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# THE BRAUER–MANIN OBSTRUCTION AND Ш[2]

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*Abstract*

We discuss the Brauer–Manin obstruction on del Pezzo surfaces of degree 4. We outline a detailed algorithm for computing the obstruction and provide associated programs in Magma. This is illustrated with the computation of an example with an irreducible cubic factor in the singular locus of the defining pencil of quadrics (in contrast to previous examples, which had at worst quadratic irreducible factors). We exploit the relationship with the Tate–Shafarevich group to give new types of examples of Ш[2], for families of curves of genus 2 of the form $y^2 = f(x)$, where $f(x)$ is a quintic containing an irreducible cubic factor.

## 1. *Introduction*

The main aim of this article is to provide a detailed description and implementation of the algorithm for computing the Brauer–Manin obstruction of del Pezzo surfaces of degree 4. The algorithm is in general terms well known (see, for example, [**9, 19**]), but we shall build up a series of geometric and group-theoretic lemmas which allow us to make the procedure more explicit and to implement the algorithm in Magma.

We begin by discussing current knowledge of the Brauer–Manin obstruction for rational surfaces. An excellent account of the history of this topic, together with a description of contemporary methods, can be found in [**7**]. More recent advances, including some important ones for del Pezzo surfaces of degree 4, are described in [**20**]. The *Brauer group* of a scheme is a generalisation of the Brauer group of a field. For background information on the Brauer group, see [**8**]. In particular, it is shown there that the Brauer group of a smooth variety $X$ is the subgroup of the Brauer group of its function field $k(X)$ consisting of those elements which are 'unramified' in a certain sense; a central simple algebra over $k(X)$ satisfying this property is called an *Azumaya algebra* on X. Furthermore, it is shown that the Brauer group is a birational invariant of smooth projective varieties. One deduces that the Brauer group of a smooth projective rational variety *over an algebraically closed field* is trivial, since this is true for projective space. Let $k$ be a number field. It follows from the Hochschild–Serre spectral sequence and the fact that $H^3(k, \bar{k}^*) = 0$ that, for any rational variety $X$ over $k$, there is an exact sequence

$$\operatorname{Br} k \to \operatorname{Br} X \to H^1(k, \operatorname{Pic} \bar{X}) \to 0,$$

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which will be our tool for calculating $\operatorname{Br} X$. Here $\bar{X}$ means the base extension of $X$ to the algebraic closure $\bar{k}$ of $k$. When $X$ has points everywhere locally, the first arrow is injective. For a rational surface, $H^1(k, \operatorname{Pic} \bar{X})$ is a finite group; for the surfaces we are considering, it is always of order 1, 2 or 4.

It was noticed by Manin [12] that the Brauer group could be used to define an obstruction to the existence of $k$-rational points on a variety $X$, as follows. An element of $\operatorname{Br} X$ can be evaluated at any $K$-valued point of $X$ (where $K$ is any field containing $k$) to obtain an element of $\operatorname{Br} K$. Denote by $X(A_k)$ the set of adelic points of $X$. When $X$ is a complete variety, this is equal to $\prod_v X(k_v)$, the product being over all places of $k$. The set $X(k)$ of rational points is contained in $X(A_k)$ under the diagonal embedding. For the existence of a rational point on $X$, it is clearly necessary that $X(A_k)$ be non-empty. Let $\operatorname{inv}_v : \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$ be the local invariant map. Manin considered the pairing $\operatorname{Br} X \times X(A_k) \to \mathbb{Q}/\mathbb{Z}$ defined by

$$(\mathcal{A}, (x_v)) \mapsto \sum_v \operatorname{inv}_v(\mathcal{A}(x_v))$$

and observed that, by the well-known local-global principle for Brauer groups, the rational points of $X$ must be contained in the right kernel of this map, which we denote $X(A_k)^{\mathrm{Br}}$. If $X(A_k)^{\mathrm{Br}}$ can be shown to be empty, then we say that there is a *Brauer–Manin obstruction* to the existence of rational points on $X$. This obstruction accounted for all known counterexamples to the Hasse principle at that time.[1] In principle, the Brauer–Manin obstruction is computable, at least for rational varieties since $\operatorname{Pic} \bar{X}$ is finitely generated. In particular, Colliot-Thélène, Kanevsky and Sansuc [4] described in detail how to compute the obstruction for diagonal cubic surfaces, which are del Pezzo surfaces of degree 3. This article describes an algorithm along the same lines for computing the Brauer–Manin obstruction on del Pezzo surfaces of degree 4.

A natural question is whether the Brauer–Manin obstruction is the *only* obstruction to the Hasse principle for certain classes of varieties: that is, does $X(A_k)^{\mathrm{Br}} \neq \emptyset$ imply $X(k) \neq \emptyset$? This has been shown not to be true for all surfaces [16], but is conjectured to be true for all rational varieties [7, p. 319]. If the conjecture is correct, then the algorithm described in this article gives a method for determining the existence of rational points on del Pezzo surfaces of degree 4.

Progress towards proving this conjecture has been made in certain special cases satisfying geometric conditions, under which certain general techniques can be applied; two main techniques are the fibration method and the method of descent, both of which are described in [7]. For example, in the articles [5, 6], Colliot-Thélène, Sansuc and Swinnerton-Dyer studied intersections of two quadrics in general dimension, of which del Pezzo surfaces of degree 4 are a special case; they proved, among several other results, that such a surface satisfies the Hasse principle if it contains a pair of skew conjugate lines.

The fibration method can be applied much more widely if one assumes Schinzel's hypothesis, which is a far-reaching conjecture generalising both the twin prime conjecture and Dirichlet's theorem on primes in arithmetic progression. Under that hypothesis, much can be proved about surfaces fibred as a pencil of curves of genus 1,

[1] The fact that the Cassels–Guy counterexample is explained by the Manin obstruction was not proved until later.

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of which our del Pezzo surfaces are an example; for the state of the art in this direction, see [20].

In Section 2, we discuss some of the prerequisite geometry of a del Pezzo surface of degree 4: the families of conics and rational quartic curves, and the configuration of lines. In Section 3, we describe in detail the algorithm itself, which involves determining the Brauer group (via the Galois group of the lines), determining the rational conics on the surface, and finding representative Azumaya algebras. We illustrate the algorithm with a worked example in Section 4, which differs from previous examples of the Brauer–Manin obstruction by having an irreducible cubic factor in the singular locus of the defining pencil of quadrics.

In Section 5, we discuss the relationship between del Pezzo surfaces of degree 4 and the homogeneous spaces arising from a 2-descent on the Jacobian of a curve of genus 2. When the del Pezzo surface violates the Hasse principle, this can give rise to members of Ш[2] of the Jacobian. When the del Pezzo surface has rational points, other methods for computing Ш[2] are possible, and Ш[2] may still be exhibited via field extensions and product varieties. In Section 6, we conclude by giving new types of examples of Ш[2] on Jacobians of genus 2 curves, including an infinite family of twists.

## 2. *Geometry of del Pezzo surfaces of degree* 4

In this section we review some of the classical geometry of del Pezzo surfaces of degree 4. There are several good references for the contents of this section: Chapter 4 of Manin's book [13]; Chapter 2 of Reid's thesis [14]; and Section 1 of [5]. While discussing geometry, we work over an algebraically closed field.

Definition 1. A *del Pezzo surface* is a rational surface $S$ such that $-K_S$ is ample, where $K_S$ is the canonical divisor class of $S$.

We will be concerned with the case $K_S^2 = 4$. Geometrically a del Pezzo surface with this property is the blowup of $\mathbb{P}^2$ in five points, of which no three lie on a single line. Also, in this case $-K_S$ is in fact very ample, so that $S$ can be embedded in $\mathbb{P}^4$ as a surface of degree 4 which is an intersection of two quadrics. We will always assume that our surface is embedded like this.

Under the embedding given by $-K_S$, the straight lines on $S$ are the exceptional curves, which follows from the adjunction formula. There are 16 exceptional curves on $S$, which we identify as follows: the 5 exceptional divisors of the blow-up, $e_1, \ldots, e_5$; the 10 strict transforms of lines joining two of the blown-up points, $l_{ij}$; and the strict transform of the conic through the five points, $q$. We will use as basis for the Picard group the classes $L$ of the inverse image of a general line in $\mathbb{P}^2$ and $E_1, \ldots, E_5$, the classes of $e_1, \ldots, e_5$. Then $K_S = -3L + \sum E_i$ and $H$, the hyperplane class of $S$, is equal to $-K_S$. In this basis, the class of $l_{ij}$ is $L - E_i - E_j$ and the class of $q$ is $2L - \sum E_i$. Hence, or directly from the description of the lines, we can state which of the lines intersect which others: $e_i$ does not meet $e_j$ if $i \neq j$, $e_i$ meets $l_{jk}$ if and only if $i \in \{j, k\}$, $e_i$ meets $q$, $l_{ij}$ meets $l_{kl}$ if and only if $\{i, j\} \cap \{k, l\} = \emptyset$ or $\{i, j\} = \{k, l\}$, and $l_{ij}$ does not meet $q$.

We will also need to describe the families of conics lying on a del Pezzo surface. This may be implicit in the discussion at the beginning of Chapter 3 of [10].

THEOREM 2. *There are* 10 *families of conics lying on a del Pezzo surface. Their classes in the Picard group are of the form* $C_i = L - E_i$ *and* $C_i' = H - C_i$*. Over an algebraically closed field, the conics in each class are parametrized by a* $\mathbb{P}^1$*.*

*Proof.* A conic $C$ is a curve of degree 2, so $C \cdot H = 2$. On the other hand, it has genus 0, so by the adjunction formula $C \cdot (C + K_S) = -2$. Since $K_S = -H$, it follows that $C^2 = 0$.

If a divisor class $D = aL - \sum b_i E_i$ satisfies these criteria, then so does $H - D$. Hence, it suffices to consider the case $a \geqslant 2$.

We have $a^2 - \sum b_i^2 = 0$ and $3a - \sum b_i = 2$. If $a = 2$, the equality $\sum b_i^2 = 4$ implies that either one of the $b_i$ is $\pm 2$ and the rest 0, which is inconsistent with the second equality, or that four of the $b_i$ are $\pm 1$ and the other one 0. For the second equality to be satisfied, we need four of the $b_i$ to be 1, and we obtain the class $H - L - E_i = H - C_i$.

Now suppose that $a \geqslant 3$.By the Cauchy–Schwartz inequality, $\sum |b_i| \leqslant \sqrt{5 \sum b_i^2} = \sqrt{5}a$. However, $3a - \sum b_i \geqslant 3a - \sum |b_i|$. Since $3a - \sum b_i = 2$, it follows that $3a - \sqrt{5}a \leqslant 2$, a contradiction.

We conclude that the only possible conic classes with $a \geqslant 2$ are those given, and hence that the only ones with $a < 2$ are $H - (H - C_i) = C_i$.

Now, in fact, each of these classes is represented by an irreducible conic; for let $C$ be such a class. The residual class $H - C$ is the sum of two intersecting, and hence coplanar, lines ($l_{ij} + e_j$ in the case $L - E_i$, and $q + e_i$ for $H - L + E_i$). The divisors residual to this one in hyperplanes are again of the class $C$, and are parametrized by hyperplanes containing a given plane, which form a $\mathbb{P}^1$. Since there are only finitely many lines on $S$, only finitely many of these divisors can degenerate. $\square$

DEFINITION 3. Let $S$ be a nonsingular del Pezzo surface of degree 4, given as the base locus of a pencil of quadrics in $\mathbb{P}^4$, spanned by, say, $Q_1, Q_2$. Then the *characteristic form* of $S$ is $\chi_S(t, u) = \det(tM_1 + uM_2)$, where $M_1, M_2$ are the symmetric matrices describing $Q_1, Q_2$ respectively. We call $\chi_S(t, -1)$ the *characteristic polynomial* of $S$.

If $Q_1$ is nonsingular then $\chi_S(t, -1) = \det(M_1)\det(tI - M_1^{-1}M_2)$ is (up to a scalar) the characteristic polynomial of $M_1^{-1}M_2$. Note that $\chi_S$ is only defined up to linear transformation of the variables, corresponding to changing the basis of the pencil of quadrics.

The characteristic form describes the subscheme of singular quadrics in the pencil with coordinates $(t : u)$. If $S$ is nonsingular, then the characteristic form is squarefree: see [**14**, Proposition 2.1]. This implies that there are 5 distinct singular quadrics containing $S$ over the splitting field of the characteristic polynomial, each of rank 4. Hence, such a quadric $Q$ is a cone over a non-singular quadric $Q'$ in $\mathbb{P}^3$. The two families of lines on $Q'$ induce two families of planes on $Q$ by taking the span of a line on $Q'$ together with the singular point of $Q$. This gives ten families of planes in total.

THEOREM 4. *Let* $tQ_1 + uQ_2$ *be a pencil of quadrics in* $\mathbb{P}^4$ *whose intersection is the del Pezzo surface* $S$*, let* $Q$ *be a singular quadric in the pencil, and let* $P$ *be a plane lying on* $Q$*. Then* $P \cap S$ *is a conic, and every conic on a del Pezzo surface of degree* 4 *arises in this way.*


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*Proof.* To see that $P \cap S$ is a conic, simply observe that $P \cap S = P \cap Q \cap Q_2 = P \cap Q_2$, the intersection of a quadric threefold with a plane. Thus we obtain $5 \cdot 2 = 10$ irreducible one-parameter families of conics lying on $S$. Theorem 2 implies that there can be no more. □

Similar results hold for degree 4 curves of genus 0 on a del Pezzo surface of degree 4.

THEOREM 5. *There are* 40 *families of reduced curves of degree* 4 *and arithmetic genus* 0 *on a del Pezzo of degree* 4. *Their divisor classes are those of the form* $C_1 + C_2$, *where* $C_1$ *and* $C_2$ *are distinct classes of conics such that* $C_1 + C_2 \neq H$.

*Proof.* A rational quartic curve $C$ on a del Pezzo surface $S$ of degree 4 has $C \cdot H = 4$ and $C \cdot (C - H) = -2$, so that $C^2 = 2$. First, let $C_1$ and $C_2$ be distinct classes of conics on $S$, with $C_1 \neq H - C_2$. Then $C_1 \cdot C_2 = 1$. To check this, we calculate $(L - E_i) \cdot (L - E_j) = L^2 - E_i \cdot L - L \cdot E_j + E_i \cdot E_j = 1 - 0 - 0 + 0 = 1$. Then $H \cdot (L - E_j) = 2$, so $(L - E_i) \cdot (H - (L - E_j)) = 2$, and likewise $(H - (L - E_i)) \cdot (H - (L - E_j)) = 2$. This gives 40 distinct divisor classes: 10 of the form $2L - E_i - E_j$, 20 of the form $H + E_i - E_j$, and 10 of the form $2H - 2L + E_i + E_j$.

Conversely, let $C$ be a divisor class represented by rational quartic curves; in other words, suppose that $C = aL - \sum b_i E_i$, so that $C \cdot H = 4$ and $C^2 = 2$. Similarly to the proof of Theorem 2, if $C$ has these properties so does $2H - C$, so it suffices to consider the case $a \geqslant 3$. If $a = 3$, then $\sum b_i = 5$ and $\sum b_i^2 = 7$, so the $b_i$ in some order are $2, 1, 1, 1, 0$, giving $H + E_i - E_j$ as claimed. If $a = 4$, then $\sum b_i = 8$ and $\sum b_i^2 = 14$. This means that no $b_i$ may be greater in absolute value than 3; but 3 is impossible, for then we would have $3, \pm 2, \pm 1, 0, 0$ in some order, and these cannot add to 8. Therefore, the $b_i$ must be $\pm 2, \pm 2, \pm 2, \pm 1, \pm 1$ in some order; for them to add to 8, all must be positive, and so we get $2H - 2L + E_i + E_j$. If $a = 5$, then $\sum b_i = 11$ and $\sum b_i^2 = 23$. If one of the $b_i$ is 4, then the rest must be $\pm 2, \pm 1, \pm 1, \pm 1$, which does not work; otherwise we must have at least two that are $\pm 3$, and then the others are $\pm 2, \pm 1$, which are too small. Finally, if $a > 5$, then as before $\sum b_i = 3a - 4 \leqslant \sum |b_i| \leqslant \sqrt{5 \sum b_i^2} = \sqrt{5}a$, a contradiction.

Note that every such class contains irreducible curves; for let $D$ be such a class. Then $2H - D$ is as well, and it contains a curve which is the union of two conics. There is then a 3-parameter family of quadrics vanishing on this curve modulo those vanishing on $S$, and the residual intersection is a curve in $D$. But only a 2-dimensional subfamily of these are reducible, because the families of conics are 1-dimensional. □

We now need to discuss the configuration of lines on a del Pezzo surface of degree 4. First, we note its automorphism group.

THEOREM 6. *The automorphism group of the configuration of sixteen lines with their intersections is the Weyl group* $\mathcal{W}(D_5)$.

*Proof.* This is part of [**13**, Theorem 25.4]. □

We now return to working over a number field $k$. The action of $\mathrm{Gal}(\bar{k}/k)$ on the sixteen lines preserves intersections, and hence we get a representation $\phi : \mathrm{Gal}(\bar{k}/k) \to \mathcal{W}$. We write $L$ for the field corresponding to the kernel of this representation. It turns out that the Weyl group has four normal subgroups: the trivial subgroup, the Weyl group itself, the subgroup of words of even length, and a fourth subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ by which the quotient is isomorphic to $S_5$. The subfield corresponding to this quotient is easily identified. (Compare the proof of [**10**, Theorem H].)

THEOREM 7. *Let $\sigma$ be the quotient map $\mathcal{W}(D_5) \to S_5$. Then the kernel of $\sigma \circ \phi$ is the absolute Galois group of the splitting field of the characteristic polynomial of the given del Pezzo surface.*

*Proof.* First we prove that the field of definition of the lines of the generic del Pezzo surface defined over $k(r_{11}, r_{12}, r_{22}, \ldots, r_{55}, s_{11}, \ldots, s_{55})$ by equations $\sum r_{ij}x_ix_j = \sum s_{ij}x_ix_j = 0$ is an extension of degree $1920 = \#\mathcal{W}(D_5)$. It suffices to do this for a single example. Thus, consider the nonsingular surface defined over $\mathbb{Q}$ by

$$\begin{cases} x_1x_4 + x_1x_5 + x_2^2 + x_2x_3 + x_3x_4 + x_3x_5 + x_4^2 = 0 \\ x_1x_2 + x_2^2 + x_2x_3 + x_2x_4 + x_3^2 + x_3x_4 + x_3x_5 + x_4^2 = 0. \end{cases}$$

Let $\alpha$ be a root of the polynomial

$$\begin{aligned} p(x) = x^{16} &+ 4x^{15} + 4x^{14} + 6x^{13} + 19x^{12} + 11x^{11} + 7x^{10} \\ &+ 27x^9 + 18x^8 - 18x^7 + 22x^6 + 16x^5 - 11x^4 + 6x^3 + 4x^2 - x + 1. \end{aligned}$$

Using MAGMA it can be shown that the Galois group of $p(x)$ is isomorphic to $\mathcal{W}(D_5)$ and that a line defined over $\mathbb{Q}(\alpha)$ lies on this surface. It follows that the field of definition of the lines is an extension of $\mathbb{Q}$ with the full Galois group $\mathcal{W}(D_5)$. (For the transcript of the verification of these statements, see Appendix A and [**3**].)

Now we will show that the sixteen lines are all defined over the compositum of the splitting field of the characteristic polynomial with the field of definition of one line. Since this compositum plainly has degree at most $120 \cdot 16 = 1920$, while the lines are defined over an extension of degree $\#\mathcal{W}(D_5) = 1920$, this is sufficient to prove the theorem.

Thus, let us consider a rational line $l_0$ over a field $k'$ in which the characteristic polynomial splits. For each of the five lines $l_i$ meeting $l_0$, we have a singular conic $l_0 \cup l_i$ on $S$. By Theorem 4, this conic lies on a plane contained in one of the 5 singular quadrics in $\mathbb{P}^4$ containing $S$. Since two planes on such a conic do not meet in a line contained in $S$, we see that each $l_i$ thus corresponds to a unique singular quadric, and thus they are not conjugate. Repeating the argument shows that each of the lines meeting one of the $l_i$ must be defined over $k'$ as well, and the lines meeting those too. But that includes all lines on $S$. $\square$

As discussed in [**18**], a certain type of collection of eight lines is particularly important; see the following definition.

DEFINITION 8. A *four* is a set of four skew lines on the surface that do not all meet a fifth one. A *double-four* is a four together with the four lines that meet three lines from the four. (Indeed, these four lines also constitute a four, and starting from this four produces the same double-four: see [**18**, p. 458].) In the notation introduced previously, for example, the lines $e_1, e_2, e_3, l_{45}$ constitute a four, and

$$e_1, e_2, e_3, l_{12}, l_{13}, l_{23}, l_{45}, q$$

make up a double-four.

Let us count the fours. Since the lines are permuted transitively by the Weyl group, and each four contains 1/4 of the lines, each line is contained in 1/4 of the fours, so it suffices to count fours containing $q$. The lines not meeting $q$ are the $l_{ij}$. Let us take $q$ and $l_{ij}$; then, rearranging subscripts if necessary, the third line is $l_{ik}$. These lines together with $l_{jk}$ constitute a four, but together with $l_{im}$ they do not, because those four lines all meet $e_i$. In particular, the fours containing $q$ are in bijection with the 10 subsets of $\{1, 2, 3, 4, 5\}$ of cardinality 3, and so there are 40 fours in all.

The fours pair up to form 20 double-fours, and using a computer it is readily checked that the Weyl group acts transitively on both fours and double-fours. The complement of a double-four is also a double-four, and each double-four is uniquely the union of two fours. Furthermore, the Picard classes containing rational quartic curves are the same as the classes of the form $l_1+m_2+m_3+m_4$, where $\{l_1, l_2, l_3, l_4\}$ is a four, $\{m_1, m_2, m_3, m_4\}$ is the complementary four, and $m_2, m_3, m_4$ are the lines in it that meet $l_1$. (It was pointed out in [**19**, Example 2] that this Picard class depends only on the four $\{l_1, l_2, l_3, l_4\}$, not on the choice of line from it.) In particular these classes are sums of two Picard classes of conics in a unique way.

Remark 9. In fact, there is a clear relation between double-fours and the singular quadrics containing $S$. Let $v_1, \ldots, v_5$ be the singular points of the 5 singular quadrics containing $S$.

The plane through $q$ and $v_i$ intersects $S$ in another line meeting $q$. Upon adjusting the labelling, we can assume it is $e_i$. The hyperplane through $q, v_i, v_j$ intersects $S$ in a fourth line, $l_{ij}$, that meets $e_i, e_j$ but not $q$.

One can check that, starting from $q$ and three vertices, say $v_1, v_2, v_3$, we can obtain 8 lines $q, e_1, e_2, e_3, l_{12}, l_{23}, l_{13}, l_{45}$, where $l_{45}$ for instance, lies in the hyperplane spanned by $e_3, v_1, v_2$.

The intersection graph, where the lines are the vertices and the edges join vertices if the corresponding lines intersect, is a cube. The two fours constituting this double-four can be recovered from the unique way in which a cube is a bipartite graph.

This shows how the 10 ways of choosing 3 of the 5 singular quadrics correspond to the 10 ways of splitting the lines into two disjoint double fours.

Finally, we review some results of Manin and Swinnerton-Dyer on the Brauer group of a del Pezzo surface of degree 4.

As in [**13**], we may compute $\operatorname{Br} S/\operatorname{Br} k$ as $H^1(k, \operatorname{Pic} \overline{S})$. Recall that $L/k$ was defined to be the fixed field of the kernel of $\phi : \operatorname{Gal}(\bar{k}/k) \to \mathcal{W}$; this is the smallest Galois extension over which all of the lines in $S$ are individually defined. The entire Picard group of $S$ is defined over $L$, because the lines generate the Picard group. By an easy calculation using the inflation-restriction sequence, we get

$$\operatorname{Br} S/\operatorname{Br} k \cong H^1\left(L/k, \operatorname{Pic} S \bigotimes\nolimits_k L\right).$$

Note in particular that $\operatorname{Br} S/\operatorname{Br} k$ is determined by $\operatorname{Gal}(L/k)$ as a subgroup of the Weyl group up to conjugacy. We must compute this group; to do so, it is sometimes necessary to determine $L$ and the action of $\operatorname{Gal}(L/k)$ on the sixteen lines. Sometimes, though, an alternative description of the Brauer group can be used to advantage.

THEOREM 10 (Swinnerton-Dyer). *Let $\alpha$ be a non-zero element of* $\operatorname{Br} S/\operatorname{Br} k$. *Then $\alpha$ can be represented by an Azumaya algebra in the following way: there is a double-four defined over $k$ whose constituent fours are not $k$-rational but defined over $k(\sqrt{b})$, for some $b \in k^*$. Further, let $V$ be a divisor defined over $k(\sqrt{b})$ whose class is the sum of the classes of one line in the double-four and the classes of the three lines in the double-four that meet it, and let $V'$ be the Galois conjugate of $V$. Let $H$ be a hyperplane section of $S$. Then the $k$-rational divisor $D = V + V' - 2H$ is principal, and if $f$ is a function whose divisor is $D$ then $\alpha$ is represented, modulo* $\operatorname{Br} k$, *by the quaternion algebra $(b, f)$.*

*Proof.* This is [**19**, Lemma 1 and Example 2]. As pointed out above, the divisor class of $V$ depends only on the four that the initial line belongs to, and since the fours are defined over $k(\sqrt{b})$, the divisor class itself is. Then, since the surface $S$ has points everywhere locally, this class is represented by a $k(\sqrt{b})$-rational divisor everywhere locally; and because the divisors in a class constitute a Brauer–Severi variety, there is such a divisor over $k(\sqrt{b})$. □

A converse of Theorem 10, determining when an algebra $(b, f)$ of the above type represents a non-trivial element in $\operatorname{Br} S/\operatorname{Br} k$, is also stated in [**19**], but we will not need the result here.

COROLLARY 11. *If $[L : k] > 96$ or if the characteristic form of $S$ has an irreducible factor of degree at least 4 then* $\operatorname{Br} S/\operatorname{Br} k$ *is trivial.*

*Proof.* The stabilizer of a double-four in $\mathcal{W}$ has order $1920/20 = 96$, so if $[L : k] > 96$ then there is no Galois-stable double-four. By Theorem 10, $\operatorname{Br} S/\operatorname{Br} k$ must be trivial.

If the characteristic form of $S$ has an irreducible factor of degree at least 4 then there is no Galois-stable way of selecting 3 singular quadrics containing $S$. From Remark 9 it follows that there is no Galois-stable pair of complementary double-fours, so no Galois-stable double-four either. By Theorem 10, $\operatorname{Br} S/\operatorname{Br} k$ must be trivial. □

It can be shown, either by explicit enumeration of the subgroups of the Weyl group or by careful consideration of the different cases, that $\operatorname{Br} S/\operatorname{Br} k$ is always either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$.

## 3. *The algorithm*

In this section we will explain how to use the theory of the Brauer group in combination with some results of Swinnerton-Dyer and the geometry described above to calculate the Brauer obstruction in concrete situations. We have implemented the following algorithm in MAGMA and made it available at [**11**]. For simplicity, we now assume that $k = \mathbb{Q}$. As input, we consider a del Pezzo surface $S$ of degree 4, given as the intersection of two quadrics $Q_1, Q_2$ in $\mathbb{P}^4$ over $\mathbb{Q}$. As output we return whether there is a Brauer–Manin obstruction for $S$ having rational points.

(0) Decide local solvability of $S$.

(1) Determine the Fano scheme and thus the individual lines.

(2) Find the Galois group of the lines, and hence the Brauer group.

(3) Determine the fours associated to the non-trivial elements of $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$.

(4) Find representative Azumaya algebras.

(5) Evaluate the Brauer–Manin obstruction.

In the early steps, it is possible to discover that the later steps will be unnecessary. For example, if the lines on the surface are all conjugate, there can be no interesting Brauer group, and so it is unnecessary to find the Galois group of the lines.

Let us start by giving the generic Galois action on the lines explicitly, since it is used several times in what follows.

PROPOSITION 12. *Let the sixteen lines on a del Pezzo surface of degree 4 be given in the order $e_1, \dots, e_5, l_{12}, l_{13}, \dots, l_{45}, q$. Then $\mathcal{W}$ is generated by elements $w_1, \dots, w_5$ which act by the following permutations:*

$$(1,2)(7,10)(8,11)(9,12), \quad (2,3)(6,7)(11,13)(12,14), \quad (3,4)(7,8)(10,11)(14,15),$$
$$(1,10)(2,7)(3,6)(15,16), \quad (4,5)(8,9)(11,12)(13,14).$$

*Proof.* This follows from the description of the permutations in [**13**, Section 25.5.7], and the enumeration of the roots in [**13**, Proposition 25.5.3]. □

### 3.1. *Step* 0: *Local solvability*

We decide local solvability over $\mathbb{R}$ by using Lagrange multipliers to search for maxima and minima of a function on the real locus. This locus is compact; if it is also nonempty, then any bounded continuous function must attain an extremum there.

Using the Weil bounds, we know that $S$ has points locally at all but finitely many, known, primes. For those primes $p$, we search for points modulo increasing powers of $p$ until either we find an obstruction to such points existing or we find an approximation to a point for which Hensel's lemma guarantees that it will lift to a point over $\mathbb{Q}_p$.

If $S$ does not have points locally at some place of $\mathbb{Q}$ then $S(\mathbb{Q})$ is empty and the Brauer–Manin obstruction is meaningless.

### 3.2. *Step* 1: *The Fano scheme*

(a) Construct the subscheme of $\mathbb{P}^4 \times \mathbb{P}^4$ consisting of pairs of points on $Q_1$ such that the line through them also lies on $Q_1$. Determine its image in $\mathbb{P}^9$ via the Plücker embedding.

(b) Repeat step (a) for $Q_2$.

(c) Intersect the schemes from (a) and (b) to obtain the 0-dimensional, degree 16 subscheme $F$ of the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$ of lines in $\mathbb{P}^4$ that lie on $S$.

### 3.3. *Step* 2: *Galois action on the lines*

Firstly, we consider two propositions that allow us to conclude that $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$ is trivial in many cases.

Proposition 13. *If* $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$ *is nontrivial, then the orbits of lines under the Galois group have one of the following sequences of sizes:*

$$(2,2,2,2,2,2,2,2), \quad (2,2,2,2,4,4), \quad (4,4,4,4), \quad (4,4,8), \quad (8,8).$$

*Proof.* This can be deduced from [13, Tables 2 and 3]. These tables cover only minimal del Pezzo surfaces of degree 4. However, blowing down a divisor on a non-minimal del Pezzo surface yields a del Pezzo surface of larger degree [13, Corollary 24.5.2] and the Brauer group of a del Pezzo surface of degree greater than 4 is trivial [13, Theorem 29.3]. □

Proposition 14. *Let $M$ be the field of least degree over which representatives of all $\mathbb{Q}$-orbits of lines are defined. If $[M:\mathbb{Q}] > 16$, then $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$ is trivial.*

*Proof.* Under the Galois correspondence, the field $M$ corresponds to the intersection $H$ of stabilizers of lines in each orbit. Using Magma, this can be computed for every conjugacy class of subgroups $G$ of $\mathcal{W}$ and it can be verified that $H^1(G, \operatorname{Pic}\bar{S})$ is trivial whenever the index of $H$ is greater than 16. Note also that the intersection of all conjugates of $H$ is the intersection of the stabilizers of all of the lines, and is therefore trivial; in other words, the Galois closure of $M$ is equal to $L$. □

We consider the splitting field $L$ of the scheme $F$ computed in Step 1 and the action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $F$.

(a) Decompose $F$ into Galois orbits using factorisation.

(b) *Early exit*: If the orbit lengths are not as listed in Proposition 13 then $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$ is trivial and we are done: there is no Brauer–Manin obstruction to $S$ having rational points.

(c) *Small case*: If $[L:\mathbb{Q}] \leqslant 16$ then we explicitly construct $L$ and determine all lines on $S$ over $L$ explicitly. We compute their intersections and label them according to a standard labelling to reconstruct $\operatorname{Gal}(L/\mathbb{Q})$ as a subgroup of $\mathcal{W}$.

(d) *Large case*: If $[L:\mathbb{Q}] > 16$, we will work backwards. We start from the large subgroups of $\mathcal{W}$ that produce nontrivial Brauer groups and determine whether $\operatorname{Gal}(L/\mathbb{Q})$ is one of them.
The group $\mathcal{W}$ has 78 conjugacy classes of subgroups of order greater than 16, but only 8 of these give nontrivial Brauer groups, all of which have $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$ cyclic of order 2.
These subgroups of the Weyl group are isomorphic to $C_2 \times A_4$, $S_4$, $C_2^2 \times D_8$, $C_2 \times S_4$ (three groups), $C_2^2 \times A_4$, and $C_2^2 \times S_4$. We consider each case individually. The assertions are presented without justification, but they are easily checked with, for instance, Magma.

$C_2 \times A_4$. There are 4 conjugacy classes of subgroups isomorphic to this one. It turns out that there is a nontrivial Brauer group if and only if there are 2 orbits of lines and there are 4 lines defined over a minimal field of definition of one line, of which the pairs of conjugate lines do not intersect. For completeness, we give the details of this case. The 4 conjugacy classes of subgroups are given by the following lists of generators:

$$\langle w_2w_1w_2w_3w_5w_3, w_2w_4w_3w_4w_5w_3w_2w_4,$$
$$w_3w_4w_5w_3, w_1w_2w_3w_4w_5w_3w_2w_1\rangle,$$
$$\langle w_3w_2, w_1w_2w_3w_5w_3w_2w_1, w_2w_4w_3w_4w_5w_3w_2w_4,$$
$$w_2w_3w_2w_4w_3w_5w_3w_2w_4w_3\rangle,$$
$$\langle w_2w_1w_2w_3w_5w_3, w_4w_5, w_3w_4w_5w_3, w_1w_2w_3w_4w_5w_3w_2w_1\rangle,$$
$$\langle w_2w_1w_2w_3w_5w_3, w_2w_1w_2w_3w_5w_3w_2w_1, w_2w_1w_3w_5w_3w_2,$$
$$w_2w_1w_2w_3w_2w_1w_4w_3w_2w_5w_3w_2w_1w_4w_3w_2w_5w_3\rangle,$$

and only the first of these has nontrivial cohomology. Now, the sizes of orbits for these groups are $(8,8),(4,4,8),(8,8),(2,6,8)$ respectively, so if the orbits do not have sizes $(8,8)$ there is no Brauer group. For the two groups for which the orbits do have sizes $(8,8)$, we consider the stabilizer of 1, which in fact also fixes $6,8,14$. In the first group, 1 and 14 are in the same orbit; these correspond to $e_1$ and $l_{35}$, which do not intersect. On the other hand, in the second group, 1 and 6 are in the same orbit; these correspond to the intersecting lines $e_1$ and $l_{12}$. Likewise, 6 and 8 do not intersect, whereas 8 and 14 do. We conclude that if $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to $C_2 \times A_4$, then the Brauer group is nontrivial if and only if there are two orbits of lines of size 8, and over the field of definition of a single line the two rational lines that were in the same $\mathbb{Q}$-orbit of lines do not intersect.

Similar arguments are required for the five remaining isomorphism classes of groups; in order to save space and avoid trying the reader's patience, these are not presented here. Instead, we only show the answers. See electronic resources in Appendix A and at [**3**] for an exhaustive treatment.

$S_4$. There are 6 conjugacy classes of subgroups isomorphic to this one. Again, there is a nontrivial Brauer group if and only if there are 2 orbits of lines and there are 4 lines defined over a minimal field of definition, of which the pairs of conjugate lines do not intersect.

$C_2^2 \times D_8$. There are 2 classes of subgroups of this type. The Brauer group is nontrivial if and only if there are 2 orbits of lines.

$C_2^2 \times A_4$. There are 2 such classes of subgroups. The Brauer group is nontrivial if and only if there are 2 orbits of lines.

$C_2 \times S_4$. There are 8 such conjugacy classes of subgroups. The Brauer group is trivial unless there are exactly 2 orbits of lines, which is the case for 4 of the classes. If there are 2 lines defined over the minimal field of definition, then the Brauer group is nontrivial. But if there are 4, we must consider the two lines in the same orbit defined over this minimal field. If they intersect, the Brauer group is trivial; if not, it is nontrivial.

$C_2^2 \times S_4$. Again, there are 2 conjugacy classes of subgroups of this isomorphism type; the Brauer group is nontrivial if and only if there are 2 orbits of lines.

(e) *Early exit*: Whether we are in the large case or the small case, we now know $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$. If it is trivial, we are done.



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### 3.4. *Step* 3: *Represent the Brauer group with double-fours*

We find several divisors $V$ of the type mentioned in Theorem 10 such that we know that for each non-trivial $\alpha \in \operatorname{Br} S/\operatorname{Br}\mathbb{Q}$, we have a representative divisor $V$.

(a) For each double-four $D$ representing an element $\alpha \in \operatorname{Br} S/\operatorname{Br}\mathbb{Q}$ we find the field $\mathbb{Q}(\sqrt{b})$ over which the fours making up $D$ are defined, which we call the *Brauer field* of $D$ (not of $\alpha$; it need not be the same for different double-fours representing the same Brauer class). In the small case this is straightforward, because the double-four consists of eight lines for which we have explicit equations and we can determine the field over which each of the fours is defined. In the large case it is more difficult, because we do not have the fours explicitly. Instead we must use some group theory. In most cases the Brauer field is the unique quadratic subfield of the field of definition of a single line. To be precise, this occurs in all of the large cases except that where the Galois group is of order 32. In that case, the orbits of lines are the double-fours themselves, and the lines in each double-four are defined over an extension of degree 16. Hence we may again determine the Brauer field by finding explicit equations for the lines in each double-four.

(b) Since $V$ is a class of degree 4 curves of arithmetic genus 0, Theorem 5 implies that $V$ is uniquely the sum of two classes $C_1, C_2$ of conics, defined over a field $K'$. When $[L : \mathbb{Q}] \leqslant 16$, we know the lines explicitly and we can determine $K'$. In the large case, we must use group theory: it turns out that when the Galois group has order 24, and in one of the cases when it is $C_2 \times S_4$, the conics are defined over the Brauer field. In the other cases, they are defined over a quadratic extension of this field. Fortunately, in the large cases where the conics are not defined over the Brauer field, the factorization type of the characteristic polynomial is either $(1, 1, 3)$ or $(2, 3)$, so the relevant conics are determined in advance as those corresponding to the factors of degree 1 or 2, and there is no ambiguity about $K'$.

(c) Consider the roots of the characteristic polynomial over $K'$. Each root corresponds to a singular quadric with vertex $T$ over a nonsingular quadric in $\mathbb{P}^3$. We determine the Fano scheme in the Grassmannian $\mathbb{G}(1, 3)$ of lines on this quadric and, by taking the planes spanned by these lines and $T$, map this to $\mathbb{G}(2, 4)$. This gives two conics in $\mathbb{P}^3$ that parametrize the planes on the singular quadric threefold, and hence the conics in the relevant divisor class on $S$.

d) Choose one of the classes, say $C_1$, from (b). According to Theorem 4, the members of that class correspond to planes on a singular quadric $Q$, which are themselves parametrized by a conic, say $Y$. The planes parametrized by $Y$ only meet in the singular point of $Q$, which is outside of $S$. Therefore, every point on $S$ has a unique plane $P$ from $Y$ through it. Since $S$ has points everywhere locally, it follows that $Y$ has points everywhere locally as well and hence is isomorphic to $\mathbb{P}^1$ over $K'$. By parametrizing $Y$, we can generate a number of conics on $S$ representing $C_1$ over $K'$.

### 3.5. *Step* 4: *Representative Azumaya algebras*

Our method of finding a representative Azumaya algebra depends on whether $K' = \mathbb{Q}(\sqrt{b})$ or not.

(a) If $V$ is represented by the sum of two conics defined over the Brauer field, then for each conic we find a linear form (unique up to scaling) over $\mathbb{Q}$ that vanishes on it and hence also on the conjugate. Let $l_1$ and $l_2$ be the forms for conics from different classes. The divisor of the section $l_1 l_2$ of $\mathcal{O}(2)$ is $V + V'$, and so for any linear form $h$ the quaternion algebra $(b, l_1 l_2/h^2)$ represents the element of $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$.

(b) $V$ is represented by the sum of two conics defined over $K'$, a quadratic extension of the Brauer field. Suppose that $C$ is one representative conic. The union $E$ of $C$ with its conjugate is defined over the Brauer field and represents $V$. Then $V'$, the conjugate of $V$ over $\mathbb{Q}$, is represented by the conjugate $E'$ of $E$. The curve $E \cup E'$ is a curve of arithmetic genus 5 whose divisor class on $S$ is equal to $2H$. It is therefore the intersection of three rational quadrics: the two that define $S$ and a new one. On the other hand, the curve $E$ lies on a $\mathbb{P}^5$ of quadrics defined over the Brauer field, and taking this modulo those defining $S$ gives us a $\mathbb{P}^3$. For any such quadric, we consider the residual intersection with the surface $S$: it is a new curve $F$ whose divisor class is $V'$. Again, we get a rational divisor of the form $V + V'$ by taking $F$ together with its conjugate. Let $q$ be a quadratic form that vanishes on this divisor but not on all of $S$: then the divisor of $q$ is $F + F'$, and so $(b, q/h^2)$ represents the element of the Brauer group.

### 3.6. *Step* 5: *Computing the Brauer–Manin obstruction*

At this point we have a number of representative quaternion Azumaya algebras that define each element of the Brauer group, modulo constant algebras. We must evaluate the local invariants at each prime $p$. For simplicity, in the following description we will only consider the case $\#(\operatorname{Br} S/\operatorname{Br}\mathbb{Q}) = 2$ and leave the reader to imagine the modifications needed for $\#(\operatorname{Br} S/\operatorname{Br}\mathbb{Q}) = 4$.

(a) For each finite prime $p$, proceed as follows.

1. Set $e = 1$, and consider all points on $S$ modulo $p^e$.
2. For each Azumaya algebra $\mathcal{A}$, evaluate the local invariant $\operatorname{inv}_p \mathcal{A}(P) \in \mathbb{Q}/\mathbb{Z}$ at each point $P$. Since the algebras are quaternion algebras, this is done by evaluating a Hilbert symbol. Note the points for which more precision is needed to compute the local invariant.
3. If there are points modulo $p^e$ with different local invariants for the same algebra, then that algebra cannot give a Brauer–Manin obstruction.
4. If there are points for which we could not yet compute the local invariant, increase $e$ by 1, lift all those points to $\mathbb{Z}/p^e\mathbb{Z}$ in all possible ways, and go to step 2.

(b) For real points, use Lagrange multipliers to find the maxima and minima of our functions on the real locus, if the Brauer field is not real.

(c) Once we have established that the local invariant is constant at each place for every algebra, we can check whether their sum is 0. If it is, then there is no Brauer obstruction from the algebra considered. If, however, the sum is $\frac{1}{2}$, then we have found a non-trivial Brauer–Manin obstruction.

There are three possible difficulties in this.

- If there is a large prime $p$ of bad reduction, it may take a very long time to find all the points mod $p$. To some extent this can be mitigated by noting that if a linear or quadratic form does not vanish on a point mod $p$ and if $p$ is unramified in the Brauer field, then the local invariant at that point is 0, so we may assume that all the forms vanish on the points. This greatly reduces the number of points to consider.
- Note that checking completely for each candidate point mod $p$ whether it lifts to a $\mathbb{Q}_p$-point can be computationally very expensive. One can leave out this check, at the expense of possibly erroneously concluding that the local invariant is not constant. This actually happens in one of the examples described below.
- Although different quaternion algebras representing the same class in $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$ do of course give the same sum of local invariants, the functions must in general be multiplied by constants (that is, the quaternion algebra must be tensored with $(c, d)$ for some rational $c$ and $d$) for them to give the same invariants at each place. This amounts to ensuring that our Azumaya algebras, which as computed are known only to represent the same element of $\operatorname{Br} S/\operatorname{Br}\mathbb{Q}$, actually represent the same element of $\operatorname{Br} S$. These constants cannot be determined *a priori*. Thus, when we consider a new prime, we consider different algebras at the same point to compare their local invariants, and multiply by a constant to make sure that the algebras have the same invariants at the prime. In fact, it is enough to tensor the algebras with $(b, h)$, where $\mathbb{Q}(\sqrt{b})$ is the Brauer field corresponding to the given algebra: see [1].

## 4. *Worked examples*

We will demonstrate the application of the algorithm in two examples, both of which violate the Hasse principle.

EXAMPLE 15. Let $Q_1 = vw+10wx+5y^2-x^2$ and let $Q_2 = (v+w)(v+2w)+5z^2-x^2$. We will refer to this surface as $S_1$. It is routine to prove that the surface $S_1$ where $Q_1$ and $Q_2$ vanish has points everywhere locally.

Let us give an elementary proof that it has no rational points. We may assume that $v, w, x, y, z$ are integers with no common factor among the five.

First suppose that $5 \mid v$. Then, because $Q_1 = 0$, we have $5 \mid x$. But then, considering $Q_2$, we deduce that $5 \mid 2w^2$ and so that $5 \mid w$. This implies that $25 \mid 5y^2$ and that $25 \mid 5z^2$, so $5 \mid y$ and $5 \mid z$, a contradiction. Similarly, 5 cannot divide $w$. If $5 \mid x$, then from $Q_1 = 0$ we get $5 \mid vw$, contradicting the above. Thus 5 does not divide $vwx$, and so 5 does not divide $(v+w)(v+2w)$ either.

Therefore $w/v \pmod 5$ is either 1 or 3. If it is 3, the first equation is 5-adically impossible, so let us assume that it is 1. Then either $v$ and $w$ are both 5-adic squares and $v+w$ and $v+2w$ are not, or vice versa. Let us only consider the first case, since the second is very similar.

Let $p$ be a prime that divides $v+w$ to odd exponent and is congruent to 2 or 3 mod 5. Now, $p$ must divide $x^2-5z^2$ to even exponent, so it must divide $v+2w$

to odd exponent as well. It therefore divides $v$ and $w$. If it divides $x$ and $z$, then it divides $y$, a contradiction. But the only such prime that can divide $x^2-5z^2$ without dividing both $x$ and $z$ is 2, so $p=2$.

Now let us consider the equations 2-adically. We have seen that $v$ and $w$ are even, while $x$ and $z$ are odd; $y$ must then be odd as well, and both $x^2-5y^2$ and $x^2-5z^2$ are congruent to $4 \pmod 8$. If $(v+w)(v+2w) \equiv 4 \pmod 8$, then $v+w, v+2w \equiv 2 \pmod 4$, so that $4 \mid w$. But then $8 \mid vw+10wx$, which is a contradiction because 8 does not divide $x^2-5y^2$.

We now show how the algorithm described in Section 3 can show that $S_1$ has no rational points. We find that the characteristic polynomial of $S_1$ is $t^4u+7t^3u^2-93t^2u^3+tu^4$ up to scaling, and this factors as

$$t \cdot u \cdot (t^3+7t^2u-93tu^2+u^3).$$

Further, the Fano scheme is a union of two irreducible components of degree 8, and both orbits have representatives that are defined over the number field $\mathbb{Q}(\alpha)$, where $\alpha$ is a root of

$$x^8-4x^7-2x^6+40x^5-7x^4-64x^3+446x^2+90x+405.$$

The Galois group of this polynomial has order 48 (being isomorphic to $C_2 \times S_4$), so it would be undesirable to compute directly in the splitting field. On the other hand, there are two orbits of lines, and for this Galois group this means that the Brauer group modulo constants is isomorphic to $C_2$. Also, in this case, the Brauer group is described by conics rational over the Brauer field, and this field is the unique quadratic subfield of $\mathbb{Q}(\alpha)$, which is $\mathbb{Q}(\sqrt{5})$.

The two rational singular quadrics are $Q_1$ and $Q_2$ themselves. It turns out that, when the Fano scheme of planes on $Q_1$ is mapped to $\mathbb{P}^9$ by the Plücker embedding, we get two curves of degree 2 in $\mathbb{P}^9$, defined and conjugate over $\mathbb{Q}(\sqrt{5})$, each of which projects to a conic of the form $xy+10yz-z^2$. This conic has an obvious rational point $(-9:1:1)$, and by intersecting the conic with lines through this point we find other points, such as the even more obvious $(1:0:0)$. Pulling back the point $(-9:1:1)$ to the Plücker embedding of the Grassmannian gives

$$(0:0:-8\sqrt{5}:0:\sqrt{5}:-9:0:-\sqrt{5}:1:1),$$

and the corresponding plane is the one containing the points

$$(-9:1:1:0:0), \quad (-\sqrt{5}:\sqrt{5}:0:1:0), \quad (0:0:0:0:1).$$

Thus the rational linear form $v+w+8x$ vanishes on this plane. Similarly, the obvious point $(1:0:0)$ gives us the linear form $w$.

On the other hand, the Fano scheme of planes on $Q_2$ projects to $x^2-5y^2-20z^2$, which has obvious points such as $(5:1:1)$. The linear forms obtained from these points include $w-x-5z$ and $w-11x-25z$.

Now we come to evaluate the obstruction. The bad primes of $S_1$ are 2, 5, 23, and 9859. Fortunately, the Brauer field is a subfield of $\mathbb{Q}_{9859}$, so the obstruction there is automatically trivial. There are 553 points on the surface mod 23. Considering, for example, the point $(3:15:22:9:1)$, we find that all of the linear forms take non-square values there (for example, $v+w+8x$ evaluates to 10, and $w-x-5z$ evaluates to 11). Thus the local invariant there is 0, and it is not necessary to

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multiply the linear forms by constants. Similarly at all but one of the remaining points, the exception being the singular point $(2:2:1:0:0)$ which does not lift mod $23^2$ and so need not be considered. We thus find that the local invariant at 23 is 0.

At 5 it is similar, except that there are 25 points mod 5 that need to be lifted mod 25. We find that the local invariant at 5 is $\frac{1}{2}$, and similarly that it is 0 at 2. The real place need not be considered, because the Brauer field is real. So the sum of the invariants is $\frac{1}{2}$ at every adelic point, and therefore there is a Brauer–Manin obstruction to the existence of a rational point. □

Example 16. Let $Q_1 = vy - 17wx$ and let $Q_2 = -12v^2 + 204w^2 + 408x^2 + 25y^2 - 2z^2$. This surface will be called $S_2$. It follows from [9, Theorem 3.17 and Proposition 4.25] that the surface $S_2$ has a point over a given field $F$ if and only if $-1$ is a norm from $F(\sqrt{2}, \sqrt{17})$. Clearly this is true if $F = \mathbb{R}$ or if $F = \mathbb{Q}_p$ with $p \neq 2, 17$, because then the étale algebra $F(\sqrt{2}, \sqrt{17})/F$ is unramified, and so the unit $-1$ is a norm. Putting $p = 2$, we find that 17 is a square in $\mathbb{Q}_p$ and $-1 = N(1 + \sqrt{2})$. Putting $p = 17$, we find that 2 is a square in $\mathbb{Q}_p$ and that $-1 = N(4 + \sqrt{17})$. Thus there are points everywhere locally. On the other hand, it is well known that $-1$ is not a norm from $K = \mathbb{Q}(\sqrt{2}, \sqrt{17})$ to $\mathbb{Q}$. Thus the surface $S_2$ does not have a rational point.

The characteristic polynomial of $S_2$ is $1382400t^5 - 48t^3u^2 - tu^4$, again up to scaling, and this factors as

$$t \cdot (1152t^2 - u^2) \cdot (1200t^2 + u^2).$$

All the lines on $S_2$ are defined over a number field of degree 8, namely

$$\mathbb{Q}(\sqrt{2}, \sqrt{-3}, \sqrt{17}).$$

Explicitly, for example, one of the lines passes through the points

$$(0 : 0 : \sqrt{34}/119 : \sqrt{2}/7 : 1) \quad \text{and} \quad (\sqrt{17} : 1 : 0 : 0 : 0).$$

So $\mathrm{Gal}(L/\mathbb{Q})$ belongs to one of the eleven conjugacy classes of rank-3 elementary abelian 2-subgroups of $\mathcal{W}$. We can now compute the cohomology and hence find that, as an abstract group, the Brauer group is isomorphic modulo constants to $\mathbb{Z}/2\mathbb{Z}$. In addition, we find that the Brauer field is $\mathbb{Q}(\sqrt{2})$. This time we are in the quartic case. One singular quadric is defined over $\mathbb{Q}$, two over $\mathbb{Q}(\sqrt{2})$, and as in the previous example we find rational points on the conics and recover conics on the original surface that are defined over the quadratic extension $\mathbb{Q}(\sqrt{2}, \sqrt{-3})$. This time there are three families, and as we only need two we consider their Picard classes to remove the one defined over $\mathbb{Q}$. We now take the conjugates of two remaining conics, obtaining subschemes of $\mathbb{P}^4$ of degree 4 with rather large coefficients. Using quadrics with coefficients in $\mathbb{Q}(\sqrt{2})$ that vanish on these, taking the residual intersection with the $\mathbb{Q}$-conjugate, and recording a third rational quadric form that vanishes on the curve of genus 5, we obtain forms again with large coefficients. The smallest comes from taking the curve of degree 4 with its conjugate, and is

$$461503262821129x_2^2 + 34012498093512x_2x_5 \\ + 5083647716998227x_3^2 + 626674890384x_5^2;$$

the coefficients of most others have about 30 digits.

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The rest of the calculation is similar to that of the previous example; since 2 is a square in $\mathbb{Q}_{17}$, we need not consider 17, and the only other primes of bad reduction are 2, 3, and 5. We find straightforwardly that the local invariant is 0 at 3 and 5, and that it is $\frac{1}{2}$ at 2.

## 5. *Relationship with* Ш[2]

We recall the basic setup for a 2-descent on the Jacobian of a curve of genus 2. See for instance [**17**]. Consider a curve of genus 2 defined over a number field $K$ in the form

$$C : y^2 = f(x) = x^5 + f_4x^4 + f_3x^3 + f_2x^2 + f_1x + f_0,$$

where $f(x) \in K[x]$ is a square-free polynomial, and let $J = \mathrm{Jac}(C)$, the Jacobian of $C$. Note that, by taking $f$ to be of degree 5, we are requiring that $C$ have a rational Weierstrass point. In particular, this implies that every rational divisor class on $C$ contains a rational divisor.

Following [**2**], we first note that a non-zero element $D \in J(K)$ can be represented by polynomials $g_D, h_D \in K[x]$, where $h_D$ is at most linear, and $g_D$ is monic and at most quadratic, such that

$$D = \{g(x) = 0, y = h(x)\} \cdot C - (\deg(g))\infty.$$

For a typical $D = [(x_1, y_1), (x_2, y_2) - 2\infty] \in J(K)$, where $[\ \ ]$ represents the divisor class modulo linear equivalence, these are simply $g_D(x) = (x - x_1)(x - x_2)$ and $y = h_D(x)$ is the line through $(x_1, y_1), (x_2, y_2)$. We define $A = K[\theta] = K[x]/(f(x))$, so that $\{1, \theta, \ldots, \theta^4\}$ is a basis of $A$ as a $K$-vector space. Also define

$$A' = \mathrm{Ker}(N_{A/K} : A^* \to K^*/K^{*2}).$$

Then $H^1(K, J[2]) \simeq A'/A^{*2}$ and, making this identification, the connecting homomorphism can be given as

$$\begin{array}{rccll} \mu : & J(K) & \to & H^1(K, J[2]) & \\ & D & \mapsto & (-1)^{\deg(g_D)} g_D(\theta) & \text{if } \gcd(g_D, f) = 1. \end{array} \tag{1}$$

If $\delta = \sum \delta_i\theta^i \in A$ represents an element in $\mu(J(K))$, then there exists a monic $g \in K[x]$ of degree at most 2 and $u_0, \ldots, u_4 \in K$ such that

$$g(\theta) = (-1)^{\deg(g)}\delta \left(\sum_{i=0}^{4} u_i\theta^i\right)^2, \tag{2}$$

and such that there is an $h(x) \in K[x]$ for which

$$\{g(x) = 0\} \cdot C = \{g(x) = 0, y = h(x)\} \cup \{g(x) = 0, y = -h(x)\}.$$

For $\delta \in A^*$ we define $Q_{\delta,i} \in K[u_0, \ldots, u_4]$ by

$$\delta \left(\sum_{i=0}^{4} u_i\theta^i\right)^2 = \sum_{i=0}^{4} Q_{\delta,i}(\underline{u})\theta^i.$$

For $\delta \in \mu(J(K))$ there must then be points on the variety

$$V_\delta : \left\{ \begin{array}{l} Q_{\delta,3}(u_0, u_1, u_2, u_3, u_4) = 0, \\ Q_{\delta,4}(u_0, u_1, u_2, u_3, u_4) = 0. \end{array} \right. \tag{3}$$

LEMMA 17. *Let $C : y^2 = f(x)$ be a curve of genus 2 over a number field $K$, with $f(x)$ quintic, let $\delta \in A^*$, where $A = K[\theta] = K[x]/(f(x))$, and let $V_\delta$ be as in (3). Then $V_\delta$ is smooth, and so is a degree 4 del Pezzo surface. Furthermore, the characteristic polynomial of $V_\delta$ is $f(x)$, up to multiplication by a nonzero constant and invertible linear change in variable.*

*Proof.* Let $\delta = \delta_0 + \delta_1\theta + \ldots + \delta_4\theta^4$, where $\delta_0, \ldots, \delta_4 \in K$. Let $L$ be the splitting field of $f(x)$, so that $f(x) = k\prod_{i=0}^{4}(x - e_i)$, with $e_0, \ldots, e_4 \in L$ distinct. Note that an invertible linear change of variable $x$ induces an invertible linear change of variables $u_0, \ldots, u_4$ on $V_\delta$; similarly any invertible linear change of of variables $u_0, \ldots, u_4$ or any change in choice of defining equations for $V_\delta$ induces an invertible linear change of variable on the characteristic form $\det(tM_1 + uM_2)$ of $V_\delta$ (where, as usual, $M_1, M_2$ are the matrices representing the two quadratic forms which give the defining equations for $V_\delta$). Therefore, it is permissible to perform invertible linear changes of variable and to make any choice for the pair of defining equations of $V_\delta$.

First inject $A^* = K[\theta]^*$ into five copies of $L^*$ via $\theta \mapsto [e_0, \ldots, e_4]$. Then our equation (2), which we write as

$$x_1x_2 - (x_1 + x_2)\theta + \theta^2 = \delta\left(\sum_{i=0}^{4} u_i\theta^i\right)^2,$$

becomes the system of equations

$$x_1x_2 - (x_1 + x_2)e_j + e_j^2 = d_j\left(\sum_{i=0}^{4} u_ie_j^i\right)^2, \qquad \text{for } j = 0, \ldots, 4,$$

where $d_j = \delta_0 + \delta_1e_j + \ldots + \delta_4e_j^4 \neq 0$ for all $j$, since $\delta \in A^*$. We now perform the invertible linear change of variable from the $u_i$ to $v_i$, given by

$$(v_0, \ldots, v_4) = (u_0, \ldots, u_4)N,$$

where $N$ is the $5 \times 5$ Vandermonde matrix $N = \left(e_j^i\right)$. Then the above system of equations becomes

$$(x_1x_2, -(x_1 + x_2), 1, 0, 0)N = (d_0v_0^2, \ldots, d_4v_4^2).$$

Multiplying both sides on the right by $N^{-1}$ and equating the last two entries then gives the defining equations of $V_\delta$ as

$$\sum_{i=0}^{4} d_iv_i^2\left(N^{-1}\right)_{i+1,3} = 0, \qquad \sum_{i=0}^{4} d_iv_i^2\left(N^{-1}\right)_{i+1,4} = 0,$$

which is in diagonal form, and the coefficients are the last two columns of $N^{-1}$. Computing the last two columns of $N^{-1}$ and substituting these into the above equation then gives $V_\delta$ defined by

$$\sum_{i=0}^{4} d_iv_i^2(e_i - s)/\prod_{j\neq i}(e_i - e_j) = 0, \qquad \sum_{i=0}^{4} d_iv_i^2/\prod_{j\neq i}(e_i - e_j) = 0,$$

where $s = e_0 + \ldots + e_4$. Any singularity on $V_\delta$ would require the two vectors

$$\big(2d_0v_0(e_0 - s), \ldots, 2d_4v_4(e_4 - s)\big), \quad \big(2d_0v_0, \ldots, 2d_4v_4\big)$$

to be linearly dependent. Since the second defining equation of $V_\delta$ has all coefficients nonzero, any such point must have at least two $v_i$ nonzero, and so the above linear dependency would force two $e_i$ to be the same, a contradiction. Hence $V_\delta$ is a smooth intersection of quadrics in $\mathbb{P}^4$, and so is a degree 4 del Pezzo surface.

If we let $M_1, M_2$ be the diagonal matrices representing the quadratic forms defining $V_\delta$, then the characteristic polynomial is given in affine form by

$$\det(M_1 + tM_2) = \frac{\prod_{i=0}^4 d_i}{\Delta} \prod_{i=0}^{4} (t + e_i - s),$$

where $\Delta$ is the discriminant of $(x - e_0) \dots (x - e_4)$. After performing the invertible linear change in variable $x = s - t$ this becomes a nonzero constant times $f(x)$, as required. $\square$

If $\delta \in S^{(2)}(J/K)$, the 2-Selmer group, then $V_\delta$ has points everywhere locally. If $V_\delta(K) = \emptyset$ due to the Brauer–Manin obstruction, it follows that $\delta \notin \mu(J(K))$ and will give a member of the 2-part of the Shafarevich–Tate group. We shall give a family of examples of this type in Section 6.

When $\delta \in S^{(2)}(J/K)$ and $V_\delta(K) \neq \emptyset$, and one suspects that $\delta \notin \mu(J(K))$, then one can attempt to prove this by finding a field extension $L$ such that $\delta \in \mu(J(L))$. One can then attempt to visualise the member of the Shafarevich–Tate group, as described in [**2**]. We first note that the method in Section 3 has consequences for the bound on the required degree of the extension.

LEMMA 18. *Let $C : y^2 = f(x)$ be a curve of genus 2 over a number field $K$, such that $f(x)$ contains an irreducible factor of degree at least 4. Let $\delta \in S^{(2)}(J/K)$ and assume that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for del Pezzo surfaces of degree 4, as conjectured by Colliot-Thélène and Sansuc. Then*

1. *there exists an imprimitive quartic extension[2] $L$ of $K$ such that $\delta \in \mu(J(L))$,*
2. *there exists a hyperelliptic genus 4 cover $\pi : D \to C$ such that $\pi^*(\delta) \in \mu(J_D(K)) \subset H^1(K, J_D[2])$; that is, $\delta$ is* visible *in a 4-dimensional abelian variety.*

*Proof.* Let $\delta \in S^{(2)}(J/K)$. Then $V_\delta$ of (3) has points everywhere locally, and its characteristic polynomial contains an irreducible factor of degree at least 4. By Theorem 10, we need a double-four (see Definition 8) defined over $\mathbb{Q}$ and by Remark 9, this requires a Galois-stable set of 3 singular quadrics containing $S$. This would imply a factor of degree 3 of the characteristic polynomial. Therefore $\operatorname{Br} V_\delta / \operatorname{Br} \mathbb{Q}$ is trivial and so $V_\delta(K) \neq \emptyset$.

From Section 5 of [**2**] we recall some notation. Let $\mathcal{K} = J/\langle -1 \rangle$ be the Kummer surface of $J$. Let $T_\delta$ be the torsor over $J$ under $J[2]$ corresponding to $\delta$. Therefore, $\delta \in \mu(J(K))$ precisely if $T_\delta$ has a $K$-rational point.

In [**2**] it is shown than $\mathcal{K}_\delta = V_\delta \times_{\mathbb{P}^2} \mathcal{K}$ is a double cover of $V_\delta$ and that $T_\delta$ is a double cover of $\mathcal{K}_\delta$. In part 1 we can take $L$ to be the field of definition of a point over $V_\delta(K)$. For part 2, we simply invoke [**2**, Proposition 3]. $\square$

[2]An imprimitive quartic extension is a quartic extension with a quadratic subfield. The Galois group of its splitting field is a subgroup of the dihedral group of order 8.

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## 6. *Examples of* Ш

In this section, we derive examples of Ш[2] on Jacobians of curves of genus 2, both in the case where the associated del Pezzo surface of degree 4 violates the Hasse principle, and when this surface has a rational point and Ш[2] is exhibited via product varieties. Our examples will all be of the type $y^2 = f(x)$, where $f(x)$ is quintic and has an irreducible factor of degree at least 3.

EXAMPLE 19. Let $H_t$ be the hyperelliptic curve

$$ty^2 = f(x) = -(x-1)(x+1)(25x^3 + 23x^2 - 25x - 21).$$

Then there exists a subset $T \subseteq \mathbb{Z}$ with infinite image in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ such that, for $t \in T$, there is an element of exact order 2 in $\text{Ш}(\mathrm{Jac}(H_t))$.

*Proof.* As usual, define $A = \mathbb{Q}[\theta] = \mathbb{Q}[x]/(f(x))$, and let

$$\delta = 1298000\theta^4 - 3068890\theta^3 - 22403341\theta^2 + 3068890\theta + 942342.$$

The del Pezzo surface of degree 4 associated to this choice of $\delta$ is equivalent to that of Example 15 by a rational change of coordinates, and so contains no $\mathbb{Q}$-rational points, as must then also be true of the principal homogeneous space.

We first claim that this principal homogeneous space is everywhere locally trivial if:

(a) $t > 0$, $t$ is a square in $\mathbb{Q}_3$, $10t$ or $15t$ is a square in $\mathbb{Q}_5$, $t$ or $2t$ is a square in $\mathbb{Q}_2$ and $t$ is a square in $\mathbb{Q}_{23}$ and $\mathbb{Q}_{9859}$, and

(b) for every prime $q$ greater than 5 dividing $t$ to odd order, the image of

$$d = 438625\beta^2 - 103840\beta - 318540$$

is a square in every completion at a prime above $q$ of the number field $\mathbb{Q}(\beta)$, where $25\beta^3 + 23\beta^2 - 25\beta - 21 = 0$.

We prove this claim as follows. The condition $t > 0$ is sufficient (and necessary) to ensure that the principal homogeneous space be trivial at infinity. To find, for example, that twisting by 2 gives a locally trivial principal homogeneous space at 2, we start by generating 2-adic points on the twist by 2 and finding their images under the local descent map. These turn out not to generate the whole space, so, using MAGMA, we next compute 2-Selmer groups of twists by $n$, where $n$ is 2 mod 16. All elements obtained must map to the image of the local 2-descent map, which has dimension 4 by [**15**, Proposition 2.4]. At this point we find that $(1, 1, d)$ is in the local image mod scalars, so the principal homogeneous space is locally trivial. Similar considerations apply at the other primes. For the larger primes dividing $t$, this condition ensures that $\delta = (1, 1, d)$ is a square in $\mathbb{Q}[x]/(f(x))$. In fact, this $\delta$ might be in the image of the local descent map even if $d$ is not a square, so condition (b) is likely to be too restrictive. In any case, we have proved that (a), (b) above are sufficient for the principal homogeneous space of $\delta$ to be locally trivial.

The fact that there are infinitely many square-free $t$ satisfying conditions (a), (b) above is a consequence of Dirichlet's theorem on primes in arithmetic progressions. For example, we could take $t = 10p$, where $p$ is a prime that is a square modulo 3 and 5, is not a square modulo 23 or 9859, and is congruent to 5 mod 8. It is easily seen that such a $t$ will give principal homogeneous spaces with points locally at all the bad primes and at $\infty$. The smallest such $t$ is 610. □

For cases when the del Pezzo surface has rational points, there remain the two other approaches mentioned in the last section: via field extensions and product varieties. There is already an example computed via field extensions in [2]; however, it does not seem that any examples have been computed via product varieties. We provide such an example here.

Example 20. Let $C : y^2 = x^6-5x^5+4x^4+x-4$. Then $Ш(\mathrm{Jac}(C)/\mathbb{Q})[2] \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and $\mathrm{Jac}(C)(\mathbb{Q}) \simeq \mathbb{Z}$.

*Proof.* First, we put this curve in monic quintic form:

$$y^2 = x^5 + 2^{10}x^4 + 2^5 \cdot 17 \cdot 769x^3 + 2^4 \cdot 3^2 \cdot 769^2x^2 + 19 \cdot 769^3x + 769^4.$$

We find $S^{(2)}(\mathrm{Jac}(C)/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^3$ and obviously the divisor $[(0, 769^2) - \infty]$ is a non-torsion point in $\mathrm{Jac}(C)(\mathbb{Q})$. Writing $A = \mathbb{Q}[\theta]$ for the algebra where we adjoin a root of the monic quintic above, we see that two elements of $S^{(2)}(\mathrm{Jac}(C)/\mathbb{Q})$ that do not correspond to $[(0, 769^2) - \infty]$ are

$$\delta_1 = 17\theta^2 + 61152\theta + 591361 \quad \text{and} \quad \delta_2 = -15\theta^2 - 61528\theta - 591361.$$

A direct computation of the Selmer group will probably not produce these representatives, but note that in our case, the corresponding del Pezzo surfaces have an irreducible quintic characteristic polynomial and hence no Brauer–Manin obstruction. We would therefore expect them to have a rational point and hence $\delta_1$ and $\delta_2$ to have representatives that are only quadratic in $\theta$.

In fact, if we put $u = x/(4x + 769)$ we find another model for $C$:

$$C_1 : (v_1)^2 = -(4u - 1)(u^5 - u + 1).$$

In order to represent $S^{(2)}$ relative to this $u$-coordinate, we consider $\beta = \theta/(769 + 4\theta)$. We find that $\beta^5 - \beta + 1 = 0$ and $\delta_1 = \beta^2 + 1$ and $\delta_2 = \beta^2 - 1$. This choice of coordinates is motivated by the fact that for

$$C_2 : v^2 = u^5 - u + 1$$

we find that $[(1, 1) + (-1, 1) - 2\infty]$ and $[(i, 1) + (-i, 1) - 2\infty]$ are rational divisors and hence that $\delta_1, \delta_2 \in S^{(2)}(\mathrm{Jac}(C_2)/\mathbb{Q})$ as well.

We consider the fibre product $D := C_1 \times_u C_2$. Putting $(w/2)^2 = -(u - 1/4)$, we obtain a model

$$D : (v/2^5)^2 = -w^{10} + 5w^8 - 10w^6 + 10w^4 + 251w^2 + 769.$$

Using [2, Proposition 3] we see that if $\delta_1, \delta_2 \in S^{(2)}(\mathrm{Jac}(C)/\mathbb{Q})$ come from $\mathrm{Jac}(C)(\mathbb{Q})$ then $\delta_1, \delta_2 \in N_{w/u}(S^{(2)}(\mathrm{Jac}(D)/\mathbb{Q}))$.

In principle, doing a 2-descent on the Jacobian of a hyperelliptic curve of genus 4 is completely analogous to doing that for a genus 2 curve. The main difficulty is in determining whether the *Cassels kernel*, that is, the kernel of the map from the proper 2-Selmer group to the fake 2-Selmer group (see [17]), is trivial or not: it is trivial if the curve has an odd degree Weierstrass place or when the Weierstrass locus consists of two quadratic conjugate loci. The latter can be checked by determining whether the following resolvent has any roots.

Let $\theta_1, \ldots, \theta_{10}$ be the points of the Weierstrass locus over an algebraic closure. Then we define

$$R = \prod \left(X - (\theta_{i_1} \cdots \theta_{i_5} + \theta_{j_1} \cdots \theta_{j_5})\right),$$

where the product is taken over all possible partitions of $\{1,\ldots,10\}$ into two unlabeled sets $\{i_1,\ldots,i_5\}$ and $\{j_1,\ldots,j_5\}$. The polynomial $R$ is symmetric in the $\theta_i$ and hence will have rational coefficients. However, $\deg(R) = 126$, which makes it rather hard to compute generically. Instead, we compute $R$ over several finite fields (where the splitting fields have manageable degrees) and use the Chinese Remainder Theorem to reconstruct the polynomial $R$ over $\mathbb{Z}$.

If $R$ is square-free then the degree 10 polynomial splits in two quadratic conjugate factors if and only if $R$ has a root. If $R$ is not square-free, then we apply a transformation to $u$, which does not change the factorization type, and compute $R$ again.

We can use the resolvent thus obtained and perform a 2-descent on $\mathrm{Jac}(D)$ as usual to compute that

$$S^{(2)}(\mathrm{Jac}(D)/\mathbb{Q}) = (\mathbb{Z}/2)^3,$$

and that

$$N_{w/u}(S^{(2)}(\mathrm{Jac}(D)(\mathbb{Q})) = \{1\}.$$

We conclude, either by using the fact that the rank of $\mathrm{Jac}(D)$ is the sum of the ranks of $\mathrm{Jac}(C)$ and $\mathrm{Jac}(C_1)$, or by using [**2**, Proposition 6], that $\delta_1$ and $\delta_2$ do not represent elements from the Mordell–Weil group in $S^{(2)}(\mathrm{Jac}(C)/\mathbb{Q})$. See [**3**] for a transcript of a MAGMA session performing these calculations. □

## Appendix A. *Additional files*

We have included the following additional files as an electronic appendix to this article:

- a text file explaining step 2(d) of the algorithm (Section 3);
- MAGMA programs to compute the Brauer–Manin obstruction on a degree 4 Del Pezzo surface;
- a transcript of computations involving visibility of Ш[2] in the Jacobian of a genus 4 curve.

The files can be found at:

`http://www.lms.ac.uk/jcm/10/lms2007-021/appendix-a/`,

and at [**3**].

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