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Bounded Proper Forcing Axiom and Well Orderings of the Reals

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The bounded proper forcing axiom and well orderings of the reals

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Abstract

We show that the bounded proper forcing axiom BPFA implies that there is a well-ordering of $\mathcal{P}(\omega_1)$ which is Δ_1 definable with parameter a subset of ω_1. Our proof shows that if BPFA holds then any inner model of the universe of sets that correctly computes $\aleph_2$ and also satisfies BPFA must contain all subsets of ω_1. We show as applications how to build minimal models of BPFA and that BPFA implies that the decision problem for the Härtig quantifier is not lightface projective.

1 Introduction

Forcing axioms are natural combinatorial statements which decide many of the questions left open by the usual axioms ZFC of set theory. The study of these axioms was started by Martin and Solovay [13] who introduced Martin's axiom MA as an abstraction of Solovay and Tennenbaum's approach to solving Suslin's problem [17], a question about uncountable trees. It was soon realized that MA together with the negation of the continuum hypothesis CH provides a rich structure theory for the reals. MA states a form of saturation of the universe of sets under possible sets added by forcing notions that satisfy the countable chain condition. As the method of forcing was further developed, generalizations of MA to larger classes of forcing posets were considered as well, most notably the proper forcing axiom PFA introduced by Baumgartner and Shelah (see for example Baumgartner's survey paper [3]) and Martin's maximum MM, the provably strongest forcing axiom, introduced by Foreman, Magidor and Shelah [7].

In recent years, bounded versions of traditional forcing axioms have received a considerable amount of attention as they have many of the same consequences, yet require

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much smaller large cardinal assumptions. These statements were first considered by Goldstern and Shelah in [8] who showed that the bounded proper forcing axiom BPFA is equiconsistent with a relatively modest large cardinal axiom, the existence of a Σ_1-reflecting cardinal. An appealing formulation of bounded forcing axioms as principles of generic absoluteness was provided by Bagaria [1]. Namely, suppose $\mathcal{K}$ is a class of forcing notions. The bounded forcing axiom $\mathsf{BFA}(\mathcal{K})$ is the statement asserting that for every $\mathcal{P} \in \mathcal{K}$,

$$(H_{\aleph_2}, \in) \prec_{\Sigma_1} (V^{\mathcal{P}}, \in).$$

Here $H_{\aleph_2}$ denotes the collection of all sets whose transitive closure has size at most $\aleph_1$. Thus, $\mathsf{BFA}(\mathcal{K})$ states that for every Σ_0 formula $\psi(x, a)$ with parameter $a \in H_{\aleph_2}$, if some forcing notion from $\mathcal{K}$ introduces a witness x for $\psi(x, a)$, then such an x already exists. For example, $\mathsf{MA}_{\aleph_1}$ is $\mathsf{BFA}(\mathsf{ccc})$, BPFA is $\mathsf{BFA}(\mathsf{Proper})$ and BMM is $\mathsf{BFA}(\mathcal{K})$ where $\mathcal{K}$ is the class of forcing notions that preserve stationary subsets of ω_1.

One of the key questions left open by ZFC and resolved by forcing axioms is the value of the continuum $\mathfrak{c}$. Thus, Foreman, Magidor and Shelah [7] showed that MM implies that $\mathfrak{c} = \aleph_2$. The same conclusion was obtained from the weaker proper forcing axiom PFA by the second author [21] and Todorčević [4]. In [22] Woodin identified a statement ψ_{AC} which follows from both Woodin's $\mathcal{P}_{\max}$-axiom $(*)$ and from MM, and implies that $\mathfrak{c} = 2^{\aleph_1} = \aleph_2$ and that there is a well-ordering of the reals definable with parameters in $(H_{\aleph_2}, \in)$. Moreover, Woodin showed that BMM together with the existence of a measurable cardinal implies that the continuum is $\aleph_2$. The assumption of the existence of a measurable cardinal was later eliminated by Todorčević [20] who deduced these consequences of ψ_{AC} from a statement he called θ_{AC} that he showed follows from BMM. Recently, Moore [14] introduced the mapping reflection principle MRP and deduced it from PFA. Although MRP does not follow from BPFA, Moore [14] used similar ideas to show that BPFA implies a certain statement υ_{AC} which in turn implies that there is a well ordering of the reals, and in fact of $\mathcal{P}(\omega_1)$, of order type ω_2 which is Δ_2-definable in the structure $(H_{\aleph_2}, \in)$ with parameter a subset of ω_1.

This paper continues this line of research. Our results were motivated by an attempt to extend the result of the second author [21] who showed that if MM holds and M is an inner model such that $\omega_2^M = \omega_2$, then $\mathcal{P}(\omega_1) \subseteq M$. The interest in this study comes from the question, initially considered by Laver and Carlson (unpublished), of whether PFA can be preserved by some forcing notion in a nontrivial way. The best published result in this direction is the result of König and Yoshinobu [11, Theorem 6.1] who showed that PFA is preserved by ω_2-closed forcing. The same holds for BPFA. In fact, BPFA is preserved by any proper forcing that does not add subsets of ω_1. In this paper we use some of the ideas introduced by Moore [14] and design a robust coding of reals by triples of ordinals smaller than ω_2. This allows us to show (Theorem 1) that if M is an inner model, BPFA holds in both M and V, and $\aleph_2^M = \aleph_2$, then $\mathcal{P}(\omega_1) \subseteq M$. This answers the question considered by Laver and Carlson negatively. Moreover, our coding allows us to show (Theorem 2) that BPFA implies the existence of a well ordering of the reals of optimal complexity, namely, Δ_1 with parameter a

subset of ω_1. We then use these ideas to build a minimal model of BPFA and to show that under PFA the set of validities of the logic with the Härtig quantifier is not ordinal definable in $L(\mathbb{R})$. This last application was suggested to us by Jouko Väänänen in private communication.

Our notation is mostly standard. If κ is an infinite cardinal, then H_κ is the family of sets whose transitive closure has cardinality smaller than κ, i.e., $H_\kappa = \{\, x : |\mathtt{tc}(x)| < \kappa \,\}$, where $\mathtt{tc}(\cdot)$ denotes transitive closure. We use $V^{\mathcal{P}}$ to denote the extension of V by the forcing notion $\mathcal{P}$. A forcing $\mathcal{Q}$ is a *factor* of $\mathcal{P}$ iff $V^{\mathcal{Q}} \subseteq V^{\mathcal{P}}$, i.e., iff every $\mathcal{P}$-generic extension of V contains an object $\mathcal{Q}$-generic over V. In this case, there is a forcing $\mathcal{B} \in V^{\mathcal{Q}}$ such that $\mathtt{ro}(\mathcal{P}) \cong \mathtt{ro}(\mathcal{Q}) * \dot{\mathcal{B}}$, and we write $\mathcal{B} \cong \mathtt{ro}(\mathcal{P})/\mathtt{ro}(\mathcal{Q})$. We may on occasion abuse language and write $\mathcal{P}$ instead of $\mathtt{ro}(\mathcal{P})$, etc. Here, $\mathtt{ro}(\mathcal{P})$ denotes the regular open algebra of $\mathcal{P}$, i.e., the unique complete Boolean algebra into which $\mathcal{P}$ embeds densely. Suppose $V \subseteq M \subseteq V^{\mathcal{P}}$ where $M \models \mathsf{ZFC}$. Then there is a factor $\mathcal{Q}$ of $\mathcal{P}$ such that $M = V^{\mathcal{Q}}$ ([10, Lemma 15.43]). By a real we mean an element of the Cantor set 2^ω. If X is a set and $(X, \in)$ satisfies extensionality, we let $\pi : X \to \bar{X}$ denote the Mostowski collapse of X, i.e., the unique isomorphism between X and its transitive collapse. For all other notation, an introduction to set theory, and forcing see [10] to which we also refer for all concepts not defined explicitly in this paper.

This paper is organized as follows. In section 2 we reduce Theorem 1 and Theorem 2 to the verification of two facts. The proof of these two facts occupies sections 3 and 4. Sections 5 and 6 contain applications of Theorem 1. In section 5 we build a minimal model of BPFA, solving a question of Sy Friedman. In section 6 we show that the set V_I of validities of the logic with the Härtig quantifier is not lightface projective under BPFA and not even ordinal definable in $L(\mathbb{R})$ under PFA; this last result follows from a suggestion by Hugh Woodin that allowed us to improve our original result that PFA implies that V_I is not definable in $L(\mathbb{R})$ without parameters.

The main results, Theorem 1 and Theorem 2, were obtained during the Fall of 2004, while the second author was visiting the Erwin Schrödinger Institute in Vienna. The results in section 5 and 6 were obtained during the Spring of 2005 while the first author was visiting the Université Denis-Diderot in Paris. We want to thank our respective hosts for their hospitality and generosity. We would also like to thank Jouko Väänänen for many useful conversations involving the content of this paper and Hugh Woodin for his suggestions on section 6.

2 Coding reals

The main results of this paper can be seen as consequences of Moore's *mapping reflection principle* MRP introduced in [14]. In [14, Theorem 1.1] it is shown that MRP is a consequence of PFA. Although BPFA is not sufficient to imply the full version of MRP we argue that it does imply the relevant consequences of MRP and thus we obtain the desired conclusions.

We start by describing a coding of reals by sets of ordinals reminiscent of Moore's

υ_{AC}. We fix a *C-sequence* $\vec{C} = (\, C_\xi : \xi < \omega_1 \text{ a limit ordinal }\,)$, i.e., C_ξ is an unbounded subset of ξ of order type ω for all limit ordinals $\xi < \omega_1$. Given x, y, z sets of natural numbers, define an equivalence relation $\sim_x$ on $\omega \setminus x$ by setting $n \sim_x m$ (for $n \leq m$) iff $[n, m] \cap x = \emptyset$. Thus the equivalence classes of $\sim_x$ are simply the intervals between the consecutive members of x. Let $(I_k)_{k \leq t}$ $(t \leq \omega)$ be the natural enumeration of those equivalence classes which intersect both y and z. In the cases that interest us t will be finite, but, in general, $t = \omega$ is possible. Let the *oscillation* of x, y, z be the function $o(x, y, z) : t \to 2$ defined by letting for all $k < t$,

$$o(x, y, z)(k) = 0 \text{ iff } \min(I_k \cap y) \leq \min(I_k \cap z).$$

Let $\omega_1 < \beta < \gamma < \delta$ be fixed limit ordinals and suppose $N \subseteq M \subseteq \delta$ are countable sets of ordinals. Assume that $\{\omega_1, \beta, \gamma\} \subset N$, that $\sup(\xi \cap N) < \sup(\xi \cap M)$ and $\sup(\xi \cap M)$ is a limit ordinal, for every $\xi \in \{\omega_1, \beta, \gamma, \delta\}$. Then the pair (N, M) codes a finite binary sequence as follows: Take the transitive collapse $\bar{M}$ of M and let π be the collapsing map. Let $\alpha_M = \pi(\omega_1), \beta_M = \pi(\beta), \gamma_M = \pi(\gamma), \delta_M = \bar{M}$, each of these is a countable limit ordinal. Let the *height* of $\alpha_N = \sup(\pi[\omega_1 \cap N])$ in α_M be the integer

$$n = n(N, M) = \operatorname{card}(\alpha_N \cap C_{\alpha_M}).$$

Define three sets x, y and z of integers by

$$x = \{\, \operatorname{card}(\pi(\xi) \cap C_{\beta_M}) : \xi \in \beta \cap N \,\}$$

and similarly for y and z with γ and δ, respectively, instead of β. Notice that x, y, z are finite by our assumption on N and M. Now, we look at the oscillation of $x \setminus n$, $y \setminus n$ and $z \setminus n$, which is a binary sequence, and if its length is at least n then we let

$$s_{\beta\gamma\delta}(N, M) = o(x \setminus n, y \setminus n, z \setminus n) \restriction n.$$

In all other cases we let $s_{\beta\gamma\delta}(N, M) = *$. We similarly write $s_{\beta\gamma\delta}(N, M) \restriction l = *$ if $l > n(N, M)$.

Remark 1 Notice that there is a finite $T \subset N$ such that for any $S \subset N$, if $T \subset S$ then $s(S, M) = s(N, M)$. In effect, it suffices that T contains $\{\omega_1, \beta, \gamma\}$, $\pi^{-1}[\alpha_N \cap C_{\alpha_M}]$, one point of N for each interval in $\beta \cap M$ determined by $\pi^{-1}[C_{\beta_M}]$ that N meets, and similarly for γ and δ.

Finally, we say that the triple (β, γ, δ) *codes* a real r if there is a continuous increasing sequence $(N_\xi : \xi < \omega_1)$ of countable sets whose union is δ such that for every countable limit ordinal ξ there is $\nu < \xi$ such that

$$r = \bigcup_{\nu < \eta < \xi} s_{\beta\gamma\delta}(N_\eta, N_\xi).$$

We are going to show that BPFA implies the following two facts:

(1) Given ordinals $\omega_1 < \beta < \gamma < \delta < \omega_2$ of cofinality ω_1 there is an increasing continuous sequence $(N_\xi : \xi < \omega_1)$ of countable sets whose union is δ such that for every limit ordinal $\xi < \omega_1$ and every integer n there is $\nu < \xi$ and $s^n_\xi \in \{0,1\}^n \cup \{*\}$ such that $s_{\beta\gamma\delta}(N_\eta, N_\xi) \restriction n = s^n_\xi$ for every η such that $\nu < \eta < \xi$.

(2) For each real r there are ordinals $\omega_1 < \beta < \gamma < \delta < \omega_2$ of cofinality ω_1 such that the triple (β, γ, δ) codes r.

Remark 2 Notice that in (1) we are not claiming that for a fixed n the values of s^n_ξ cohere as ξ varies. However, by the way we have defined our coding it follows that for a fixed limit ordinal ξ the values of s^n_ξ (other than $*$) cohere as n varies.

Before we proceed to the proof that BPFA implies (1) and (2) let us point out some consequences.

Theorem 1 *Assume M is an inner model,* BPFA *holds in both M and V, and $\omega_2^M = \omega_2$. Then $\mathcal{P}(\omega_1) \subset M$.*

Proof: We claim that the theorem follows from (1) and (2). To see this, assume that M is an inner model of V, that BPFA holds in both M and V and $\omega_1^M = \omega_1$. Fix a C-sequence in M to carry out the codings just described. Suppose $\omega_1 < \beta < \gamma < \delta < \omega_2^M$ are ordinals of cofinality ω_1 and the triple (β, γ, δ) codes in V a real r. Let $(N_\xi : \xi < \omega_1)$ be a continuous increasing sequence of countable sets with union δ witnessing this. In M there is a sequence $(P_\xi : \xi < \omega_1)$ witnessing (1) for (β, γ, δ). There is a club $C \subseteq \omega_1$ in V such that $N_\xi = P_\xi$, for every $\xi \in C$. Then it follows that for any ξ which is a limit point of C, $r = \bigcup_n s^n_\xi$, as computed in M relative to the sequence $(P_\xi : \xi < \omega_1)$. It follows that $r \in M$. If, moreover, $\omega_2 = \omega_2^M$, then any real is coded in V by some triple of ordinals less than ω_2^M, thus all reals are in M. But then $\mathcal{P}(\omega_1) \subseteq M$, since, given any ω_1-sequence in M of almost disjoint reals, BPFA, and in fact $\mathsf{MA}_{\aleph_1}$, allows us to code any subset of ω_1 via this sequence and a real. □

Recall that Moore showed in [14] that BPFA, in fact, the coding principle he named υ_{AC}, implies that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. Since BPFA implies $\mathsf{MA}_{\aleph_1}$, this also follows from our result.

Theorem 2 BPFA *implies that there is a Δ_1 well ordering of $\mathcal{P}(\omega_1)$ with parameter a subset of ω_1. The length of the well ordering is ω_2.*

Proof: Fix as parameter a C-sequence $\vec{C} = (C_\xi : \xi < \omega_1 \text{ a limit ordinal})$. Let T be the theory "$\mathsf{ZFC}-\mathsf{Power\ set}+\mathsf{MA}_{\aleph_1} + (1) + (2) + \forall x\,(|x| \leq \aleph_1)$". Notice that any transitive model M of T which contains $\vec{C}$ is uniquely determined by $\mathrm{Ord} \cap M$. In effect, notice that since $\vec{C} \in M$, M computes ω_1 correctly. Suppose a real r is coded by some triple (β, γ, δ) of ordinals in M. Then, arguing as in the proof of Theorem 1, we see that $r \in M$. Notice that we are not claiming that M knows that (β, γ, δ)

codes r, just that $r \in M$. Since M also satisfies (2) it follows that the reals in M are precisely the reals coded by some triple of ordinals which belong to M. Since $\mathsf{MA}_{\aleph_1}$ holds in M it follows that $\mathcal{P}(\omega_1)^M$ is completely determined as well. Namely, from $\vec{C}$ we can define a canonical ω_1-sequence $\vec{r}$ of almost disjoint reals, and we can use the standard almost disjoint coding to code a subset of ω_1 by the sequence $\vec{r}$ and a real. Now, for an ordinal $\theta < \omega_2$, let M_θ be the unique transitive model M of T containing $\vec{C}$ such that $\mathrm{Ord} \cap M = \theta$, if it exists; otherwise let $M_\theta = \emptyset$. Notice that the function $\theta \mapsto M_\theta$ is Δ_1 in the parameter $\vec{C}$.

Now, let $<_*$ be the antilexicographic ordering on the class $[\mathrm{Ord}]^3$ of increasing triples of ordinals. For a real r let θ_r be the least θ such that $r \in M_\theta$ and let $(\beta_r, \gamma_r, \delta_r)$ be the $<_*$-least triple of ordinals smaller than θ_r such that $M_{\theta_r} \models (\beta_r, \gamma_r, \delta_r)$ codes r. Finally, let $r \lhd s$ iff either $\theta_r < \theta_s$ or $\theta_r = \theta_s$ and $(\beta_r, \gamma_r, \delta_r) <_* (\beta_s, \gamma_s, \delta_s)$.

We can now define a well ordering of $\mathcal{P}(\omega_1)$ as follows. First, from $\vec{C}$ we can define a canonical ω_1-sequence $\vec{r} = (r_\alpha : \alpha < \omega_1)$ of almost disjoint reals. Then we can use the standard almost disjoint coding to code a subset of ω_1 by the sequence $\vec{r}$ and a real. Finally, for a and b subsets of ω_1, we define $a \prec b$ iff the $\lhd$-least real coding a is $\lhd$-smaller than the $\lhd$-least real coding b. By an argument similar to the above, $\prec$ is also Δ_1 in the parameter $\vec{C}$.

As an additional bonus, notice that the set of possible parameters in the definition of the well ordering, i.e., the set of C-sequences, is Δ_0 in the parameter ω_1. $\square$

Remark 3 The readers familiar with [14] may find in order a few words contrasting our Theorems 1 and 2 with the results in [14], specifically Moore's result that BPFA implies that there is a well-ordering of $\mathcal{P}(\omega_1)$ definable over $(H_{\aleph_2}, \in, \mathrm{NS}_{\omega_1})$ from a C-sequence. In [14] the principle υ_{AC} is introduced and shown to follow from BPFA. From υ_{AC} it follows that each equivalence class $[S] \in \mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ is coded by an ordinal $\delta < \omega_2$ of cofinality ω_1, where NS_{ω_1} denotes the nonstationary ideal on ω_1. This corresponds to the statement we numbered (2) above. The statement corresponding to (1), namely that any such δ codes some equivalence class $[S]$, is also a consequence of BPFA as can be easily seen from the arguments in [14] even though it is not explicitly mentioned there, alternately one can argue as in section 3 below. However, it seems that these versions of (1) and (2) do not suffice to prove Theorem 1. The standard way in which $\mathcal{P}(\omega)$ is coded once we have υ_{AC} (or Woodin's ψ_{AC}, or Todorčević's θ_{AC}) consists of fixing a partition $\vec{S} = (S_n : n < \omega)$ of ω_1 into stationary sets and assigning to a real $r \in 2^\omega$ the ordinal corresponding to $[S_r]$, where $S_r = \bigcup_{r(n)=1} S_n$.

Suppose that M is an inner model of V, that $\omega_2^M = \omega_2$, and that BPFA holds in both M and V. Suppose that a partition $\vec{S} \in M$ as above is given such that for each n, S_n is still stationary in V. Such partitions actually exist, as Larson shows in [12]. In V let r be a real and let δ be an ordinal coding r relative to the partition $\vec{S}$. In M, δ codes the equivalence class of some set A. It follows that *in* V, the symmetric difference of A and S_r is nonstationary. However, this is not enough to conclude that $r \in M$. Namely, for any n such that $r(n) = 0$, $S_n \setminus A$ is stationary in M since it is

stationary in V, but, for some n such that $r(n) = 1$, it may be a priori that $S_n \setminus A$ is a stationary set in M whose stationarity is destroyed in V. This means that in M, we may not be able to recover r from A. And even if $r \in M$, it may be that $A \setminus S_r$ is stationary in M but not in V, so we may fail to identify correctly in M the ordinals that code r in V.

These considerations lead us to look for a coding of reals by triple of ordinals similar to the one given by υ_{AC} but not so explicitly dependent on NS_{ω_1}, and this ultimately resulted in the versions of (1) and (2) mentioned above.

Now consider the complexity of the well-ordering obtained in Theorem 2 and the one obtained in [14]. Using υ_{AC} and fixing a partition $\vec{S}$ of ω_1 into stationary sets, for each real r let δ_r be the least ordinal which codes the equivalence class of S_r. We say $r < s$ iff $\delta_r < \delta_s$. To say that $r < s$ one needs to express that there is an ordinal δ which codes S_r (this part is Σ_1) and such that for all ordinals $\mu < \delta$ the set coded by μ is not equal to S_s modulo NS_{ω_1}. The point is that this second part adds another quantifier, since expressing that the symmetric difference of the set coded by μ and S_s is stationary is a Π_1 statement.

Remark 4 It is open whether even MM implies that there is a definable well ordering of the reals without parameters.

Corollary 1 *Assume* BPFA *holds and* $\omega_1 = \omega_1^L$. *Then there is a projective well-ordering of the reals.*

Proof: The argument is very similar to the one in [5], so we only sketch the main points. Just notice that in L there is a C-sequence $\vec{C}$ that is Δ^1_2 in the codes. By Shoenfield absoluteness, $\vec{C}$ is still Δ^1_2 in V and, since $\omega_1 = \omega_1^L$, it is still a C-sequence. From $\vec{C}$, we can define a well-ordering of $\mathbb{R}$ by looking at ordinals in $S^{\omega_2}_{\omega_1}$, but they can be coded as subsets of ω_1, which can be coded by reals using almost disjoint forcing, for this we can use a Δ^1_2 sequence in L of almost disjoint reals as the ω_1-sequence relative to which the almost disjoint coding takes place. As shown in [5], these codings are projective. □

Remark 5 The well ordering obtained in Corollary 1 is in general not optimal. Friedman has shown the consistency of BPFA together with a Σ^1_3 well ordering of the reals by class forcing techniques, see [6].

3 Open partitions

The purpose of this section is to show that BPFA implies (1). We start by recalling the relevant definitions from [14].

Definition 1 ([14, Definition 2.1]) *Let θ be a regular cardinal, let X be uncountable, and let $M \prec H_\theta$ be countable such that $[X]^\omega \in M$. A subset Σ of $[X]^\omega$ is* M-stationary *iff for all $E \in M$ such that $E \subseteq [X]^\omega$ is club, $\Sigma \cap E \cap M \neq \emptyset$.*

Recall that the Ellentuck topology on $[X]^\omega$ is obtained by declaring a set open iff it is the union of sets of the form

$$[x, N] = \{Y \in [X]^\omega : x \subseteq Y \subseteq N\}$$

where $N \in [X]^\omega$ and $x \subseteq N$ is finite. When we say 'open' in this paper we refer to this topology.

Definition 2 ([14, Definition 2.3]) *A set mapping Σ is* open stationary *iff there is an uncountable set $X = X_\Sigma$ and a regular cardinal $\theta = \theta_\Sigma$ such that $[X]^\omega \in H_\theta$, $dom(\Sigma)$ is a club in $[H_\theta]^\omega$ and $\Sigma(M) \subseteq [X]^\omega$ is open and M-stationary, for every M in the domain of Σ.*

Recall the Mapping Reflection Principle (MRP) introduced in [14].

> If Σ is an open stationary set mapping whose domain is a club, there is a continuous $\in$-chain $\vec{N} = (N_\xi : \xi < \omega_1)$ of elements in the domain of Σ such that for all limit ordinals $\xi < \omega_1$ there is $\nu < \xi$ such that $N_\eta \cap X_\Sigma \in \Sigma(N_\xi)$ for all η such that $\nu < \eta < \xi$.

If $(N_\xi : \xi < \omega_1)$ satisfies the conclusion of MRP for Σ then it is said to be a *reflecting sequence* for Σ. It is shown in [14] that MRP is a consequence of PFA. In fact, what is proved in [14] is that for a given open stationary set mapping Σ there is a proper forcing notion which introduces a reflecting sequence for Σ.

Lemma 1 *Assume* MRP. *Suppose that for each $\alpha < \omega_1$ we have a partition $\alpha = K^0_\alpha \cup K^1_\alpha$ into clopen sets in the standard order topology on α. Then there a club $C \subset \omega_1$ such that for every limit point ξ of C there is $i \in 2$ such that $C \setminus K^i_\xi$ is bounded in ξ. Moreover, this conclusion follows from* BPFA.

Proof: Let $X = \omega_1$ and let θ be a sufficiently large regular cardinal. Notice that ω_1 is a closed subset of $[\omega_1]^\omega$ in the Ellentuck topology and the relativization of this topology to ω_1 is just the usual order topology on ω_1.

Let $M \prec H_\theta$ be countable and let $\alpha = M \cap \omega_1$. We claim that one of the K^i_α $(i \in 2)$ is M-stationary. Otherwise, there are $E_0, E_1 \in M$ club subsets of ω_1 such that $E_i \cap K^i_\alpha = \emptyset$ $(i \in 2)$ and therefore $E_0 \cap E_1 = \emptyset$ by elementarity. Define Σ by letting $\Sigma(M) = [\omega_1]^\omega \setminus K^i_\alpha$ for the i such that K^{1-i}_α is M-stationary. Now, applying MRP we obtain a reflecting sequence $(N_\xi : \xi < \omega_1)$ for Σ. Let $C = \{N_\xi \cap \omega_1 : \xi < \omega_1\}$. It is clear that C is as required. To see that the conclusion follows from BPFA notice that the existence of C as stated is a Σ_1 statement with parameter $(K^i_\alpha : \alpha < \omega_1, i \in 2)$. Since any particular instance of MRP can be forced by a proper forcing it follows that BPFA implies the conclusion of the lemma. $\square$

Lemma 2 *Assume* BPFA. *Then* (1) *holds.*

PROOF: Fix a triple of ordinals $\omega_1 < \beta < \gamma < \delta < \omega_2$, each of cofinality ω_1. We first find an increasing continuous sequence $(N_\xi : \xi < \omega_1)$ of countable sets with union δ. By thinning out the sequence, if necessary, we may assume that $\{\omega_1, \beta, \gamma\} \subseteq N_0$, that $\sup(\rho \cap N_\xi)$ is a limit ordinal and $\sup(\rho \cap N_\eta) < \sup(\rho \cap N_\xi)$, for every $\eta < \xi < \omega_1$ and $\rho \in \{\omega_1, \beta, \gamma, \delta\}$. Fix an integer n and define, for each countable limit ordinal ξ, a partition of ξ into $2^n + 1$ disjoint sets, $\xi = \bigcup_{s \in 2^n \cup \{*\}} K^s_\xi$ by

$$\eta \in K^s_\xi \qquad \text{iff} \qquad s_{\beta\gamma\delta}(N_\eta, N_\xi) \restriction n = s.$$

By Remark 1 each piece of the partition is open in the order topology on ω_1 and since the pieces are disjoint and cover ξ they are also closed. By applying Lemma 1 we find a closed unbounded subset C_n of ω_1 and for each limit point ξ of C_n, an element s^n_ξ of $2^n \cup \{*\}$ such that there is $\nu < \xi$ such that $s_{\beta\gamma\delta}(N_\eta, N_\xi) \restriction n = s^n_\xi$ for every $\eta \in C_n$ such that $\nu < \eta < \xi$. Finally, let $C = \bigcap_n C_n$. Then the sequence $(N_\xi : \xi \in C)$ witnesses (1) for the triple (β, γ, δ). $\square$

4 Playing games

Now we show that BPFA implies (2). As before we first show that MRP implies (2) and then argue that BPFA suffices. In order to show that the appropriate set mapping is open stationary we shall use a game argument reminiscent of [21, Lemma 3.7]. For a given algebra $F : [\omega_4]^{<\omega} \to \omega_4$, we define a sequence of games $\mathcal{G}^F_\nu$ indexed by countable ordinals ν. These are ω-length perfect information games between two players I and II playing alternately. We show that these games are determined and that player II has a winning strategy for almost all of them. Playing against these strategies allows us to build models where any desired oscillation pattern occurs. This implies that for every real r a certain map Σ^r to be defined below is open stationary, from which (2) follows immediately in the presence of MRP. We then argue as in Lemma 1 to show that BPFA is sufficient to imply (2). The game $\mathcal{G}^F_\nu$ is defined as follows:

I	β_0		γ_0		δ_0		β_1		γ_1	
II		κ_0, ε_0		λ_0, ϑ_0		μ_0, ϖ_0		κ_1, ε_1		$\ldots$

As indicated in the diagram, player I plays, alternately, ordinals β_i, γ_i or δ_i such that $(\beta_i)_i$, $(\gamma_i)_i$ and $(\delta_i)_i$ are increasing sequences of ordinals in ω_2, ω_3 and ω_4 respectively.

Player II responds in turn by playing, alternately,

- ordinals κ_i, ε_i such that $\beta_i \leq \kappa_i \leq \varepsilon_i < \aleph_2$.
- ordinals λ_i, ϑ_i such that $\gamma_i \leq \lambda_i \leq \vartheta_i < \aleph_3$.
- ordinals μ_i, ϖ_i such that $\delta_i \leq \mu_i \leq \varpi_i < \aleph_4$.

We also require, although this is not really essential, that

- $\varepsilon_i < \beta_{i+1}$, $\vartheta_i < \gamma_{i+1}$ and $\varpi_i < \delta_{i+1}$, for all $i < \omega$.

The first player to violate these rules loses. Otherwise, let

$$X = \mathrm{cl}_F(\{\, \kappa_n, \lambda_n, \mu_n : n < \omega\} \cup \nu),$$

where cl_F denotes closure under F. Then we say that player II wins iff

- $X \cap \omega_1 = \nu$,
- $X \cap [\beta_n, \beta_{n+1}) \subseteq [\beta_n, \varepsilon_n)$ for all n,
- $X \cap [\gamma_n, \gamma_{n+1}) \subseteq [\gamma_n, \vartheta_n)$ for all n, and
- $X \cap [\delta_n, \delta_{n+1}) \subseteq [\delta_n, \varpi_n)$ for all n.

It follows that if player I wins a particular run of $\mathcal{G}^F_\nu$ then this happens at some finite stage. By the Gale-Stewart theorem $\mathcal{G}^F_\nu$ is determined, for all ν. We now show that player II wins $\mathcal{G}^F_\nu$ for club many ν. Let

$$A_F = \{\, \nu < \omega_1 : \text{ I has a winning strategy in } \mathcal{G}^F_\nu \,\}.$$

Lemma 3 *A_F is nonstationary.*

Proof: This is similar to [21, Lemma 3.7]. Assume towards contradiction that A_F is stationary and fix, for each $\nu \in A_F$, a winning strategy σ_ν for I in $\mathcal{G}^F_\nu$. Since A_F has cardinality $\aleph_1$ there is a strategy σ which dominates all the σ_ν. Namely, at any given position when it is player I's turn, σ looks at what the strategies σ_ν would play, for $\nu \in A_F$, and plays the supremum of these ordinals. Then clearly σ is a winning strategy for I in $\mathcal{G}^F_\nu$ for all $\nu \in A_F$. Moreover, we can assume that at every stage when it is player I's turn to play an ordinal $\gamma_n < \omega_3$ the move played by σ does not depend on the previous move of player II. This is possible since there are only $\aleph_2$ possible previous moves of player II, so we look at what σ would play against each of them and play the supremum of these ordinals. Similarly, we may assume that at every stage when it is player I's turn to play an ordinal $\delta_n < \omega_4$, the move played by σ does not depend on the two previous moves of player II. We call this property of σ *stability*.

We now describe a play of II against σ. Fix a sufficiently large regular cardinal θ and build an increasing continuous chain

$$P_0 \prec P_1 \prec \cdots \prec P_\xi \prec \cdots \prec H_\theta, \quad \xi < \omega_1 \cdot \omega$$

of elementary submodels of H_θ of size $\aleph_1$ such that $F, A_F, \sigma \in P_0$. Let $N_n = P_{\omega_1 \cdot n}$. Notice that the ordinals $\zeta_n = \sup(N_n \cap \omega_2)$, $\eta_n = \sup(N_n \cap \omega_3)$ and $\theta_n = \sup(N_n \cap \omega_4)$ all have cofinality ω_1 and N_n contains an ω_1-club in each of these ordinals, for all n. Since we assumed that A_F is stationary we can find a countable $M \prec N$ such that $\{\zeta_n, \eta_n, \theta_n\}_n \subseteq M$, $F, \sigma, A_F \in M$ and $\alpha = M \cap \omega_1 \in A_F$. We describe a run of $\mathcal{G}^F_\alpha$

in which player I plays following σ. We consider the game as consisting of rounds of three moves of each player, one in each of ω_2, ω_3 and ω_4.

Suppose that we are at the start of the n-th round, the current position is $p_{n-1} = \langle \beta_0, (\kappa_0, \varepsilon_0), \dots, \delta_{n-1}, (\mu_{n-1}, \varpi_{n-1}) \rangle$, and we have arranged that $p_{n-1} \in N_n$. Player I then plays $\beta_n = \sigma(p_{n-1})$. Since $\sigma, p_{n-1} \in N_n$ it follows that $\beta_n \in N_n$. Player II picks an ordinal $\kappa_n \in M \cap N_{n+1}$ such that $\zeta_n \leq \kappa_n < \omega_2$. Let $\varepsilon_n = \sup(M \cap N_{n+1} \cap \omega_2)$. Since M is countable and $\zeta_{n+1} = \sup(N_{n+1} \cap \omega_2)$ has cofinality ω_1 it follows that $\varepsilon_n < \zeta_{n+1}$. In fact, since N_{n+1} contains a club in ζ_{n+1} it follows that $\varepsilon_n \in N_{n+1}$. Player II then plays $(\kappa_n, \varepsilon_n)$. Player I responds by playing $\sigma(p_{n-1}{}^\frown\langle \beta_n, (\kappa_n, \varepsilon_n) \rangle) = \gamma_n$. By elementarity of N_n and stability of σ it follows that $\gamma_n \in N_n$. Player II then picks some $\lambda_n \in M \cap N_{n+1}$ such that $\eta_n \leq \lambda_n < \omega_3$. Let $\vartheta_n = \sup(M \cap N_{n+1} \cap \omega_3)$. By the same argument as above, $\vartheta_n < \eta_{n+1}$ and $\vartheta_n \in N_{n+1}$. Player II then plays (λ_n, ϑ_n). Player I responds by playing $\delta_n = \sigma(p_{n-1}{}^\frown\langle \beta_n, (\kappa_n, \varepsilon_n), \gamma_n, (\lambda_n, \vartheta_n) \rangle)$. Again, by elementarity of N_n and stability of σ it follows that $\delta_n \in N_n$. Player II picks some ordinal $\mu_n \in M \cap N_{n+1}$ such that $\theta_n \leq \mu_n < \omega_4$. Let $\varpi_n = \sup(M \cap N_{n+1} \cap \omega_4)$. Again, we have that $\varpi_n < \theta_{n+1}$ and, in fact, $\varpi_n \in N_{n+1}$. Player II then plays (μ_n, ϖ_n). Let $p_n = p_{n-1}{}^\frown\langle \beta_n, (\kappa_n, \varepsilon_n), \gamma_n, (\lambda_n, \vartheta_n), \delta_n, (\mu_n, \varpi_n) \rangle$. It follows that $p_n \in N_{n+1}$. This completes the inductive construction of the play. Let

$$X = \mathrm{cl}_F(\{\, \kappa_i, \lambda_i, \mu_i : i < \omega \} \cup \alpha).$$

Notice that $X \subseteq M$. This implies that $X \cap \omega_1 = \alpha$. Also since $\beta_{n+1} \in N_{n+1}$ and $\varepsilon_n = \sup(M \cap N_{n+1} \cap \omega_2)$ it follows that $X \cap [\beta_n, \beta_{n+1}) \subseteq [\beta_n, \varepsilon_n)$, for all n. Similarly, we have that $X \cap [\gamma_n, \gamma_{n+1}) \subseteq [\gamma_n, \vartheta_n)$, and $X \cap [\delta_n, \delta_{n+1}) \subseteq [\delta_n, \varpi_n)$, for all n. Thus, player II has won this run of $\mathcal{G}^F_\alpha$, a contradiction. $\square$

Lemma 4 *Assume* BPFA. *Then for every real r there is a triple of ordinals less than ω_2 coding r, i.e.,* (2) *holds.*

Proof: Fix a real r. As before we will first derive the desired conclusion from MRP and then argue that it also follows from BPFA. We start by defining an open stationary set mapping Σ^r. Fix a sufficiently large regular cardinal θ. For a countable elementary submodel M of H_θ define

$$\Sigma^r(M) = \{\, N \in [\omega_4]^\omega : N \subseteq M \text{ and } s_{\omega_2\omega_3\omega_4}(N, M \cap \omega_4) \text{ is an initial segment of } r \,\}.$$

By Remark 1, $\Sigma^r(M)$ is open in the Ellentuck topology. We will show that it is also M-stationary. Assume this for a moment and let us see how to finish the proof.

Assume first MRP and let $(M_\xi : \xi < \omega_1)$ be a reflecting sequence for Σ^r. Let $M = \bigcup_{\xi<\omega_1} M_\xi$, let $\bar{M}$ be the transitive collapse of M and let π be the collapsing map. Let $\beta = \pi(\omega_2)$, $\gamma = \pi(\omega_3)$ and $\delta = \pi(\omega_4)$. Then $\omega_1 < \beta < \gamma < \delta < \omega_2$ and $\mathrm{cof}(\beta) = \mathrm{cof}(\gamma) = \mathrm{cof}(\delta) = \omega_1$. Let $N_\xi = \pi(M_\xi \cap \omega_4)$. Note that $s_{\beta\gamma\delta}(N_\eta, N_\xi) =$

$s_{\omega_2\omega_3\omega_4}(M_\eta \cap \omega_4, M_\xi \cap \omega_4)$ for every $\eta < \xi < \omega_1$. Moreover, for ξ a limit ordinal, $N_\xi = \bigcup_{\eta<\xi} N_\eta$ and therefore $\lim_{\eta<\xi} n(N_\eta, N_\xi) = \omega$. It follows that

$$r = \bigcup_{\eta<\xi} s_{\beta\gamma\delta}(N_\eta, N_\xi).$$

Therefore the triple (β, γ, δ) codes r as desired. To see that the same conclusion follows from BPFA note that there is a proper forcing notion adding a reflecting sequence for Σ^r and the existence of a triple coding r is a Σ_1 statement in parameter the C-sequence used for the coding and the real r. Therefore the existence of such a triple of ordinals follows from BPFA. To complete the proof of Lemma 4 it remains to show the following:

Lemma 5 *Σ^r is a stationary set mapping.*

PROOF: Let M be a countable elementary submodel of H_θ. We have to show that $\Sigma^r(M)$ is M-stationary. We work in M and fix an algebra F on ω_4. By Lemma 3 we can find a club C in ω_1 and for every $\nu \in C$ a winning strategy σ_ν for player II in $\mathcal{G}^F_\nu$. We show that we can fix an ordinal $\nu \in C \cap M$ and play finitely many moves of the game $\mathcal{G}^F_\nu$ with ordinals inside M such that if T is the finite set of relevant ordinals played by player II and $X = \mathrm{cl}_F(T \cup \nu)$ then $s_{\omega_2\omega_3\omega_4}(X, M \cap \omega_4)$ is an initial segment of r. We will never leave M, even though we refer to objects which do not belong to M to assist us with our choices.

Let $\bar{M}$ be the transitive collapse of M and let π be the collapsing map. Let $\alpha_M = \pi(\omega_1)$, $\beta_M = \pi(\omega_2)$, $\gamma_M = \pi(\omega_3)$ $\delta_M = \pi(\omega_4)$. Recall that we have fixed a C-sequence $(C_\xi : \xi < \omega_1)$. If $\rho \in M \cap \omega_4$, the *height* of ρ in ω_i, $\mathrm{ht}_{\omega_i}(\rho)$, is $\mathrm{card}(\pi(\rho) \cap C_{\pi(\omega_i)})$, for $i = 1, \ldots, 4$. Of course, the height of ρ in ω_i is known only outside of M, however we only need to refer to the heights of finitely many ordinals.

Now, look at what the strategies σ_ν do at the first step when player I plays 0. For each $\nu \in C$, $\sigma_\nu(\langle 0\rangle) = (\kappa^\nu_0, \varepsilon^\nu_0)$ is a pair of ordinals below ω_2 with $\kappa^\nu_0 \leq \varepsilon^\nu_0$. Let $\varepsilon_0 \in \omega_2 \cap M$ be the supremum of all the ε^ν_0. Similarly, let $(\lambda^\nu_0, \vartheta^\nu_0) = \sigma_\nu(\langle 0, \sigma_\nu(\langle 0\rangle), 0\rangle)$, and let $\vartheta_0 \in M \cap \omega_3$ be the supremum of all the ϑ^ν_0. Define ϖ_0 analogously. Stepping out of M for a moment, look at the height of ε_0 in ω_2 and similarly with ϑ_0 and ϖ_0. Let k be the largest of these integers. We now choose $\nu \in C \cap M$ be such that its height in ω_1, say n, is bigger than k. We are going to play the game $\mathcal{G}^F_\nu$ and player II is going to use his winning strategy σ_ν.

The first round of the play, consisting of three moves for each player, is a bit special. In it, player I simply plays 0 three times and player II responds according to σ_ν. Let us say that the three moves of player II in this round are $(\kappa_0, \varepsilon_0)$, (λ_0, ϑ_0) and (μ_0, ϖ_0). By the choice of ν we have that the height of ε_0 in ω_2, the height of ϑ_0 in ω_3 and the height of ϖ_0 in ω_4 are all smaller than n.

Suppose we have played rounds $0, \ldots, i-1$ and the moves of player II in the last round are $(\kappa_{i-1}, \varepsilon_{i-1})$, $(\lambda_{i-1}, \vartheta_{i-1})$ and $(\mu_{i-1}, \varpi_{i-1})$. In the next round we have to code $r(i-1)$. Player I first chooses β_i such that

$$\mathrm{ht}_{\omega_2}(\beta_i) > \max\{\mathrm{ht}_{\omega_3}(\vartheta_{i-1}), \mathrm{ht}_{\omega_4}(\varpi_{i-1}), n\}.$$

Player II responds using σ_ν and plays, say, $(\kappa_i, \varepsilon_i)$. We have $\beta_i \leq \kappa_i \leq \varepsilon_i$. Then there are two cases.

Case 1 $r(i-1) = 0$. Player I then chooses γ_i such that $\mathrm{ht}_{\omega_3}(\gamma_i) > \mathrm{ht}_{\omega_2}(\varepsilon_i)$. Player II follows σ_ν and plays, say, (λ_i, ϑ_i). Of course, we have that $\gamma_i \leq \lambda_i \leq \vartheta_i$. Player I then plays δ_i such that $\mathrm{ht}_{\omega_4}(\delta_i) > \mathrm{ht}_{\omega_3}(\vartheta_i)$ and player II, using σ_ν, plays, say, (μ_i, ϖ_i). Of course, we have $\delta_i \leq \mu_i \leq \varpi_i$. Therefore, the height in ω_2 of any point of the interval $[\beta_i, \varepsilon_i)$ is smaller than the height in ω_3 of any point of the interval $[\gamma_i, \vartheta_i)$ which in turn is smaller than the height in ω_4 of any point of the interval $[\delta_i, \varpi_i)$.

Case 2. $r(i-1) = 1$. First fix an ordinal $\xi \in \omega_4 \cap M$ such that $\mathrm{ht}_{\omega_4}(\xi) > \mathrm{ht}_{\omega_2}(\varepsilon_i)$. Now we go back to the ω_3-board. For each move, say, $\gamma < \omega_3$ of player I, suppose the strategy σ_ν responds by $(\lambda(\gamma), \vartheta(\gamma))$. Now, suppose that if player I then plays ξ the strategy σ_ν responds by $(\mu(\gamma), \varpi(\gamma))$. Let $\varpi = \sup\{\varpi(\gamma) : \gamma < \omega_3\}$. Player I then plays some $\gamma_i \in \omega_3 \cap M$ such that $\mathrm{ht}_{\omega_3}(\gamma_i) > \mathrm{ht}_{\omega_4}(\varpi)$. Player II then responds according to σ_ν and player I then plays ξ and player II responds by using σ_ν and plays, say, (μ_i, ϖ_i). In this way we ensure that $\mathrm{ht}_{\omega_3}(\gamma_i) > \mathrm{ht}_{\omega_4}(\varpi_i)$. Therefore, the height in ω_2 of any point of the interval $[\beta_i, \varepsilon_i)$ is smaller than the height in ω_4 of any point of the interval $[\delta_i, \varpi_i)$ which is smaller than the height in ω_3 of any point of the interval $[\gamma_i, \vartheta_i)$.

We play this game for $n+1$ rounds and code in this way the first n digits of r. Finally, let $X = \mathrm{cl}_F(\{\kappa_i, \lambda_i, \mu_i : i \leq n+1\} \cup \nu)$. Since σ_ν is a winning strategy for player II in $\mathcal{G}^F_\alpha$, by our construction it follows that $s_{\omega_2\omega_3\omega_4}(X, M \cap \omega_4) = r \restriction n$. Thus, $X \in \Sigma^r(M)$. Since F is an arbitrary algebra on ω_4 and X is closed under F it follows that $\Sigma^r(M)$ is M-stationary. This completes the proof of Lemma 5 and Lemma 4. □

5 A minimal model of BPFA

Let ϕ be a sentence in the language of set theory. Let us say that a model M of $\mathsf{ZFC}+\phi$ is *minimal for* ϕ iff for any inner model $N \subseteq M$ with the same ordinals, if N is also a model of $\mathsf{ZFC}+\phi$ then $N = M$.

Recall ([8]) that an uncountable regular cardinal κ is Σ_1-*reflecting* (or H_κ-reflecting) iff for every $a \in H_\kappa$ and every formula $\varphi(x)$, if there is a regular cardinal θ such that $H_\theta \models \varphi(a)$ then there is a $\theta' < \kappa$ such that $a \in H_{\theta'}$ and $H_{\theta'} \models \varphi(a)$.

In [8, Theorem 2.11], Goldstern and Shelah show that if κ is Σ_1-reflecting, then there is a countable support iteration of length κ of proper forcing notions of size smaller than κ that forces BPFA. For future reference, we sketch the construction. We define a countable support iteration of proper forcing notions of size less than κ, $\langle \mathcal{P}_\alpha; \dot{\mathcal{Q}}_\alpha : \alpha < \kappa\rangle$, and let $\mathcal{P}_\kappa$ be the limit. At stage α we have some Σ_1 formula and a parameter which is a subset of ω_1, and ask if there is a proper poset which adds a solution to this formula. If the answer is yes then, since κ is Σ_1-reflecting, there

is such a poset of size less than κ and we choose one as $\mathcal{Q}_\alpha$. We do this using some standard bookkeeping device so every subset of ω_1 appearing along the iteration and every Σ_1 formula are considered at cofinally many stages. We will refer to this $\mathcal{P}_\kappa$ as *the standard iteration for* BPFA. Conversely, by [8, Theorem 4.1], if BPFA holds then $\aleph_2$ is Σ_1-reflecting in L.

Let κ be the smallest Σ_1-reflecting cardinal in L and let G be generic for the standard iteration $\mathcal{P}_\kappa$ for BPFA. We argue in this section that there is an inner model M of $L[G]$ which is a minimal model for BPFA. This answers a question of Sy Friedman (private communication).

Theorem 3 *Suppose that κ is a Σ_1-reflecting cardinal in V and that $\mathcal{P}_\kappa$ is the standard iteration for* BPFA. *Let $M = V(\mathcal{P}(\omega_1)^{V^{\mathcal{P}_\kappa}})$. Then $M \models$ BPFA.*

Proof: Recall that if N is an inner model of the universe V and X is a set, then $N(X) = \bigcup_\alpha L(N_\alpha \cup \mathtt{tc}(\{X\}))$, where $N_\alpha = N \cap V_\alpha$ and the union is over all the ordinals. This is a model of ZF. In particular, M is a model of ZF. It is in fact a model of Choice, by the argument of Theorem 2. Let $\mathcal{Q}$ be the complete subalgebra of $\mathtt{ro}(\mathcal{P}_\kappa)$ such that $V^{\mathcal{Q}} = M$. Then $\mathcal{Q}$ is proper. Let $a \subseteq \omega_1$ (so $a \in M$) and let φ be a Σ_1 formula, say φ is $\exists x \psi(x, a)$, where ψ is Σ_0. Since $\mathcal{P}_\kappa$ is κ-cc, there is an $\alpha < \kappa$ such that $a \in V^{\mathcal{P}_\alpha}$ and a has size $\aleph_1$ there. The formula φ and the parameter a are considered at some stage $\beta > \alpha$ of the iteration. Notice that $V^{\mathcal{P}_\beta} \subseteq V^{\mathcal{Q}}$. Since the tail of the iteration $\mathcal{P}_\kappa/\mathcal{P}_\beta$ is proper in $V^{\mathcal{P}_\beta}$ and $\mathcal{Q}/\mathcal{P}_\beta$ is a factor of it, $\mathcal{Q}/\mathcal{P}_\beta$ is proper in $V^{\mathcal{P}_\beta}$, as well. It follows that if there is a proper forcing $\mathcal{R} \in M$ adding to M a witness to $\varphi(a)$ then, since $\mathcal{Q}/\mathcal{P}_\beta * \dot{\mathcal{R}}$ is proper, at stage β there was a proper forcing adding a witness to $\varphi(a)$. Therefore, $\mathcal{Q}_\alpha$ was chosen to add such a witness. It follows that M satisfies BPFA. □

Corollary 2 *In L, let κ be the smallest Σ_1-reflecting cardinal and let $\mathcal{P}_\kappa$ be the standard iteration for* BPFA *of length κ. Let $M = L(\mathcal{P}(\omega_1))^{L^{\mathcal{P}_\kappa}}$. Then M is a minimal model of* BPFA.

Proof: By Theorem 3, M is a model of ZFC + BPFA. Assume that N is an inner model of M and that $N \models$ BPFA. By minimality of κ, $\omega_2^N = \kappa = \omega_2^M$, so Theorem 1 applies, and $\mathcal{P}(\omega_1)^M \subset N$, so $N = M$. □

Remark 6 Let κ be Σ_1-reflecting over V and suppose that $\mathcal{P}_\kappa$ is given by the standard iteration for BPFA. Let G be $\mathcal{P}_\kappa$-generic over V. It would be interesting to know whether $V(\mathcal{P}(\omega_1)^{V[G]})$ is actually different from $V[G]$. Notice that the forcing extending $V(\mathcal{P}(\omega_1)^{V[G]})$ to $V[G]$ is ω-distributive and ω_2-cc. If the ground model is L, it is even ω_1-distributive. The problem is that, even though the initial parts $G_\alpha = G \cap \mathcal{P}_\alpha$ of the generic G belong to $V(\mathcal{P}(\omega_1))$, they may not be the only $\mathcal{P}_\alpha$-generic filters in $V(\mathcal{P}(\omega_1))$. The question is whether we can glue them together to get the whole generic G, i.e. we have to choose generic filters for initial segments of the iteration which cohere.

Working over L, or any other definable generically invariant class with definable well-orderings, like, for example, the Dodd-Jensen core model K_{DJ}, one can modify slightly the forcing $\mathcal{P}_\kappa$ so that it has in addition a certain amount of rigidity: if in V there are two different $\mathcal{P}_\kappa$-generic filters over L then ω_1^L is collapsed. We use the following fact that we proved in section 4. Given a C-sequence and any real r, there is a proper forcing $\mathcal{P}_r$ of size $\aleph_4^{\aleph_0}$ collapsing $\aleph_4$ to $\aleph_1$ such that in $V^{\mathcal{P}_r}$ the triple $(\omega_2^V, \omega_3^V, \omega_4^V)$ codes r.

Let κ be a reflecting cardinal. We define $\langle \mathcal{P}_\alpha; \dot{\mathcal{Q}}_\alpha : \alpha < \kappa \rangle$ and let $\mathcal{P}_\kappa$ be their countably support limit. At stage α odd we proceed as before, i.e., we choose the forcing $\mathcal{Q}_\alpha$ of least $\mathcal{P}_\alpha$-name in the sense of the natural well ordering of L. If α is even, $\mathcal{Q}_\alpha$ does the following. Let λ be $|\mathcal{P}_\alpha|^L$, so $\lambda < \kappa$. We first collapse λ to ω_1 with countable conditions. Now, the whole generic up to this point, including the collapse, can be coded by a subset a_α of ω_1, and then using almost disjoint coding relative to a fixed definable sequence of almost disjoint reals in L, we code a_α by a real, say $r = r_\alpha$. We know the values of $\aleph_2$, $\aleph_3$ and $\aleph_4$ of this model (actually we know all the cardinals, but we only need these ones), since they are just the L-successors of λ. For simplicity let us call them $\beta < \gamma < \delta$. We then force with $\mathcal{P}_r$. It is easy to see that $\mathcal{P}_\kappa$ thus obtained forces BPFA.

Notice that at any even stage α, the real coded by (β, γ, δ) cannot change in any outer model with the same ω_1, as being a club in ω_1 is upwards absolute between models with the same ω_1, and this real is seen to be r in the extension by $\mathcal{P}_\kappa$. In fact, this coding will be done anyway at some triple (not necessarily (β, γ, δ)) when we force BPFA in the usual way, but the point is that we code the whole generic up to stage α by these three ordinals which moreover we can identify in the ground model. Now, in the end, the generic G will actually be definable without parameters, as it can be reconstructed inductively from its initial segments, which can be recovered using the reals r_α. Moreover, G will be unique in any outer model which has the same ω_1 as L. It follows that $V = L(\mathcal{P}(\omega_1))$ holds in $L[G]$.

The canonicity in the choice of $\mathcal{Q}_\alpha$ for odd stages α and a careful choice of bookkeeping gives us that if $\lambda < \kappa$ is a smaller reflecting cardinal in L, then the λ^{th} stage of the iteration $\mathcal{P}_\lambda$ also forces BPFA over L. In particular, $L[G]$ will not be a minimal model for BPFA unless κ is the L-least reflecting cardinal. This leads naturally to the question of whether if M is a minimal model for BPFA, then $\kappa = \omega_2^M$ is the least reflecting cardinal in L. The argument above does not answer this question, as there may be of course a poset $\mathcal{Q} \in L$ forcing BPFA or even $\mathsf{BPFA} + V = L(\mathcal{P}(\omega_1))$ but such that no $\mathcal{P}_\kappa$ as above is a factor of $\mathcal{Q}$.

6 The complexity of the logic with the Härtig quantifier

In this section we present as an application of the main result, an observation suggested by Jouko Väänänen on the complexity of the set of validities of the Härtig quantifier.

Definition 3 *The logic $\mathcal{L}(I)$ is obtained by augmenting first-order logic with the* Härtig

quantifier I*; this is a binary quantifier. Formulas in* $\mathcal{L}(I)$ *are defined recursively as in first-order logic, with the additional clause:*

- *If* $\varphi(x, \vec{z})$ *and* $\psi(y, \vec{z})$ *are formulas in* $\mathcal{L}(I)$*, then so is* $Ixy\,(\phi(x, \vec{z}), \psi(y, \vec{z}))$*, where* x *and* y *are considered bound variables.*

The semantics of $\mathcal{L}(I)$ *is defined recursively as in first-order logic, with the additional clause:*

- $\mathcal{M} = (M, \ldots) \models Ixy\,(\phi(x, \vec{a}), \psi(y, \vec{a}))$ *iff* $|\{\, b \in M : \mathcal{M} \models \phi(b, \vec{a})\,\}| = |\{\, c \in M : \mathcal{M} \models \psi(c, \vec{a})\,\}|$.

In a straightforward fashion, we can assign Gödel numbers to the formulas in $\mathcal{L}(I)$. Let V_I be the set of *validities* of the logic $\mathcal{L}(I)$, i.e., the set of (Gödel numbers for) sentences in $\mathcal{L}(I)$ valid in all models. We refer the reader to [9] for an introduction to the Härtig quantifier and for the results we mention in the argument below. This proof is closely modeled on the argument given in [9] that V_I is neither Σ^1_2 nor Π^1_2.

Corollary 3 *Suppose that* BPFA *holds. Then* V_I *is not lightface projective.*

Proof: We show that in fact V_I is not first-order definable (without parameters) over $N = L_{\aleph_2^V}(\mathcal{P}(\omega_1))$ so, in particular, it is not lightface projective.

Let n be a sufficiently large integer and denote by ZFC_n the finitely axiomatizable theory that results from restricting the replacement schema of ZFC to Σ_n-formulas. The integer n is chosen so Theorem 1 is provable in ZFC_n and also the argument below goes through.

Let φ be a first order sentence without parameters in the language of set theory. We find an $\mathcal{L}(I)$-formula ψ_φ such that $N \models \varphi$ iff $\psi_\varphi \in V_I$. Since φ is arbitrary, it follows that V_I cannot be first order definable over N.

Our sentence ψ_φ has the form $\phi \to \theta_\varphi$, where θ_φ is a first order sentence (in the language of set theory) expressing the statement $N \models \varphi$. ϕ will be such that its models are precisely the well-founded models of $\mathsf{ZFC}_n + \mathsf{BPFA}$ whose transitive collapse correctly computes $\aleph_2$. Suppose M is such a model. Then, by Theorem 1, it correctly computes N, and the equivalence follows.

All that remains is to exhibit ϕ. In [9] it is explained how to produce, for any finite extension T of a sufficiently strong fragment of ZFC, an $\mathcal{L}(I)$-sentence ϕ_T such that a structure (M, E) is a *well-founded* model of T iff it admits an expansion to a model of ϕ_T. Let $\phi_{\mathbb{N}}(x)$, $\phi_{\omega_1}(x)$ and $\phi_{\omega_2}(x)$ be the first order formulas stating, respectively, that x is a natural number, that x is a countable ordinal, and that x is an ordinal below ω_2. Our ϕ is the conjunction of $\phi_{\mathsf{ZFC}_n + \mathsf{BPFA}}$ and $\neg Ixy(\phi_{\mathbb{N}}(x), \phi_{\omega_1}(y)) \wedge \neg Ixy(\phi_{\omega_1}(x), \phi_{\omega_2}(y))$, and it is easy to see that ϕ is as wanted. This completes the proof. Notice that the map $\varphi \mapsto \psi_\varphi$ is recursive. $\square$

Remark 7 It is easy to see by condensation and a similar argument, that $V = L$ implies that V_I is not Σ^m_n for any n and any $m < \omega$ (or even Σ^α_n for any n and any $\alpha < \omega_1^{CK}$). This has also been observed by Väänänen (private communication).

Assume now that PFA holds; what follows incorporates a suggestion of Woodin that strengthens the result we originally obtained and simplifies our original argument. By the covering lemma and Todorčević's result [19] that PFA implies $\neg\square_\kappa$ for all $\kappa \geq \omega_1$, it follows that $X^\sharp$ exists for all sets X. In particular, $\mathbb{R}^\sharp$ exists. By Theorem 1, if N is an inner model of BPFA (N may be a set) that correctly computes ω_2, then $H_{\aleph_2} \subseteq N$. If N is also a model of PFA then there is an $X \in N$ such that $N \models X = \mathbb{R}^\sharp$, but then X really is $\mathbb{R}^\sharp$ (see [16] for the definition and basic properties of $\mathbb{R}^\sharp$, from which this follows). Since $\mathbb{R}^\sharp$ codes the theory of $L(\mathbb{R})$, this easily implies as above that the theory of $L(\mathbb{R})$ without parameters is recursive in V_I and therefore V_I is not definable in $L(\mathbb{R})$ without parameters.

But something stronger in fact holds. By Steel [18], PFA implies that $\mathsf{AD}^{L(\mathbb{R})}$ holds. Solovay [16] has shown that $\mathsf{AD}^{L(\mathbb{R})}$ and the existence of $\mathbb{R}^\sharp$ imply that there is a real which is ordinal definable in $L(\mathbb{R}^\sharp)$ but not ordinal definable in $L(\mathbb{R})$. Since the least such real in the natural well-ordering of $\mathrm{HOD}^{L(\mathbb{R}^\sharp)}$ is obviously recursive in V_I by the arguments above, it follows that V_I itself cannot be ordinal definable in $L(\mathbb{R})$.

Corollary 4 *Assume* PFA. *Then V_I is not ordinal definable in $L(\mathbb{R})$.* □

References

[1] J. Bagaria. *Bounded forcing axioms as principles of generic absoluteness*, Archive for Mathematical Logic **39 (6)** (2000), 393–401.

[2] J. Baumgartner. *Iterated forcing*, in **Surveys in Set Theory**, A.R.D. Mathias, ed., Cambridge University Press (1983), 1–59.

[3] J. Baumgartner. *Applications of the proper forcing axiom*, in **Handbook of set-theoretic topology** K. Kunen, J. Vaughan, eds., North-Holland (1984), 913–959.

[4] M. Bekkali. **Topics in set theory**, Springer (1991).

[5] A. Caicedo. *Projective well-orderings and bounded forcing axioms*, The Journal of Symbolic Logic **70 (2)** (2005), 557–572.

[6] A. Caicedo, S. Friedman. *Projective well-orderings of the reals: A survey*, in preparation.

[7] M. Foreman, M. Magidor, S. Shelah. *Martin's maximum, saturated ideals, and nonregular ultrafilters. I*, Annals of Mathematics **127 (1)** (1988), 1–47.

[8] M. Goldstern, S. Shelah. *The Bounded Proper Forcing Axiom*, The Journal of Symbolic Logic **60 (1)** (1995), 58–73.

[9] H. Herre, M. Krynicky, A. Pinus, J. Väänänen. *The Härtig quantifier: A survey*, The Journal of Symbolic Logic **56 (4)** (1991), 1153–1183.

[10] T. Jech. **Set theory, the third millenium edition**, Springer (2003).

[11] B. König, Y. Yoshinobu. *Fragments of Martin's Maximum in generic extensions*, Mathematical Logic Quarterly **50 (3)** (2004), 297–302.

[12] P. Larson. *The nonstationary ideal in $\mathcal{P}_{\max}$ extensions*, in preparation.

[13] D. Martin, R. Solovay. *Internal Cohen extensions*, Annals of Mathematical Logic **2 (2)** (1970), 143–178.

[14] J. Moore. *Set mapping reflection*, Journal of Mathematical Logic **5 (1)** (2005), 87–98.

[15] S. Shelah. **Proper and improper forcing**, Springer (1998).

[16] R. Solovay. *The independence of* DC *from* AD, in **Cabal Seminar 76–77 (Proceedings of the UCLA-Caltech logic seminar, 1976–77)**, A. Kechris, Y. Moschovakis, eds., Springer (1978), 171–183.

[17] R. Solovay, S. Tennenbaum. *Iterated Cohen extensions and Souslin's problem*, Annals of Mathematics **94 (2)** (1971), 201–245.

[18] J. Steel. PFA *implies* $\mathsf{AD}^{L(\mathbb{R})}$, preprint.

[19] S. Todorčević. *A note on the proper forcing axiom*, in **Axiomatic set theory (Boulder, Colorado, 1983)**, J. Baumgartner, D. Martin, eds., American Mathematical Society (1984), 209-218.

[20] S. Todorčević. *Generic absoluteness and the continuum*, Mathematical Research Letters **9** (2002), 1–7.

[21] B. Veličković. *Forcing axioms and stationary sets*, Advances in Mathematics **94 (2)** (1992), 256–284.

[22] H. Woodin. **The Axiom of Determinacy, forcing axioms, and the nonstationary ideal**, Walter de Gruyter (1999).