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The Boussinesq equation revisited

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Abstract

The continuous spectrum and soliton solutions for the Boussinesq equation are investigated using the $\bar{\partial}$ -dressing method. Solitons demonstrate quite extraordinary behavior; they may decay or form a singularity in a finite time. Formation of singularity (collapse of solitons) for the Boussinesq equation was discovered several years ago. Systematic study of the solitonic sector is presented. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Our renewed interest in the Boussinesq equation is explained mostly by the unusual behavior of the soliton solutions of this equation. A common consideration regarding solitons in integrable systems is that they are stable objects interacting trivially, changing only phase as a result of interaction. However, the behavior of solitons of the Boussinesq equation destroys this stereotype. Solitons of the Boussinesq equation may decay under the action of perturbation or form a singularity in a finite time. One would probably think that the Boussinesq equation is itself rather unusual. Not at all, it is a typical example of dimensional reduction in the framework of the KP hierarchy (the KdV equation being the simplest), and it is also a physically relevant equation, representing a nonlinear integrable generalization of the wave equation [1]. Formation of singularity (collapse) for Boussinesq equation solitons was first observed several years ago [2] (see also [3]). In this work we perform a systematic study of the solitonic sector of the Boussinesq equation and also sum up the results concerning the continuous spectrum obtained in the framework of the $\bar{\partial}$ -dressing method [4–6].

The plan of the paper is the following. First, we sum up basic facts concerning the Boussinesq equation.

Then, we briefly review the technique of the $\bar{\partial}$ -dressing method [7–10], restricting ourselves to the scalar case as the simplest. We would like to emphasize that most of the contents of this part is not original and is in the main described in the papers mentioned above. We concentrate on the generally less known features of the method, namely on the technique of dimensional reduction and on the characterization of the continuous spectrum [4–6]. We will

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discuss different types of problems in the complex plane that arise in this context. We also derive the determinant formula for soliton solutions.

Using the developed technique, we investigate the continuous spectrum for all four versions of the Boussinesq equation and obtain Carleman type problems in the complex plane and integral equations describing them.

Finally, we study the behavior of solutions defined by the determinant formula, which gives a solitonic sector for the Boussinesq equation. To illustrate behavior of soliton solutions, we will use the pictures obtained from analytical formulae by Mathematica.

2. Boussinesq equation

The Boussinesq equation describes propagation of waves in weakly nonlinear and weakly dispersive media [1]. To derive the Boussinesq equation for some physical model, one should start from a Lagrangian

$$L = \int dx \left(\frac{3}{4} \alpha^2 (u_t)^2 - \beta (u_x)^2 + \frac{1}{4} (u_{xx})^2 + 3 (u_x)^3 \right), \quad (1)$$

where $\alpha^2, \beta \in \mathbb{R}$. The equation of motion corresponding to Lagrangian (1) is the Boussinesq equation for the function $v = u_x$

$$\left(\frac{3}{4} \alpha^2 v_{tt} - \beta v_{xx} \right) = - \left(\frac{1}{4} v_{xx} + \frac{3}{2} v^2 \right)_{xx}. \quad (2)$$

This equation describes waves moving in both directions. One-wave approximation reduces the Boussinesq equation to the Korteweg–de Vries equation.

In fact there are four different cases of Boussinesq equation (2). The coefficients can be rescaled to get $\beta = \pm 1$, $\alpha^2 = \pm 1$, therefore the only choice is the choice of the two signs. The properties of the Boussinesq equation depend essentially on this choice. The primary choice for us will be the choice of the sign of β . According to this choice, we will distinguish between the ‘plus’ Boussinesq equation and the ‘minus’ Boussinesq equation. The ‘plus’ Boussinesq equation reads

$$\pm \frac{3}{4} v_{tt} - v_{xx} + \frac{1}{4} v_{xxx} + \left(\frac{3}{2} v^2 \right)_{xx} = 0. \quad (3)$$

In a linear approximation the monochromatic solution of this equation is

$$v \simeq e^{i(\omega y + kx)}, \quad \omega^2 = \pm \frac{4}{3} \left(k^2 + \frac{1}{4} k^4 \right),$$

where the sign (plus or minus) corresponds to the sign before the first term in (3). In the case of sign plus it is a nonlinear wave equation, in the minus case it is a nonlinear elliptic equation.

The ‘minus’ Boussinesq equation is

$$\pm \frac{3}{4} v_{tt} + v_{xx} + \frac{1}{4} v_{xxx} + \left(\frac{3}{2} v^2 \right)_{xx} = 0. \quad (4)$$

Dispersion law for this equation is given by the expression

$$\omega^2 = \pm \frac{4}{3} \left(-k^2 + \frac{1}{4} k^4 \right).$$

For the sign plus this dispersion law is unstable for short waves and stable for long waves, for the sign minus it is stable for short waves and unstable for long waves.

So, four cases of the Boussinesq equation can be characterized in the following way: wave case, elliptic case, Boussinesq equation with long-wave instability and Boussinesq equation with short-wave instability.

The Boussinesq equation is integrable by the inverse problem method (see, e.g. [11]), the Lax pair for this equation was constructed in [12]. Our interest in this equation is explained by nontrivial properties of both the continuous and discrete spectrum for this equation.

Technically, the Boussinesq equation is a result of dimensional reduction of the KP equation taken in the moving frame. The initial KP equation in the moving frame reads

$$\frac{\partial}{\partial x} \left((v_t - \beta v_x) + \frac{1}{4} v_{xxx} + 3v_x v \right) = -\frac{3}{4} \alpha^2 v_{yy}, \quad (5)$$

where the constant α defines the choice between KP1 ($\alpha = i$) and KP2 ($\alpha = 1$) equations, and the constant β is the velocity of the frame (we take $\beta = \pm 1$). Considering stationary solutions of $(2 + 1)$ -dimensional equation (5), we get $(1 + 1)$ -dimensional Boussinesq equation (2),

$$\left(\frac{3}{4} \alpha^2 v_{yy} - \beta v_{xx} \right) = -\left(\frac{1}{4} v_{xx} + \frac{3}{2} v^2 \right)_{xx}, \quad (6)$$

where the role of the time variable t is played by KP variable y .

To investigate the continuous spectrum, we use the $\bar{\partial}$ -dressing method [7–10], in which very effective apparatus to describe dimensional reductions and continuous spectrum was developed [4–6]. We get information about the structure of the continuous spectrum and the problems in the complex plane corresponding to all four versions of the Boussinesq equation. Geometry of the spectrum is rather interesting, the spectral data are localized on the hyperbola in the complex plane and on the segment of the real axis and small decreasing solutions are given by the Riemann problem with a shift on this curve (see another approach in [13,14]).

Behavior of solutions of the Boussinesq equation belonging to the solitonic sector is also rather unusual. The formula for the multisoliton solution of the Boussinesq equation can be obtained from the formula for the plain solitons of the KP equation [11]

$$v = \frac{\partial^2}{\partial x^2} \log \det(A), \quad A_{ij} = \delta_{ij} - \frac{R_i}{\mu_i - \lambda_j}, \quad (7)$$

here

$$R_k = ic_k \exp(i(\mu_k - \lambda_k)(x - i\alpha^{-1}(\mu_k + \lambda_k)y)), \quad (\lambda_k^3 \pm \lambda_k - (\mu_k^3 \pm \mu_k) = 0, \quad (8)$$

$\lambda_k \neq \mu_j$, where λ_k, μ_k are two arbitrary sets of points of the complex plane satisfying the condition (8), which characterizes stationary KP solutions in the moving reference frame (c_k, λ_k, μ_k should also satisfy some reduction conditions to get a real solution). Formula (7) can be obtained in many different ways, in our work we will derive it using the $\bar{\partial}$ -dressing method.

We will treat mostly the case of the ‘plus’ Boussinesq equation with $\alpha = 1$. This equation has a stable ‘wave sector’ (i.e., in the linear limit it is a wave equation). There are two soliton sectors for this equation: ‘usual’ solitons, running with the velocity limited from above, and soliton configurations, forming a singularity in a finite time. The latter may be considered as bounded states of several singular solitons.

But even ‘usual’ solitons demonstrate quite extraordinary behavior in this case. Slow solitons are unstable with respect to small perturbations and may decay into two solitons or two singular solitons (that means formation of a singularity in finite time). Interaction of slow solitons unavoidably leads to formation of the singularity. Rapid solitons moving in the same direction behave as it is usually expected from the system of solitons; they do not decay and their interaction does not lead to formation of singularities.

In this work we present a systematic study of the solitonic sector of the Boussinesq equation.

3. $\bar{\partial}$ -dressing: the basic technique

The main technical tool of our work is the dressing method based on the nonlocal $\bar{\partial}$ -problem [7–10]. This is a powerful method of constructing $(2 + 1)$ -dimensional integrable equations together with a broad class of their solutions.

The Boussinesq equation may be considered as a dimensional reduction of the KP equation in the moving frame. To apply $\bar{\partial}$ -dressing method to the Boussinesq equation, we will use the scheme of dimensional reduction for the $\bar{\partial}$ -dressing method developed in [5]. It leads us to the problem with a special kind of nonlocality—the $\bar{\partial}$ -problem with a shift and to the Riemann problem with a shift. It appears that these *scalar nonlocal* problems are a general and natural technical tool in the $(1 + 1)$ -dimensional case.

The construction developed in [4] gives a simple and straightforward description of solutions belonging to the continuous spectrum (i.e., small decreasing solutions) in the framework of the $\bar{\partial}$ -dressing method. Continuous spectrum is characterized in terms of conditions which single out some special classes of the kernels of the general nonlocal $\bar{\partial}$ -problem.

Taking into account conditions of dimensional reduction, for small decreasing solutions of $(1 + 1)$ -dimensional equations we arrive at Carleman type problems in the complex plane.

The scheme of the dressing method uses the nonlocal $\bar{\partial}$ -problem with a special dependence of the kernel on additional (space and time) variables

$$\bar{\partial}(\psi(\mathbf{x}, \lambda) - \eta(\mathbf{x}, \lambda)) = \hat{R}\psi(\mathbf{x}, \lambda), \quad (9)$$

$$\hat{R}\psi(\mathbf{x}, \lambda) = \iint \psi(\mathbf{x}, \mu) R(\lambda, \mu) \exp\left(\sum_i \phi_i x_i\right) d\mu \wedge d\bar{\mu}, \quad \phi_i = K_i(\mu) - K_i(\lambda), \quad (10)$$

where $\lambda \in \mathbb{C}$, $\bar{\partial} = \partial/\partial\bar{\lambda}$, $\eta(\mathbf{x}, \lambda)$ is a rational function of λ (normalization), $K_i(\lambda)$ are rational functions, the choice of which determines the equations that can be solved using the problem (9). We suppose that the kernel $R(\lambda, \mu)$ is equal to zero in a neighborhood with respect to λ and to μ of the divisor of poles of functions $K_i(\lambda)$, tends to zero as $\lambda, \mu \rightarrow \infty$ and that for the chosen kernel $R(\lambda, \mu)$ problem (9) is uniquely solvable (at least for sufficiently small \mathbf{x}). The solution of problem (9) normalized by η is the function

$$\psi(\mathbf{x}, \lambda) = \eta(\mathbf{x}, \lambda) + \varphi(\mathbf{x}, \lambda),$$

where $\eta(\mathbf{x}, \lambda)$ is a rational function of λ (normalization), $\varphi(\mathbf{x}, \lambda)$ decreases as $\lambda \rightarrow \infty$ and is *analytic* in a neighborhood of poles of $K_i(\lambda)$.

Problem (9) reduces to an integral equation for the function φ

$$\varphi(\mathbf{x}, \lambda) = \bar{\partial}^{-1} \hat{R}(\varphi(\mathbf{x}, \lambda) + \eta(\mathbf{x}, \lambda)), \quad (11)$$

here

$$(\bar{\partial}^{-1}\varphi)(\lambda) = (2\pi i)^{-1} \iint \frac{\varphi(\lambda')}{(\lambda' - \lambda)} d\lambda' \wedge d\bar{\lambda}' = (2\pi i)^{-1} \lim_{\epsilon \rightarrow 0} \iint \frac{\varphi(\lambda')(\bar{\lambda}' - \bar{\lambda})}{(|\lambda' - \lambda|^2 + \epsilon)} d\lambda' \wedge d\bar{\lambda}',$$

which is supposed to be uniquely solvable for given R . Solvability is guaranteed if operator $\bar{\partial}^{-1}R$ is ‘small enough’ (i.e., the norm of this operator is less than 1 for some properly chosen space of functions).

Let us introduce $\rho(\lambda, \bar{\lambda}) = \bar{\partial}\varphi$. Now

$$\psi(\mathbf{x}, \lambda) = \eta + (2\pi i)^{-1} \iint \frac{\rho(\lambda')}{(\lambda' - \lambda)} d\lambda' \wedge d\bar{\lambda}'. \quad (12)$$

Substituting (12) into (11), we can get another form of the basic integral equation, resolving the nonlocal $\bar{\partial}$ -problem

$$\rho(\mathbf{x}, \lambda) = \hat{R}(\eta + \bar{\partial}^{-1}\rho). \quad (13)$$

The nonlocal $\bar{\partial}$ -problem and its special cases ($\bar{\partial}$ -problem with a shift, nonlocal Riemann problem, Riemann problem with a shift) are powerful tools for constructing integrable nonlinear equations and their solutions (see [7–10]).

The algebraic scheme of constructing equations is based on the following property of problem (9): if $\psi(\mathbf{x}, \lambda)$ is a solution of the problem (9), then the functions

$$u(\mathbf{x})\psi, \quad D_i\psi = \left(\frac{\partial}{\partial x_i} + K_i \right) \psi \quad (14)$$

are also solutions. Combining this property with unique solvability of problem (9), one obtains differential relations between the coefficients of expansion of functions $\psi(\mathbf{x}, \lambda)$ into powers of $(\lambda - \lambda_p)$ at the poles of $K_i(\lambda)$. Let us outline the basic steps of this scheme for the KP equation that will be used in this work.

For the KP equation

$$K_1(\lambda) = i\lambda, \quad K_2(\lambda) = \alpha^{-1}\lambda^2, \quad K_3(\lambda) = i\lambda^3,$$

respectively,

$$D_1 = \frac{\partial}{\partial x} + i\lambda, \quad D_2 = \frac{\partial}{\partial y} + \alpha^{-1}\lambda^2 \quad (\alpha = 1; i), \quad D_3 = \frac{\partial}{\partial t} + i\lambda^3.$$

Let us introduce the solution of problem (9) normalized by 1 ($\eta = 1$),

$$\psi(\lambda, x, y, t)_{\lambda \rightarrow \infty} \rightarrow 1 + \psi_0(x, y, t)\lambda^{-1} + \dots$$

The basis in the space of solutions of problem (9) with polynomial normalization is constituted by the set of functions $D_1^n \psi$, $0 \leq n < \infty$. It follows from unique solvability of problem (9) that ψ satisfies the relations

$$(D_3 + D_1^3 + g(x, y, t)D_1 + h(x, y, t))\psi = 0, \quad (15)$$

$$(\alpha D_2 + D_1^2 + 2v(x, y, t))\psi = 0. \quad (16)$$

The successive use of coefficients of expansion of these relations as $\lambda \rightarrow \infty$ allows us to define the functions v, g, h

$$v = -i \frac{\partial}{\partial x} \psi_0, \quad g = 3v, \quad h_x = \frac{3}{2}(v_{xx} - \alpha v_y),$$

and to derive KP equation for the first coefficient of expansion of the function ψ as $\lambda \rightarrow \infty$:

$$\frac{\partial}{\partial x} \left(v_t + \frac{1}{4}v_{xxx} + 3v_x v \right) = -\frac{3}{4}\alpha^2 v_{yy}. \quad (17)$$

3.1. Special cases of the nonlocal $\bar{\partial}$ -problem

In the most important cases the kernel $R(\lambda, \mu)$ is a singular function localized on some manifold in \mathbb{C}^2 . That means that the kernel contains the δ -function localized on the corresponding manifold, or in other words that the measure of integration in the operator $\bar{\partial}^{-1}\hat{R}$ is localized on this manifold. The operator $\bar{\partial}^{-1}\hat{R}$ in this case is still well defined.

3.1.1. $\bar{\partial}$ -problem with a shift

In a typical situation this manifold is a covering of the complex λ -plane, defined by the equation

$$f(\lambda, \bar{\lambda}, \mu, \bar{\mu}) = 0, \quad (18)$$

where f is some function in \mathbb{C}^2 . Eq. (18) defines a multi-valued shift function $\mu = \mu_i(\lambda, \bar{\lambda})$. The kernel of problem (9) in this case reads

$$R = \sum_i R_i(\lambda, \bar{\lambda}) \delta(\mu - \mu_i(\lambda, \bar{\lambda})).$$

We will call this case a $\bar{\partial}$ -problem with a shift.

3.1.2. Nonlocal Riemann problem

Another special case of problem (9) is a nonlocal Riemann problem. Let $\gamma = \lambda(\xi)$, $\xi \in \mathbb{R}$ be an oriented curve in the complex plane (may be not connected), and the kernel of problem (9) be localized on the product of couple of these curves in λ and in μ planes. In other words,

$$R(\lambda, \mu) = \delta_\gamma(\lambda) R_\gamma(\lambda, \mu) \delta_\gamma(\mu), \quad (19)$$

where $\delta_\gamma(\lambda)$ is a δ -function picking out points on γ . The solution ψ of problem (9) with the kernel (19) is rational outside γ and has boundary values ψ^+ , ψ^- on γ . After regularizing δ_γ we obtain from problem (9) with the kernel (19) a nonlocal Riemann problem

$$\psi^+ - \psi^- = \frac{1}{2} \int_\gamma (\psi^+ + \psi^-) R_\gamma(\lambda, \mu) d\mu, \quad (20)$$

the integration in (20) goes along the curve γ .

3.1.3. Riemann problem with a shift

A combination of these two special cases leads to the Riemann problem with a shift (or Carleman's problem). The shift function $\mu = \mu_i(\lambda)$ is defined now on the curve γ ($\lambda, \mu \in \gamma$). In this case

$$R_\gamma(\lambda, \mu) = \sum_i R_\gamma^i(\lambda) \delta_\gamma(\mu - \mu_i(\lambda)),$$

and

$$\psi^+ - \psi^- = \frac{1}{2} \sum_i (\psi^+(\mu_i(\lambda)) + \psi^-(\mu_i(\lambda))) R_\gamma^i(\lambda), \quad (21)$$

where $\mu_i(\lambda)$ is a multi-valued shift function on the curve $\lambda(\xi)$. We will write problem (21) symbolically in the form

$$\Delta(\psi(\lambda(\xi))) = R_\gamma(\lambda, \mu(\lambda)) \psi(\mu(\lambda(\xi))), \quad (22)$$

where $\gamma = \lambda(\xi)$ ($\xi \in \mathbb{R}$) is a curve in the complex plane, Δ a jump of the function across the curve, the value of the function on the curve is the half-sum of the boundary values, $\mu(\lambda)$ the shift function (may be multi-valued).

3.1.4. Integral equations

In all these three cases problem (9) is equivalent to a certain integral equation which can be obtained by a proper reduction of Eqs. (11) and (13). Let us do that for a Riemann problem with a shift. Introducing

$$\rho_\gamma(\lambda) = \psi^+ - \psi^-|_{\lambda \in \gamma},$$

we can restore the function ψ in a form

$$\psi = \eta + \frac{1}{2\pi i} \int_{\gamma} \frac{\rho_{\gamma}(\lambda')}{(\lambda - \lambda')} d\lambda'.$$

Hence

$$\frac{1}{2}(\psi^+ + \psi^-)|_{\lambda \in \gamma} = \eta(\lambda) + \frac{1}{2\pi i} v.p. \int \frac{\rho_{\gamma}(\lambda')}{(\lambda - \lambda')} d\lambda',$$

and from Eq. (21) one gets

$$\rho_{\gamma}(\lambda) = \sum_i \left(\eta(\mu_i(\lambda)) + \frac{1}{2\pi i} v.p. \int \frac{\rho_{\gamma}(\lambda')}{(\mu_i(\lambda) - \lambda')} d\lambda' \right) R_{\gamma}^i(\lambda), \quad \lambda \in \gamma. \quad (23)$$

Let the curve γ consist of n connected branches $\gamma_i = \lambda_i(\xi)$, $\xi \in \mathbb{R}$, and $\rho_i(\xi)$ be the jump of the function ψ across the corresponding branch. Then the expression for the function ψ takes the form

$$\psi = \eta + \frac{1}{2\pi i} \sum_{i=1}^n \int \frac{\rho_i(\xi')}{(\lambda - \lambda_i(\xi'))} \frac{d\lambda_i}{d\xi'} d\xi', \quad (24)$$

and integral equation (23) reads

$$\rho_k(\xi) = \sum_i \left(\eta(f_i(\xi)) + \frac{1}{2\pi i} \sum_{j=1}^n v.p. \int \frac{\rho_j(\xi')}{(\mu_i(\lambda_k(\xi)) - \lambda_j(\xi'))} \frac{d\lambda_j}{d\xi'} d\xi' \right) R^{ik}(\xi). \quad (25)$$

Thus we have obtained a system of n singular integral equations.

3.1.5. The δ -functional kernels

There is one important special case of the nonlocal $\bar{\partial}$ -problem which is exactly solvable, which corresponds to soliton solutions and discrete spectrum (in some broad sense). This is a case of δ -functional kernels

$$R(\lambda, \mu) = 2\pi i \sum_{i=1}^N R_i \delta(\lambda - \lambda_i) \delta(\mu - \mu_i), \quad (26)$$

where λ_i, μ_i is a set of points in the complex plane, $\lambda_i \neq \mu_j$,

$$R_i = c_i \exp \left(\sum_n (K_n(\lambda_i) - K_n(\mu_i)) x_n \right).$$

In this case the solution of problem (9) is a rational function, and problem (9) reduces to a system of linear equations. The formula for the solution normalized by $(\lambda - \mu)^{-1}$ is

$$\psi(\lambda, \mu) = \frac{1}{\lambda - \mu} + ((A)^{-1})_{ij} \frac{R_j}{(\mu_j - \mu)(\lambda - \lambda_i)}, \quad A_{ij} = \delta_{ij} - \frac{R_i}{\mu_i - \lambda_j}, \quad (27)$$

or, in a more symmetric form with respect to λ and μ

$$\psi(\lambda, \mu) = \frac{1}{\lambda - \mu} + ((A')^{-1})_{ij} \frac{1}{(\mu_j - \mu)(\lambda - \lambda_i)}, \quad A'_{ij} = R_i^{-1} \delta_{ij} - \frac{1}{\mu_i - \lambda_j}. \quad (28)$$

In the limit when a pair of poles λ_i, μ_j coincide, factors rational with respect to x_q appear in the formula for ψ . The limit $\lambda_i \rightarrow \mu_i$ for all $0 < i \leq N$ corresponds to a solution rational with respect to x_q .

An expression for solution with canonic normalization ($\eta = 1$) can be obtained from the formula (27),

$$\psi(\lambda, \mathbf{x}) = \lim_{\mu \rightarrow \infty} -\mu \psi(\lambda, \mu, \mathbf{x}) = 1 + \sum_j ((A')^{-1})_{ij}(\lambda - \lambda_j), \quad (29)$$

and potential

$$\psi_0(\mathbf{x}) = \lim_{\mu \rightarrow \infty} \lambda \psi(\lambda, \mathbf{x})$$

reads

$$\psi_0(\mathbf{x}) = \sum_j \sum_i ((A')^{-1})_{ij}. \quad (30)$$

Introducing variable x with $K(\lambda) = i\lambda$, it is easy to check that expression (30) can be rewritten in the form

$$\psi_0(\mathbf{x}) \sim -i \frac{\partial}{\partial x} \log \det(A) \sim -i \frac{\partial}{\partial x} \log \det(A') \quad (31)$$

(up to a constant). Indeed,

$$\begin{aligned} -i \frac{\partial}{\partial x} \log \det(A') &\sim -i \frac{\partial}{\partial x} \log \det \left(c_i^{-1} \delta_{ij} - \frac{\exp(i(\mu_i - \lambda_j)x)}{(\mu_i - \lambda_j)} \right) \\ &= \sum_i \sum_j \exp(i(\mu_i - \lambda_j)) \left(c_p^{-1} \delta_{pq} - \frac{\exp(i(\mu_p - \lambda_q)x)}{(\mu_p - \lambda_q)} \right)_{ji}^{-1} = \sum_j \sum_i ((A')^{-1})_{ij} \end{aligned} \quad (32)$$

(for simplicity, we have omitted other dynamical variables x_q in this calculation).

3.2. Solutions with special properties

3.2.1. Small decreasing solutions (continuous spectrum)

A solution given by the problem (9) in a general case is defined only locally in a vicinity of the point $\mathbf{x} = 0$, where the $\bar{\delta}$ -problem is uniquely solvable. Solvability may be lost on some manifold in a space (x_1, x_2, x_3) , where the solution has a singularity. To get ‘good enough’ solutions having no singularities and bounded (decreasing) as $|\mathbf{x}| \rightarrow \infty$ one should put some restrictions on the kernel $R(\lambda, \mu)$. These restrictions were discussed in our paper [4]. The main result of this paper can be formulated as follows. Let us choose a unit vector n_i ($\sum n_i^2 = 1$) defining a direction in the \mathbf{x} -space. The solution given by problem (9) is regular in a neighborhood of straight line $x_i = n_i \xi$ and decreasing along this line as $\xi \rightarrow \pm\infty$ if the condition

$$\operatorname{Re} \sum_{i=1}^3 n_i (K_i(\lambda) - K_i(\mu)) = 0 \quad (33)$$

is satisfied (this condition is in fact the condition for the kernel $R(\lambda, \mu)$, it means that we should use the kernel localized on the manifold (33)), and the kernel $R(\lambda, \mu)$ is ‘small enough’.

To get a solution which is ‘good enough’ in a neighborhood of some plane, defined by two vectors n_i, m_i , one has to satisfy two conditions

$$\operatorname{Re} \sum_{i=1}^3 n_i (K_i(\lambda) - K_i(\mu)) = 0, \quad \operatorname{Re} \sum_{i=1}^3 m_i (K_i(\lambda) - K_i(\mu)) = 0.$$

In a generic case a pair of conditions (33) define some manifold with real dimension 2 in the space \mathbb{C}^2 of complex variables λ, μ .

Let us illustrate this result using the simple example of the KP equation. To obtain a small nonsingular solution decreasing in the plane (x, y) it is sufficient to use the problem (9) with the kernel localized on the manifold defined by the system of conditions (33)

$$\operatorname{Im}(\lambda - \mu) = 0, \quad (34)$$

$$\operatorname{Re} \alpha^{-1}(\lambda^2 - \mu^2) = 0. \quad (35)$$

If $\alpha = i$, the solution of the system (34) and (35) is $\lambda, \mu \in \mathbb{R}$, which defines a nonlocal Riemann problem on the real axis. So small decreasing solutions of the KP1 equation are given by the nonlocal Riemann problem

$$\psi^+ - \psi^- = \int_{\gamma} (\psi^+ + \psi^-) R_{\gamma}(\lambda, \mu) \exp(\phi_i x_i) d\mu, \quad (36)$$

that was originally used by Manakov [15] to integrate KP1 equation.

If $\alpha = 1$, the solution of the system (35) is $\mu = -\bar{\lambda}$. Thus small decreasing solutions of KP2 equation are given by the $\bar{\partial}$ -problem with a conjugation

$$\bar{\partial} \psi(x, y, t, \lambda) = R(\lambda, -\bar{\lambda}) \exp(\phi_i x_i) \psi(x, y, t, -\bar{\lambda}), \quad (37)$$

and we reproduce the problem used by Ablowitz et al. [16] to integrate KP2 equation.

3.2.2. Dimensional reduction

Solutions independent of the variable x_j can be obtained from problem (9) with the kernel localized on the manifold

$$K_j(\lambda) - K_j(\mu) = 0. \quad (38)$$

This observation allows us to use $(2 + 1)$ -dimensional dressing method for $(1 + 1)$ -dimensional equations and leads us naturally to the $\bar{\partial}$ -problem with a shift and, for decreasing solutions, to the Riemann problem with a shift. Let us consider this observation in more detail.

If we have $(2 + 1)$ -dimensional integrable equation, defined by the functions $K_i(\lambda)$, we can descend to the $(1 + 1)$ -dimensional case, using condition (38) for some coordinate x_i in the original or rotated coordinate system. For example, the y -independent KP equation gives the KdV equation

$$(v_t + \frac{1}{4} v_{xxx} + 3v_x v) = 0.$$

Condition (38) in this case reads

$$\lambda^2 - \mu^2 = 0,$$

and solutions of the KdV equation are given by the $\bar{\partial}$ -problem with a shift [8]

$$\bar{\partial} \psi(\lambda) = R(\lambda, -\lambda) \exp(\phi_i x_i) \psi(-\lambda), \quad (39)$$

the shift function for this case is quite simple ($\mu = -\lambda$), and it is easy to transform problem (39) to a local matrix (2×2) Riemann problem.

We may also consider the case of the t -independent KP equation, which corresponds to the simplified Boussinesq equation

$$\frac{3}{4} \alpha^2 v_{yy} = -(\frac{1}{4} v_{xx} + \frac{3}{2} v^2)_{xx}. \quad (40)$$

Condition (38) in this case reads

$$\lambda^3 - \mu^3 = 0,$$

and solutions of simplified Boussinesq equation (40) are given by the $\bar{\partial}$ -problem

$$\bar{\partial}\psi(\lambda) = \sum_{i=1}^3 R_i \psi(e_i \lambda),$$

where $e_i^3 = 1$. The simplified version of the Boussinesq equation was considered in [14]. Let us show that for decreasing solutions our approach leads to the Riemann problem with a shift for the functions analytic in sectors (such a geometry for local matrix Riemann problem arose in [14] from analytical properties of the direct scattering problem). Combining condition (38) with condition (33)

$$\text{Im}(\lambda - \mu) = 0,$$

we obtain

$$\lambda - e_i \mu = 0, \quad \lambda - \mu = \xi, \quad \xi \in \mathbb{R}.$$

The solution of this system is

$$\lambda = \xi(1 - e_i)^{-1}, \quad \mu = -\xi(1 - e_i^{-1})^{-1},$$

it defines a Riemann problem with a shift on the pair of straight lines with the vectors $\exp(i\pi/6)$, $\exp(-i\pi/6)$, the shift function is $\mu = -\bar{\lambda}$. So we arrived at the problem for the function analytic in corresponding sectors.

For an arbitrary rational function $K_i(\lambda)$ condition (38) defines a multi-valued shift function $\mu_i(\lambda)$, and the corresponding $\bar{\partial}$ -problem reads

$$\bar{\partial}\psi(\lambda) = \sum_{i=1}^n R_i \psi(\lambda_i(\mu)). \quad (41)$$

4. Boussinesq equation via the $\bar{\partial}$ -dressing method

Let us consider the KP equation in the moving frame

$$\frac{\partial}{\partial x} \left((v_t - \beta v_x) + \frac{1}{4} v_{xxx} + 3v_x v \right) = -\frac{3}{4} \alpha^2 v_{yy}, \quad \beta^2 = 1. \quad (42)$$

Solutions of this equation are given by problem (9) with the dependence of the kernel on variables x, y, t defined by the expressions (compare (10) and (14))

$$D_1 = \frac{\partial}{\partial x} + i\lambda, \quad D_2 = \frac{\partial}{\partial y} + \alpha^{-1} \lambda^2 \quad (\alpha = 1; i), \quad D_3 = \frac{\partial}{\partial t} + i\lambda^3 + i\beta\lambda. \quad (43)$$

Time-independent solutions of Eq. (42) satisfy the Boussinesq equation

$$\left(\frac{3}{4} \alpha^2 v_{yy} - \beta v_{xx} \right) = -\left(\frac{1}{4} v_{xx} + \frac{3}{2} v^2 \right)_{xx}. \quad (44)$$

Such solutions are given by problem (9) ($v = -i(\partial/\partial x)\psi_0$), if the support of kernel $R(\lambda, \mu)$ belongs to the manifold defined by condition (38)

$$(\lambda^3 + \beta\lambda - \mu^3 - \beta\mu) = 0, \quad \lambda \neq \mu, \quad (45)$$

or

$$\lambda^2 + \lambda\mu + \mu^2 + \beta = 0.$$

This relation defines a $\bar{\partial}$ -problem with a shift

$$\bar{\partial}\psi(\lambda, x, y) = R(\lambda, \mu(\lambda)) \exp(\phi_i x_i) \psi(\mu(\lambda), x, y), \quad \mu = \frac{1}{2}(-\lambda \pm (4\beta - 3\lambda^2)^{1/2}). \quad (46)$$

Solutions of the Boussinesq equation, given by problem (46) ($v = -i(\partial/\partial x)\psi_0$), are defined locally in the neighborhood of the point $x = 0, y = 0$. We consider the Boussinesq equation as a dynamical equation with respect to the variable y . To obtain small solutions decreasing as $|x| \rightarrow \infty$, we should investigate the intersection of manifold (38) with the manifold, defined by condition (33):

$$\text{Im}(\lambda - \mu) = 0. \quad (47)$$

Conditions (45) and (47) define a Riemann problem with a shift (the Carleman's problem) which is a proper tool to solve the Boussinesq equation. Introducing $\xi = (1/2)(\lambda - \mu)$, $v = -i(1/2)(\lambda + \mu)$, $\xi \in \mathbb{R}$, one can get

$$\beta + \xi^2 - 3v^2 = 0. \quad (48)$$

4.1. About the reduction

Let us make a remark about the reduction. For $\alpha = 1$ $v(x, y)$ is real if the kernel of the problem (9) satisfies the condition

$$R(\lambda, \mu) = \bar{R}(-\bar{\lambda}, -\bar{\mu}) \quad (49)$$

for $\alpha = i$ if

$$R(\lambda, \mu) = \bar{R}(\bar{\mu}, \bar{\lambda}). \quad (50)$$

4.2. Continuous spectrum

4.2.1. 'Plus' Boussinesq equation

One can see that the properties of the Boussinesq equation depend essentially on the sign of β . Let $\beta = 1$. The corresponding equation ('plus' Boussinesq equation) reads

$$\frac{3}{4}\alpha^2 v_{yy} - v_{xx} + \frac{1}{4}v_{xxxx} + (\frac{3}{2}v^2)_{xx} = 0. \quad (51)$$

In the case $\alpha^2 = 1$ it is a nonlinear wave equation, having in a linear approximation monochromatic solution

$$v \simeq e^{i(\omega y + kx)}, \quad \omega^2 = \frac{4}{3}(k^2 + \frac{1}{4}k^4).$$

In the case $\alpha^2 = -1$ it is a nonlinear elliptic equation. In both cases Eq. (51) can be solved by the following Riemann problem with a shift

$$-3v^2 + \xi^2 + 1 = 0, \quad \lambda = -\bar{\mu}, \quad \lambda = \xi + iv, \quad \mu = -\xi + iv. \quad (52)$$

Eq. (52) defines a hyperbola with the branches belonging respectively to upper and lower half-planes (Fig. 1). The shift is defined as change of sign of the real part of λ . Let us introduce

$$\rho_{\pm}(\xi) = \Delta\psi|_{\pm}$$

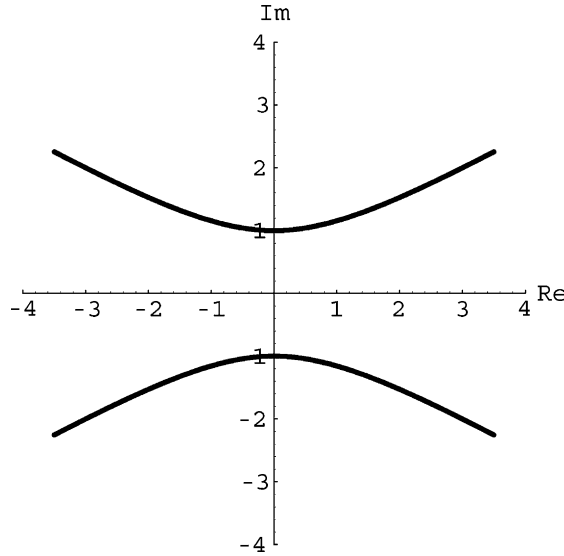


Fig. 1. Localization of continuous spectrum for ‘plus’ Boussinesq equation.

jumps of the function $\psi(\lambda)$ across upper and lower branches of the hyperbola. The function ψ can be represented in the form

$$\psi = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda - \lambda_-(\xi'))} \frac{d\lambda_-}{d\xi'} d\xi',$$

where

$$\lambda_{\pm}(\xi) = \xi \pm i \sqrt{\frac{1 + \xi^2}{3}}.$$

The Riemann problem with a shift (52) is equivalent to the system of two integral equations (25)

$$\begin{aligned} \rho_+(\xi) &= \left(1 + \frac{1}{2\pi i} v.p. \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_+(-\xi) - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_+(-\xi) - \lambda_-(\xi'))} \frac{d\lambda_-}{d\xi'} d\xi' \right) \\ &\quad \times R^+(\xi) e^{(4i/\sqrt{3}\alpha)\xi\sqrt{1+\xi^2}y+2i\xi x}, \\ \rho_-(\xi) &= \left(1 + \frac{1}{2\pi i} v.p. \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_-(-\xi) - \lambda_-(\xi'))} \frac{d\lambda_-}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_-(-\xi) - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' \right) \\ &\quad \times R^-(\xi) e^{(-4i/\sqrt{3}\alpha)\xi\sqrt{1+\xi^2}y-2i\xi x}. \end{aligned}$$

The solution of the Boussinesq equation is given by the formula

$$u = -\frac{\partial}{\partial x} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\rho_+(\xi) \frac{d\lambda_+}{d\xi} + \rho_-(\xi) \frac{d\lambda_-}{d\xi} \right) d\xi.$$

4.2.2. ‘Minus’ Boussinesq equation

This equation,

$$\frac{3}{4}\alpha^2 v_{yy} + v_{xx} + \frac{1}{4}v_{xxx} + \left(\frac{3}{2}v^2\right)_{xx} = 0$$

arises after putting $\beta = -1$. The reduced $\bar{\partial}$ -problem for this equation is described by the conditions

$$\lambda^2 + \lambda\mu + \mu^2 = 1 \tag{53}$$

(time independence) and

$$\text{Im}(\lambda - \mu) = 0 \tag{54}$$

(decreasing at infinity in x -direction). There are two possibilities to satisfy these conditions.

1. λ and μ are real ($\lambda^2 < 4/3, \mu^2 < 4/3$) and

$$\mu = -\frac{1}{2}\lambda \pm \sqrt{1 - \frac{3}{4}\lambda^2}. \tag{55}$$

We have a Riemann problem on the cut $-\sqrt{4/3} < \text{Re } \lambda < \sqrt{4/3}$ with the twofold shift (55).

2. λ and μ are complex, $\lambda = \nu + i\xi, \mu = -\nu + i\xi, \xi, \nu \in \mathbb{R}$,

$$\nu^2 - 3\xi^2 = 1. \tag{56}$$

Both λ and μ belong to the hyperbola (see Fig. 2). The shift as for the ‘plus’ Boussinesq equation is reflection with respect to imaginary axis.

Let us parameterize the curves, on which the solution ψ of the Riemann problem with a shift has a discontinuity, in the following way

$$\begin{aligned} \gamma_+ = \lambda_+(\xi) &= i\xi + \sqrt{1 + 3\xi^2}, & -\infty < \xi < \infty, & & \gamma_- = \lambda_-(\xi) &= i\xi - \sqrt{1 + 3\xi^2}, & \infty < \xi < \infty, \\ \gamma_0 = \lambda_0(\xi) &= \xi, & \xi^2 < \frac{4}{3}, \end{aligned}$$

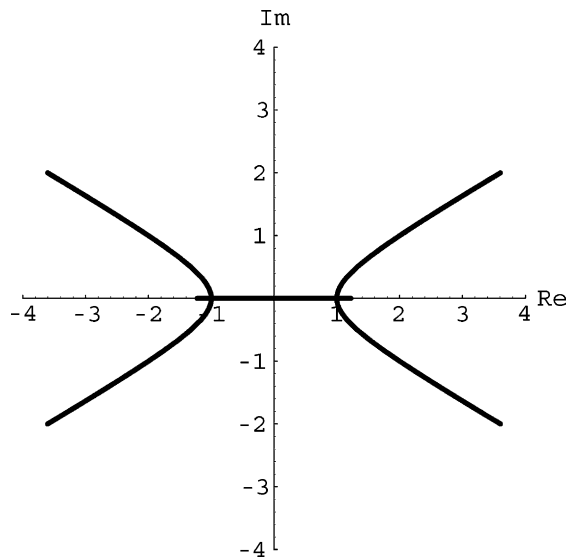


Fig. 2. Localization of continuous spectrum for ‘minus’ Boussinesq equation.

and introduce the jumps $\rho_+(\xi)$, $\rho_-(\xi)$, $\rho_0(\xi)$ of the function ψ across the curves. Then the function ψ can be represented in the form

$$\psi = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda - \lambda_-(\xi'))} \frac{d\lambda_-}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_0(\xi')}{(\lambda - \xi')} d\xi'.$$

The Riemann problem in this case is equivalent to the system of three integral equations

$$\begin{aligned} \rho_0(\xi) = 1 + & \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\mu_+(\xi) - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\mu_+(\xi) - \lambda_-(\xi'))} \frac{d\lambda_-}{d\xi'} d\xi' \right. \\ & \left. + \frac{1}{2\pi i} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_0(\xi')}{(\mu_+(\xi) - \xi')} d\xi' \right) R_0^+(\xi) e^{i((1/2)\xi - \sqrt{1-(3/4)\xi^2})x + (1/\alpha)((3/2)\xi^2 - \xi\sqrt{1-(3/4)\xi^2} - 1)y} \\ & + \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\mu_-(\xi) - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\mu_-(\xi) - \lambda_-(\xi'))} \frac{d\lambda_-}{d\xi'} d\xi' \right. \\ & \left. + \frac{1}{2\pi i} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_0(\xi')}{(\mu_-(\xi) - \xi')} d\xi' \right) R_0^-(\xi) e^{i((1/2)\xi + \sqrt{1-(3/4)\xi^2})x + (1/\alpha)((3/2)\xi^2 + \xi\sqrt{1-(3/4)\xi^2} - 1)y}, \end{aligned}$$

where

$$\mu_{\pm} = \frac{1}{2}\xi \pm \sqrt{1 - \frac{3}{4}\xi^2},$$

$$\begin{aligned} \rho_+(\xi) = 1 + & \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_-(\xi) - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_-(\xi) - \lambda_-(\xi'))} \frac{d\lambda_-}{d\xi'} d\xi' \right. \\ & \left. + \frac{1}{2\pi i} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_0(\xi')}{(\lambda_-(\xi) - \xi')} d\xi' \right) R^+(\xi) e^{2i\sqrt{1-3\xi^2}x + (4i/\alpha)\xi\sqrt{1-3\xi^2}y}, \end{aligned}$$

$$\begin{aligned} \rho_-(\xi) = 1 + & \left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_+(\xi')}{(\lambda_+(\xi) - \lambda_+(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\rho_-(\xi')}{(\lambda_+(\xi) - \lambda_-(\xi'))} \frac{d\lambda_+}{d\xi'} d\xi' \right. \\ & \left. + \frac{1}{2\pi i} \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \frac{\rho_0(\xi')}{(\lambda_+(\xi) - \xi')} d\xi' \right) R^-(\xi) e^{-2i\sqrt{1-3\xi^2}x - (4i/\alpha)\xi\sqrt{1-3\xi^2}y}. \end{aligned}$$

The solution of the Boussinesq equation is given by the formula

$$u = -\frac{\partial}{\partial x} \frac{1}{2\pi} \left[\int_{-\infty}^{\infty} \left(\rho_+(\xi) \frac{d\lambda_+}{d\xi} + \rho_-(\xi) \frac{d\lambda_-}{d\xi} \right) d\xi + \int_{-\sqrt{4/3}}^{\sqrt{4/3}} \rho_0(\xi) \frac{d\lambda_0}{d\xi} d\xi \right].$$

In this case the spectral data R_γ split into two parts; the short-wave part of continuous spectrum is localized on the hyperbola (56), and the long-wave part of the spectrum on the segment of the real axis (in fact on the covering of this segment); see Fig. 2. For $\alpha = 1$ hyperbola corresponds to the stable part of the spectrum (exponent (10) for y is imaginary) and the segment to the unstable part (exponent is real), for $\alpha = i$ the situation is reversed, i.e., long-wave instability takes place for $\alpha = 1$, and short-wave instability for $\alpha = i$.

4.3. Soliton solutions

Behavior of solitons in the case of the Boussinesq equation is very unusual for integrable systems. We will mostly study the case of the ‘plus’ Boussinesq equation with $\alpha^2 = 1$. This equation has a stable ‘wave sector’ (i.e., in the linear limit it is a wave equation), and it may be considered as integrable nonlinear generalization of the wave equation. There are two soliton sectors for this equation: ‘usual’ solitons, running with the velocity limited from above, and soliton configurations, forming a singularity in a finite time. The latter may be considered as bounded states of several singular solitons.

But even ‘usual’ solitons demonstrate quite extraordinary behavior in this case. They are unstable with respect to small perturbations and may decay into two solitons or two singular solitons (that means a formation of singularity).

This phenomenon was discovered by Orlov [2] several years ago, but it is not well known even in the soliton community, so we would like to investigate it here in detail in the framework of our general approach.

To study soliton solutions of the Boussinesq equation, we start from the general determinant formula (31). For the KP equation in the moving frame (42) from this formula we get (see similar expression for KP in [11])

$$v = \frac{\partial^2}{\partial x^2} \log \det(A), \quad A_{ij} = \delta_{ij} - \frac{R_i}{\mu_i - \lambda_j}, \quad (57)$$

where

$$R_k = -ic_k \exp\left(i(\mu_k - \lambda_k) \left(x - \frac{i}{\alpha}(\mu_k + \lambda_k)y\right)\right).$$

We will use an equivalent form of this formula

$$v = \frac{\partial^2}{\partial x^2} \log \det(\tilde{A}), \quad (58)$$

$$\tilde{A}_{ij} = \frac{1}{-ic_i} \exp(\Phi_i) \delta_{ij} + \frac{1}{\lambda_i - \mu_j}, \quad (59)$$

where

$$\Phi_k = i(\lambda_k - \mu_k)(x - i(\mu_k + \lambda_k)y).$$

To get solutions for the Boussinesq equation (44), the pairs (λ_k, μ_k) should satisfy the condition of dimensional reduction (45)

$$\lambda^2 + \lambda\mu + \mu^2 + 1 = 0,$$

and $\lambda_k \neq \mu_j$. The reduction (49) is to be taken into account.

We should also put some restrictions to get from the formula (57) solutions having no singularities at least for some values of y (y is dynamical variable, ‘time’, in our treatment). The prescription we will use is to put the condition

$$\operatorname{Re}(\lambda_i - \mu_i) = 0. \quad (60)$$

Then the exponents containing x are real, and at $y = 0$ we can provide the absence of singularities by the choice of coefficients. This condition together with condition (45) define a curve to which the points λ_i, μ_i should belong. This curve is identical to the curve we studied in the case of the ‘minus’ Boussinesq equation with the interchanged real and imaginary axes (see Fig. 3). So we have the curve consisting of two parts:

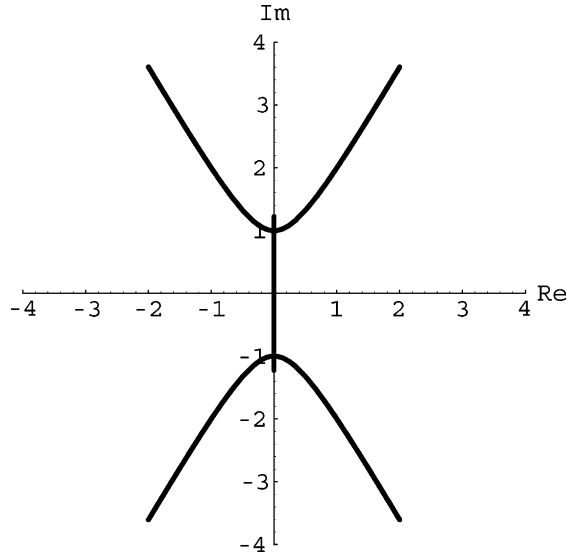


Fig. 3. Localization of discrete spectrum for 'plus' Boussinesq equation.

the segment of the imaginary axis and hyperbola. Let us consider the simplest solutions corresponding to these parts.

First, in complete analogy with formulae (55) and (56), we introduce parameterization of the pairs of points λ_k, μ_k in the following way: for the segment of the imaginary axis

$$\lambda = -i\xi, \quad \mu = -i\eta, \quad \eta = -\frac{1}{2}(\xi \pm \sqrt{4 - 3\xi^2}) \quad (61)$$

for pairs belonging to hyperbola $\lambda = \xi - i\nu, \mu = \xi + i\nu, \xi, \nu \in \mathbb{R}$,

$$\nu^2 - 3\xi^2 = 1. \quad (62)$$

Let us start with solutions corresponding to the points on the hyperbola.

We should take two pairs of points on the hyperbola

$$\begin{aligned} \lambda_1 &= \xi - i\sqrt{1 + 3\xi^2}, & +\mu_1 &= \xi + i\sqrt{1 + 3\xi^2}, & \lambda_2 &= -\bar{\lambda}_1 = -\xi - i\sqrt{1 + 3\xi^2}, \\ \mu_2 &= -\bar{\mu}_1 = -\xi + i\sqrt{1 + 3\xi^2} \end{aligned} \quad (63)$$

to satisfy reduction condition (49)

$$R(\lambda, \mu) = \bar{R}(-\bar{\lambda}, -\bar{\mu}).$$

The general formula for determinant solution (59) corresponding to two pairs of points $(\lambda_1, \mu_1), (\lambda_2, \mu_2)$ is

$$\begin{aligned} \det(\tilde{A}) &= \frac{(\lambda_1 - \lambda_2)(\mu_1 - \mu_2)}{(\lambda_1 - \mu_2)(\mu_1 - \lambda_2)} + \frac{\lambda_1 - \mu_1}{-ic_1} \exp(\Phi_1) + \frac{\lambda_2 - \mu_2}{-ic_2} \exp(\Phi_2) \\ &+ \frac{\lambda_1 - \mu_1}{ic_1} \exp(\Phi_1) \frac{\lambda_2 - \mu_2}{-ic_2} \exp(\Phi_2). \end{aligned} \quad (64)$$

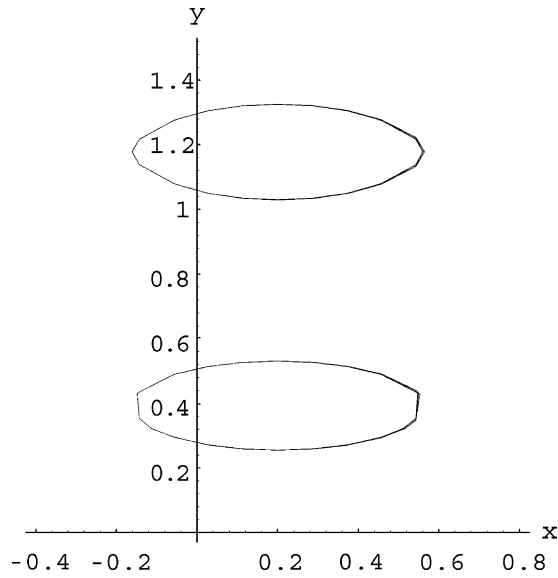


Fig. 4. The curve in x, y plane where solution has a singularity.

Let us take two pairs of points on the hyperbola parameterized by formulae (63) and $c_1, c_2 = c \in \mathbb{R}$. Then determinant (64) is real, and it is given by the expression

$$\det(\tilde{A}) = \frac{\xi^2}{1 + 4\xi^2} + \frac{2\sqrt{1 + 3\xi^2}}{c} (e^{2\sqrt{1 + 3\xi^2}(x - 2i\xi y)} + e^{2\sqrt{1 + 3\xi^2}(x + 2i\xi y)}) + \frac{4(1 + 3\xi^2)}{c^2} e^{4\sqrt{1 + 3\xi^2}x}. \quad (65)$$

For positive c this expression has no zeroes at initial moment, so the solution is nonsingular and decreasing. But then at some moment zeroes appear in this expression, so the singularities are formed. Let us illustrate this process by several figures corresponding to some special choice of parameters ($c = -20, \xi = 1$). Fig. 4 shows the lines on the plane x, y , where the determinant is equal to zero. The general form of the solution is given by Fig. 5. Fig. 6 illustrates development of singularity for the solution (dynamics is considered with respect to y variable). Fig. 7 shows the solution after creation of singularity. Then the solution behaves like two singular solitons (see Fig. 8), first they go away from each other to some maximal distance, then they come close and the singularity disappears (see Fig. 9). The process is periodic with respect to y . Qualitatively this picture is the game for arbitrary values of parameters. The change of c just shifts the picture. Parameter ξ defines the period of the process and the characteristic length. Maximal distance between the singularities is

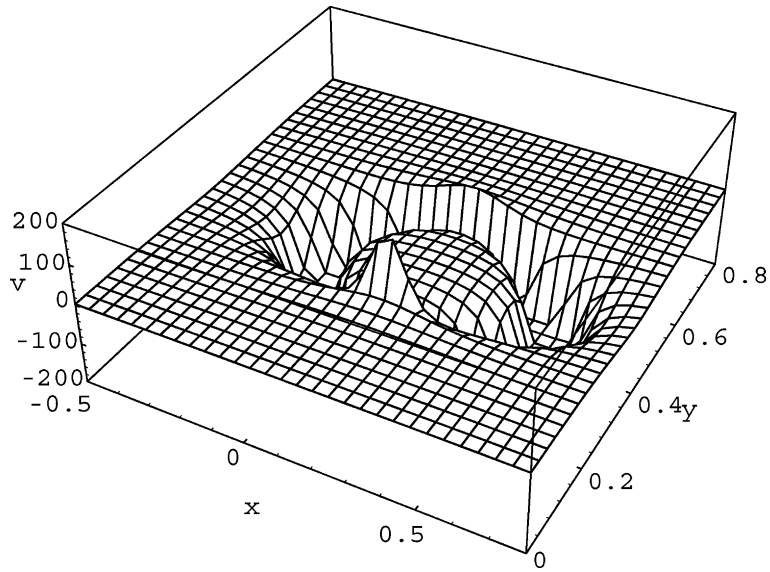
$$l_{\max} = \frac{1}{\sqrt{1 + 3\xi^2}} \operatorname{arccosh} \left(\frac{1}{F(\xi)} \right), \quad (66)$$

time between creation and disappearance of singularities is

$$t = \frac{1}{2\xi} \operatorname{arccosh}(F(\xi)), \quad (67)$$

where

$$F(\xi) = 2\xi \sqrt{\frac{1}{1 + 4\xi^2}}. \quad (68)$$

Fig. 5. General form of solution in x, y plane.

The function $F(\xi)$ is a monotone increasing function equal to zero at $\xi = 0$ with the limit 1 at infinity and -1 at minus infinity.

Dynamics of singularities becomes more complicated when we consider a solution corresponding to a set of several points λ_i, μ_i . The case of four pairs of points is illustrated by Fig. 10.

4.3.1. Decay of solitons

A pair of points belonging to the segment of the imaginary axis gives us a soliton solution

$$\det \tilde{A} = 1 + \frac{\lambda - \mu}{-ic} \exp(i(\lambda - \mu)(x - i(\mu + \lambda)y)).$$

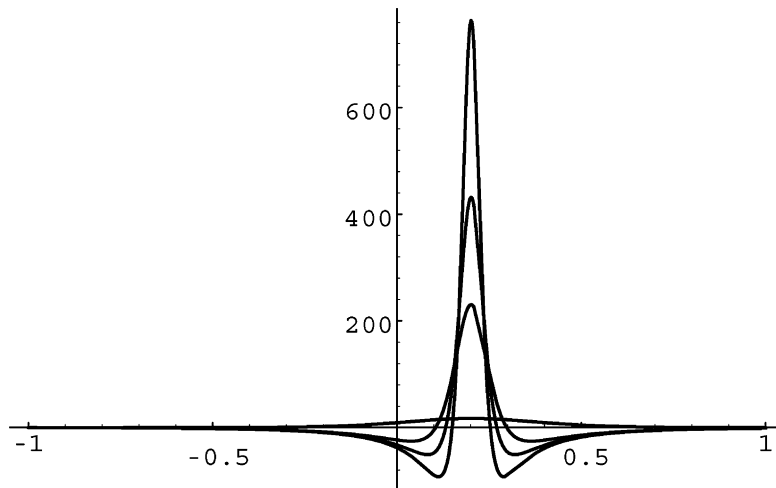


Fig. 6. Development of singularity.

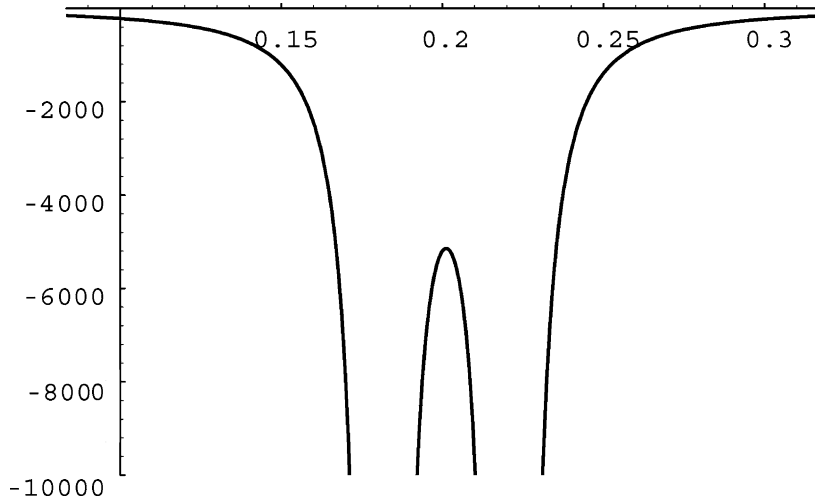


Fig. 7. Singularity formed.

Using parameterization (63), we get the formula

$$\det \tilde{A} = 1 + \frac{\xi - \eta}{c} \exp((\xi - \eta)(x - (\xi + \eta)y)) = 1 + \frac{3\xi \pm \sqrt{4 - 3\xi^2}}{2c} e^{(1/4)(3\xi \pm \sqrt{4 - 3\xi^2})(2x - (\xi \mp \sqrt{4 - 3\xi^2})y)}, \quad (69)$$

$$v = \frac{\partial^2}{\partial x^2} \log \cosh \frac{3\xi \pm \sqrt{4 - 3\xi^2}}{8} (2(x - x_0) - (\xi \mp \sqrt{4 - 3\xi^2})y). \quad (70)$$

To understand the dependence of the soliton on parameters ξ, η , it is useful to recall that these parameters belong to the ellipse

$$\xi^2 + \xi\eta + \eta^2 = 1. \quad (71)$$

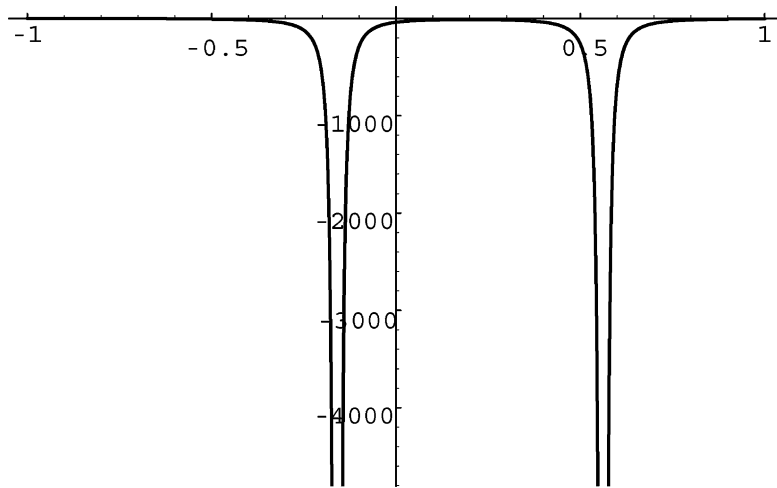


Fig. 8. Two singular solitons stage of development of singularity.

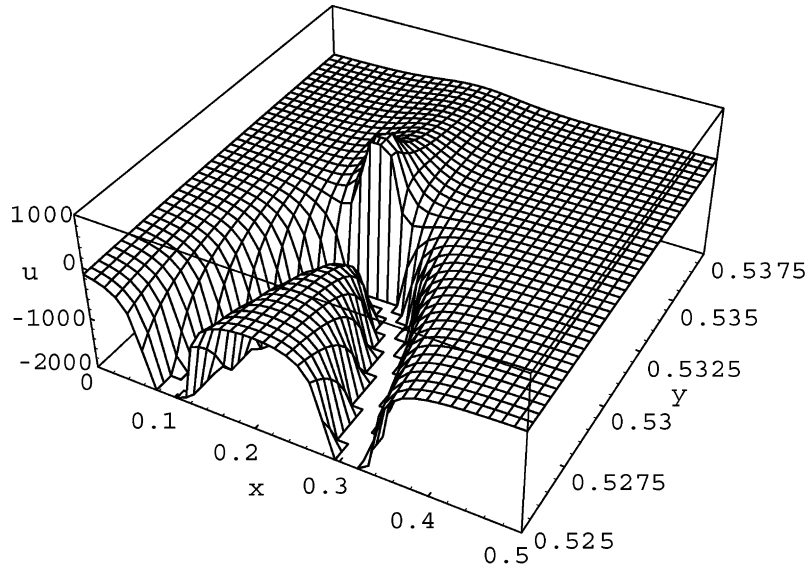


Fig. 9. Disappearance of singularity.

Introducing velocity of the soliton

$$v = \xi + \eta,$$

and a parameter defining amplitude of the soliton

$$A = \xi - \eta,$$

we rewrite the equation of this ellipse in the form

$$3v^2 + A^2 = 4.$$

Then it is easy to see that the absolute value of velocity of the soliton $v = \xi + \eta$ is limited by $\sqrt{4/3}$. For $(\xi - \eta)/c > 0$ the soliton is nonsingular. As $\xi - \eta \rightarrow 0$, the amplitude of the soliton goes to zero, and velocity reaches its maximum $|v| \rightarrow \sqrt{4/3}$. Two points with $\xi = \eta$ belonging to the ellipse are the points of the change of sign, where (for fixed c) the nonsingular soliton becomes singular and vice versa.

There are some unusual features concerning the behavior of the soliton under small perturbations, which come to light when we study two-soliton solutions. The general formula of two-soliton interaction (64) rewritten for the segment of the imaginary axis in terms of parameters ξ, η looks like

$$\det(\tilde{A}) = \frac{(\xi_1 - \xi_2)(\eta_1 - \eta_2)}{(\xi_1 - \eta_2)(\eta_1 - \xi_2)} + \frac{\xi_1 - \eta_1}{c_1} \exp(\Phi_1) + \frac{\xi_2 - \eta_2}{c_2} \exp(\Phi_2) + \frac{\xi_1 - \eta_1}{c_1} \exp(\Phi_1) \frac{\xi_2 - \eta_2}{c_2} \exp(\Phi_2), \quad (72)$$

where

$$\Phi_i = (\xi_i - \eta_i)(x - (\xi_i + \eta_i)y) = A_i(x - v_i y).$$

Considering formula (63) defining η_k through ξ_k , one remarks that there are two possible choices of η corresponding to the same ξ (and also two possible ξ corresponding to the same η). It is natural to ask what kind of solution we get if we consider two pairs of points with the same ξ (or the same η). The formula (72) in this case degenerates,

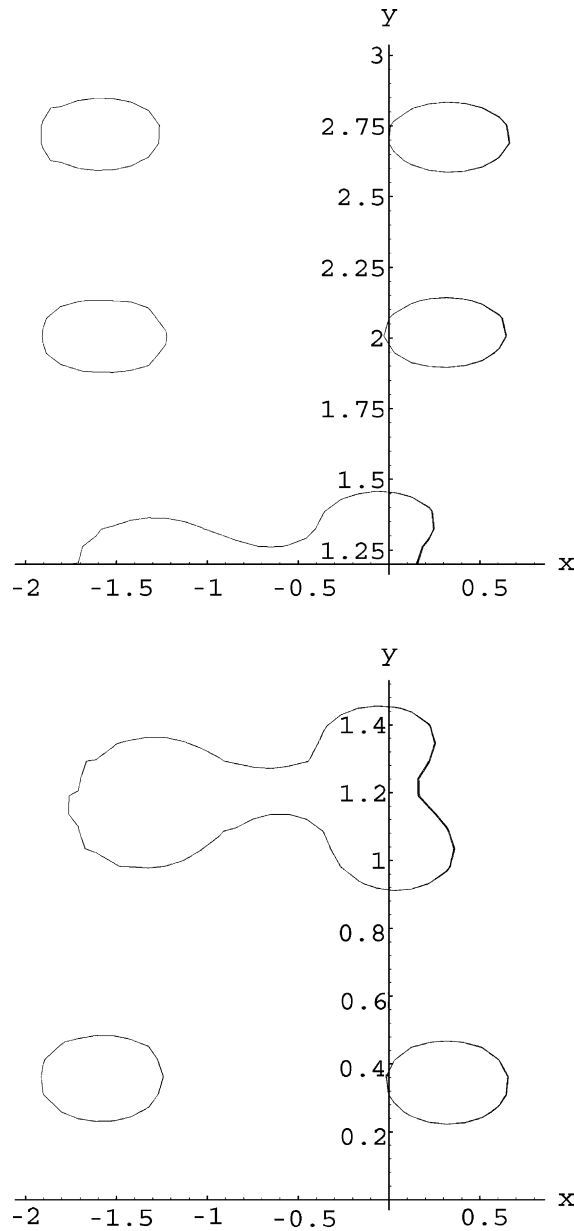


Fig. 10. The curve in x, y plane where solution has a singularity, the case of four pairs of points λ_i, μ_i on the hyperbola.

the first term in it is equal to zero. Naively, we expect this solution to be a two-soliton solution. But further study shows that this solution possesses rather unusual properties. It describes the decay of soliton (70) (or fusion of two solitons).

Considering formula (72) with $\xi_1 = \xi_2 = \xi$, we get

$$\det(\tilde{A}) = +\frac{\xi - \eta_1}{c_1} \exp(\Phi_1) + \frac{\xi - \eta_2}{c_2} \exp(\Phi_2) + \frac{\xi - \eta_1}{c_1} \exp(\Phi_1) \frac{\xi - \eta_2}{c_2} \exp(\Phi_2), \quad (73)$$

or, using explicit parameterization (63) of η_1, η_2 through ξ ,

$$\det(\tilde{A}) \sim 1 + \frac{2c_2 e^{(1/4)(3\xi - \sqrt{4-3\xi^2})(-2x + \xi y + \sqrt{4-3\xi^2}y)}}{3\xi - \sqrt{4-3\xi^2}} + \frac{2c_1 e^{(1/4)(3\xi + \sqrt{4-3\xi^2})(-2x + \xi y - \sqrt{4-3\xi^2}y)}}{3\xi + \sqrt{4-3\xi^2}}, \quad (74)$$

here $c_1, c_2 \in \mathbb{R}$, and the determinant is written up to an exponential factor that does not change the solution. Experimenting with the plots obtained by Mathematica from this formula for some choice of parameters, one discovers two solitons for some (big positive) values of y , and one soliton for other (big negative) values. Analytic study of formula (74) confirms this impression.

Let us consider the simplest case of the staying soliton (soliton with velocity zero). In this case the value of parameter ξ is equal to 1, and formula (69) (with the sign +) takes the form

$$\det \tilde{A} \sim 1 + \frac{1}{2}c \exp(-2x), \quad (75)$$

that corresponds to the standard soliton solution with zero velocity

$$v = \cosh^{-2}(x - x_0).$$

Substituting $\xi = 1$ to the ‘two-soliton’ formula (74), we get

$$\det(\tilde{A}) \sim 1 + c_2 \exp(y - x) + \frac{1}{2}c_1 \exp(-2x). \quad (76)$$

To study asymptotic behavior of the solution corresponding to this determinant, we should take into account that the solution is given by the second derivative of the logarithm of the determinant (formula (57)). At $y = -\infty$ we discover only a staying soliton of the form (75)

$$v \approx \frac{\partial^2}{\partial x^2} \log \left(1 + \frac{1}{2}c_1 \exp(-2x) \right),$$

which is nonsingular if c_1 is positive. Multiplying the determinant (76) by $\exp(x - y)$ (that does not change the solution), we get another representation of the ‘two-soliton’ solution

$$v = \frac{\partial^2}{\partial x^2} \log \left(\exp(x - y) + c_2 + \frac{1}{2}c_1 \exp(-x - y) \right).$$

Using this representation to study asymptotic behavior of the solution at $y = \infty$ in the arbitrary moving frame $x = \tilde{x} + v y$, we discover that for $v = \pm 1$ asymptotics is nontrivial, corresponding to two solitons moving with velocities $v = \pm 1$,

$$v \approx \frac{\partial^2}{\partial x^2} \log(\exp(x - y) + c_2) + \frac{\partial^2}{\partial x^2} \log \left(\frac{1}{2}c_1 \exp(-x - y) + c_2 \right). \quad (77)$$

As we have mentioned before, for a nonsingular staying soliton c_1 is positive. If c_2 is also positive, formula (77) gives two nonsingular solitons, and negative c_2 corresponds to two singular solitons.

The initial data for the solution v corresponding to the determinant (76) may be made infinitely close to the one-soliton solution by the choice of constants. In fact at $y = -\infty$ this is exactly a soliton. But then this slightly disturbed soliton solution decays! It may decay into two solitons or into two singular solitons, depending on the initial perturbation (see Figs. 11 and 12). So the staying soliton solution for the Boussinesq equation is unstable with respect to small perturbations, it may develop a singularity or decay into two solitons.

A natural question to ask next is whether an arbitrary soliton may decay. To answer it, we start from some general remarks concerning the decay formula (73)

$$\det(\tilde{A}) \sim 1 + \frac{c_1}{\xi - \eta_1} \exp(-\Phi_1) + \frac{c_2}{\xi - \eta_2} \exp(-\Phi_2). \quad (78)$$

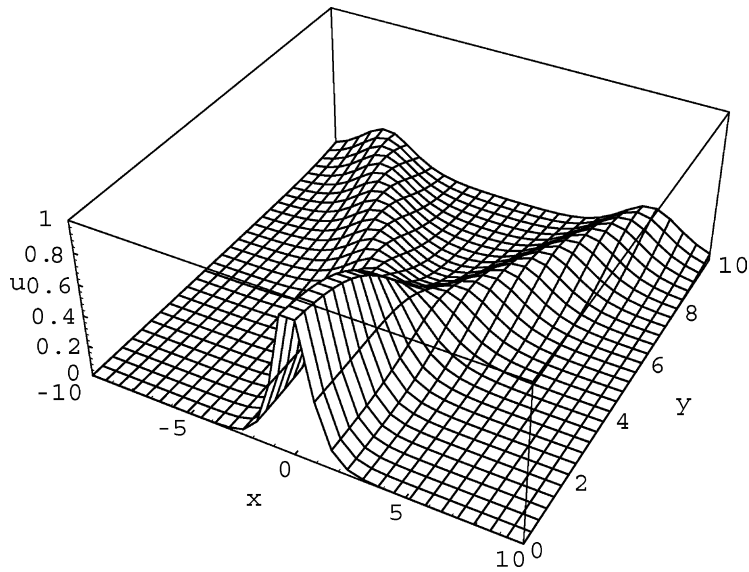


Fig. 11. Decay of soliton into two solitons.

Using the simple example of the staying soliton, we have shown that three different solitons enter this formula. The soliton is defined by a pair of real parameters ξ, η satisfying equation (71), or, in other words, by the point of ellipse (71). The point η, ξ defines the same soliton (up to a change of constant c). Deriving formula (78), we start from two solitons having the same ξ . To understand the appearance of the third soliton, it is easy to show that if $\eta = \eta_1, \rho = \eta_2$ satisfy Eq. (71) with the same ξ , then the point (η, ρ) also belongs to the ellipse (71). So the formula (78)

$$\det(\tilde{A}) \sim 1 + \frac{c_1}{\xi - \eta} e^{-(\xi - \eta)(x - (\xi + \eta)y)} + \frac{c_2}{\xi - \rho} e^{-(\xi - \rho)(x - (\xi + \rho)y)} \tag{79}$$

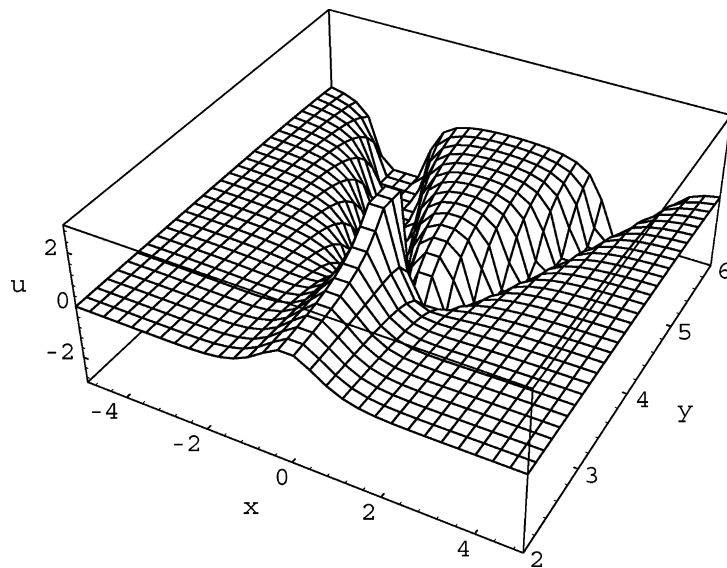


Fig. 12. Decay of soliton into two singular solitons.

contains *three* solitons with the parameters (ξ, η) , (ξ, ρ) , (η, ρ) . Thus, the decay process is characterized by three real parameters ξ, η, ρ , possessing the property that each pair of parameters defines the point of the ellipse (71). Depending on ξ (which defines η and ρ through the formula (63)), each soliton is present only at $y = \infty$ or at $y = -\infty$.

Considering formula (79) in the moving frame

$$x = \tilde{x} + (\xi + \eta)y,$$

we discover that a soliton with the parameters (ξ, η) is present at

$$y = \text{sign}((\xi - \rho)(\eta - \rho))\infty.$$

Similarly, we come to the conclusion that a soliton with the parameters (ξ, ρ) is present at

$$y = \text{sign}((\xi - \eta)(\rho - \eta))\infty.$$

Rewriting expression (79) in equivalent form

$$\det(\tilde{A}) \sim e^{-(\eta-\xi)(x-(\xi+\eta)y)} + \frac{c_1}{\xi - \eta} + \frac{c_2}{\xi - \rho} e^{-(\eta-\rho)(x-(\eta+\rho)y)}, \quad (80)$$

and considering the moving frame

$$x = \tilde{x} + (\eta + \rho)y,$$

we show that a soliton with the parameters (η, ρ) is present at

$$y = \text{sign}((\eta - \xi)(\rho - \xi))\infty.$$

Choosing to be definite $\xi > \rho > \eta$, we discover that formula (79) describes decay of a soliton with the parameters (ξ, η) , i.e., the smallest and the largest of parameters ξ, ρ, η . Let us use explicit parameterization (63)

$$\eta = -\frac{1}{2}(\xi + \sqrt{4 - 3\xi^2}), \quad \rho = -\frac{1}{2}(\xi - \sqrt{4 - 3\xi^2}).$$

If we start from the maximal value of $\xi = \sqrt{4/3}$, formula (79) describes decay of a soliton with the parameters (ξ, η) ; the velocity of this soliton is

$$v = \frac{1}{2}(\xi - \sqrt{4 - 3\xi^2}).$$

As ρ comes close to ξ (at $\xi = \sqrt{1/3}$), the velocity of the soliton (ξ, ρ) reaches maximal velocity $v = \sqrt{4/3}$, and the velocity of decaying soliton (ξ, η) reaches the value $v = -\sqrt{1/3}$. At $\rho = \xi$ formula (79) degenerates, it describes one soliton with velocity $v = -\sqrt{1/3}$. As ξ becomes smaller than $\sqrt{1/3}$, ρ becomes larger than ξ , and formula (79) describes decay of soliton (ρ, η) with the velocity

$$v = -\xi.$$

At $\xi = -\sqrt{1/3}$, ξ becomes equal to η , formula (79) degenerates once again, the velocity of decaying soliton reaches the value $\sqrt{1/3}$, and for $-\sqrt{1/3} > \xi \geq -\sqrt{4/3}$ it describes decay of soliton ξ, ρ with the velocity

$$v = \frac{1}{2}(\xi + \sqrt{4 - 3\xi^2}).$$

Thus, the velocity of the decaying soliton changes in the range

$$-\sqrt{\frac{1}{3}} < v_{\text{dec}} < \sqrt{\frac{1}{3}}.$$

There are decay processes into two solitons or two singular solitons, depending on the choice of constants c_1, c_2 in the formula (79).

There are no decay processes for the solitons with $|v| > \sqrt{1/3}$, so these solitons are stable (with respect to decay). Thus, we have answered our first questions, and the answer is negative, not all solitons may decay.

4.3.2. Interaction of solitons

The next question is about soliton systems and interaction of two solitons. It is whether singularities may appear as a result of interaction, and are there any stable soliton systems (not forming singularities as a result of interaction).

Remark. In general, studying stability of systems of solitons, one should take into account not only the interaction between solitons, but also the interaction of solitons with the continuum spectrum also. We would like to emphasize that here we will take into account only the soliton sector, so we are able to give only a partial answer to these questions. Indeed, systems of solitons stable with respect to interaction between solitons may, in principle, under arbitrary small perturbation develop instability and even singularity. However, for the Boussinesq equation the questions we consider are nontrivial even for pure soliton interactions.

First we would like to formulate two results concerning these questions.

Statement 1. Solitons moving in one direction with velocities $|v| > \sqrt{1/3}$ do not form singularities as a result of two-soliton interaction.

Statement 2. Two-soliton interaction of solitons with velocities $|v| < \sqrt{1/3}$ necessarily leads to formation of a singularity (i.e., the result of interaction of two solitons always is two singular solitons).

The proof of both statements is based on formula (72). Interaction of two solitons is much more standard in soliton theory than the decay process, so we will not consider it in detail. Using moving reference frames and considering asymptotical behavior of solution (72) at $y = \pm\infty$, it is easy to show that the character of the interaction process is defined by the sign of the first term in the formula (72)

$$c = \frac{(\xi_1 - \xi_2)(\eta_1 - \eta_2)}{(\xi_1 - \eta_2)(\eta_1 - \xi_2)}. \quad (81)$$

If $c > 0$, the result of interaction is a pair of solitons, and for $c < 0$ the result is a pair of singular solitons (that means that singularity is formed in the process of interaction). The points of change of sign $\xi_1 = \xi_2, \eta_1 = \eta_2, \xi_1 = \eta_2, \eta_1 = \xi_2$ correspond to degeneration of formula (72) into fusion process and appearance of third soliton. The results of analysis of triple soliton diagram given before show that it always contains two solitons moving in the same direction; one with velocity $|v_1| < \sqrt{1/3}$ (the decaying soliton, or, for the inversed time, the result of fusion), and another with velocity $|v_2| > \sqrt{1/3}$. The third soliton moves in the opposite direction with the velocity $|v_3| > \sqrt{1/3}$.

Let us take two solitons moving in the same direction with velocities $|v| > \sqrt{1/3}$. There are no fusion diagrams containing these solitons, and expression (81) does not change sign when we change parameters of solitons. It is easy to check that in this case the sign is positive, and the result of interaction of two solitons is two (nonsingular) solitons, that proves Statement 1. This statement can be easily generalized to the case of N -soliton interaction, and thus the system of solitons moving in the same direction with velocities $|v| > \sqrt{1/3}$ does not form singularities and is stable with respect to decay processes. In other words, this system demonstrates a ‘standard’ behavior usually associated with a system of solitons.

Similarly, considering two solitons moving with velocities $|v| < \sqrt{1/3}$, we come to the conclusion that the sign of expression (81) is negative. Interaction of two solitons in this case always results in two singular solitons, i.e., in the formation of singularity, that proves Statement 2. Thus, the system of solitons moving with velocities $|v| < \sqrt{1/3}$ demonstrates rather extraordinary behavior. First, the solitons are unstable under perturbation and may decay into two solitons or two singular solitons. And second, interaction of two solitons unavoidably leads to formation of a singularity in a finite time.

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