{11}^T)^{-1}a_{21}^T\vec{\xi} \\ \vec{\xi}.\end{pmatrix} \tag{4.23}$$

The case $\lambda = 0$ and hence $\underline{a}\vec{\xi} = 0$ is, in particular, included in the previous formulation.

Remark 4.1. For simplicity, in the following we assume that $\underline{b}^{-1}\underline{a}$ is diagonalizable. Hence, in particular, one can find $(r-q)$ independent vectors $\Xi_1, \dots \Xi_{r-q}$ such that

$$(A - \lambda_i B)\Xi_i = 0.$$

To handle the general case, one should work with generalized eigenspaces.

4.2 The hyperbolic limit in the case of non invertible viscosity matrix

4.2.1 The hyperbolic limit in the case of the Cauchy problem

In this section for completeness we review the construction of the Riemann solver for the Cauchy problem in the case the viscosity matrix B is not invertible. We refer to [7] for the complete analysis. Also, some of the steps in the construction are the same as in the case of an invertible viscosity matrix, which is discussed in Section 3.2.1. Thus, here we will focus only on the points where the singularity of the viscosity matrix plays an important role.

The goal is the characterization of the limit of

$$\begin{cases} E(u^\varepsilon)u^\varepsilon_t + A(u^\varepsilon,\, u^\varepsilon_x)u^\varepsilon_x = \varepsilon B(u^\varepsilon)u^\varepsilon_{xx} \\ u^\varepsilon(0,\,,x) = \begin{cases} u^- & x \le 0 \\ \bar u_0 & x > 0 \end{cases} \end{cases} \tag{4.24}$$

under Hypotheses 2, 3, 5 and 6.

The construction is as follows. Thanks to Lemma 4.4, the equation of travelling waves

$$B(U)U'' = \Big(A(U,\, U') - \sigma E(U)\Big)U', \tag{4.25}$$

is equivalent to the system

$$\begin{cases} u' = \Big(\bar w(u,\, \tilde z,\, \sigma),\ \tilde w(u,\, \tilde z,\, \sigma),\ \bar z(u,\, \tilde z,\, \sigma),\ \tilde z\Big) \\ \tilde z' = f(u,\, \tilde z,\, \sigma) \\ \sigma' = 0. \end{cases} \tag{4.26}$$

When $\tilde z = \vec 0$ then for every u and σ $f(u,\, \vec 0,\, \sigma) = \vec 0$, $\bar w(u,\, \vec 0,\, \sigma) = \vec 0$, $\tilde w(u,\, \vec 0,\, \sigma) = \vec 0$, $\bar z(u,\, \vec 0,\, \sigma) = \vec 0$. Thus, $(\bar u_0,\, \vec 0,\, \lambda_i(\bar u_0))$ is an equilibrium for (3.13), where $\lambda_i(\bar u_0)$ is an eigenvalue of $E^{-1}(\bar u_0)A(\bar u_0,\, 0)$. Linearizing around such an equilibrium point one obtains

$$\begin{pmatrix} 0 & \underline{c} & 0 \\ 0 & \underline{b}^{-1}\bar a(\bar u_0,\, 0,\, \sigma_i) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.27}$$

where the exact expression of the matrix $\underline{c} \in \mathbb{R}^{N \times (r-q)}$ is not important here. Let $\Xi_i(\bar u_0)$ denote the eigenvector corresponding to $\lambda_i(\bar u_0)$, then $\Xi_i(\bar u_0$ is in the form given by Lemma 4.5 for some vector $\vec\xi(\bar u_0) \in \mathbb{R}^{r-q}$. Then the eigenspace of (4.27) associated to the eigenvalue 0 is

$$V^c := \Big\{ \Big(u,\ v_i\vec\xi_i(\bar u_0),\, \sigma\Big) :\ u \in \mathbb{R}^N,\ v_i,\, \sigma \in \mathbb{R} \Big\}.$$

One can then proceed as in the case of an invertible viscosity matrix (see Section 3.2.1). Fix a center manifold $\mathcal{M}^c$, then one can verify that a point$(u,\, \tilde z,\, \sigma)$ belongs to $\mathcal{M}^c$ if and only if

$$\tilde z = v_i \tilde\xi_i(u,\, v_i,\, \sigma),$$

where $\tilde\xi_i \in \mathbb{R}^{r-q}$ is a suitable vector valued function such that

$$\tilde\xi_i\Big(\bar u_0,\, 0,\, \lambda_i(\bar u_0)\Big) = \vec\xi_i(\bar u_0).$$

Thus, on the center manifold

$$u' = \Big(\bar w(u,\, \tilde z,\, \sigma_i),\ \tilde w(u,\, \tilde z,\, \sigma_i),\ \bar z(u,\, \tilde z,\, \sigma_i),\ \tilde z\Big) = \tilde\Xi_i(u,\, v_i,\, \sigma_i)v_i$$

for some vector valued function $\tilde\Xi_i \in \mathbb{R}^N$ such that

$$\tilde\Xi_i\Big(\bar u_0,\, 0,\, \lambda_i(\bar u_0)\Big) = \Xi_i(\bar u_0).$$

Plugging the relation $u' = \tilde{\Xi}_i v_i$ into the equation

$$B(u)u'' = [A(u, u') - \sigma E(u)]u'$$

one obtains

$$[B\tilde{\Xi}_i + \tilde{\Xi}_{iv} v_i]v_{ix} = \Big([A - \sigma E]\tilde{\Xi}_i - D_u\tilde{\Xi}_i\tilde{\Xi}_i v_i\Big)v_i.$$

Taking both members dot product $\tilde{\Xi}_i$ one gets

$$c_i v_{ix} = a_i v_i$$

for suitable functions $c_i(u, v_i, \sigma)$ and $a_i(u, v_i, \sigma)$. Thanks to Lemma 4.5, $\Xi_i(\bar{u}_0)$ is not in the kernel of B and hence

$$c_i\Big(\bar{u}_0, 0, \lambda_i(\bar{u}_0)\Big) = \langle \Xi_i(\bar{u}_0), B(\bar{u}_0)\Xi_i(\bar{u}_0)\rangle > 0$$

Thus, $c_i > 0$ in a neighborhood and hence one can introduce $\phi_i := a_i/c_i$. Also,

$$\frac{\partial \phi_i}{\partial \sigma}\Bigg|_{(\bar{u}_0, 0, \lambda_i(\bar{u}_0))} = -\langle \Xi_i(\bar{u}_0), E\Xi_i(\bar{u}_0)\rangle < 0. \tag{4.28}$$

Thus, system the solutions of (3.13) laying on $\mathcal{M}^c$ satisfy

$$\begin{cases} u_i' = v_i\tilde{\Xi}_i(u, v_i, \sigma_i) \\ v_i' = \phi_i(u, v_i, \sigma_i) \\ \sigma_i' = 0 \end{cases} \tag{4.29}$$

with ϕ satisfying (4.28). One can therefore apply the results in [7]. Proceeding as in Section 3.2.1 one eventually defines the i-th curve of admissible states $T^i_{s_i}\bar{u}_0$, which satisfies

$$\frac{\partial T^i \bar{u}_0}{\partial s_i}\Bigg|_{s_i=0} = \Xi_i(\bar{u}_0)$$

and all the other properties listed in Section 3.2.1.

Consider the composite function

$$\psi(\bar{u}_0, s_1 \dots s_N) = T^1_{s_1} \circ \dots T^N_{s_N}\bar{u}_0$$

With the previous expression we mean that the starting point for T^{N-1}_s is $T^N_{s_N}\bar{u}_0$. It turns out that the map $\phi(\bar{u}_0, \cdot)$ is invertible in a neighbourhood of $(s_1 \dots s_N) = (0 \dots 0)$. In other words, if u^- is fixed and is sufficiently close to $\bar{u}_0$, then the values of $s_1 \dots s_N$ are uniquely determined by the equation

$$u^- = \psi(\bar{u}_0, s_1 \dots s_N). \tag{4.30}$$

Taking the same u^- as in (4.24), one obtains the parameters $(s_1 \dots s_N)$ which can be used to reconstruct the hyperbolic limit u of (4.24). Indeed, once $(s_1 \dots s_N)$ are known then u can be obtained in the way described in Section 3.2.1.

Remark 4.2. To construct the curves $T^i_{s_i}\bar{u}_0$ one proceeds as follows. Suppose that $s_i > 0$, then one considers the fixed point problem

$$\begin{cases} u(\tau) = \bar{u}_k + \displaystyle\int_0^\tau \tilde{r}_k(u(\xi), v_k(\xi), \sigma_k(\xi))d\xi \\ v_k(\tau) = f_k(\tau, u, v_k, \sigma_k) - \text{conc} f_k(\tau, u, v_k, \sigma_k) \\ \sigma_k(\tau) = \frac{1}{c_E(\bar{u}_0)}\dfrac{d}{d\tau}\text{conc} f_k(\tau, u, v_k, \sigma_k). \end{cases} \tag{4.31}$$

Denote by (u_i, v_i, σ_i) the solution of (4.31), whose existence is ensured by results in [7]. Then we define

$$T^i_{s_i}\bar{u}_0 = u_i(s_i).$$

The value $u_i(s_i)$ is connected by $\bar{u}_0$ by a sequence of rarefaction and shocks. This is actually true only if system (4.28) is equivalent to (4.25), i.e. if Lemma 4.4 holds true. The problem here is that in (4.28) σ is a parameter, while in (4.31) can vary. Actually, it turns out that we can still apply Lemma 4.4 provided that the dimension of the kernel of $A_{11}(u) - \sigma E_{11}(u)$ does not vary in a small enough neighborhood of $\sigma = \lambda_i(\bar{u}_0)$, i.e. if the kernel of $A_{11}(u) - \lambda_i(\bar{u}_0)E_{11}(u)$ is trivial. This is certainly a restrictive condition. In [13] the authors will address the problem of finding less restrictive conditions.

4.2.2 A lemma on the dimension of the stable manifold

The aim of this section is to introduce Lemma 4.7, which determines the dimension of the stable manifold of the system

$$B(U)U'' = A(U, U')U', \tag{4.32}$$

satisfied by the steady solutions of

$$E(u)u_t + A(u, u_x)u_x = u_{xx}.$$

In this section we thus give an answer to a question introduced in [44]: more precisely, it is shown that the condition called there Hypothesis (H5) is actually a consequence of Kawashima condition.

Thanks to Lemma (4.4), system (4.32) is equivalent to

$$\begin{cases} u_x = \Big(\bar{w}(u, \tilde{z}, 0),\ \tilde{w}(u, \tilde{z}, 0),\ \bar{z}(u, \tilde{z}, 0),\ \tilde{z}\Big)^T \\ \tilde{z}_x = f(u, \tilde{z}, 0). \end{cases} \tag{4.33}$$

Hence, linearizing around the equilibrium point $(\bar{u}, 0)$, one finds that the Jacobian is represented by the matrix

$$\begin{pmatrix} 0 & \underline{m} \\ 0 & \underline{b}^{-1}\underline{a}(\bar{u}, 0, 0) \end{pmatrix}$$

where $\underline{a}$ and $\underline{b}$ are defined by (4.21) and $\underline{m}$ is a $N \times (r-q)$ dimensional matrix whose exact expression is not important here. Hence the stable manifold of system (4.33) is tangent in $(\bar{u}, 0)$ to

$$\tilde{V}^s = \left\{ \Big(\bar{u} + \sum_{i=1}^{d} v_i \underline{m} \vec{\theta}_i(\bar{u}),\ \sum_{i=1}^{d} \vec{\theta}_i(\bar{u}) \Big) :\ v_i \in \mathbb{R} \right\} \subseteq \mathbb{R}^{N+r-q}, \tag{4.34}$$

In the previous expression $\vec{\theta}_1, \dots, \vec{\theta}_d \in \mathbb{R}^{r-q}$ are the eigenvectors of $\underline{b}^{-1}\underline{a}$ associated to eigenvalues with strictly negative real part. What we want to determine in Lemma (4.4) is the number d, i.e. the number of eigenvalues of $\underline{b}^{-1}\underline{a}$ with strictly negative real part.

Before stating the lemma, we recall some notations: $k-1$ denotes the number of eigenvalues of $E^{-1}(u)A(u, 0)$, that satisfy

$$\lambda_1(u) < \lambda_2(u) < \dots < \lambda_{k-1}(u) < -c < 0$$

for some constant $c > 0$. The number $(k-1)$ does not depend on u and u_x because of the strict hyperbolicity (Hypothesis 3). Moreover, let

$$A(u, u_x) = \begin{pmatrix} A_{11}(u) & A_{21}^T(u) \\ A_{21}(u) & A_{22}(u, u_x) \end{pmatrix} \qquad E(u) = \begin{pmatrix} E_{11}(u) & E_{21}^T(u) \\ E_{21}(u) & E_{22}(u) \end{pmatrix} \tag{4.35}$$

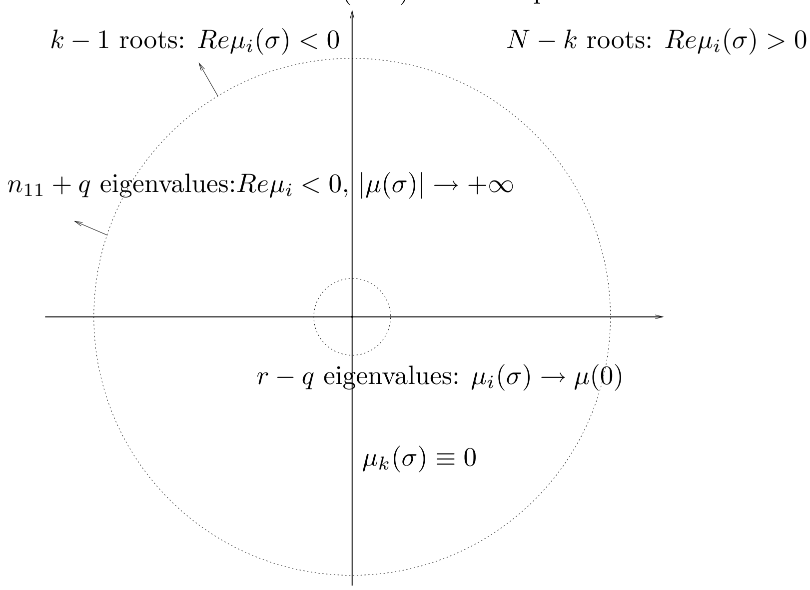
be the block decompositions of A and E corresponding to the block decomposition

$$B(u) = \begin{pmatrix} 0 & 0 \\ 0 & b(u) \end{pmatrix}$$

Also, n_{11} denotes the number of negative eigenvalues of A_{11}. Since E is positive definite, also E_{11} is and hence by Lemma 3.1 n_{11} is equal to the number of strictly negative eigenvalues of $E_{11}^{-1}(u)A_{11}(u)$. Hence, by the third assumption in Hypothesis 2, n_{11} does not depend on u. Finally, q denotes the dimension of the kernel of $A_{11}(u)$, which as well does not depend on u.

Before introducing Lemma 4.7, it is necessary to state a preliminary result, which is exploited in the proof of the lemma:

Figure 6: the behavior of the roots of (4.38) when the perturbation $\sigma \to 0^+$



Lemma 4.6. *Let Hypothesis 2 hold. Then there exists $\delta > 0$ such that if u belongs to a neighborhood of $\bar{u}_0$ of size δ then the following holds. The polynomial*

$$\det\Big(A(\bar{u},\,0) - \mu B(\bar{u})\Big) \tag{4.36}$$

has degree $(r-q)$ with respect to μ.

The proof follows immediately from Lemma 4.5, which guarantees that

$$\det\Big(A(\bar{u},\,0) - \mu B(\bar{u})\Big) = 0 \iff \det\Big(\underline{b}^{-1}(\bar{u},\,0)\underline{a}(\bar{u},\,0,\,0) - \mu I_{r-q}\Big) = 0,$$

where the matrices $\underline{a}$, $\underline{b} \in \mathbb{M}^{(r-q)\times(r-q)}$ are defined by (4.21) and $\underline{b}$ is invertible by Lemma 4.3. The symbol I_{r-q} denotes the identity matrix of dimension $(r-q)$.

Lemma 4.7. *Let Hypotheses 2 and 3 hold. Then the dimension d of the stable space* (4.34) *of system* (4.32) *is $d = k - 1 - n_{11} - q$.*

Proof. Since the state $\bar{u}$ is fixed, in the following for simplicity the matrices $A(\bar{u},\,0)$ and $B(\bar{u})$ will be denoted by A and B respectively. To prove the lemma one has to show that the equation

$$\det\Big(A - \mu_{\underline{i}} B\Big) = 0 \tag{4.37}$$

admits exactly $(k - 1 - n_{11} - q)$ roots with negative real part.

The proof is organized into six steps:

1. First, it is introduced a perturbation technique. More precisely, it is considered the problem

$$\det\Big(A - \mu_i(\sigma)(B + \sigma I)\Big) = 0. \qquad \sigma > 0, \tag{4.38}$$

where σ is a positive parameter which is allowed to go to 0. The symbol I denotes the N-dimensional identity matrix. Just to fix the ideas and to consider the most general case, it will be supposed that the matrix A is singular:

$$\lambda_1 < \cdots < \lambda_{k-1} < 0 = \lambda_k < \lambda_{k+1} < \cdots < \lambda_N$$

Once the value of the parameter σ is fixed, the solutions $\mu_i(\sigma)$ of (4.38) are the roots of a polynomial of degree N. Moreover, the matrix $(B+\sigma I)$ is positive definite and hence Lemma 3.1 guarantees that $(k-1)$ roots have strictly negative real part, $(N-k)$ roots have strictly positive real part and one root is zero.

If one let the parameter σ vary, (4.38) defines N algebraic functions $\mu_1(\sigma), \dots \mu_N(\sigma)$: since the coefficients of the polynomial (4.38) are analytic with respect to σ, it is known (see for example [38]) that the functions μ_i are analytic everywhere in the complex plane except for some so called exceptional points that are certainly of finite number in every compact subset. Moreover, for every fixed function μ_i only two behaviors are possible in an exceptional point $\sigma = \bar{\sigma}$:

- $\mu_i(\sigma)$ is continuous in $\bar{\sigma}$.
- $\lim_{\sigma \to \bar{\sigma}} |\mu_i(\sigma)| = +\infty$.

In the case of the algebraic functions defined by (4.38), the point $\sigma = 0$ is an exceptional point since in $\sigma = 0$ the degree of the polynomial drops from N to $r-q$: we will denote by $\mu_{\underline{i}}(0)$, $\underline{i} = 1, \dots (r-q)$ the $(r-q)$ roots of (4.37).

Thanks to the known results recalled before, however, one deduces that there are

- $r-q$ functions $\mu_{\underline{i}}(\sigma)$ such that

$$\lim_{\sigma \to 0} \mu_{\underline{i}}(\sigma) = \mu_{\underline{i}}(0) \qquad i = 1, \dots r.$$

- $N-r+q$ functions $\mu_{\underline{i}}(\sigma)$ such that

$$\lim_{\sigma \to 0} |\mu_{\underline{i}}(\sigma)| = +\infty \qquad i = r+1, \dots N.$$

The situation is summarized in Figure 6.

Moreover, restricting to a small enough neighborhood, it is not restrictive to suppose that $\sigma = 0$ is the only singular point.

2. As a second step we prove that to obtain the lemma it is sufficient to show that the number of functions $\mu_{\underline{i}}(\sigma)$ continuous in $\sigma = 0$ and such that $Re\mu_{\underline{i}}(\sigma) < 0$ when $\sigma \in \mathbb{R}^+$ is equal to $k-1-n_{11}-q$. Since the number of roots of (4.38) with negative is equal to $k-1$, this is equivalent to show that there are exactly $n_{11}+q$ functions $\mu_{\underline{i}}(\sigma)$ such that $Re\mu_{\underline{i}}(\sigma) < 0$ and $|\mu_{\underline{i}}(\sigma)| \uparrow +\infty$ when $\sigma \to 0^+$.

   The only case that has to be excluded is the possibility that in the limit a function $\mu_{\underline{i}}(\sigma)$ with $Re\mu_{\underline{i}}(\sigma) < 0$ has zero real part.

   First, one observes that equation (4.37) does not admit purely imaginary roots: by contradiction, assume that there exists $\mu \in \mathbb{R}$, $\mu \neq 0$ and $v = v_r + iv_i$, v_r, $v_i \in \mathbb{R}^N$ such that

$$A(v_r + iv_i) = i\mu B(v_r + iv_i). \tag{4.39}$$

   Just to fix the ideas, one can assume $\mu > 0$: as in the proof of Lemma 3.1, it turns out that

$$0 \geq -\mu \langle Bv_i, v_i \rangle = \langle Av_r, v_i \rangle = \langle Av_i, v_r \rangle = \mu \langle Bv_r, v_r \rangle \geq 0 \tag{4.40}$$

   because B is positive semidefinite. Hence the only possibility is that v_i and v_r both belong to $\ker B$, but this contradicts (4.39) and hence there are no purely imaginary roots of (4.37).

   Hence one is left to exclude the possibility that a function $\mu_{\underline{i}}(\sigma)$ continuous in $\sigma = 0$ and such that $Re\mu_{\underline{i}}(\sigma) < 0$ converges to 0. From the strict hyperbolicity of the matrix $E^{-1}A$ (Hypothesis 3) and from Lemma 3.1 it follows that 0 is an eigenvalue of A with multiplicity one and hence $\mu_{\underline{k}} = 0$ is a root of (4.37) with multiplicity one. Moreover, again from Lemma 3.1, one deduces that the algebraic function $\mu_{\underline{k}}(\sigma)$ is identically equal to 0. Since the multiplicity of 0 as a root of (4.37) is one, it cannot happen that other functions $\mu_{\underline{i}}(\sigma)$, $\underline{i} \neq \underline{k}$ converge to zero when $\sigma \to 0^+$.

3. As a third step, we perform a change of variable. More precisely, the number of solutions $\mu_{\underline{i}}(\sigma)$ of (4.38) such that $Re\mu_{\underline{i}}(\sigma) < 0$ and $|\mu_{\underline{i}}(\sigma)| \uparrow +\infty$ when $\sigma \to 0^+$ is equal to the number of roots $x_{\underline{i}}(\sigma)$ of

$$\det\Big(B + \sigma I - x_{\underline{i}}(\sigma)A\Big) = 0 \qquad \underline{i} = 1, \dots N - r + q \tag{4.41}$$

such that $Rex_i(\sigma) < 0$ and $x_i(\sigma) \to 0$ when $\sigma \to 0^+$. This is clear if one defines $x_{\underline{i}}(\sigma) := 1/\mu_{\underline{i}}(\sigma)$.

4. In the fourth part of the proof, we study the inverse problem to (4.41), i.e. the eigenvalue problem

$$\det\Big(B - xA + \sigma_{\underline{j}}(x)I\Big) = 0 \qquad \underline{j} = 1, \dots N. \tag{4.42}$$

The meaning of the notation is that in this case one fixes x and then finds σ.

More precisely, it will be studied the behavior of the $N - r$ eigenvalues $-\sigma_{\underline{j}}(x)$ such that $\sigma_{\underline{j}}(x) \to 0$ when $x \to 0$. From the analysis in [12] it follows that

$$F(x) = -A_{11}x - A_{21}^T b^{-1} A_{21} x^2 + o(x^2) \qquad x \to 0$$

where $F(x)$ denotes the projection of $B - xA$ on the generalized subspaces converging to the kernel of B. The blocks A_{11} and A_{21} are defined by (4.35). Let $F_0(x)$ be the projection of $F(x)$ on the kernel of A_{11} and $F_\perp(x)$ be the projection on the subspace orthogonal to $\ker A_{11}$.

Thus again the analysis in [12] guarantees that

$$F_0(x) = -(A_{21}^I)^T b^{-1} A_{21}^I x^2 + o(x^2) \qquad x \to 0$$

and

$$F_\perp(x) = -\tilde{A}_{11}x + o(x) \qquad x \to 0.$$

In the previous expression it has been used the notations introduced in (4.9):

$$\begin{pmatrix} A_{11}(u,\sigma) & A_{21}(u\,\sigma)^T \\ A_{21}(u\,\sigma) & A_{22}(u,u_x,\sigma) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \big(A_{21}^I\big)^T(u,\sigma) \\ 0 & \tilde{A}_{11}(u,\sigma) & \big(A_{21}^{II}\big)^T(u,\sigma) \\ A_{21}^I(u) & A_{21}^{II}(u,\sigma) & A_{22}(u,\sigma,u_x) \end{pmatrix}$$

From Kawashima condition it follows that the matrix $(A_{21}^I)^T b^{-1} A_{21}^I$ is positive definite: one can proceed in the same way as in the proof of Lemma (4.2).

Hence projecting on suitable subspaces (we refer again to [12] for the precise computations) it follows that the following expansions hold:

$$\sigma_{\underline{j}}(x) = \mu_{\underline{j}}x^2 + o(x) \qquad x \to 0, \qquad j = 1, \dots q \tag{4.43}$$

and

$$\sigma_{\underline{j}}(x) = \lambda_{\underline{j}}x + o(x) \qquad x \to 0 \qquad j = q+1, \dots N - r. \tag{4.44}$$

In the previous expression, $\mu_{\underline{j}} > 0$ is an eigenvalue of $(A_{21}^I)^T b^{-1} A_{21}^I$ and $\lambda_{\underline{j}} \neq 0$ is an eigenvalue of $\tilde{A}_{11}$.

5. As a fifth step, we analyze equation (4.44): for every fixed $\underline{j} = q+1, \dots N-r$, the expansion guarantees that the function $\sigma_{\underline{j}}(x)$ is invertible in a neighborhood of $x = 0$. The inverse function $x_{\underline{j}}(\sigma)$ satisfies (4.41) and because of (4.44) the condition that $Rex_{\underline{j}}(\sigma) < 0$ when $\sigma \in \mathbb{R}^+$ is satisfied if and only if $\lambda_{\underline{j}}$ is an eigenvalue of $\tilde{A}_{11}$ with negative real part. This implies that there are exactly n_{11} functions $x_{\underline{j}}(\sigma)$ satisfying (4.44) and such that $Rex_{\underline{j}}(\sigma) < 0$ when $\sigma \in \mathbb{R}^+$.

Moreover, there are $N - r - q$ functions $x_{\underline{j}}(\sigma)$ satisfying (4.41) and (4.44).

6. Since there are exactly $2q$ functions $x_{\underline{j}}(\sigma)$ satisfying (4.41) and (4.43), one deduces that there are q functions with strictly positive real part and q functions with strictly negative real part. The possibility of a root with zero real part is excluded on the basis of the same considerations as in step 2.

   Hence from steps 5 and 6 one obtains that there are exactly $n_{11} + q$ solutions $x_{\underline{i}}(\sigma)$ of (4.41) such that $Rex_{\underline{i}}(\sigma) < 0$ and $x_{\underline{i}}(\sigma) \to 0$ when $\sigma \to 0^+$. Thanks to the considerations in steps 1, 2 and 3 this concludes the proof of the lemma.

□

To introduce Lemma 4.8, it is useful to introduce some further notations: let

$$V^s(u) := \text{span}\langle \Theta_{n_{11}-q+1}, \dots \Theta_{k-1} \rangle \tag{4.45}$$

be the space generated by the vectors that satisfy

$$\Big(A(\bar{u}, 0) - \mu_i(\bar{u}, 0) B(\bar{u}) \Big) \Theta_i(\bar{u}, 0) = 0 \qquad \mu_i(\bar{u}, 0) < 0. \tag{4.46}$$

Let

$$V^u(u) := \text{span}\langle \Xi_{k+1}, \dots \Xi_N \rangle \tag{4.47}$$

the subspace generated by the Ξ_i satisfying

$$\Big(A(u, 0) - \lambda_i E(u) \Big) \Xi_i = 0 \qquad \lambda_i > 0. \tag{4.48}$$

When $A(u, 0)$ is invertible the dimension of V^u is equal to $(N - n)$, where n is the number of negative eigenvalues of A defined as in (3.19). On the other side, when $A(u, 0)$ is singular the dimension of V^u is equal to $(N - k)$, where $(k - 1)$ is the number of strictly negative eigenvalues of A as in (3.25). In this case, the following subspace is non trivial:

$$V^c(u) := \text{span}\langle \Theta_k \rangle, \;\; \text{where } A(u, 0)\, \Theta_k = 0. \tag{4.49}$$

The proof of the following result is analogous to that of Lemma 7.1 in [6]:

Lemma 4.8. *Let Hypotheses 2 and 3 hold. Then*

$$V^s \cap V^u = \big\{0\big\} \qquad V^s \cap V^c = \big\{0\big\} \qquad V^c \cap V^u = \big\{0\big\},$$

where V^s, V^c and V^u are defined by (4.45), (4.49) *and* (4.47) *respectively.*

4.2.3 The hyperbolic limit in the non characteristic case

In this section we will provide a characterization of the limit of the parabolic approximation (4.1) when the boundary non characteristic, i.e. when none of the eigenvalues of $E^{-1}(u)A(u, u_x)$ can attain the value 0 (Hypothesis 8). As in Section 3.2.2, n will denote the number of eigenvalues of $E^{-1}(u)A(u, u_x)$ with strictly negative negative real part and $N - p$ the number of eigenvalues with strictly positive real part. Also, we recall that n_{11} is the number of strictly negative eigenvalues of $A_{11}(u)$, while the eigenvalue 0 has multiplicity q.

To give a characterization of the limit of (4.1) we will follow the same steps Section 3.2.2. Thus, in the exposition we will focus only on the points at which the singularity of the viscosity matrix B plays an important role.

The characterization of the limit works as follows. We will construct a map $\phi(\bar{u}_0, s_{n_{11}+q} \dots s_N)$ which describes, as $(s_{n_{11}+q} \dots s_N)$ vary, states that can be connected to $\bar{u}_0$. We will compose ϕ with the function β, which is used to assign the boundary condition in (4.1). We will the show that the composite map is locally invertible. This means that, given $\bar{u}_0$ and $\bar{g}$ such that $|\beta(\bar{u}_0 - \bar{g}|$ is sufficiently small, the values of $(s_{n_{11}+q} \dots s_N)$ are uniquely determined by the equation

$$\bar{g} = \phi(\bar{u}_0, \, s_1 \dots s_N).$$

Once $(s_{n_{11}+q} \dots s_N)$ are known the limit of (3.1) is completely characterized. The construction of the map ϕ is divided in some steps:

1. *Waves with positive speed*

   Consider the Cauchy datum $\bar{u}_0$, fix $(N-n)$ parameters $(s_{n+1} \dots s_N)$ and consider the value

   $$\bar{u} = T^n_{s_n} \circ \dots T^N_{s_N} \bar{u}_0.$$

   The curves $T^n_{s_n} \dots T^N_{s_N}$ are, as in Section 4.2.1, the *curves of admissible states* introduced in [7]. The state $\bar{u}_0$ is then connected to $\bar{u}$ by a sequence of rarefaction and travelling waves with positive speed.

2. *Boundary layers*

   We have now to characterize the set of values u such that the following problem admits a solution:

   $$\begin{cases} A(U, U_x)U_x = B(U)U_{xx} \\ U(0) = u \\ \lim_{x \to +\infty} U(x) = \bar{u}. \end{cases} \tag{4.50}$$

   Because of Lemma 4.4, one has to study

   $$\begin{cases} U_x = \Big(\bar{w}(U, \tilde{z}, 0),\, \tilde{w}(U, \tilde{z}, 0),\, \bar{z}(U, \tilde{z}, 0),\, \tilde{z}\Big) \\ \tilde{z}_x = f(U, \tilde{z}, 0) \end{cases} \tag{4.51}$$

   Consider the equilibrium point $(\bar{u}, 0)$, linearize at that point and denote by V^s the stable space, i.e. the eigenspace associated to the eigenvalues with strictly negative real part. Thanks to Lemma 4.7, the dimension of V^s is equal to $n - n_{11} - q$.

   $$V^s = \Big\{ \big(\bar{u} + \sum_{i=1}^n \frac{x_i}{\mu_i(\bar{u})} \vec{\xi}_i(\bar{u}),\, \sum_{i=1}^n x_i \vec{\xi}_i(\bar{u})\big),\ x_1 \dots x_n \in \mathbb{R} \Big\},$$

   where $\mu_1(\bar{u}) \dots \mu_n(\bar{u})$ are the eigenvalues of $\underline{b}^{-1}(\bar{u},\, \vec{0},\, 0)\underline{a}((\bar{u}),\, \vec{0},\, 0)$ with negative real part and $\vec{\xi_1}(\bar{u}) \dots \vec{\xi_n}$ are the corresponding eigenvectors.

   Denote by $\mathcal{M}^s$ the stable manifold, which is parameterized by V^s. Also, denote by ϕ_s a parameterization of $\mathcal{M}^s$:

   $$\phi_s : V^s \to \mathbb{R}^N.$$

   Let π_u be the projection

   $$\begin{aligned} \pi_u : \mathbb{R}^N \times \mathbb{R}^{r-q} &\to \mathbb{R}^N \\ (u,\, \tilde{z}) &\mapsto u \end{aligned}$$

   One can then proceed as in the proof of Section 3.2.2 and conclude that system (4.50) admits a solution if $u \in \pi_u\big(\phi_s(s_{n_{11}+q+1} \dots s_n)\big)$ for some $s_{n_{11}+q+1} \dots s_n$. Also, thanks to (4.22) the columns of the jacobian of $\pi_u \circ \phi_s$ computed at $s_{n_{11}+q+1} = 0 \dots s_n = 0$ are $\Xi_{n_{11}+q+1} \dots \Xi_n$.

   Note that the map $\pi_u \circ \phi_s$ actually depends also on the point $\bar{u}$ and it does in a Lipschitz continuos way:

   $$|\pi_u \circ \phi_s(\bar{u}_1,\, s_{n_{11}+q+1} \dots s_n) - \pi_u \circ \phi_s(\bar{u}_2,\, s_{n_{11}+q+1} \dots s_n)| \le L|\bar{u}_1 - \bar{u}_2|.$$

3. *Conclusion*

   Define the map ϕ as follows:

   $$\phi(\bar{u}_0,\, s_{n_{11}+q+1} \dots s_N) = \pi_u \circ \phi_s \Big(T^{N-k}_{s_{N-k}} \circ \dots T^N_{s_N} \bar{u}_0,\, s_{n_{11}+q+1} \dots s_n \Big) \tag{4.52}$$

   From the previous steps it follows that ϕ is Lipschitz continuos and that it is differentiable at $s_{n_{11}+q+1} = 0 \dots s_N = 0$. Also, the columns of the jacobian are $\Xi_{n_{11}+q+1}(\bar{u}_0) \dots \Xi_n(\bar{u}_0)$, $\Theta_{n+1}(\bar{u}_0) \dots \Theta_N$, where

   $$\Big(A(\bar{u}_0,\, 0) - \lambda_i(\bar{u}_0 E(\bar{u}_0)\Big)\Theta_i(\bar{u}_0) = 0$$

for $\lambda_i(\bar{u}_0) > 0$ and

$$\Big(A(\bar{u}_0,\, 0) - \mu_i(\bar{u}_0 E(\bar{u}_0)\Big)\Xi_i(\bar{u}_0) = B(\bar{u}_0)\Xi_i(\bar{u}_0)$$

for $\mu_i(\bar{u}_0)$ with strictly negative real part.

In the case of an invertible viscosity matrix (Section 3.2.2) the definition of the map ϕ is the final step in the construction. Here, instead, one has to take into account the function ß, which is used to assign the boundary condition and is defined in Section 2.2.1. Consider

$$ß \circ \phi(\bar{u}_0,\, s_{n_{11}+q+1} \dots s_N).$$

Thanks to the regularity of ß and to the previous remarks, $ß \circ \phi$ is Lipschitz continous and differentiable at $s_{n_{11}+q+1} = 0 \dots s_N = 0$. Denote by

$$\mathcal{V}(\bar{u}_0) = \text{span}\langle \Xi_1(\bar{u}_0) \dots \Xi_n(\bar{u}_0),\, \Theta_{n+1}(\bar{u}_0) \dots \Theta_N(\bar{u}_0) \rangle.$$

Lemma 4.9, which is introduced in Section 4.3, ensures that for every $\vec{V} \in \mathcal{V}(\bar{u}_0)$

$$Dß(\bar{u}_0)\vec{V} = 0 \implies \vec{V} = \vec{0}$$

Thus, the jacobian of $ß \circ \phi$ at $s_{n_{11}+q+1} = 0 \dots s_N = 0$ is an invertible matrix. Thanks to the extension of the implicit function theorem discussed in [24] (page 253) one can conclude that the map $ß \circ \phi(\bar{u}_0, \cdot)$ is invertible in a neighbourhood of $(s_{n_{11}+q+1} \dots s_N) = (0 \dots 0)$. In particular, if one takes $\bar{u}_b$ as in (3.1) and assumes that $|ß(\bar{u}_0) - \bar{g}|$ is sufficiently small, then the values of $s_{n_{11}+q+1} \dots s_N$ are uniquely determined by the equation

$$\bar{g} = ß \circ \phi(\bar{u}_0,\, s_{n_{11}+q+1} \dots s_n) \tag{4.53}$$

Once the values of $s_{n_{11}+q+1} \dots s_N$ are known, then the limit $u(t,\, x)$ can be reconstructed. In particular, the trace of u on the axis $x = 0$ is given by

$$\bar{u} := T^{n+1}_{s_{n+1}} \circ \dots T^N_{s_n} \bar{u}_0. \tag{4.54}$$

Also, the self similar function u can be obtained gluing together pieces like (3.18) .

Here is a summary of the results obtained in this section:

Theorem 4.1. *Let Hypotheses 2, 3, 4, 5, 6 , 7 and 8 hold. Then there exists $\delta > 0$ small enough such that the following holds. If $|ß(\bar{u}_0) - \bar{g}| < \delta$, then the limit of the parabolic approximation* (3.1) *satisfies*

$$\bar{g} = ß\phi(\bar{u}_0,\, s_{n_{11}+q+1} \dots s_N)$$

for a suitable vector $(s_{n_{11}+q+1} \dots s_N)$. The map ϕ is defined by (4.1)*. Given $\bar{u}_0$ and $\bar{g}$, one can invert $ß \circ \phi$ and determine uniquely $(s_{n_{11}+q+1} \dots s_N)$. Once $(s_{n_{11}+q+1} \dots s_N)$ are known the value $u(t,\, x)$ assumed by the limit function is determined a.e. $(t,\, x)$. In particular, the trace $\bar{u}$ of the hyperbolic limit in the axis $x = 0$ is given by* (4.54)*.*

4.2.4 The hyperbolic limit in the boundary characteristic case

This section deals with the limit of the parabolic approximation (4.1) in the boundary characteristic case, i.e. when one of the eigenvalues of $E^{-1}A$ can attain the value 0. More precisely, we will assume Hypothesis 9, which is introduced at the beginning of Section 3.2.3. As in Section 3.2.3, $k-1$ is the number of eigenvalues of $E^{-1}(\bar{u}_0)A(\bar{u}_0,\, 0)$ that are less or equal then $-c$, where c is the separation speed introduced in Hypothesis 3. Also, n_{11} is the number of strictly negative eigenvalues of $A_{11}(\bar{u}_0)$, while q is the dimension of the kernel of $A_{11}(\bar{u}_0)$.

To give a characterization of the limit of (4.1) one can follow the same steps Section 3.2.3. Also, in Section 4.2.3 we explain how to tackle the difficulties due to the fact that the viscosity matrix is not invertible. Thus, in the following we give only a quick overview of the key points in the characterization.

The characterization works as follows. We construct a map $\phi(\bar{u}_0, s_{n_{11}+q} \dots s_N)$ such that the following holds. As $(s_{n_{11}+q} \dots s_N)$ vary, $\phi(\bar{u}_0, s_{n_{11}+q} \dots s_N)$ describes states that can be connected to $\bar{u}_0$. We compose ϕ with the function β, which is used to assign the boundary condition in (4.1). We then show that the composite map is locally invertible. This means that, given $\bar{u}_0$ and $\bar{g}$ such that $|\beta(\bar{u}_0 - \bar{g}|$ is sufficiently small, then the values of $(s_{n_{11}+q} \dots s_N)$ are uniquely determined by the equation

$$\bar{g} = ß \circ \phi(\bar{u}_0, \, s_1 \dots s_N).$$

Once $(s_{n_{11}+q} \dots s_N)$ are known the limit of (3.1) is completely characterized and it is obtained gluing together pieces like 3.18. The construction of the map ϕ is divided in some steps:

1. *Waves with positive speed*

   Given the Cauchy datum $\bar{u}_0$, we fix $(N-k)$ parameters $(s_{k+1} \dots s_N$ and consider the value

   $$\bar{u}_k = T^{k+1}_{s_{k+1}} \circ \dots T^N_{s_N} \bar{u}_0.$$

   The curves $T^{k+1}_{s_{k+1}} \dots T^N_{s_N}$ are, as in Section 4.2.1, the *curves of admissible states* introduced in [7]. The state $\bar{u}_0$ is then connected to $\bar{u}$ by a sequence of rarefaction and travelling waves with positive speed.

2. *Analysis of the center stable manifold*

   Consider the equation satisfied by travelling waves:

   $$B(U)U'' = \big(A - \sigma E\big)U'.$$

   Thanks to Lemma 4.4, this is equivalent to system

   $$\begin{cases} U_x = \Big(\bar{w}(U, \tilde{z}, \, 0), \, \tilde{w}(U, \tilde{z}, \, 0), \, \bar{z}(U, \tilde{z}, \, 0), \, \tilde{z}\Big) \\ \tilde{z}_x = f(U, \tilde{z}, \, 0) \\ \sigma_x = 0 \end{cases} \tag{4.55}$$

   The point $(\bar{u}_k, \, \vec{0}, \, \lambda_k)$ is an equilibrium. One can than define an invariant *center stable* $\mathcal{M}^{cs}$ manifold with the same properties listed in Section 3.2.3. Thanks to Lemma 4.7, the dimension of every center stable manifold is $k - n_{11} - q$. Also, one can proceed again as in Section 3.2.3 and find the equation satisfied by the solutions of 4.2.1 laying on $\mathcal{M}^{cs}$. Moreover, every solution laying on $\mathcal{M}^{cs}$ can be decomposed in a purely center component, a purely stable component and a component of perturbation, in the same way described in Section 3.2.3. Eventually, one is able to define a map $F(\bar{u}_k, \, s_{n_{11}+q}, \dots s_K)$ which is Lischitz continous with respect to both $\bar{u}_k$ and $s_{n_{11}+q}, \dots s_K$. Also, it is differentiable at $s_{n_{11}+q} = 0, \dots s_K = 0$ and the columns of the jacobian matrix are the vectors $\Xi_{n_{11}+q}(\bar{u}_k) \dots \Xi_k(\bar{u}_k)$. For every $i = n_{11} + q \dots K$ it holds

   $$\Big[A(\bar{u}_k, \, 0) - \mu_i E(\bar{u}_k\Big]\Xi_i = B(\bar{u}_k)\Xi_i$$

   with the real part of μ_i less or equal to zero.

3. *Conclusion*

   Define the map ϕ as

   $$\phi(\bar{u}_0, \, s_{n_{11}+q+1} \dots s_N) = \pi_u \circ \phi_s \Big(T^{N-k}_{s_{N-k}} \circ \dots T^N_{s_N} \bar{u}_0, \, s_{n_{11}+q+1} \dots s_n\Big) \tag{4.56}$$

   From the previous steps it follows that ϕ is Lipschitz continuos and that it is differentiable at $s_{n_{11}+q+1} = 0 \dots s_N = 0$. Also, the columns of the jacobian are $\Xi_1(\bar{u}_0) \dots \Xi_k(\bar{u}_0)$, $\Theta_{k+1}(\bar{u}_0) \dots \Theta_N$, where

   $$\Big(A(\bar{u}_0, \, 0) - \lambda_i(\bar{u}_0 E(\bar{u}_0)\Big)\Theta_i(\bar{u}_0) = 0$$

with $\lambda_i > 0$ for every i. The vectors Ξ are as before.

To take into account the function β, which is used to assign the boundary condition and is defined in Section 2.2.1., one considers

$$ß \circ \phi(\bar{u}_0,\, s_{n_{11}+q+1} \dots s_N).$$

Thanks to the regularity of ß and to the previous remarks, $ß \circ \phi$ is Lipschitz continous and differentiable at $s_{n_{11}+q+1} = 0 \dots s_N = 0$. Denote by

$$\mathcal{V}(\bar{u}_0) = \text{span}\langle \Xi_1(\bar{u}_0) \dots \Xi_k(\bar{u}_0),\, \Theta_{k+1}(\bar{u}_0) \dots \Theta_N \rangle.$$

Lemma 4.9, which is introduced in Section 4.3, ensures that for every $\vec{V} \in \mathcal{V}(\bar{u}_0)$

$$Dß(\bar{u}_0)\vec{V} = 0 \implies \vec{V} = \vec{0}$$

Thus, the jacobian of $ß \circ \phi$ at $s_{n_{11}+q+1} = 0 \dots s_N = 0$ is an invertible matrix. Thanks to the extension of the implicit function theorem discussed in [24] (page 253) one can conclude that the map $ß \circ \phi(\bar{u}_0, \cdot)$ is invertible in a neighbourhood of $(s_{n_{11}+q+1} \dots s_N) = (0 \dots 0)$. In particular, if one takes $\bar{u}_b$ as in (3.1) and assumes that $|ß(\bar{u}_0) - \bar{g}|$ is sufficiently small, then the values of $s_{n_{11}+q+1} \dots s_N$ are uniquely determined by the equation

$$\bar{g} = ß \circ \phi(\bar{u}_0,\, s_{n_{11}+q+1} \dots s_n) \tag{4.57}$$

Once the values of $s_{n_{11}+q+1} \dots s_N$ are known, then the limit $u(t,\, x)$ can be reconstructed gluing together pieces like 3.18. In particular, the value of the trace $\bar{u}$ on the axis $x = 0$ can be determined in the same way described in Section 3.2.3

Here is a summary of the results obtained in this section:

Theorem 4.2. *Let Hypotheses 2, 3, 4, 5, 6 , 7 and 9 hold. Then there exists $\delta > 0$ small enough such that the following holds. If $|ß(\bar{u}_0) - \bar{g}| < \delta$, then the limit of the parabolic approximation* (3.1) *satisfies*

$$\bar{g} = ß \circ \phi(\bar{u}_0,\, s_{n_{11}+q+1} \dots s_N)$$

for a suitable vector $(s_{n_{11}+q+1} \dots s_N)$. The map ϕ is defined by (4.56)*. Given $\bar{u}_0$ and $\bar{g}$, one can invert $ß \circ \phi$ and determine uniquely $(s_{n_{11}+q+1} \dots s_N)$. Once $(s_{n_{11}+q+1} \dots s_N)$ are known, then the value $u(t,\, x)$ assumed by the limit function is determined a.e. $(t,\, x)$.*

4.3 A transversality lemma

In the first part of this section we state and prove Lemma 4.9. It is a technical result and it guarantees that the map $ß \circ \phi$ that appears in both Theorems 4.1 and 4.2 is indeed locally invertible. In the second part of the section we introduce a new definition for the map ß which is used to assign the boundary condition in

$$\begin{cases} E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx} \\ ß(u(t,\, 0)) = \bar{g} \qquad u(0,\, x) = \bar{u}_0. \end{cases} \tag{4.58}$$

This definition is an extension of Definiton 2.1 and it guarantees both the local invertibily of the map $ß \circ \phi$ and the well posedness of the initial boundary value problem (4.58).

Before stating Lemma 4.9, we recall some notations. We denote by A_{11}, E_{11} the blocks of A, E defined by (4.35). Let $\vec{\zeta}_i(u) \in \mathbb{R}^{N-r}$ be an eigenvector of $E_{11}^{-1}(u)A_{11}(u)$ associated to an eigenvalue with non positive real part. Also, let $Z_i(u) \in \mathbb{R}^N$ be defined by

$$Z_i := \begin{pmatrix} \vec{\zeta}_i \\ 0 \end{pmatrix} \tag{4.59}$$

and

$$\mathcal{Z}(u) := \text{span}\langle Z_1(u), \dots, Z_{n_{11}+q}(u) \rangle.$$

We define the complementary subspace

$$\mathcal{W}(u) := \text{span}\Big\langle \begin{pmatrix} 0 \\ \vec{e}_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vec{e}_r \end{pmatrix}, \begin{pmatrix} \vec{w}_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \vec{w}_{N-r-n_{11}-q} \\ 0 \end{pmatrix} \Big\rangle,$$

where $\vec{e}_i \in \mathbb{R}^r$ are the vectors of a basis in $\mathbb{R}^r$ and $\vec{w}_j \in \mathbb{R}^{N-r}$ are the eigenvectors of $E_{11}^{-1}A_{11}$ associated to eigenvalue with strictly positive real part. By Definition 2.1, the function ß(u) is the component u_1 of the vector u in the decomposition

$$u = u_1 + u_2, \quad u_1 \in \mathcal{W}(u), u_2 \in \mathcal{V}(u). \tag{4.60}$$

Also, we define the space $\mathcal{V}(u)$ as

$$\mathcal{V}(u) := \text{span}\langle \Theta_{n_{11}+q+1}(u), \dots, \Theta_{k-1}(u), \Theta_k(u), \Xi_{k+1}(u), \dots \Xi_N(u)\rangle, \tag{4.61}$$

when the matrix $A(u,\, 0)$ can be singular (boundary characteristic case). The vectors Θ_i satisfy

$$A(u,\, 0)\Theta_i(u) = \mu_i B\Theta_i \qquad Re(\mu_i) < 0,$$

while the vectors Ξ_i satisfy

$$\Big(A(u,\, 0) - \lambda_i(u)E(u)\Big)\Xi_i = \vec{0} \qquad \lambda_i(u) \geq 0$$

In the case of a non characteristic boundary (i.e when $A(u,\, 0)$ is always invertible) the subspace $\mathcal{V}(u)$ is defined as

$$\mathcal{V}(u) := \text{span}\langle \Theta_{n_{11}+q+1}(u,\, 0), \dots, \Theta_n(u,\, 0),\, \Xi_{n+1}, \dots \Xi_N(u,\, 0)\rangle,$$

With n we denote the number of negative eigenvalues of A, according to (3.19). According to the results in Sections 4.2.3 and 4.2.4, $\mathcal{V}(\bar{u}_0)$ is the space generated by the columns of the jacobian of the map ϕ at $(s_{n_{11}+q+1} \dots s_N) = \vec{0}$.

The following lemma guarantees that, even if $\mathcal{W}(u)$ and $\mathcal{V}(u)$ do not coincide in general, nevertheless they are transversal to the same subspace $\mathcal{Z}(u)$. This result is already known, but we repeat the proof for completeness.

Lemma 4.9. *The following holds:*

$$\mathcal{V}(\bar{u}_0) \oplus \mathcal{Z}(\bar{u}_0) = \mathbb{R}^N \qquad \mathcal{W}(\bar{u}_0) \oplus \mathcal{Z}(\bar{u}_0) = \mathbb{R}^N$$

Proof. The second part of the statement is trivial.

In the proof of the first part for simplicity we consider only the boundary characteristic case ($A(u,\, 0)$ can be singular), being the other case absolutely analogous. Also, we write A instead of $A(\bar{(}u_0,\,)$. The proof is organized into four steps:

1. First, we prove that if V is a non zero vector in $\text{span}\langle \Xi_{k+1}, \dots, \Xi_N\rangle$, then

   $$\langle V,\, AV\rangle > 0. \tag{4.62}$$

   The steps are analogous to those in the proof of Lemma 7.1 in [6], but for completeness we repeat them.

   If $V \in \text{span}\langle \Xi_{k+1}, \dots, \Xi_N\rangle$, then V belongs to the unstable manifold of the linear system

   $$u_x = Au$$

   and hence there exists a solution of

   $$\begin{cases} u_x = Au \\ u(0) = V \qquad \lim_{x \to -\infty} u(x) = 0. \end{cases}$$

   Thanks to the symmetry of A, such a solution satisfies

   $$\frac{d}{dx}\langle u,\, Au\rangle = 2\langle u_x,\, Au\rangle = 2|Au|^2 > 0$$

   and hence to conclude it is enough to observe that

   $$\lim_{x \to -\infty} \langle u(x),\, Au(x)\rangle = 0.$$

2. if
$$Z = \sum_{i=1}^{n_{11}+q} x_i Z_i,$$
then
$$\langle Z,\, AZ\rangle \leq 0. \tag{4.63}$$
Indeed, the matrices A_{11}, E_{11} are symmetric and E_{11} is positive definite, hence they admit an orthogonal basis of eigenvectors $\zeta_1 \dots \zeta_{N-r}$ such that $\langle \zeta_j,\, A_{11}\zeta_i\rangle = \eta_i \delta_{i\,j}\, \langle \zeta_i,\, E_{11}\zeta_i\rangle$. In particular,
$$\langle Z,\, AZ\rangle = \sum_{i=1}^{n_{11}+q} \eta_i x_i^2 \langle \zeta_i, E_{11}\zeta_i\rangle \leq 0.$$

3. if
$$\Theta = \sum_{j=n_{11}+q+1}^{k} y_j \Theta_j,$$
then
$$\langle A\Theta,\, \Theta\rangle \leq 0. \tag{4.64}$$
Indeed, by Lemma 4.4 one deduces that the following system admits a solution:
$$\begin{cases} Bu_x = Au \\ u(0) = \Theta \qquad \lim_{x\to+\infty} u(x) = 0. \end{cases}$$
Hence, by considerations analogous to those performed in the first step, one concludes that (4.64) holds true.

4. it holds
$$\text{span}\langle Z_1, \dots, Z_{n_{11}+q}, \Theta_{n_{11}+q+1}, \dots, \Theta_k\rangle \cap \text{span}\langle \Xi_{k+1}, \dots, \Xi_N\rangle = \{0\}. \tag{4.65}$$
To prove it, it is enough to show that if
$$Z \in \text{span}\langle Z_1, \dots, Z_{n_{11}+q}\rangle, \qquad \Theta \in \text{span}\langle \Theta_{n_{11}+q+1}, \dots, \Theta_k\rangle, \qquad \Big(Z+\Theta\Big) \neq 0, \tag{4.66}$$
then
$$\Big(Z+\Theta\Big) \notin \text{span}\langle \Xi_{k+1}, \dots, \Xi_N\rangle.$$
If
$$\Theta = \sum_{i=n_{11}+q+1}^{N} x_i \Theta_i,$$
then
$$\langle AZ,\, \Theta\rangle = \langle Z,\, A\Theta\rangle == \sum_{i=n_{11}+q+1}^{N} x_i \langle Z,\, A\Theta_i\rangle = \sum_{i=n_{11}+q+1}^{N} \mu_i x_i \langle Z,\, B\Theta_i\rangle = 0$$
and
$$\langle Z+\Theta,\, AZ + A\Theta\rangle = \langle Z,\, AZ\rangle + 2\langle Z,\, A\Theta\rangle + \langle \Theta,\, A\Theta\rangle,$$
from (4.63), (4.62) and (4.64) it follows that (4.66) holds true.

5. to conclude, it is enough to show that for every $i = 1, \dots, (n_{11}+q)$
$$Z_i \notin \text{ span}\langle \Theta_{n_{11}+q+1}(u,\, 0), \dots, \Theta_k(u,\, 0)\rangle. \tag{4.67}$$
Suppose by contradiction that
$$Z_i = \sum_{j=n_{11}+q+1}^{k} c_j \Theta_j \tag{4.68}$$

for suitable numbers $c_{n_{11}+q+1}, \dots c_k$. Because of Lemma 4.5, for every j, Θ_j has the structure described by (4.23), where $\vec{\xi}_j$ is an eigenvector of $\underline{b}^{-1}\underline{a}$. The matrices $\underline{b}$ and $\underline{a}$ have dimension $(r-q)$ and are defined by (4.21). Considering the last $(r-q)$ lines of the equality (4.68) one obtains

$$0 = \sum_j c_j \vec{\xi}_j,$$

which implies $c_j = 0$ for every j, because the $\vec{\xi}_j$ are all independent. Hence (4.68) cannot hold and (4.67) is proved.

□

From Lemma 4.9 we deduce that the map $ß \circ \phi$ which appears in Theorems 4.1 and 4.2 is locally invertible. We proceed as follows.

By construction the kernel of the jacobian $D\, ß$ is $\mathcal{Z}(\bar{u}_0)$. Thus,

$$D\, ß(\bar{u}_0) V = \vec{0} \implies V \in \mathcal{Z}(\bar{u}_0).$$

The columns of the matrix

$$D\, ß(\bar{u}_0)\, D\, \phi(\bar{u}_0)$$

are $D\, ß(\bar{u}_0)\Theta_{n_{11}+q+1} \dots D\, ß(\bar{u}_0)\Theta_{k-1}, D\, ß(\bar{u}_0)\Xi_k \dots D\, ß(\bar{u}_0)\Xi_N$. To prove that the columns are all indipendent it is enough to show that

$$\sum_{i=n_{11}+q+1}^{k-1} x_i D\, ß(\bar{u}_0)\Theta_i + \sum_{i=k}^{N} x_i D\, ß(\bar{u}_0)\Xi_i = \vec{0} \implies x_{n_{11}+q+1} = \dots x_N = 0. \tag{4.69}$$

Since $\Theta_{n_{11}+q+1} \dots \Theta_{k-1}$, $\Xi_k \dots \Xi_N$ are all indipendent, to prove (4.69) is is enough to show that if $V \in \mathcal{V}(\bar{u}_0)$ and

$$D\, ß(\bar{u}_0) V = \vec{0}$$

then $V = \vec{0}$. This a consequence of Lemma 4.9, which states that

$$\mathcal{V}(\bar{u}_0) \oplus \mathcal{Z}(\bar{u}_0) = \mathbb{R}^N$$

We now introduce a generalization of Definition 2.1.

Definition 4.1. The function

$$ß : \mathbb{R}^N \to \mathbb{R}^{N-n_{11}-q}$$

is smooth and satifies

$$\text{kernel}\Big(Dß(\bar{u}_0)\Big) \oplus \mathcal{W}(\bar{u}_0) = \mathbb{R}^N \tag{4.70}$$

and

$$\text{kernel}\Big(Dß(\bar{u}_0)\Big) \oplus \mathcal{V}(\bar{u}_0) = \mathbb{R}^N \tag{4.71}$$

Thanks to (4.70), the initial boundary value problem

$$\begin{cases} E(u)u_t + A(u,\, u_x)u_x = B(u)u_{xx} \\ ß(u(t,\, 0)) = \bar{g} \qquad u(0,\, x) = \bar{u}_0 \end{cases}$$

is well posed. This is a consequence of the same considerations as in Section 2.2.1.

Also, because of (4.71) the matrix

$$D\, ß(\bar{u}_0)\, D\, \phi(\bar{u}_0)$$

is invertible. Thus, the function $ß \circ \phi$ which appears in Theorems 4.1 and 4.2 is locally invertible. In other words, the analysis in Section 4 is still valid if we use Definition 4.1 instead of Definition 2.1.

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