

THE BRAID GROUPS

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1. Introduction.

The braid groups B_n , $n = 1, 2, 3, \dots$, were introduced in 1926 by E. Artin [1] and have been the subject of numerous investigations. Although there is a well-known presentation of B_n that has been derived several times the derivations that appear in the literature e.g. [1], [2] are all, in one way or another, somewhat devious. Our principal object is to give a straightforward derivation of this presentation, based on the previously unnoted remark that B_n may be considered as the fundamental group of the space \tilde{E}^{2n} of configurations of n undifferentiated points in the plane.

Our derivation uses a method of computation that has never been published, although knowledge of it is probably widely distributed. It is proposed to publish the details of this method in a later paper; however the ideas involved are transparent enough to be believably communicated very briefly, and this we do in § 2 of the present paper.

By examining a certain covering of \tilde{E}^{2n} and using the results of [3] it is shown that \tilde{E}^{2n} is aspherical, and certain consequences of this fact are noted. In particular it follows immediately that B_n has no elements of finite order; we believe that this was not previously known.

2. Computation of π_1 .

If X is a regular cell-complex, then we consider mappings of X onto X/R where R is a relation obtained from a family Φ of identifications of the cells of X . Φ is required to satisfy the following conditions:

- 0) Each φ in Φ is a homeomorphism with domain a closed cell of X .
- i) If U is a cell of X , $\varphi: \bar{U} \rightarrow \bar{U}$ is in Φ if and only if φ is the identity.
- ii) If $\varphi \in \Phi$, $\varphi: \bar{U}_1 \rightarrow \bar{U}_2$ then $\varphi^{-1}: \bar{U}_2 \rightarrow \bar{U}_1$ is in Φ .
- iii) If $\varphi: \bar{U}_1 \rightarrow \bar{U}_2$ and $\varphi^1: \bar{U}_2 \rightarrow \bar{U}_3$ are in Φ , so is $\varphi^1\varphi: \bar{U}_1 \rightarrow \bar{U}_3$.
- iv) If $\varphi: \bar{U}_1 \rightarrow \bar{U}_2$ is in Φ and V_1 is a cell contained in \bar{U}_1 then $V_2 = \varphi(V_1)$ is also a cell, and $\varphi|_{\bar{V}_1}: \bar{V}_1 \rightarrow \bar{V}_2$ is in Φ .

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In what follows X and X/R will be manifolds of dimension n , and we shall compute π_1 of the complement of an $(n - 2)$ -dimensional subcomplex K of X/R .

The algorithm for the computation is roughly as follows: Select in X/R a maximal n -dimensional "cave" \mathcal{C} of n -dimensional, and oriented $(n - 1)$ -dimensional cells (this will be dual to a maximal tree in a dual cell decomposition). To each oriented $(n - 1)$ -cell not in the cave will correspond a generator of π_1 . This generator is represented by a loop that penetrates the $(n - 1)$ cell once with intersection number 1 but otherwise lies entirely in \mathcal{C} . To each $(n - 2)$ -cell of X/R that does not belong to K will correspond a relation, obtained from the "non-abelian coboundary" of the $(n - 2)$ -cell in question. More precisely, a small loop about an $(n - 2)$ -cell σ will intersect, in a certain order and sense, all the $(n - 1)$ -cells having σ on their boundary. Joining this loop to the base point will give a representative of an element of the fundamental group of the union of the n , and $(n - 1)$ -cells of $X/R - K$. In this way a set of elements of the free group generated by the $(n - 1)$ -cells not in \mathcal{C} is defined. This set of elements, one for each $(n - 2)$ -cell not in K , will be a complete set of relations for $\pi_1(X/R - K)$.

3. A cellular decomposition of S^{2n} .

An ordered n -tuple (p_1, \dots, p_n) of points of the plane E^2 may be considered to be a point p of $2n$ -dimensional space E^{2n} . If the coordinates of p_i are x_i, y_i , the coordinates of the corresponding point p are

$$x_1, y_1, x_2, y_2, \dots, x_n, y_n .$$

Let us write $i_1 < i_2$ whenever the abscissa of p_{i_1} is smaller than the abscissa of p_{i_2} , $i_1 \underline{\leq} i_2$ whenever p_{i_1} and p_{i_2} have the same abscissa, and the ordinate of p_{i_1} is smaller than the ordinate of p_{i_2} , and $i_1 = i_2$ whenever p_{i_1} coincides with p_{i_2} . Information of this sort can be condensed into a single symbol, θ , describing a point set in E^{2n} . Thus, for example, the symbol $(3 < 5 = 1 < 6 \underline{\leq} 4 \underline{\leq} 2 = 7)$ denotes the set of all points $(x_1, y_1, \dots, x_7, y_7)$ in E^{14} such that

$$\begin{aligned} x_3 < x_5 = x_1 < x_6 = x_4 = x_2 = x_7 , \\ y_5 = y_1, \quad y_6 < y_4 < y_2 = y_7 . \end{aligned}$$

(Of course the same information is indicated by each of the symbols

$$\begin{aligned} (3 < 1 = 5 < 6 \underline{\leq} 4 \underline{\leq} 2 = 7) , \\ (3 < 5 = 1 < 6 \underline{\leq} 4 \underline{\leq} 7 = 2) , \\ (3 < 1 = 5 < 6 \underline{\leq} 4 \underline{\leq} 7 = 2) ; \end{aligned}$$

we shall not distinguish between such equivalent symbols). The same symbol θ will be used to denote the set of all those points p satisfying the indicated conditions.

It is easy to see that each θ is a convex subset of E^{2n} and that, together with the point at infinity, they are the (open) cells of a regular cell-subdivision of the $2n$ -dimensional sphere $S^{2n} = E^{2n} \cup \infty$. The dimension of the cell θ is obviously equal to $2n$ minus the sum of the number of occurrences of $\underline{\leq}$ and twice the number of occurrences of $=$. The lower dimensional cells that are on the boundary of θ are obtained by replacing instances of $i_1 < i_2$ by $i_1 \underline{\leq} i_2$ or $i_2 \underline{\leq} i_1$ and/or replacing instances of $j_1 \underline{\leq} j_2$ by $j_1 = j_2$ (or $j_2 = j_1$). For example the boundary of the 5-dimensional cell $(1 < 2 \underline{\leq} 3)$ consists of the 4-dimensional cells $(1 \underline{\leq} 2 \underline{\leq} 3)$, $(2 \underline{\leq} 1 \underline{\leq} 3)$, $(1 < 2 = 3)$, the 3-dimensional cells $(1 = 2 \underline{\leq} 3)$, $(1 \underline{\leq} 2 = 3)$, $(2 \underline{\leq} 1 = 3)$, the 2-dimensional cell $(1 = 2 = 3)$, and the vertex ∞ .

In what follows we shall be concerned especially with the cells of dimension $\geq 2n - 2$. There are $n!$ cells of dimension $2n$. One of them is $(1 < 2 < \dots < n)$, and the others may be obtained from this by permuting the indices $1, 2, \dots, n$. The $(2n - 1)$ -cells on the boundary of

$$(1 < 2 < \dots < n)$$

are

$$\begin{aligned} &(1 \underline{\leq} 2 < 3 < \dots < n), \\ &(2 \underline{\leq} 1 < 3 < \dots < n), \\ &(1 < 2 \underline{\leq} 3 < \dots < n), \\ &(1 < 3 \underline{\leq} 2 < \dots < n) \text{ etc.}, \end{aligned}$$

and the $(2n - 2)$ -cells on the boundary of, say, $(1 \underline{\leq} 2 < 3 < \dots < n)$ are

$$\begin{aligned} &(1 = 2 < 3 < \dots < n), \\ &(1 \underline{\leq} 2 \underline{\leq} 3 < \dots < n), \\ &(1 \underline{\leq} 3 \underline{\leq} 2 < \dots < n), \\ &(3 \underline{\leq} 1 \underline{\leq} 2 < \dots < n), \\ &(1 \underline{\leq} 2 < 3 \underline{\leq} 4 < \dots < n), \\ &(1 \underline{\leq} 2 < 4 \underline{\leq} 3 < \dots < n) \text{ etc.} \end{aligned}$$

4. The action of Σ_n on S^{2n} .

To the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

associate the autohomeomorphism of S^{2n} that maps an arbitrary point

(p_1, p_2, \dots, p_n) of E^{2n} into the point $(p_{i_1}, p_{i_2}, \dots, p_{i_n})$; thus an action on S^{2n} of the symmetric group Σ_n of permutations of n symbols $1, 2, \dots, n$ is defined. Denote the collapsed space by \hat{S}^{2n} , and the image of E^{2n} under the collapsing Δ by \hat{E}^{2n} . Each of the autohomeomorphisms of S^{2n} considered maps ∞ into ∞ and permutes the cells θ ; the collapsing Δ maps one or more m -cells σ upon an m -cell τ of \hat{E}^{2n} , (not necessarily homeomorphically). The cells τ , together with the image of the point at ∞ , constitute a regular cell-subdivision with identifications of \hat{S}^{2n} . A symbolic designation of the cells τ is readily derived. For example the cells of \hat{S}^6 are $\{1 < 2 < 3\}$, $\{1 < 2 \underline{\vee} 3\}$, $\{1 \underline{\vee} 2 < 3\}$, $\{1 \underline{\vee} 2 \underline{\vee} 3\}$, $\{1 < 2 = 3\}$, $\{1 = 2 < 3\}$, $\{1 \underline{\vee} 2 = 3\}$, $\{1 = 2 \underline{\vee} 3\}$, $\{1 = 2 = 3\}$, and ∞ .

5. The subcomplex Δ .

The points p_1, \dots, p_n of E^2 are distinct if and only if, for each $i < j$, $(x_i - x_j)^2 + (y_i - y_j)^2 > 0$. Accordingly we consider the collection Δ of those cells θ of our decomposition of E^{2n} in whose symbols the sign $=$ occurs at least once. Since boundaries are obtained by changing $<$ to $\underline{\vee}$ or $\underline{\vee}$ to $=$, it is clear that Δ and ∞ together form a $(2n - 2)$ -dimensional subcomplex of the cell complex S^{2n} . Furthermore the points p_1, \dots, p_n of E^{2n} are distinct if and only if p lies in $E^{2n} - \Delta$. Let $\hat{\Delta}$ denote the image of Δ under the collapsing Δ of S^{2n} to \hat{S}^{2n} . Then $\hat{\Delta} \cup \infty$ is a subcomplex of the cell complex \hat{S}^{2n} , and p_1, \dots, p_n are distinct if and only if $\hat{p} \in \hat{E}^{2n} - \hat{\Delta}$. Note that the point \hat{p} may be considered to be an unordered n -tuple of points p_1, \dots, p_n of E^2 . Let $\hat{E}^{2n} = \hat{E}^{2n} - \hat{\Delta}$.

6. The Braid group.

Let \mathcal{B}_n denote the braid group on n strings, φ the well-known homomorphism of \mathcal{B}_n upon Σ^n , and \mathcal{S}_n the kernel of φ . If we look at the plane cross sections of a braid \mathcal{B} , we see that it may be described kinematically as a motion of n distinct points in the plane that ends with these points back in their original position but permuted as indicated by the permutation $\varphi(\mathcal{B})$. In particular \mathcal{B} belongs to \mathcal{S}_n if and only if the motion described returns each point to its original position. From these remarks it should be clear that the fundamental group of $E^{2n} - \Delta$ is \mathcal{S}_n , the fundamental group of $\hat{E}^{2n} - \hat{\Delta}$ is \mathcal{B}_n , and that $E^{2n} - \Delta$ is the unbranched covering space of $\hat{E}^{2n} - \hat{\Delta}$ that belongs to the subgroup \mathcal{S}_n of \mathcal{B}_n .

7. A presentation of \mathcal{B}_n .

To calculate $\pi_1(\hat{E}^{2n} - \hat{\Delta})$ choose the base point in the interior of the $2n$ -cell $\lambda^{2n} = \{1 < 2 < \dots < n\}$. Since this is the only $2n$ -cell of \hat{S}^{2n} , there is a generator σ_j corresponding to each $(2n - 1)$ -cell

$$\lambda_j^{2n-1} = \{ \dots < j \preceq j+1 < \dots \};$$

it is represented by a loop in $\lambda^{2n} \cup \lambda_j^{2n-1}$ that cuts λ_j^{2n-1} exactly once. Let us suppose that λ_j^{2n-1} is so oriented that the motion of $p_1 \cup \dots \cup p_n$ in E^2 described by a loop representative of σ_j causes the points p_j and p_{j+1} to interchange places (and names) by circling one another in a counterclockwise direction. The motion for σ_j^{-1} is shown in Figure 1.

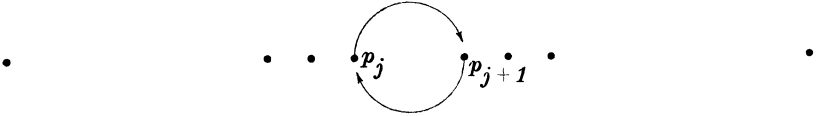


Fig. 1.

The braid σ_j^{-1} is shown in Figure 2.

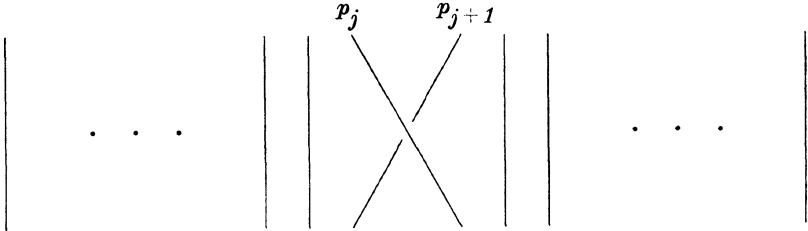


Fig. 2.

According to the general theory, a complete set of relations can be found in one to one correspondence with the cells of $\hat{E}^{2n} - \hat{J}$ of dimension $2n - 2$. These are of two sorts:

$$\lambda_{i,k} = \{ \dots < i \preceq i+1 < \dots < k \preceq k+1 < \dots \}, \quad i+1 < k,$$

$$\lambda_{i,i+1} = \{ \dots < i \preceq i+1 \preceq i+2 < \dots \}.$$

Now $\lambda_{i,k}$ is on the boundary of just the $(2n - 1)$ -cells λ_i and λ_k . Figure 3 shows a local cross section of E^{2n} by a plane perpendicular to the $(2n - 2)$ -cell

$$(1 < \dots < i \preceq i+1 < \dots < k \preceq k+1 < \dots < n).$$

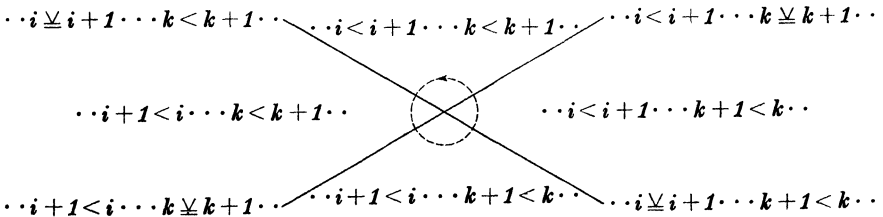


Fig. 3.

The relation $r_{i,k}$ corresponding to the cell $\lambda_{i,k}$ in E^{2n} is read as a “non-abelian coboundary” of $\lambda_{i,k}$. It is

$$r_{i,k} = \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1}$$

as may be seen by traversing the dotted loop in Figure 3. The motion of (p_1, \dots, p_n) in E^2 described by $r_{i,k}$ is shown in Figure 4 and its interpretation as a braid in Figure 5.

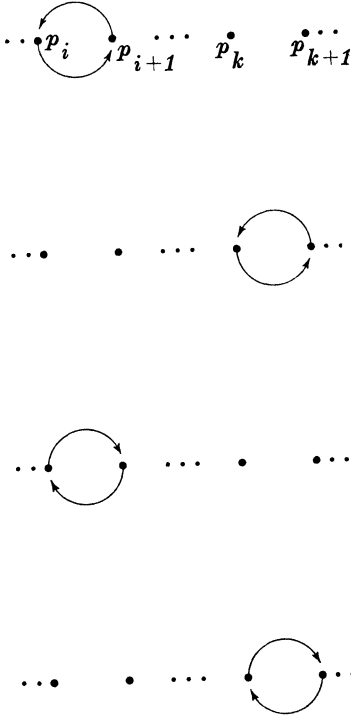


Fig. 4.

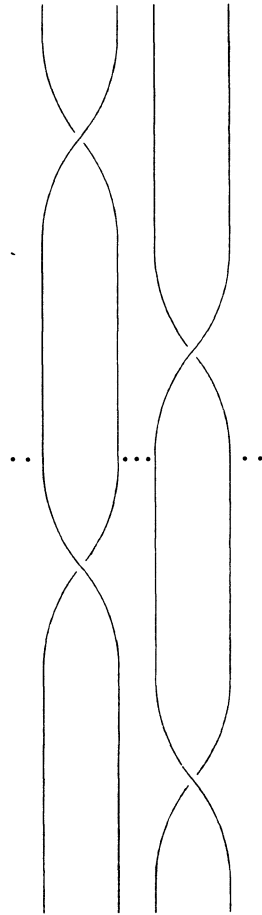


Fig. 5.

As for $\lambda_{i,i+1}$, it is on the boundary of λ_i and λ_{i+1} . A local cross section of E^{2n} by a plane perpendicular to the $(2n-2)$ -cell

$$(1 < \dots < i \leq i+1 \leq i+2 < \dots < n)$$

is shown in Figure 6.

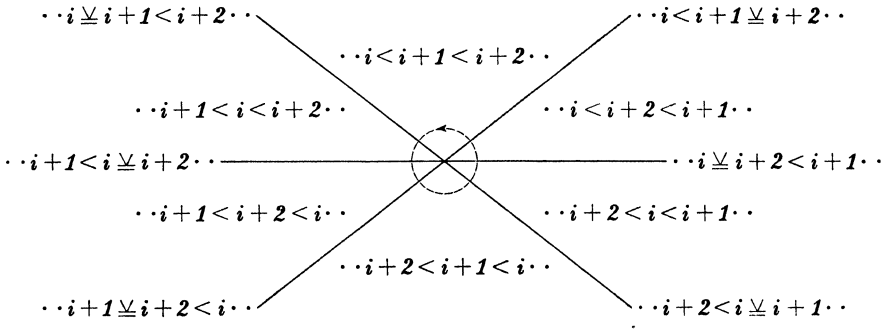


Fig. 6.

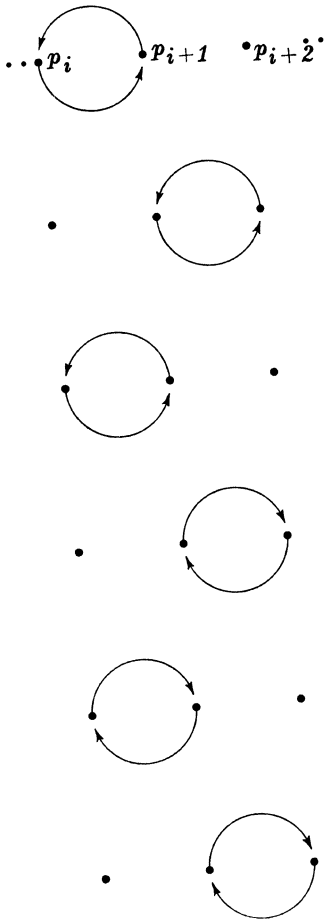


Fig. 7.

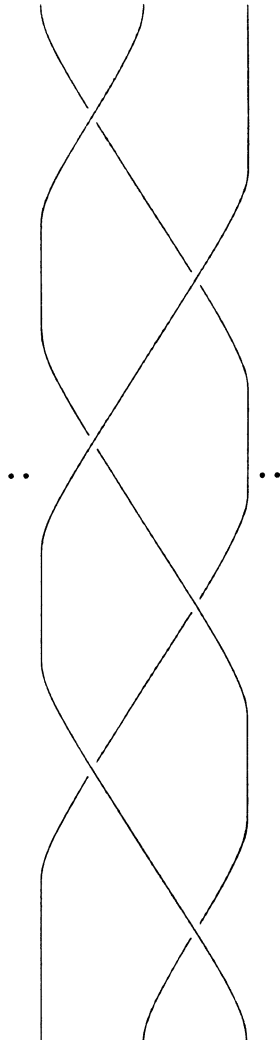


Fig. 8.

The corresponding relation $r_{i, i+1}$ is

$$r_{i, i+1} = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$$

as may be seen by traversing the dotted loop in Figure 6. The motion of (p_1, \dots, p_n) in E^2 thereby described is shown in Figure 7, and its interpretation as a braid in Figure 8.

Thus we have derived anew the well-known presentation

$$\mathcal{B}_n = (\sigma_1, \dots, \sigma_{n-1} : r_{1,2}, r_{1,3}, \dots, r_{n-2, n-1}).$$

REMARK. The same method could be used to find a presentation of \mathcal{S}_n , but the result could just as well be obtained by applying the Reidemeister-Schreier theorem.

8. Corollaries.

The covering of \tilde{E}^{2n} corresponding to the representation of \mathcal{B}_n on Σ_n (symmetric group of degree n) is just the space $F_{0,n}^2$ of [3], hence according to [3] has trivial homotopy groups above dimension 1. It follows then that \tilde{E}^{2n} is aspherical. As an immediate corollary we have:

COROLLARY 1. \mathcal{B}_n has no elements of finite order.

PROOF. \tilde{E}^{2n} is a finite dimensional $K(\mathcal{B}_n, 1)$ space, hence every subgroup of \mathcal{B}_n must be of finite geometric, hence finite cohomological dimension, but an element of finite order would generate a subgroup of infinite cohomological dimension.

Clearly \tilde{E}^{2n} is an open $2n$ -dimensional manifold so we have:

COROLLARY 2. \mathcal{B}_n has the homology groups of an open $2n$ -dimensional manifold.

REMARK. It seems reasonable to expect that the homology groups of \tilde{E}^{2n} , which by virtue of the asphericity of \tilde{E}^{2n} are those of \mathcal{B}_n , may be calculated from the cellular decomposition of \tilde{E}^{2n} which we have given.

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