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THE BRAUER GROUP OF A RING MODULO AN IDEAL

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Let R be a commutative ring. Let $B(R)$ be the Brauer group of R as defined in [2]. If I is an ideal of R the natural homomorphism from R to R/I induces a homomorphism from $B(R)$ to $B(R/I)$. We study this homomorphism in two contexts. In § 1 we show that if I is a nilpotent ideal then the homomorphism from $B(R)$ to $B(R/I)$ is an isomorphism. In § 2 we assume R is a local ring and I is its maximal ideal. We describe the kernel of the homomorphism from $B(R)$ to $B(R/I)$ and then show this homomorphism is onto if R/I is a number field, the p-adic completion of a number field, or a function field in one variable over a finite field. The results of § 1 have been proved by R. Hoobler using cohomological methods, but our proof is algebra theoretic and uses recent results of E. Ingraham [8]. Professor Ingraham provided part of the argument used in the proof of Theorem 1. The results of § 2 include some of those of V. J. Katz [9] who assumed R to be a regular local domain.

Throughout, our notational conventions will be the same as in [7] which contains the definitions of all unexplained terminology.

Section 1. In this section we prove the following theorem.

THEOREM 1. *Let I be a nilpotent ideal in the commutative ring R. Then the natural homomorphism from $B(R)$ to $B(R/I)$ is an isomorphism.*

PROOF. First show the homomorphism is one to one. Let A be a central separable R-algebra with $R/I \otimes A$ in the zero class of $B(R/I)$. Then $R/I \otimes A \simeq End_{R/I}(P)$ for P a finitely generated projective faithful R/I-module. We first show there is a finitely generated projective faithful R-module P^* with $R/I \otimes P^* = P$. Let P_1 be a finitely generated projective R/I-module with $P \oplus P_1 = (R/I)^n$. Let π be the corresponding projection of $(R/I)^n$ onto P. Then π is an idempotent in $M_n(R/I)$, the algebra of $n \times n$ matrices over R/I. Moreover, $M_n(R/I) = M_n(R)/IM_n(R)$ and $IM_n(R)$ is nilpotent so there is a projection π^* in $M_n(R)$ mapping onto π modulo $I \circ M_n(R)$. Let $P^* = \pi^*(R^{(n)})$. Then P^* is a finitely generated projective R-module and the diagram

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$$\begin{array}{ccc} R^n & \xrightarrow{\pi^*} & P^* \\ \downarrow & & \downarrow \\ (R/I)^n & \xrightarrow[\pi]{} & P \end{array}$$

commutes where the vertical arrows are natural. Thus $P^*/IP^* \simeq P$. By Nakayama's Lemma (Lemma 1.1.7 of [7]) $\text{Ann}_R(P^*)$ is generated by an idempotent. Since P is faithful, $\text{Ann}_R(P^*) \subseteq I$, so P^* is faithful by the nilpotence of I.

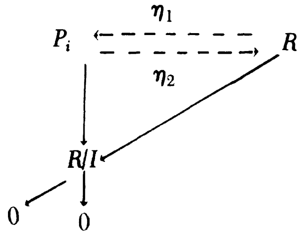
By a theorem of Bass (Corollary 16.2 of [4]) there is a finitely generated projective R/I-module Q with $P \otimes Q$ free over R/I. With Q^* constructed as P^* was in the previous paragraph we have

$$\begin{aligned} R/I \otimes \text{End}_R(Q^*) \otimes A &= (R/I \otimes A) \otimes_{R/I} \text{End}_{R/I}(Q) \\ &= \text{End}_{R/I}(P) \otimes_{R/I} \text{END}_{R/I}(Q) \\ &= \text{End}_{R/I}(P \otimes Q) \\ &= M_n(R/I). \end{aligned}$$

We next assert that since $P \otimes Q$ is free over R/I, then $P^* \otimes Q^*$ is free over R. Let F be a finitely generated projective R-module with $R/I \otimes F$ free over R/I. Note $R/I \otimes R = F/IF$ and let $\eta : F \to F/IF$ be the natural map. Then η induces the homomorphism $\phi : \text{End}_R(F) \to \text{End}_{R/I}(F/IF) = M_n(R/I)$. Let $1 = \sum_{i=1}^{n} \pi_i$, where the π_i are orthogonal projections in $M_n(R/I)$ whose images are isomorphic to R/I. Lift π_i to projections π_i^* in $\text{End}_R(F)$ so that $\phi(\pi_i^*) = \pi_i$. Then $F = \oplus \sum_{i=1}^{n} \pi_i^*(F)$. Let $P_i = \pi_i^*(F)$. The square

$$\begin{array}{ccc} F & \xrightarrow{\pi_i^*} & P_i \\ \downarrow & & \downarrow \\ F/IF & \xrightarrow[\pi]{} & R/I \end{array}$$

with vertical arrows the natural maps commutes so $P_i/IP_i = R/I$. This yields the diagram

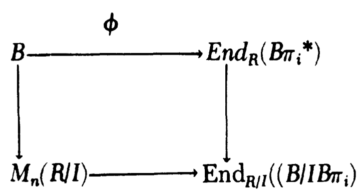


where η_1 and η_2 exist by the projectivity of R and P_i. now $\eta_2 \cdot \eta_1 \equiv 1$ mod I, so $\eta_2 \cdot \eta_1$ is invertible, $P_i \simeq R$ and F is free.

Finally, let $B = A \otimes \mathrm{End}_R(Q^*)$. Then $B/IB \simeq M_n(R/I)$. Write $1 = \sum_{i=1}^{n} \pi_i^*$ in B, where $\pi_i^* + IB = \pi_i$; the π_i are given in the previous paragraph. Now $\pi_i M_n(R/I)\pi_i \simeq R/I$ so $\pi_i^* B\pi_i^*$ is a finitely generated projective R-module with homomorphic image R/I by the map

$$\pi_i^* B\pi_i^* \to \pi_i^* B\pi_i^*/(\pi_i^* B\pi_i^*)I.$$

By the previous paragraph, $\pi_i^* B\pi_i^* \simeq R$. Let $\phi : B \to \mathrm{Hom}_R(B\pi_i^*, B\pi_i^*)$ by $\phi(b)[x\pi_i^*] = b \times \pi_i^*$, where we identify R with $\pi_i^* B\pi_i^*$. We get the diagram



$B\pi_i^*$ is finitely generated projective over R since π_i^* is idempotent and faithful since $\pi_i^* B\pi_i^* \simeq R$. The homomorphism ϕ is one to one on R, and so, by Theorem 2.3.7 of [7], is one to one. The image of ϕ is a central separable R-subalgebra of rank n^2. Since the bottom arrow is an isomorphism, the rank of $\mathrm{End}_R(B\pi_i^*)$ is n^2, so, by Theorem 2.4.3 of [7], ϕ is an isomorphism. Thus the homomorphism from $B(R)$ to $B(R/I)$ is one to one.

Next we show the homomorphism from $B(R)$ to $B(R/I)$ is onto. Let $\bar{A}$ be a central separable R/I-algebra. By an earlier paragraph there is a finitely generated projective faithful R-module P with $P/IP = \bar{A}$ as R-modules. We get the diagram

$$\begin{array}{ccc} \bar{A}\otimes\bar{A}^{0} & \xrightarrow{\ \psi\ } & \mathrm{End}_{R/I}(\bar{A}) \\ & & \Big\uparrow \phi \\ & & \mathrm{End}_R(P), \end{array}$$

where $\psi(a\otimes b)[x] = a \times b$. Identify $\bar{A}$ with $\psi(\bar{A}\otimes R/I)$ in $\mathrm{End}_{R/I}(\bar{A})$, and let $a_1, \cdots, a_n$ be elements in $\mathrm{End}_R(P)$ mapping by ϕ onto the generators of $\bar{A}$. Let B be the R-subalgebra of $\mathrm{End}_R(P)$ generated by $a_1, \cdots, a_n$. Then $B/N = \bar{A}$, where $N = B \cap \phi^{-1}(0)$. Let $g_i \in N$; then $g_i \in \mathrm{Hom}_R(P, IP)$. If n is the index of nilpotency of I, then

$$g_1 \cdots g_n(x) = i_n \cdots i_1 y_n = 0,\ i_j \in I,\ y_n \in P.$$

Thus N is nilpotent of index n. Since B is finitely generated by $a_1, \cdots, a_n$ as an algebra over R, and since B is an R-subalgebra of $\mathrm{End}_R(P)$ so that every a_i satisfies a monic polynomial over R, we have that B is finitely generated as a module over R. By Theorem 1 of [8] there is a separable R-subalgebra A of B with $A + N = B$.

We have A is a separable subalgebra of the central separable algebra $\mathrm{End}_R(P)$. Thus $\mathrm{End}_R(P)$ is finitely generated and projective as a module over $Z(A) =$ the center of A. Thus $\mathrm{End}_R(P)$ is a progenerator over $Z(A)$, and for some integer n the sequence $\mathrm{End}_R(P)^{(n)} \to Z(A) \to 0$ is exact and splits over $Z(A)$, and thus over R. We conclude that $Z(A)$ is projective over R, so A is projective over R. But projective separable algebras are finitely generated (2.2.1 of [1]), so A is finitely generated over R. The ideal $IA \subseteq \ker \phi|_A$, so ϕ induces a homomorphism $\phi : A/IA \to \bar{A}$. Under ϕ the center Z/IZ of A/IA is sent to R/I since $\bar{A}$ is central separable over R/I. Thus $Z/IZ \cong R/I \oplus Z_1/IZ_1$. Since I is nilpotent, idempotents can be lifted modulo IZ so $Z \cong R \oplus Z_1$, where $Z_1 \subseteq \ker \phi|_Z$. This decomposition of Z yields a corresponding decomposition $A = A_1 \oplus A_2$, where A_1 is central separable over R and $A_2 \subseteq \ker \phi|_A$. Thus $\phi(A_1) = R/I \otimes A_1 \cong A$, which shows the map from $B(R)$ to $B(R/I)$ is onto.

Section 2. In this section R is a local ring with maximal ideal m and residue class field $k = R/m$. Let Ω be a separable closure of R (see p. 99 of [7]), and let $\mathbf{S}$ be the collection of those finite separable projective extensions S of R in Ω so that $S/mS = \oplus \sum R/m$. Let $S_1, S_2 \in \mathbf{S}$, and let $S_1 \cdot S_2$ be the composite of S_1 and S_2 in Ω. Then $S_1 \cdot S_2/mS_1 \cdot S_2$ is an R-homomorphic image of

$$R/m \otimes (S_1 \otimes S_2) = \oplus \sum R/m.$$

Thus $S_1 \cdot S_2 \in \mathbf{S}$. This demonstrates that $\mathbf{S}$ is a directed set under inclusion. Let $N = \lim \{S \in \mathbf{S}\}$, then N is a locally separable extension of R in Ω. If $\sigma \in \mathrm{Aut}_R(\Omega)$, then $\sigma(N) \subseteq N$ since $\sigma(S) \in \mathbf{S}$ for each $S \in \mathbf{S}$. Thus N is a normal extension of R in Ω called the maximal completely split extension of R in Ω.

THEOREM 2. *Let R be a local ring with infinite residue class field R/m, and let N be the maximal completely split extension of R. Then the sequence*

$$0 \to B(N/R) \to B(R) \to B(R/m)$$

is exact.

PROOF. Let $|A| \in B(N/R)$. Then $N \otimes A = \mathrm{Hom}_N(P, P)$ for a finitely generated projective N-module P. The proof of proposition 3 in [6] shows that P is a free N-module. Let $x_1, \cdots, x_n$ be a basis for A over R. Each $1 \otimes x_i$ is an $n \times n$ martix A_i over N. Find an element $S \in \mathbf{S}$ containing the n^3 elements of N in these matrices. Then $S \otimes A \simeq M_n(S)$ by $1 \otimes x_i \to A_i$. *Now* $R/m \otimes S \otimes A \simeq (\oplus \sum R/m) \otimes A \simeq \oplus \sum (R/m \otimes A) \simeq \sum M_n(R/m)$, thus $R/m \otimes A$ is in the zero class of $B(R/m)$.

Now let $|A| \in B(R)$ with $R/m \otimes A$ in the zero class of $B(R/m)$. Then $R/m \otimes A \simeq A/mA \simeq M_n(R/m)$. Since R/m is infinite we can choose a diagonal matrix $\bar{\theta}$ in $M_n(R/m)$ with distinct entries β_i on the diagonal. Now $\bar{\theta}$ satisfies $\prod_{i=1}^{n} (\beta_i - x) = p(x)$ which is a separable polynomial over R/m. Also, $R/m(\bar{\theta})$ is a maximal commutative subalgebra of $M_n(R/m)$ since $R/m(\bar{\theta})$ includes all diagonal matrices.

Let $\theta \in A$ with θ mapping to $\bar{\theta}$ under the mapping from A to A/mA. Let $S = R \cdot 1 + R\theta + \cdots + R\theta^{n-1}$. Since $\{1, \theta, \cdots, \theta^{n-1}\}$ is a free basis for an R/m-direct summand of A/mA, Nakayama's Lemma (pg. 377 of [2]) implies that $\{1, \theta, \cdots, \theta^{n-1}\}$ is a free set of generators of S over R which extends to a free set of generators of A over R, so S is an R-module direct summand of A.

Now we show S is a subring of A by showing $\theta^n \in S$. Let $\bar{R}$ be the Henselization of R and let $\bar{A} = \bar{R} \otimes A$. Then $A/mA \simeq \bar{A}/m\bar{A}$. Since $\bar{R}$ is Henselian and $\bar{A}/m\bar{A}$ is in the zero class of $B(\bar{R}/m\bar{R})$, A is in the zero class of $B(R)$ as is shown in [3]. Thus $\bar{A} \simeq M_n(\bar{R})$, and A is a subring of $\bar{A}$. By the Cayley-Hamilton Theorem, θ satisfies a monic polynomial of degree n over R. But S is an R direct summand of A, and so, since $\theta^n \in \bar{R} \otimes S$, we have $\theta^n \in S$.

S is separable over R since $S \cap mS = S \cap mA$, and thus $S/mS = R/m(\bar{\theta})$, which is separable over R/m. Let S^* be the commutant of

S of A; by Theorem 4.3 of [7], S^* is a finitely generated projective R-algebra, and the commutant in A of S^* is S. But $S^* \cap mA$ is a two-sided ideal in S^*, so, by Corollary 3.2 of [2], there is an ideal M of S with $MS^* = S^* \cap mA$. But $MS^* \cap S^* = M = (S^* \cap mA) \cap S = mS$, so $M = mS$ and $S^* = mS^* + S$. By Nakayama's Lemma, $S^* = S$, so by Theorem 5.5 of [7], $S \otimes_R A$ is in the zero class of $B(S)$. Observe that $S \in \mathbf{S}$ since $S/mS = R/m(\bar{\theta}) = \oplus \sum R/m$. This proves the theorem.

If R/m is a finite field then $B(R/m)$ is trivial so the natural map from $B(R)$ to $B(R/m)$ is onto and every element of $B(R)$ is in the kernel. Our last theorem gives another condition under which the homomorphism from $B(R)$ to $B(R/m)$ is onto.

THEOREM 3. *Let R be a local ring with maximal ideal m and residue class field $k = R/m$. Assume that every element of $B(k)$ of order p^m for p a prime is split by an extension $k(\epsilon)$ of k where ϵ is a p^n-th root of unity. Then the homomorphism $B(R) \to B(k)$ is onto.*

PROOF. Every element in $B(k)$ is a product of elements of prime power order. Thus it suffices to show that each such element is in the image of the map from $B(R)$ to $B(k)$. Let $|A|$ be a class in $B(k)$ of order p^m, p a prime. Then $A \sim \Delta(k(\epsilon), H, \beta)$, where ϵ is a primitive p^n-th root of 1, $H = \mathrm{Gal}(k(\epsilon)/k)$, and $\beta \in Z^2(H, k(\epsilon)^*)$.

Case 1. p is odd.

In this case H is cyclic. Let $S = R(\epsilon)$, where ϵ is a primitive p^n-th root of 1 over R. Then S is a Galois extension of R with cyclic Galois group G. Moreover, $S/mS \simeq k(\epsilon) \oplus \cdots \oplus k(\epsilon)$ is a Galois extension of k with group G. Note S/mS splits $|A|$, so $|A| = |\Delta(S/mS, G, \rho)|$, where if $G = \langle \sigma \rangle$, then one can choose ρ so that

$$\rho(\sigma^i, \sigma^j) = \begin{cases} 1, & i + j < o(\langle \sigma \rangle) \\ \bar{a}, & i + j \geqq o(\langle \sigma \rangle), \bar{a} \in k. \end{cases}$$

Let $|\Delta(S, G, \alpha)|$ be in $B(R)$, where

$$\alpha(\sigma^i, \sigma^j) = \begin{cases} 1, & i + j < o(\langle \sigma \rangle) \\ a, & i + j \geqq o(\langle \sigma \rangle), a \in R, a + m = \bar{a}. \end{cases}$$

Then clearly, $|\Delta(S, G, \alpha)|$ maps onto $|A|$.

Case 2. $p = 2$.

Subcase 1. $\sqrt{-1} \in k$ and $\sqrt{-1} \in R$. In this case choose ϵ, S as in Case 1, then H, G are cyclic and one proceeds as in Case 1.

Subcase 2. $\sqrt{-1} \notin k$.

In this case $k(\epsilon)$ and $R(\epsilon)$ have the same rank over k, R, respectively, so $R(\epsilon)/mR(\epsilon) \simeq k(\epsilon)$, and $R(\epsilon)$ is a local ring. By proceeding as in the proof of Theorem 53.3 of [5], one can, by adjoining roots of polynomials of the form $x^{2^k} - \bar{a}$ to $k(\epsilon)$, write a 2-cocycle $\bar{\beta}$ on an extension K of $k(\epsilon)$ with Galois group $\bar{H}$ over k with the values of $\bar{\beta}$ in roots of unity and $|A| = |\Delta(K, \bar{H}, \bar{\beta})|$. Because $R(\epsilon)$ and $k(\epsilon)$ both have a primitive 2^n-th root of 1, one can construct by Kummer's Theory (p. 130 of [7]) an extension N of $R(\epsilon)$ obtained by adjoining roots of polynomials of the form $x^{2^k} - a$ to $R(\epsilon)$, where $a + m = \bar{a}$, $N/mN = K$, and $\mathrm{Gal}(N/R) = \bar{H}$. Consider $\Delta(N, \bar{H}, \beta)$, where β has the same root of unity values as $\bar{\beta}$. Clearly $|\Delta(N, \bar{H}, \beta)|$ maps into $|\Delta(k, \bar{H}, \beta)| = |A|$.

Subcase 3. $\sqrt{-1} \in k$, but $\sqrt{-1} \notin R$.

In this case $|A| = |\Delta(k(\epsilon), H, a)$ where H is cyclic. Also $R(\epsilon)$ is a normal separable extension of R with Galois group $G = C_2 \times C_n$ (where C_n is the cyclic group of order n and $n = 2^k$). Moreover,

$$\begin{aligned} R(\epsilon)/mR(\epsilon) &= [k(\epsilon) \oplus \cdots \oplus k(\epsilon)] \oplus [k(\epsilon) \oplus \cdots \oplus k(\epsilon)] \\ &= N \oplus N \end{aligned}$$

is a Galois extension of R/m with Galois group G, where N is identified with either term in square brackets. Write $G = \langle\sigma\rangle \times \langle\tau\rangle$ with $\sigma^2 = e$, $\tau^n = e$. Then $N \simeq \{(\alpha, \alpha) \mid \alpha \in N\} = (N \otimes N)^{\langle\sigma\rangle}$ is a Galois extension of k with cyclic Galois group $\langle\tau\rangle$, and A is split by N. Thus $A = |\Delta(N, \langle\tau\rangle, \bar{\beta})|$, where

$$\bar{\beta}(\tau^i, \tau^j) = \begin{cases} 1, & i + j < n \\ \bar{a}, & i + j \geqq n, \quad \bar{a} \in k. \end{cases}$$

Exactly as in Theorem 8.15E of [1], we have $A = |\Delta(N \oplus N, G, \beta)|$, where

$$\beta(\sigma^i\tau^j, \sigma^k\tau^l) = \begin{cases} 1, & j + l < n \\ \bar{a}, & j + l \geqq n. \end{cases}$$

Let $|\Delta(R(\epsilon), G, \alpha)|$ be an element in $B(R)$ with α given by

$$\alpha(\sigma^i\tau^j, \sigma^k\tau^l) = \begin{cases} 1, & j + l < n \\ a, & j + l \geqq n, \ a \in R, \ a + m = \bar{a}. \end{cases}$$

Again, $|A|$ is the image of $|\Delta(R(\epsilon), G, \alpha)|$. This completes the proof.

The hypothesis of Theorem 3 is satisfied by algebraic number fields, function fields in one variable over finite fields, and p-adic completions of number fields. We thus have the following theorem.

THEOREM 4. *Let R be a local ring with maximal ideal m and residue class field $R/m = k$. Let N be the maximal completely split extension of R. If k is an algebraic number field, a function field in one variable over a finite field, or the p-adic completion of a number field, then the sequence*

$$0 \to B(N/R) \to B(R) \to B(R/m) \to 0$$

is exact.

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