THE BROWN MEASURE OF THE SUM OF A SELF-ADJOINT ELEMENT AND AN IMAGINARY MULTIPLE OF A SEMICIRCULAR ELEMENT

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ABSTRACT. We compute the Brown measure of $x_0 + i\sigma_t$, where σ_t is a free semicircular Brownian motion and x_0 is a freely independent self-adjoint element that is not a multiple of the identity. The Brown measure is supported in the closure of a certain bounded region Ω_t in the plane. In Ω_t, the Brown measure is absolutely continuous with respect to Lebesgue measure, with a density that is constant in the vertical direction. Our results refine and rigorize results of Janik, Nowak, Papp, Wambach, and Zahed and of Jarosz and Nowak in the physics literature.

We also show that pushing forward the Brown measure of $x_0 + i\sigma_t$ by a certain map $Q_t : \Omega_t \to \mathbb{R}$ gives the distribution of $x_0 + \sigma_t$. We also establish a similar result relating the Brown measure of $x_0 + i\sigma_t$ to the Brown measure of $x_0 + c_t$, where c_t is the free circular Brownian motion.

CONTENTS

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1. Introduction

1.1. **Sums of independent random matrices.** A fundamental problem in random matrix theory is to understand the eigenvalue distribution of sums of independent random matrices. When the random matrices are Hermitian, the subordination method, introduced by Voiculescu [36] and further developed by Biane [3] and Voiculescu [37] gives a powerful method of analyzing the problem in the setting of free probability. (See Section 5.2 for a brief discussion of the subordination method.) For related results in the random matrix setting, see, for example, works of Pastur and Vasilchuk [30] and of Kargin [26].

A natural next step would be to consider non-normal random matrices of the form $X+iY$ where X and Y are independent Hermitian random matrices. Although a general framework has been developed for analyzing combinations of freely independent elements in free probability (see works of Belinschi, Mai, and Speicher [6] and Belinschi, Śniady, and Speicher [7]), it does not appear to be easy to apply this framework to get analytic results about the $X + iY$ case.

The $X + iY$ problem has been analyzed at a nonrigorous level in the physics literature. A highly cited paper of Stephanov [32] uses the case in which X is Bernoulli and Y is GUE to provide a model of QCD. In the case that Y is GUE, work of Janik, Nowak, Papp, Wambach, and Zahed [23] identified the domain into which the eigenvalues should cluster in the large-N limit. Then work of Jarosz and Nowak [24, 25] analyzed the limiting eigenvalue distribution for general X and Y, with explicit computations of examples when Y is GUE and X has various distributions [24, Section 6.1].

In this paper, we *compute* the Brown measure of $x_0+i\sigma_t$, where σ_t is a semicircular Brownian motion and x_0 is an arbitrary self-adjoint element freely independent of σ_t. This Brown measure is the natural candidate for the limiting eigenvalue distribution of random matrices of the form $X + iY$ where X and Y are independent and Y is GUE. We also *relate* the Brown measure of $x_0 + i\sigma_t$ to the distribution of $x_0+\sigma_t$ (without the factor of i). Our computation of the Brown measure of $x_0+i\sigma_t$ refines and rigorizes the results of [23] and [24, 25], using a different method, while the relationship between $x_0 + i\sigma_t$ and $x_0 + \sigma_t$ is a new result. See Section 1.4 for further discussion of these works and Sections 5.4 and 9 for a detailed comparison of results.

Our work extends that of Ho and Zhong [22], which (among other results) computes the Brown measure of $x_0 + i\sigma_t$ in the case $x_0 = y_0 + \tilde{\sigma}_t$, where $\tilde{\sigma}_t$ is another semicircular Brownian motion, freely independent of both y_0 and σ_t. In this case, $x_0 + i\sigma_t$ has the form of $y_0 + c_{2t}$, where c_t is a free circular Brownian motion.

Our results are based on the PDE method introduced in [12]. This method has been used in subsequent works by Ho and Zhong [22], Demni and Hamdi [11], and Hall and Ho [19], and is discussed from the physics point of view by Grela, Nowak, and Tarnowski in [16]. See also the expository article [18] of the first author for an introduction to the PDE method. Similar PDEs, in which the regularization parameter in the construction of the Brown measure becomes a variable in the PDE, have appeared in the physics literature in the work of Burda, Grela, Nowak, Tarnowski, and Warchoł [9, 10].

Since this article was posted on the arXiv, three papers have appeared that extend our results have appeared. The paper [20] of Ho examines in detail the case in which x_0 is the sum of a self-adjoint element and a freely independent semicircular element, so that $x_0 + i\sigma_t$ becomes the sum of a self-adjoint element and a freely independent elliptic element. The paper [21] extends the results of the present paper by allowing x_0 to be unbounded. Finally, the paper [39] of Zhong analyzes the Brown measure of $x_0 + g$, where g is a twisted elliptic element and x_0 is freely independent of g but otherwise arbitrary. In the case that x_0 is self-adjoint and g is an imaginary multiple of a semicircular element, Zhong's results reduce to ours.

1.2. **Statement of results.** Let σ_t be a semicircular Brownian motion living in a tracial von Neumann algebra $(\mathcal{A}, \tau)$ and let x_0 be a self-adjoint element of $\mathcal{A}$ that is freely independent of every σ_t, $t > 0$. (In particular, x_0 is a *bounded* self-adjoint operator.) Throughout the paper, we let μ be the distribution of x_0, that is, the unique compactly supported probability measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^n \; d\mu(x) = \tau(x_0^n), \quad \text{for all } n \in \mathbb{N}. \tag{1.1}$$

Our goal is then to compute the Brown measure of the element

$$x_0 + i\sigma_t \tag{1.2}$$

in $\mathcal{A}$. (See Section 2 for the definition of the Brown measure.) Throughout the paper, we impose the following standing assumption about μ.

Assumption 1.1. *The measure μ is not a δ-measure, that is, not supported at a single point.*

Of course, the case in which μ is a δ-measure is not hard to analyze—in that case, $x_0 + i\sigma_t$ has the form $a + i\sigma_t$, for some constant $a \in \mathbb{R}$, so that the Brown measure is a semicircular distribution on a vertical segment through a. But this case is *different*; in all other cases, the Brown measure is absolutely continuous with respect to the Lebesgue measure on a two-dimensional region in the plane. Thus, our main results do not hold as stated in the case that μ is a δ-measure.

The element (1.2) is the large-N limit of the following random matrix model. Let Y^N be an $N \times N$ random variable distributed according to the Gaussian unitary ensemble. Let X^N be a sequence of self-adjoint random matrices that are independent of Y^N and whose eigenvalue distributions converge almost surely to the law μ of x_0. (The X^N's may, for example, be chosen to be deterministic diagonal

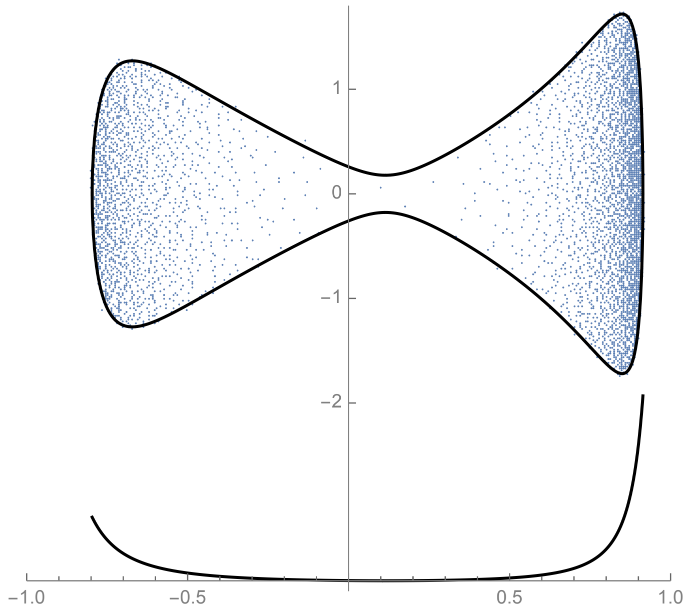

FIGURE 1. The top of the figure shows the domain Ω_t for the case $\mu = \frac{1}{3}\delta_{-1} + \frac{2}{3}\delta_1$ and $t = 1.05$, together with a simulation of the corresponding random matrix model. The bottom of the figure shows the density (in Ω_t) of the Brown measure as a function of a.

matrices, which is the case in all the simulations shown in this paper.) Then the random matrices

$$X^N + i\sqrt{t}Y^N \tag{1.3}$$

will converge in $*$-distribution to $x_0 + i\sigma_t$.

In this paper we compute the Brown measure of $x_0 + i\sigma_t$. This Brown measure is the natural candidate for the limiting empirical eigenvalue distribution of the random matrices in (1.3). Our main results are summarized briefly in the following theorem.

Theorem 1.2. *For each $t > 0$, there exists a continuous function $b_t : \mathbb{R} \to [0, \infty)$ such that the following results hold. Let*

$$\Omega_t = \{ a + ib \in \mathbb{C} | \; |b| < b_t(a) \} .$$

Then the Brown measure of $x_0+i\sigma_t$ is supported on the closure of Ω_t and Ω_t itself is a set of full Brown measure. Inside Ω_t, the Brown measure is absolutely continuous with a density that is constant in the vertical directions*. Specifically, the density $w_t(a + ib)$ is independent of b in Ω_t and has the form*

$$w_t(a + ib) = \frac{1}{2\pi t}\left(\frac{da_0^t(a)}{da} - \frac{1}{2}\right), \quad a + ib \in \Omega_t.$$

for a certain function a_0^t.

See Figures 1 and 2.

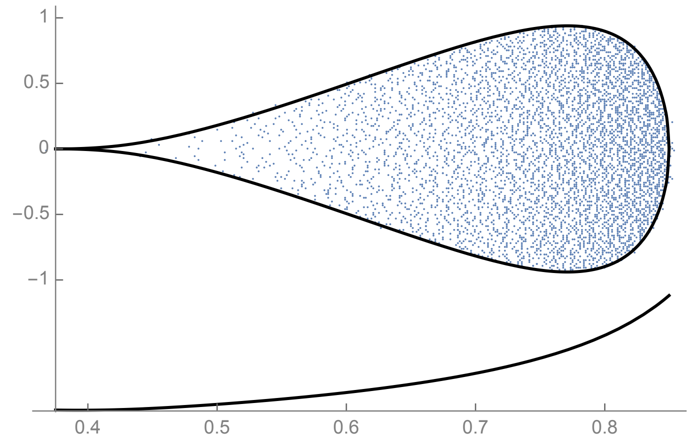

FIGURE 2. The top of the figure shows the domain Ω_t for the case in which μ has density $3x^2$ on $[0,1]$ and $t = 1/4$, together with a simulation of the corresponding random matrix model. The bottom of the figure shows the density (in Ω_t) of the Brown measure as a function of a.

We now describe how to compute the functions b_t and a_0^t in Theorem 1.2. Recall that μ is the law of x_0, as in (1.1). We then fix $t > 0$ and consider two equations:

$$\int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + v^2}\ d\mu(x) = \frac{1}{t} \tag{1.4}$$

$$\int_{\mathbb{R}} \frac{x}{(a_0 - x)^2 + v^2}\ d\mu(x) = \frac{a}{t}, \tag{1.5}$$

where we look for a solution with $v > 0$ and $a_0 \in \mathbb{R}$. We will show in Section 7.2 that there can be at most one such pair (v, a_0) for each $a \in \mathbb{R}$. If, for a given $a \in \mathbb{R}$, we can find $v > 0$ and $a_0 \in \mathbb{R}$ solving these equations, we set

$$a_0^t(a) = a_0 \tag{1.6}$$

and

$$b_t(a) = 2v. \tag{1.7}$$

If, on the other hand, no solution exists, we set $b_t(a) = 0$ and leave $a_0^t(a)$ undefined. (If $b_t(a) = 0$, there are no points of the form $a + ib$ in Ω_t and so the density of the Brown measure is undefined.)

The equations (1.4) and (1.5) can be solved explicitly for some simple choices of μ, as shown in Section 10. For any reasonable choice of μ, the equations can be easily solved numerically.

We now explain a connection between the Brown measure of $x_0 + i\sigma_t$ and two other models. In addition to the semicircular Brownian motion σ_t, we consider also a circular Brownian motion c_t. This may be constructed as

$$c_t = \sigma_{t/2} + i\tilde{\sigma}_{t/2},$$

where $\sigma.$ and $\tilde{\sigma}.$ are two freely independent semicircular Brownian motions. We now describe a remarkable direct connection between the Brown measure of $x_0 + i\sigma_t$ and the Brown measure of $x_0 + c_t$, and a similar direct connection between the the Brown measure of $x_0+i\sigma_t$ and the law of $x_0+\sigma_t$. We remark that a fascinating indication of a connection between the behavior of $x_0+\sigma_t$ and the behavior of $x_0+i\sigma_t$ were given previously in the work of Janik, Nowak, Papp, Wambach, and Zahed, discussed in Section 5.4. Note that since σ_t has the same law as $\sigma_{t/2}+\tilde{\sigma}_{t/2}$, we can describe the three random variables in question as

$$\begin{aligned} x_0+\sigma_t &\equiv x_0+\sigma_{t/2}+\tilde{\sigma}_{t/2} \\ x_0+c_t &\equiv x_0+\sigma_{t/2}+i\tilde{\sigma}_{t/2} \\ x_0+i\sigma_t &\equiv x_0+i\sigma_{t/2}+i\tilde{\sigma}_{t/2}, \end{aligned}$$

where the notation $A\equiv B$ means that A and B have the same $*$-distribution and therefore the same Brown measure.

The Brown measure of x_0+c_t was computed by the second author and Zhong in [22]. They also established that the Brown measure of x_0+c_t is related to the law of $x_0+\sigma_t$. We then show that the Brown measure of $x_0+i\sigma_t$ is related to the Brown measure of x_0+c_t. By combining our this last result with what was shown in [22, Prop. 3.14], we obtain the following result.

Theorem 1.3. *The Brown measure of x_0+c_t is supported in the closure of a certain domain Λ_t identified in* [22]. *There is a homeomorphism U_t of $\overline{\Lambda}_t$ onto $\overline{\Omega}_t$ with the property that the push-forward of* $\mathrm{Brown}(x_0+c_t)$ *under U_t is equal to* $\mathrm{Brown}(x_0+i\sigma_t)$. *Furthermore, there is a continuous map $Q_t:\overline{\Omega}_t\to\mathbb{R}$ such that the push-forward of* $\mathrm{Brown}(x_0+i\sigma_t)$ *under Q_t is the law of $x_0+\sigma_t$, as computed by Biane.*

The maps U_t and Q_t are described in Sections 7.2 and 8, respectively. The map U_t has the property that vertical line segments in $\overline{\Lambda}_t$ map linearly to vertical line segments in $\overline{\Omega}_t$, while the map Q_t has the property that vertical line segments in $\overline{\Omega}_t$ map to single points in $\mathbb{R}$. (See Figures 3 and 4.) The map Q_t is computed by first applying the inverse of the map U_t and then applying the map denoted as Ψ_t in Point 3 of Theorem 1.1 in [22].

1.3. **Method of proof.** Our proofs are based on the PDE method developed in [12] and used also in [22] and [11]. (See also [18] for a gentle introduction to the method.) For any operator A in a tracial von Neumann algebra $(\mathcal{A},\tau)$, the Brown measure of A, denoted $\mathrm{Brown}(A)$, may be computed as follows. (See Section 2 for more details.) Let

$$S(\lambda,\varepsilon)=\tau[\log((A-\lambda)^*(A-\lambda)+\varepsilon)]$$

for $\varepsilon>0$. Then the limit

$$s(\lambda):=\lim_{\varepsilon\to 0^+}S(\lambda,\varepsilon)$$

exists as a subharmonic function. The Brown measure is then defined as

$$\mathrm{Brown}(A)=\frac{1}{4\pi}\Delta s,$$

where the Laplacian is computed in the distributional sense. The general theory then guarantees that $\mathrm{Brown}(A)$ is a probability measure supported on the spectrum of A. (The closed support of $\mathrm{Brown}(A)$ can be a proper subset of the spectrum of A.)

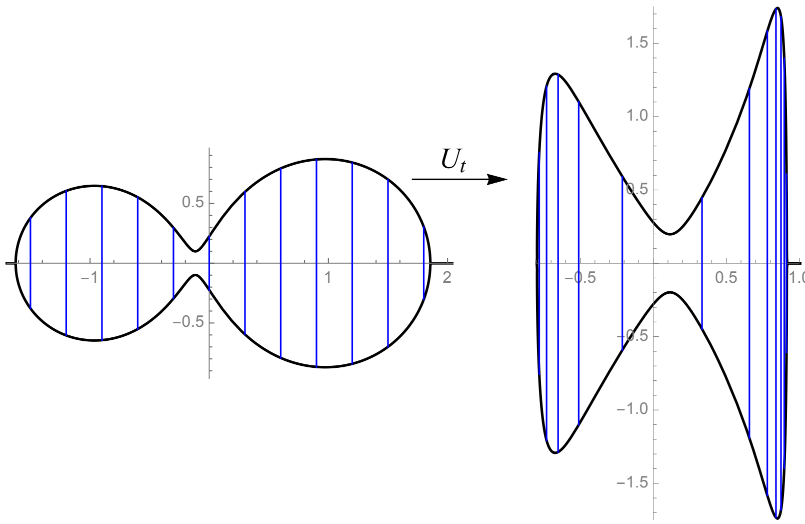

FIGURE 3. A visualization of the map $U_t : \overline{\Lambda}_t \to \overline{\Omega}_t$. The map takes vertical segments in Λ_t linearly to vertical segments in Ω_t. Shown for $\mu = \frac{1}{3}\delta_{-1} + \frac{2}{3}\delta_1$ and $t = 1.05$.

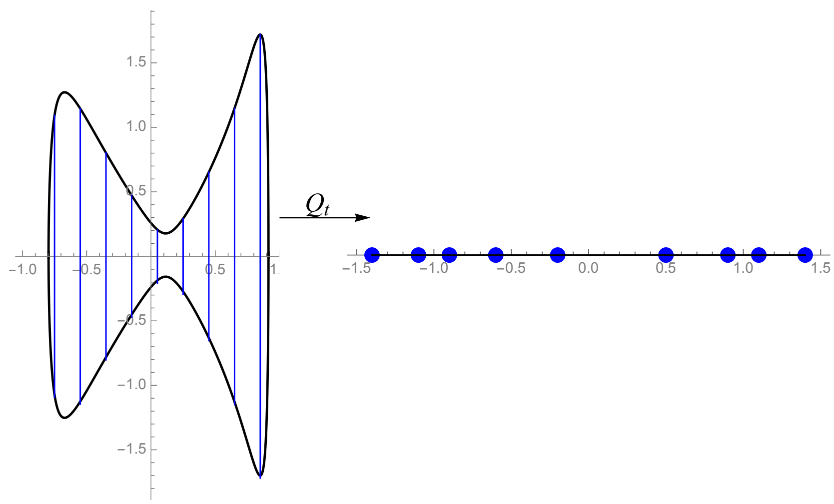

FIGURE 4. A visualization of the map $Q_t : \overline{\Omega}_t \to \mathbb{R}$. The map takes vertical segments in $\overline{\Omega}_t$ to single points in $\mathbb{R}$. Shown for $\mu = \frac{1}{3}\delta_{-1} + \frac{2}{3}\delta_1$ and $t = 1.05$.

In our case, we take $A = x_0 + i\sigma_t$, so that S also depends on t. Thus, we consider the functions

$$S(t, \lambda, \varepsilon) = \tau[\log((x_0 + i\sigma_t - \lambda)^*(x_0 + i\sigma_t - \lambda) + \varepsilon)] \tag{1.8}$$

and

$$s_t(\lambda) = \lim_{\varepsilon \to 0^+} S(t, \lambda, \varepsilon).$$

Then

$$\mathrm{Brown}(x_0 + i\sigma_t) = \frac{1}{4\pi} \Delta s_t(\lambda),$$

where the Laplacian is taken with respect to λ with t fixed.

Our first main result (Theorem 3.1) is that the function S in (1.8) satisfies a first-order nonlinear PDE of Hamilton–Jacobi type, given in Theorem 3.1. Our goal is then to solve the PDE for $S(t, \lambda, \varepsilon)$, evaluate the solution in the limit $\varepsilon \to 0$, and then take the Laplacian with respect to λ. We use two different approaches to this goal, one approach outside a certain domain Ω_t and a different approach inside Ω_t, where the Brown measure turns out to be zero outside Ω_t and nonzero inside Ω_t. See Sections 6 and 7.

1.4. **Comparison to previous results.** A different approach to the problem was previously developed in the physics literature by Jarosz and Nowak [24, 25]. Using linearization and subordination functions, they propose an algorithm for computing the Brown measure of $H_1 + iH_2$, where H_1 and H_2 are arbitrary freely independent Hermitian elements. (See, specifically, Eqs. (75)–(80) in [25].) Section 6 of [24] presents examples in which one of H_1 and H_2 is semicircular and the other has various distributions.

Although the method of [24, 25] is not rigorous as written, it is possible that the strategy used there could be made rigorous using the general framework developed by Belinschi, Mai, and Speicher [6]. (See, specifically, the very general algorithm in Section 4 of [6]. See also [7] for further rigorous developments in this direction.) We emphasize, however, that it would require considerable effort to get analytic results for the $H_1 + iH_2$ case from the general algorithm of [6]. In any case, we show in Section 9 that our results are compatible with those obtained by the algorithm of Jarosz and Nowak.

In addition to presenting a rigorous argument, we provide information about the Brown measure of $x_0 + i\sigma_t$ that is not found in [24, 25]. First, we highlight the crucial result that the density of the Brown measure, inside its support, is always constant in the vertical direction. Although this result certainly follows from the algorithm of Jarosz and Nowak (and is reflected in the examples in [24, Sect. 6]), it is not explicitly stated in their work. Second, we give significantly more explicit formulas for the support of the Brown measure and for its density when x_0 is arbitrary. Third, we obtain (Section 8) a direct relationship between the Brown measure of $x_0 + i\sigma_t$ and the distribution of $x_0 + \sigma_t$ that is not found in [24] or [25].

Meanwhile, in Section 5, we also confirm a separate, nonrigorous argument of Janik, Nowak, Papp, Wambach, and Zahed predicting the domain on which the Brown measure is supported.

Finally, as mentioned previously, Section 3 of the paper [22] of the second author and Zhong computed the Brown measure of $y_0 + c_t$, where c_t is the free circular Brownian motion (large-N limit of the Ginibre ensemble). Now, c_t can be constructed as $c_t = \tilde{\sigma}_{t/2} + i\sigma_{t/2}$, where $\sigma_\cdot$ and $\tilde{\sigma}_\cdot$ are two freely independent semicircular Brownian motions. Thus, the results of the present paper in the case where x_0 is the sum of a self-adjoint element y_0 and a freely independent semicircular element fall under the results of [22]. But actually, the connection between the present paper and [22] is deeper than that. For any choice of x_0, the region Λ_t in which the Brown measure of $x_0 + c_t$ is supported shows up in the computation of the Brown measure of $x_0 + i\sigma_t$, as the "domain in the λ_0-plane" (Section 5.1). And then we show that the Brown measure of $x_0 + i\sigma_t$ is the pushforward of the Brown measure of $x_0 + c_t$ under a certain map (Section 8). Thus, one of the notable aspect of the results of the present paper is the way they illuminate the deep connections between $x_0 + c_t$ and $x_0 + i\sigma_t$.

2. The Brown measure formalism

We present here general results about the Brown measure. For more information, the reader is referred to the original paper [8] of Brown and to Chapter 11 of the monograph of Mingo and Speicher [28].

Let $(\mathcal{A}, \tau)$ be a **tracial von Neumann algebra**, that is, a finite von Neumann algebra $\mathcal{A}$ with a faithful, normal, tracial state $\tau : \mathcal{A} \to \mathbb{C}$. Thus, τ is a norm-1 linear functional with the properties that $\tau(A^*A) > 0$ for all nonzero elements of $\mathcal{A}$ and that $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{A}$. For any $A \in \mathcal{A}$, we define a function S by

$$S(\lambda, \varepsilon) = \tau[\log((A - \lambda)^*(A - \lambda) + \varepsilon)], \quad \lambda \in \mathbb{C},\ \varepsilon > 0.$$

It is known that

$$s(\lambda) := \lim_{\varepsilon \to 0^+} S(\lambda, \varepsilon)$$

exists as a subharmonic function on $\mathbb{C}$. Then the **Brown measure** of A is defined in terms of the distributional Laplacian of s:

$$\mathrm{Brown}(A) = \frac{1}{4\pi} \Delta s.$$

The motivation for this definition comes from the case in which $\mathcal{A}$ is the algebra of all $N \times N$ matrices and τ is the normalized trace ($1/N$ time ordinary trace). In this case, if A has eigenvalues $\lambda_1, \ldots, \lambda_N$ (counted with their algebraic multiplicities), then the function s may be computed as

$$s(\lambda) = \frac{2}{N} \sum_{j=1}^{N} \log |\lambda - \lambda_j| \,.$$

That is to say, s is $2/N$ time the logarithm of the absolute value of the characteristic polynomial of A. Since $\frac{1}{2\pi} \log |\lambda|$ is the Green's function for the Laplacian on the plane, we find that

$$\mathrm{Brown}(A) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}.$$

Thus, the Brown measure of a matrix is just its empirical eigenvalue distribution.

If a sequence of random matrices A^N converges in $*$-distribution to an element A in a tracial von Neumann algebra, one generally expects that the empirical eigenvalue distribution of A^N will converge almost surely the Brown measure of A. But such a result does not *always* hold and it is a hard technical problem to prove that it does in specific examples. Works of Girko [14], Bai [1], and Tao and Vu [33] (among others) on the circular law provide techniques for establish such convergence results, while a somewhat different approach to such problems was developed by Guionnet, Krishnapur, and Zeitouni [17].

3. The differential equation for S

Let σ_t be a free semicircular Brownian motion and let x_0 be a Hermitian element freely independent of each σ_t, $t > 0$. The main result of this section is the following.

Theorem 3.1. *Let*

$$S(t,\lambda,\varepsilon) = \tau[\log((x_0 + i\sigma_t - \lambda)^*(x_0 + i\sigma_t - \lambda) + \varepsilon)] \quad \lambda \in \mathbb{C},\ \varepsilon > 0$$

and write λ as $\lambda = a + ib$ with $a, b \in \mathbb{R}$. Then the function S satisfies the PDE

$$\frac{\partial S}{\partial t} = \frac{1}{4}\left(\left(\frac{\partial S}{\partial a}\right)^2 - \left(\frac{\partial S}{\partial b}\right)^2\right) + \varepsilon\left(\frac{\partial S}{\partial \varepsilon}\right)^2 \tag{3.1}$$

subject to the initial condition

$$S(0,\lambda,\varepsilon) = \tau[\log((x_0 - \lambda)^*(x_0 - \lambda) + \varepsilon)].$$

We use the notation

$$x_t := x_0 + i\sigma_t$$
$$x_{t,\lambda} := x_t - \lambda.$$

Then the free SDE's of $x_{t,\lambda}$ and $x_{t,\lambda}^*$ are

$$dx_{t,\lambda} = i\,d\sigma_t, \qquad dx_{t,\lambda}^* = -i\,d\sigma_t. \tag{3.2}$$

The main tool of this section is the free Itô formula. The following theorem is a simpler form of Theorem 4.1.2 of [5] which states the free Itô formula. The form of the Itô formula used here is similar to what is in Lemma 2.5 and Lemma 4.3 of [27]. For a "functional" form of these free Itô formulas, see Section 4.3 of [29].

Theorem 3.2. *Let $(\mathcal{A}_t)_{t\geq 0}$ be a filtration such that $\sigma_t \in \mathcal{A}_t$ for all t and $\sigma_t - \sigma_s$ is free with $\mathcal{A}_s$ for all $s \leq t$. Also let f_t, g_t be two free Itô processes satisfying the free SDEs*

$$df_t = \sum_{k=1}^n a_t^k\ d\sigma_t\ b_t^k + c_t\ dt \tag{3.3}$$

$$dg_t = \sum_{k=1}^n \tilde{a}_t^k\ d\sigma_t\ \tilde{b}_t^k + \tilde{c}_t\ dt. \tag{3.4}$$

for some continuous adapted processes $\{a_t^k, b_t^k, c_t, \tilde{a}_t^k, \tilde{b}_t^k, \tilde{c}_t\}_{k=1}^n$. Then $f_t g_t$ satisfies the free SDE

$$d(f_t g_t) = \sum_{k=1}^n (a_t^k\ d\sigma_t\ b_t^k g_t + f_t \tilde{a}_t^k\ d\sigma_t\ \tilde{b}_t^k) + \left(c_t g_t + f_t \tilde{c}_t + \sum_{j,k=1}^n \tau[b_t^k \tilde{a}_t^j] a_t^k \tilde{b}_t^j\right) dt. \tag{3.5}$$

That is, $d(f_t g_t)$ can be informally computed using the free Itô product rule:

$$d(f_t g_t) = df_t\, g_t + f_t\, dg_t + df_t\ dg_t,$$

where $df_t\ dg_t$ is computed using the rules

$$d\sigma_t\ \theta_t\ dt = dt\ \theta_t\ d\sigma_t = dt\ d\theta_t\ dt = 0, \tag{3.6}$$

$$d\sigma_t\ \theta_t\ d\sigma_t = \tau[\theta_t]\, dt \tag{3.7}$$

for any continuous adapted process θ_t.

Furthermore, if a process f_t satisfies an SDE as in (3.3), then $\tau[f_t]$ satisfies

$$d\tau[f_t] = \tau[c_t]\ dt.$$

This result can be expressed informally as saying d commutes with τ and that

$$\tau[\theta_t\ d\sigma_t] = 0 \tag{3.8}$$

for any continuous adapted process θ_t.

The theorem stated above is applicable to our current situation. Let $\mathcal{A}_0$ be the von Neumann algebra generated by x_0, and $\mathcal{B}_t$ be the von Neumann algebra generated by $\{\sigma_r : r \leq t\}$. Then we apply Theorem 3.2 with $\mathcal{A}_t = \mathcal{A}_0 * \mathcal{B}_t$, the reduced free product of $\mathcal{A}_0$ and $\mathcal{B}_t$.

We shall use the free Itô formula to compute a partial differential equation that S satisfies. Our strategy is to first do a power series expansion of the logarithm and then apply the free Itô formula to compute the partial derivative of the powers of $x_{t,\lambda}^* x_{t,\lambda}$ with respect to t. We start by computing the time derivatives of $\tau[(x_{t,\lambda}^* x_{t,\lambda})^n]$.

Lemma 3.3. *We have*

$$\frac{\partial}{\partial t}\tau[(x_{t,\lambda}^* x_{t,\lambda})] = 1. \tag{3.9}$$

When $n \geq 2$,

$$\begin{aligned} \frac{\partial}{\partial t}\tau[(x_{t,\lambda}^* x_{t,\lambda})^n] = &-\frac{n}{2}\sum_{m=1}^{n-1}\tau[x_{t,\lambda}^*(x_{t,\lambda}^* x_{t,\lambda})^{m-1}]\tau[x_{t,\lambda}^*(x_{t,\lambda}^* x_{t,\lambda})^{n-m-1}] \\ &-\frac{n}{2}\sum_{m=1}^{n-1}\tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda})^{m-1}]\tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda})^{n-m-1}] \\ &+ n\sum_{m=1}^{n}\tau[(x_{t,\lambda}^* x_{t,\lambda})^{n-m}]\tau[(x_{t,\lambda}^* x_{t,\lambda})^{m-1}]. \end{aligned} \tag{3.10}$$

Proof. For $n = 1$, we apply the free Itô formula to get

$$d(x_{t,\lambda}^* x_{t,\lambda}) = x_{t,\lambda}^*(i\, d\sigma_t) + (-i\, d\sigma_t)x_{t,\lambda} + d\sigma_t \cdot d\sigma_t = i x_{t,\lambda}^*\, d\sigma_t - i\, d\sigma_t\, x_{t,\lambda} + dt$$

which gives (3.9), after taking trace on both sides.

Now, we assume $n \geq 2$. When we apply Theorem 3.2 repeatedly to obtain results for the product of several free Itô processes. When computing $d\tau[(x_{t,\lambda}^* x_{t,\lambda})^n]$, we obtain four types of terms, as follows.

(1) Terms involving only one differential, either of $x_{t,\lambda}^*$ or of $x_{t,\lambda}$.
(2) Terms involving two differentials of $x_{t,\lambda}$.
(3) Terms involving two differentials of $x_{t,\lambda}^*$.
(4) Terms involving a differential of $x_{t,\lambda}^*$ and a differential of $x_{t,\lambda}$.

We now compute $d\tau[(x_{t,\lambda}^* x_{t,\lambda})^n]$ by moving the d inside the trace and then applying Theorem 3.2. By (3.8), the terms in Point 1 will not contribute.

We then consider the terms in Point 2. There are exactly n factors of $x_{t,\lambda}$ in $(x_{t,\lambda}^* x_{t,\lambda})^n$. Since the terms in Point 2 involve exactly two $dx_{t,\lambda}$'s, there are precisely $\binom{n}{2}$ terms in Point 2. For the purpose of computing these terms, we label all of the $x_{t,\lambda}$'s by $x_{t,\lambda}^{(k)}$ for $k = 1, \ldots, n$. We view choosing two $x_{t,\lambda}$'s as first choosing an

$x_{t,\lambda}^{(i)}$, then another $x_{t,\lambda}^{(j)}$. We then cyclically permute the factors until $dx_{t,\lambda}^{(i)}$ is at the beginning. Using the free stochastic equation (3.2) of $x_{t,\lambda}$, this term has the form

$$\begin{aligned}&\tau[dx_{t,\lambda}^{(i)}\,(x_{t,\lambda}^*x_{t,\lambda})^m x_{t,\lambda}^*\,dx_{t,\lambda}^{(j)}\,(x_{t,\lambda}^*x_{t,\lambda})^{n-m-2}x_{t,\lambda}^*]\\ &= -\tau[(x_{t,\lambda}^*x_{t,\lambda})^m x_{t,\lambda}^*]\tau[(x_{t,\lambda}^*x_{t,\lambda})^{n-m-2}x_{t,\lambda}^*]\;dt\end{aligned}$$

where $m = j - i - 1 \bmod n$ and we omit the labeling of all $x_{t,\lambda}$'s except $x_{t,\lambda}^{(i)}$ and $x_{t,\lambda}^{(j)}$.

If we then sum over all $j \neq i$, we obtain

$$-\sum_{m=0}^{n-2}\tau[x_{t,\lambda}^*(x_{t,\lambda}^*x_{t,\lambda})^m]\tau[x_{t,\lambda}^*(x_{t,\lambda}^*x_{t,\lambda})^{n-m-2}]\;dt.$$

Since this expression is independent of i, summing over i produces a factor of n in front. But then we have counted every term exactly twice, since we can choose the i first and then the j or vice versa. Thus, the sum of all the terms in Point 2 is

$$-\frac{n}{2}\sum_{m=0}^{n-2}\tau[(x_{t,\lambda}^*(x_{t,\lambda}^*x_{t,\lambda})^m]\tau[(x_{t,\lambda}^*(x_{t,\lambda}^*x_{t,\lambda})^{n-m-2}]\;dt. \tag{3.11}$$

By a similar argument, the sum of all the terms in Point 3 is

$$-\frac{n}{2}\sum_{m=0}^{n-2}\tau[x_{t,\lambda}(x_{t,\lambda}^*x_{t,\lambda})^m]\tau[x_{t,\lambda}(x_{t,\lambda}^*x_{t,\lambda})^{n-m-2}]\;dt. \tag{3.12}$$

We now compute the terms in Point 4. We can cyclically permute the factors until $dx_{t,\lambda}^*$ is at the beginning. Thus, each of the terms in Point 4 can be written as

$$\begin{aligned}&\tau[dx_{t,\lambda}^*(x_{t,\lambda}x_{t,\lambda}^*)^m\,dx_{t,\lambda}\,(x_{t,\lambda}^*x_{t,\lambda})^{n-m-1}]\\ &= \tau[(x_{t,\lambda}^*x_{t,\lambda})^m]\tau[(x_{t,\lambda}^*x_{t,\lambda})^{n-m-1}]\;dt,\end{aligned} \tag{3.13}$$

where $m = 0, \ldots, n-1$. Now, there are a total of n^2 terms in Point 4, but from (3.13), we can see that there are only n *distinct* terms, each of which occurs n times, so that the sum of all terms from Point 4 is

$$n\sum_{m=0}^{n-1}\tau[(x_{t,\lambda}^*x_{t,\lambda})^m]\tau[(x_{t,\lambda}^*x_{t,\lambda})^{n-m-1}]\;dt. \tag{3.14}$$

We now obtain (3.10) by adding (3.11), (3.12), and (3.14) and making a change of index. □

Proposition 3.4. *The function S satisfies the equation*

$$\begin{aligned}\frac{\partial S}{\partial t} &= \frac{1}{2}\tau[x_{t,\lambda}(x_{t,\lambda}^*x_{t,\lambda}+\varepsilon)^{-1}]^2\\ &\quad+\frac{1}{2}\tau[x_{t,\lambda}^*(x_{t,\lambda}^*x_{t,\lambda}+\varepsilon)^{-1}]^2+\varepsilon\tau[(x_{t,\lambda}^*x_{t,\lambda}+\varepsilon)^{-1}]^2.\end{aligned} \tag{3.15}$$

Proof. We first show that (3.15) holds for all $\varepsilon > \|x_{t,\lambda}^*x_{t,\lambda}\|$. Let $\varepsilon > \|x_{t,\lambda}^*x_{t,\lambda}\|$. We write $\log(x+\varepsilon)$ as $\log\varepsilon + \log(1+x/\varepsilon)$ and then expand in powers of x/ε. We then substitute $x = x_{t,\lambda}^*x_{t,\lambda}$, and then apply the trace term by term, giving

$$S(t,\lambda,\varepsilon) = \log\varepsilon + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\varepsilon^n}\tau[(x_{t,\lambda}^*x_{t,\lambda})^n]. \tag{3.16}$$

We now wish to differentiate the right-hand side of (3.16) term by term in t. We will see shortly that when we differentiate inside the sum, the resulting series still converges for $\varepsilon > \|x_{t,\lambda}^* x_{t,\lambda}\|$. Furthermore, since the map $t \mapsto x_t$ is continuous in the operator norm topology, $\|x_t\|$ is a locally bounded function of t. Hence, the series of derivatives converges locally uniformly in t. This, together with the pointwise convergence of the original series, will show that term-by-term differentiation is valid.

If we differentiate inside the sum in (3.16), we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\varepsilon^n} \frac{\partial}{\partial t} \tau[(x_{t,\lambda}^* x_{t,\lambda})^n]. \tag{3.17}$$

By Lemma 3.3, the above power series becomes

$$\begin{aligned} &\frac{1}{2}\sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \frac{(-1)^n}{\varepsilon^n} \tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda})^{m-1}]\tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda})^{n-m-1}] \\ &+ \frac{1}{2}\sum_{n=2}^{\infty}\sum_{m=1}^{n-1} \frac{(-1)^n}{\varepsilon^n} \tau[x_{t,\lambda}^*(x_{t,\lambda}^* x_{t,\lambda})^{m-1}]\tau[x_{t,\lambda}^*(x_{t,\lambda}^* x_{t,\lambda})^{n-m-1}] \\ &+ \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n-1}}{\varepsilon^n} \tau[(x_{t,\lambda}^* x_{t,\lambda})^{n-m}]\tau[(x_{t,\lambda}^* x_{t,\lambda})^{m-1}]. \end{aligned} \tag{3.18}$$

Note that the constant term 1 is in the last term in (3.18). The first term in (3.18) may be rewritten as

$$\begin{aligned} &\frac{1}{2}\frac{1}{\varepsilon^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(-1)^n}{\varepsilon^n} \tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda})^{m}]\tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda})^{n-m}] \\ =&\frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{\varepsilon^{k+1}} \tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda})^{k}]\right)\left(\sum_{l=0}^{\infty} \frac{(-1)^l}{\varepsilon^{l+1}} \tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda})^{l}]\right) \\ =&\frac{1}{2}\tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda}+\varepsilon)^{-1}]^2. \end{aligned}$$

The second term in (3.18) differs from the first term only by replacing the $x_{t,\lambda}$ by $x_{t,\lambda}^*$ in the two trace terms, and is therefore computed as

$$\frac{1}{2}\tau[x_{t,\lambda}^*(x_{t,\lambda}^* x_{t,\lambda}+\varepsilon)^{-1}]^2.$$

A similar computation expresses the last term in (3.18) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{(-1)^{n-1}}{\varepsilon^n} \tau[(x_{t,\lambda}^* x_{t,\lambda})^{n-m}]\tau[(x_{t,\lambda}^* x_{t,\lambda})^{m-1}] \;= \varepsilon\tau[(x_{t,\lambda}^* x_{t,\lambda}+\varepsilon)^{-1}]^2.$$

This shows that the series in (3.17) converges to the right hand side of (3.15). It follows that (3.15) holds for all $\varepsilon > \|x_{t,\lambda}^* x_{t,\lambda}\|$.

Thus, for all $\varepsilon > \max_{s\leq t}\|x_{s,\lambda}^* x_{s,\lambda}\|$, we have

$$\begin{aligned} S(t,\lambda,\varepsilon) = S(0,\lambda,\varepsilon) + \int_0^t \Big\{ &\frac{1}{2}\tau[x_{s,\lambda}(x_{s,\lambda}^* x_{s,\lambda}+\varepsilon)^{-1}]^2 \\ &+\frac{1}{2}\tau[x_{s,\lambda}^*(x_{s,\lambda}^* x_{s,\lambda}+\varepsilon)^{-1}]^2 + \varepsilon\tau[(x_{s,\lambda}^* x_{s,\lambda}+\varepsilon)^{-1}]^2 \Big\}\, ds. \end{aligned} \tag{3.19}$$

The right hand side of (3.19) is analytic in ε for all $\varepsilon > 0$. We now claim that the left hand side of (3.19) is also analytic. At each $\varepsilon > 0$, we have the operator-valued power series expansion

$$\log(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon + h) = \log(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} h^n}{n} (x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-n}$$

for $|h| < \|(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}\|$. Taking the trace gives

$$S(t, \lambda, \varepsilon + h) = \log(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} h^n}{n} \tau[(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-n}]$$

for $|h| < \|(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}\|$. This shows $S(t, \lambda, \cdot)$ is analytic on the positive real line. Since both sides of (3.19) define an analytic function for $\varepsilon > 0$ and they agree for all large ε, they are indeed equal for all $\varepsilon > 0$. Now, the conclusion of the proposition follows from differentiating both sides of (3.19) with respect to t. □

Lemma 3.5. *The partial derivatives of S with respect to ε and λ are given by the following formulas.*

$$\begin{aligned}
\frac{\partial S}{\partial \lambda} &= -\tau[x_{t,\lambda}^* (x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}] \\
\frac{\partial S}{\partial \bar{\lambda}} &= -\tau[x_{t,\lambda} (x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}] \\
\frac{\partial S}{\partial \varepsilon} &= \tau[(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}].
\end{aligned}$$

Proof. By Lemma 1.1 in Brown's paper [8], the derivative of the trace of a logarithm is given by

$$\frac{d}{du} \tau[\log(f(u))] = \tau\left[f(u)^{-1} \frac{df}{du}\right]. \tag{3.20}$$

The lemma follows from applying this formula. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.4,

$$\frac{\partial S}{\partial t} = \frac{1}{2}\tau[x_{t,\lambda}(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}]^2 + \frac{1}{2}\tau[x_{t,\lambda}^*(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}]^2 + \varepsilon\tau[(x_{t,\lambda}^* x_{t,\lambda} + \varepsilon)^{-1}]^2.$$

Using Lemma 3.5, the above displayed equation can be written as

$$\frac{\partial S}{\partial t} = \frac{1}{2}\left(\frac{\partial S}{\partial \lambda}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial \bar{\lambda}}\right)^2 + \varepsilon\left(\frac{\partial S}{\partial \varepsilon}\right)^2.$$

Now, (3.1) follows from applying the definition of Cauchy–Riemann operators to the above equation. The initial condition holds because $x_t = x_0$ when $t = 0$. □

4. The Hamilton–Jacobi analysis

4.1. **The Hamilton–Jacobi method.** We define a "Hamiltonian" function $H : \mathbb{R}^6 \to \mathbb{R}$ by replacing the derivatives $\partial S/\partial a$, $\partial S/\partial b$, and $\partial S/\partial \varepsilon$ on the right-hand side of the PDE in Theorem 3.1 by "momentum" variables p_a, p_b, and p_ε, and then reversing the overall sign. Thus, we define

$$H(a, b, \varepsilon, p_a, p_b, p_\varepsilon) = -\frac{1}{4}(p_a^2 - p_b^2) - \varepsilon p_\varepsilon^2, \tag{4.1}$$

where in this case, H happens to be independent of a and b. We then introduce Hamilton's equations for the Hamiltonian H, namely

$$\frac{du}{dt} = \frac{\partial H}{\partial p_u}; \quad \frac{dp_u}{dt} = -\frac{\partial H}{\partial u}, \tag{4.2}$$

where u ranges over the set $\{a, b, \varepsilon\}$. We will use the notation

$$\lambda(t) = a(t) + ib(t).$$

Notation 4.1. *We use the notation*

$$p_{a,0}, \ p_{b,0}, \ p_0$$

for the initial values of p_a, p_b, and p_ε, respectively.

In the Hamilton–Jacobi analysis, the initial momenta are determined by the initial positions λ_0 and ε_0 by means of the following formula:

$$p_{a,0} = \frac{\partial}{\partial a_0} S(0, \lambda_0, \varepsilon_0); \quad p_{b,0} = \frac{\partial}{\partial b_0} S(0, \lambda_0, \varepsilon_0); \quad p_0 = \frac{\partial}{\partial \varepsilon_0} S(0, \lambda_0, \varepsilon_0). \tag{4.3}$$

Now, the formula for $S(0, \lambda, \varepsilon)$ in Theorem 3.1 may be written more explicitly as

$$S(0, \lambda, \varepsilon) = \int_{\mathbb{R}} \log(|x - \lambda|^2 + \varepsilon) \, d\mu(x),$$

where μ is the law of x_0, as in (1.1). We thus obtain the following formula for the initial momenta:

$$\begin{aligned} p_{a,0} &= \int_{\mathbb{R}} \frac{2(a_0 - x)}{(a_0 - x)^2 + b_0^2 + \varepsilon_0} \, d\mu(x) \\ p_{b,0} &= \int_{\mathbb{R}} \frac{2b_0}{(a_0 - x)^2 + b_0^2 + \varepsilon_0} \, d\mu(x) \\ p_0 &= \int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + b_0^2 + \varepsilon_0} \, d\mu(x). \end{aligned} \tag{4.4}$$

Provided we assume $\varepsilon_0 > 0$, the integrals are convergent.

Proposition 4.2. *Suppose we have a solution to the Hamiltonian system on a time interval $[0, T]$ such that $\varepsilon(t) > 0$ for all $t \in [0, T]$. Then we have*

$$S(t, \lambda(t), \varepsilon(t)) = S(0, \lambda_0, \varepsilon_0) + tH_0, \tag{4.5}$$

where

$$H_0 = H(a_0, b_0, \varepsilon_0, p_{a,0}, p_{b,0}, p_0).$$

We also have

$$\frac{\partial S}{\partial u}(t, \lambda(t), \varepsilon(t)) = p_u(t) \tag{4.6}$$

for all $u \in \{a, b, \varepsilon\}$.

We refer to (4.5) and (4.6) as the first and second Hamilton–Jacobi formulas, respectively.

Proof. The reader may consult Section 6.1 of [12] for a concise statement and derivation of the general Hamilton–Jacobi method. (See also the book of Evans

[13].) The general form of the first Hamilton–Jacobi formula, when applied to this case, reads as

$$S(t,\lambda(t),\varepsilon(t)) = S(0,\lambda_0,\varepsilon_0) - tH_0 + \int_0^t \sum_{u\in\{a,b,\varepsilon\}} p_u \frac{\partial H}{\partial p_u}\ ds.$$

In our case, because the Hamiltonian is homogeneous of degree two in the momentum variables, $\sum_{u\in\{a,b,\varepsilon\}} p_u \frac{\partial H}{\partial p_u}$ is equal to $2H$. Since H is a constant of motion, the general formula reduces to (4.5). Meanwhile, (4.6) is an immediate consequence of the general form of the second Hamilton–Jacobi formula. □

4.2. **Solving the ODEs.** We now solve the Hamiltonian system (4.2) with Hamiltonian given by (4.1). We start by noting several helpful constants of motion.

Proposition 4.3. *The quantities*

$$H,\ p_a,\ p_b,\ \varepsilon p_\varepsilon^2$$

are constants of motion, meaning that they are constant along any solution of Hamilton's equations (4.2).

Proof. The Hamiltonian is always a constant of motion in any Hamiltonian system. The quantities p_a and p_b are constants of motion because H is independent of a and b. And finally, $\varepsilon p_\varepsilon^2$ is a constant of motion because it equals $-\frac{1}{4}(p_a^2 - p_b^2) - H$. □

We now obtain solutions to (4.2), where at the moment, we allow arbitrary initial momenta, not necessarily given by (4.3).

Proposition 4.4. *Consider the Hamiltonian system (4.2) with Hamiltonian (4.1) and initial conditions*

$$(a_0, b_0, \varepsilon_0, p_{a,0}, p_{b,0}, p_0),$$

with $p_0 > 0$. *Then the solution to the system exists up to time*

$$t_* = 1/p_0.$$

Up until that time, we have

$$\begin{aligned} p_a(t) &= p_{a,0} \\ a(t) &= a_0 - \frac{1}{2}p_{a,0}t. \\ p_b(t) &= p_{b,0} \\ b(t) &= b_0 + \frac{1}{2}p_{b,0}t \\ p_\varepsilon(t) &= \frac{p_0}{1-p_0t} \\ \varepsilon(t) &= \varepsilon_0\,(1-p_0t)^2\,. \end{aligned}$$

If $\varepsilon_0 > 0$ *then* $\varepsilon(t)$ *remains positive for all* $t < t_*$.

Proof. We begin by noting that

$$\dot{p}_\varepsilon = -\frac{\partial H}{\partial \varepsilon} = p_\varepsilon^2.$$

We may solve this separable equation as

$$-\left(\frac{1}{p_\varepsilon(t)} - \frac{1}{p_0}\right) = t,$$

from which the claimed formula for $p_\varepsilon(t)$ follows. We then note that

$$\begin{aligned}\frac{d\varepsilon}{dt} &= \frac{\partial H}{\partial p_\varepsilon} \\ &= -2\varepsilon p_\varepsilon \\ &= -2\varepsilon \frac{p_0}{1-p_0 t}.\end{aligned}$$

This equation is also separable and may easily be integrated to give the claimed formula for $\varepsilon(t)$.

The formulas for p_a and p_b simply amount to saying that they are constants of motion, and the formulas for a and b are then easily obtained. □

We now specialize the initial conditions to the form occurring in the Hamilton–Jacobi method, that is, where the initial momenta are given by (4.4). We note that the formulas in (4.4) can be written as

$$p_{b,0} = 2b_0 p_0 \tag{4.7}$$

and

$$p_{a,0} = 2a_0 p_0 - 2p_1, \tag{4.8}$$

where

$$p_1 = \int_{\mathbb{R}} \frac{x}{(a_0-x)^2 + b_0^2 + \varepsilon_0}\, d\mu(x). \tag{4.9}$$

Proposition 4.5. *Suppose a_0, b_0, and ε_0 are chosen in such a way that $p_0 = 1/t$, so that the lifetime t_* of the system equals t. Then we have*

$$\begin{aligned}\lim_{s\to t^-} a(s) &= tp_1 \\ \lim_{s\to t^-} b(s) &= 2b_0 \\ \lim_{s\to t^-} \varepsilon(s) &= 0,\end{aligned}$$

where p_1 is as in (4.9).

Proof. The result follows easily from the formulas in Proposition 4.4, after using the relations (4.7) and (4.8) and setting $p_0 = 1/t$. □

Definition 4.6. *Let $t_*(\lambda_0, \varepsilon_0)$ denote the lifetime of the solution, namely*

$$t_*(\lambda_0,\varepsilon_0) = \frac{1}{p_0} = \left(\int_{\mathbb{R}} \frac{d\mu(x)}{(a_0-x)^2+b_0^2+\varepsilon_0}\right)^{-1},$$

and let

$$T(\lambda_0) := \lim_{\varepsilon_0\to 0^+} t_*(\lambda_0,\varepsilon_0) = \left(\int_{\mathbb{R}} \frac{d\mu(x)}{(a_0-x)^2+b_0^2}\right)^{-1}.$$

We note that if $b_0 = 0$ then the integral in the definition of $T(a_0+ib_0)$ may be infinite for certain values of a_0. Thus, it is possible for $T(a_0+ib_0)$ to equal 0 when $b_0 = 0$.

Proposition 4.7. *Let*

$$\lambda(t;\lambda_0,\varepsilon_0)$$

denote the solution to the system (4.2) with $\lambda(0) = \lambda_0$ *and* $\varepsilon(0) = \varepsilon_0$*, and with initial momenta given by (4.4). Suppose* λ_0 *satisfies* $T(\lambda_0) > t$*. Then*

$$\lim_{\varepsilon_0 \to 0^+} \lambda(t; \lambda_0, \varepsilon_0) = \lambda_0 - t \int_{\mathbb{R}} \frac{1}{\lambda_0 - x}\, d\mu(x),$$

provided that λ_0 *does not belong the closed support of* μ*.*

Proof. Using Proposition 4.4, we find that

$$\begin{aligned} \lambda(t; \lambda_0, \varepsilon_0) &= a(t) + ib(t) \\ &= \lambda_0 - \frac{t}{2}(p_{a,0} - ip_{b,0}). \end{aligned}$$

In the limit as ε_0 tends to zero, we have (provided λ_0 is not in $\operatorname{supp}(\mu) \subset \mathbb{R}$)

$$p_{a,0} - ip_{b,0} = \int_{\mathbb{R}} \frac{2(a_0 - x)}{(a_0 - x)^2 + b_0^2}\, d\mu(x) - i \int_{\mathbb{R}} \frac{2b_0}{(a_0 - x)^2 + b_0^2}\, d\mu(x).$$

It is then easy to check that

$$p_{a,0} - ip_{b,0} = 2 \int_{\mathbb{R}} \frac{1}{a_0 + ib_0 - x}\, d\mu(x),$$

which gives the claimed formula. □

5. The domains

5.1. **The domain in the λ_0-plane.**

We now define the first of two domains we will be interested in. When we apply the Hamilton–Jacobi method in Section 6, we will try to find solutions with $\varepsilon(t)$ very close to zero. Based on the formula for $\varepsilon(t)$ in Proposition 4.4, it seems that we can make $\varepsilon(t)$ small by making ε_0 small. The difficulty with this approach, however, is that if we fix some λ_0 and let ε_0 tend to zero, the lifetime of the path may be smaller than t. Thus, if the small-ε_0 lifetime of the path—as computed by the function T in Definition 4.6—is smaller than t, the simple approach of letting ε_0 tend to zero will not work. This observation motivates the following definition.

Definition 5.1. *Let* T *be the function defined in Definition 4.6. We then define a domain* $\Lambda_t \subset \mathbb{C}$ *by*

$$\Lambda_t = \{\lambda_0 \in \mathbb{C} |\, T(\lambda_0) < t\}.$$

Explicitly, a point $\lambda_0 = a_0 + ib_0$ *belongs to* Λ_t *if and only if*

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(a_0 - x)^2 + b_0^2} > \frac{1}{t}. \tag{5.1}$$

This domain appeared originally in the work Biane [2], for reasons that we will explain in Section 5.2. The domain Λ_t also plays a crucial role in work of the second author with Zhong [22]. In Section 5.3, we will consider another domain Ω_t, whose closure will be the support of the Brown measure of $x_0 + i\sigma_t$. See Figure 6 for plots of Λ_t and the corresponding domain Ω_t.

We give now a more explicit description of the domain Λ_t.

Proposition 5.2. *For each* $t > 0$*, define a function* $v_t : \mathbb{R} \to [0, \infty)$ *as follows. For each* $a_0 \in \mathbb{R}$*, if*

$$\int_{\mathbb{R}} \frac{1}{(a_0 - x)^2}\, d\mu(x) > \frac{1}{t}, \tag{5.2}$$

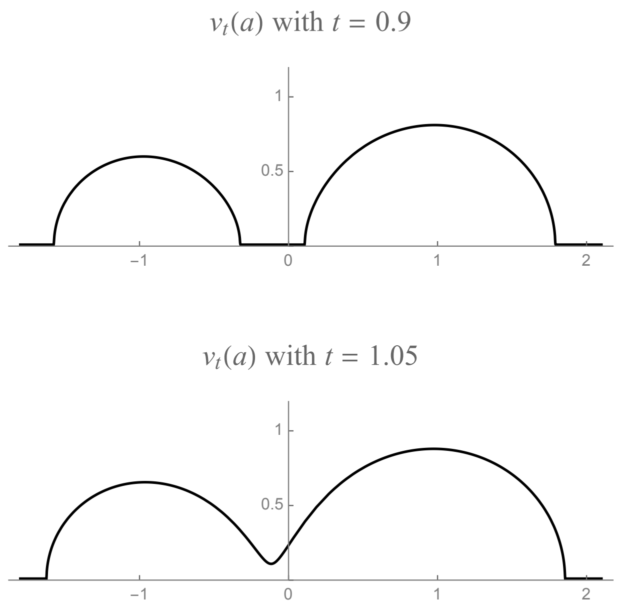

FIGURE 5. The function $v_t(a)$ for the case in which $\mu = \frac{1}{3}\delta_{-1} + \frac{2}{3}\delta_1$.

let $v_t(a_0)$ be the unique positive number such that

$$\int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + v_t(a_0)^2}\ d\mu(x) = \frac{1}{t}. \tag{5.3}$$

If, on the other hand,

$$\int_{\mathbb{R}} \frac{1}{(a_0 - x)^2}\ d\mu(x) \leq \frac{1}{t}, \tag{5.4}$$

set $v_t(a_0) = 0$.

Then the function $v_t : \mathbb{R} \to [0, \infty)$ is continuous and the domain Λ_t may be described as

$$\Lambda_t = \{\, a_0 + ib_0 \in \mathbb{C} |\ \ |b_0| < v_t(a_0) \}\,, \tag{5.5}$$

so that

$$\Lambda_t \cap \mathbb{R} = \{\, a_0 \in \mathbb{R} |\, v_t(a_0) > 0 \}\,. \tag{5.6}$$

See Figure 5 for some plots of the function v_t.

Proof. We first note that for any fixed a_0, the integral

$$\int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + v^2}\ d\mu(x) \tag{5.7}$$

is a strictly decreasing function of $v \geq 0$ and that the integral tends to zero as v tends to infinity. Thus, whenever condition (5.2) holds, it is easy to see that there is a unique positive number $v_t(a_0)$ for which (5.3) holds. Continuity of v_t is established in [2, Lemma 2].

Using the monotonicity of the integral in (5.7), it is now easy to see that the characterization of the domain Λ_t in (5.5) is equivalent to the characterization in (5.1). □

5.2. **The result of Biane.** We now explain how the domain Λ_t arose in the work of Biane [2]. The results of Biane will be needed to formulate one of our main results (Theorem 8.2).

For any operator $A \in \mathcal{A}$, we let G_A denote the Cauchy transform of A, also known as the Stieltjes transform or holomorphic Green's function, defined as

$$G_A(z) = \tau[(z - A)^{-1}] \tag{5.8}$$

for all $z \in \mathbb{C}$ outside the spectrum of A. Then G_A is holomorphic on its domain. If A is self-adjoint, we can recover the distribution of A from its Cauchy transform by the Stieltjes inversion formula. Even if A is not self-adjoint, G_A determines the holomorphic moments of the Brown measure Brown(A) of A, that is, the integrals of λ^n with respect to Brown(A). (We emphasize that these holomorphic moments do not, in general, determine the Brown measure itself.)

Let x_0 be a self-adoint element of $\mathcal{A}$ and let $\sigma_t \in \mathcal{A}$ be a semicircular Brownian motion freely independent of x_0. Define a function H_t by

$$H_t(\lambda_0) = \lambda_0 + tG_{x_0}(\lambda_0). \tag{5.9}$$

The significance of this function is from the following result of Biane [2], which shows that the Cauchy transform of $x_0 + \sigma_t$ is related to the Cauchy transform of x_0 by the formula

$$G_{x_0+\sigma_t}(H_t(\lambda_0)) = G_{x_0}(\lambda_0), \tag{5.10}$$

for λ_0 in an appropriate set, which we will specify shortly. Note that this result is for the Cauchy transform of the self-adjoint operator $x_0 + \sigma_t$, not for $x_0 + i\sigma_t$.

We now explain the precise domain (taken to be in the upper half-plane for simplicity) on which the identity (5.10) holds. Let

$$\Delta_t = \{\, a_0 + ib_0 |\, b_0 > v_t(a_0)\} \,, \tag{5.11}$$

which is just the set of points in the upper half-plane outside the closure of Λ_t. The boundary of Δ_t is then the graph of v_t:

$$\partial\Delta_t = \{a_0 + i\, v_t(a_0) |\, a_0 \in \mathbb{R}\} \,.$$

Theorem 5.3 (Biane)**.** *First, the function H_t is an injective conformal map of Δ_t onto the upper half-plane. Second, H_t maps $\partial\Delta_t$ homeomorphically onto the real line. Last, the identity (5.10) holds for all λ_0 in Δ_t. Thus, we may write*

$$G_{x_0+\sigma_t}(\lambda) = G_{x_0}(H_t^{-1}(\lambda))$$

for all λ in the upper half-plane, where the inverse function H_t^{-1} is chosen to map into Δ_t.

See Lemma 4 and Proposition 2 in [2]. In the terminology of Voiculescu [34, 35], we may say that H_t^{-1} is one of the **subordination functions** for the sum $x_0 + \sigma_t$, meaning that one can compute $G_{x_0+\sigma_t}$ from G_{x_0} by composing with H_t^{-1}. Since $x_0 + \sigma_t$ is self-adjoint, one can then compute the distribution of $x_0 + \sigma_t$ from its Cauchy transform. We remark that Biane denotes the map H_t^{-1} by F_t on p. 710 of [2].

5.3. **The domain in the λ-plane.** Our strategy in applying the Hamilton–Jacobi method will be in two stages. In the first stage, we attempt to make $\varepsilon(t)$ close to zero by taking ε_0 close to zero. For this strategy to work, we must have λ_0 outside the closure of the domain Λ_t introduced in Section 5.1. We will then solve the system of ODEs (4.2) in the limit as ε_0 approaches zero, using Proposition 4.7. Let us define a map J_t by

$$J_t(\lambda_0) = \lambda_0 - tG_{x_0}(\lambda_0), \tag{5.12}$$

which differs from the function H_t in Section 5.2 by a change of sign. (See Section 5.4 for a different perspective on how this function arises.) With this notation, Proposition 4.7 says that if $\lambda(0) = \lambda_0$ and ε_0 approaches zero, then

$$\lambda(t) = J_t(\lambda_0),$$

provided that λ_0 is outside the closure of Λ_t. Thus, the first stage of our analysis will allow us to compute the Brown measure at points of the form $J_t(\lambda_0)$ with $\lambda_0 \notin \overline{\Lambda}_t$. We will find that the Brown measure is zero in a neighborhood of any such point. A second stage of the analysis will then be required to compute the Brown measure at points inside $\overline{\Lambda}_t$.

The discussion the previous paragraph motivates the following definition.

Definition 5.4. *For each $t > 0$, define a domain Ω_t in $\mathbb{C}$ by*

$$\Omega_t = [J_t(\Lambda_t^c)]^c.$$

That is to say, the complement of Ω_t is the image under J_t of the complement of Λ_t.

See Figure 6 for plots of the domains Λ_t and Ω_t.

We recall our standing assumption that μ is not a δ-measure and we remind the reader that the set Δ_t in (5.11) is the region above the graph of v_t so that $\overline{\Delta}_t$ is the set of points on or above the graph of v_t.

Proposition 5.5. *The following results hold.*

1. *The map J_t is well-defined, continuous, and injective on $\overline{\Delta}_t$.*
2. *Define a function $a_t : \mathbb{R} \to \mathbb{R}$ by*
$$a_t(a_0) = \mathrm{Re}[J_t(a_0 + iv_t(a_0))]. \tag{5.13}$$
Then at any point a_0 with $v_t(a_0) > 0$, the function a_t is differentiable and satisfies
$$0 < \frac{da_t}{da_0} < 2.$$
3. *The function a_t is continuous and strictly increasing and maps $\mathbb{R}$ onto $\mathbb{R}$.*
4. *The map J_t maps the graph of v_t to the graph of a function, which we denote by b_t. The function b_t satisfies*
$$b_t(a_t(a_0)) = 2v_t(a_0) \tag{5.14}$$
for all $a_0 \in \mathbb{R}$.
5. *The map J_t takes the region above the graph of v_t* onto *the region above the graph of b_t.*
6. *The set Ω_t defined in Definition 5.4 may be computed as*
$$\Omega_t = \{ a + ib \in \mathbb{C} |\ \ |b| < b_t(a) \}.$$

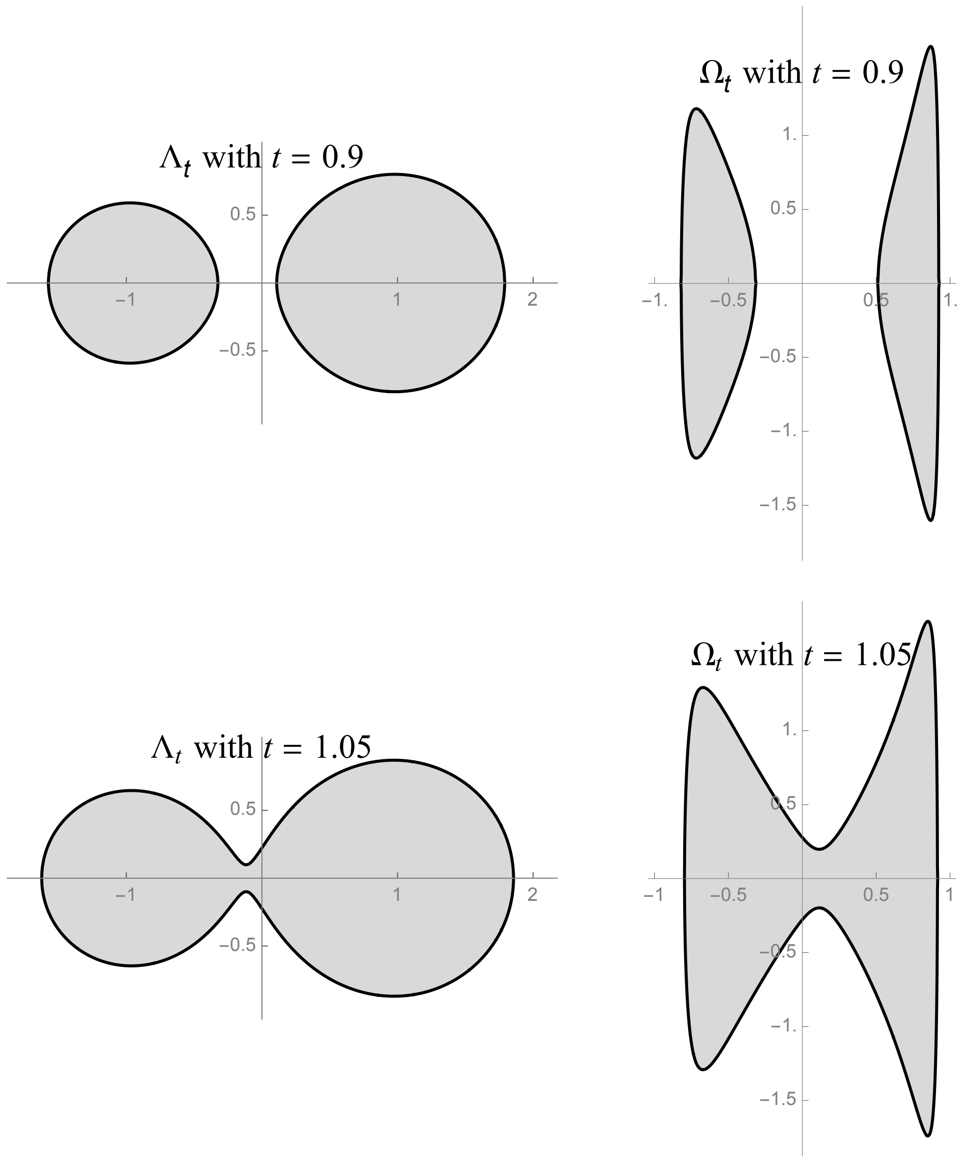

FIGURE 6. The regions Λ_t and Ω_t for $\mu = \frac{1}{3}\delta_{-1} + \frac{2}{3}\delta_1$.

Since $J_t(z) = 2z - H_t(z)$ and $H_t(a_0 + iv_t(a_0))$ is real, we see that $a_t(a_0) = 2a_0 - H_t(a_0 + iv_t(a_0))$. Lemma 5 of [2] and Theorem 3.14 of [22] show that $0 < H_t'(a_0 + iv_t(a_0)) \leq 2$, which means $0 \leq a_t'(a_0) < 2$. Thus, Point 2 improves the result to $0 < a_t'(a_0) < 2$.

The proof requires μ to have more than one point in its support in order to prove $a_t'(a_0) \neq 0$. When $\mu = \delta_0$, it can be computed that $J_t(z) = z - \frac{t}{z}$ and $a_0 + iv_t(a_0)$ is the upper semicircle of radius $\sqrt{t}$. Therefore, $\mathrm{Re}[J_t(a_0 + iv_t(a_0))] = 0$ for all $a_0 \in \Lambda_t \cap \mathbb{R}$, and its derivative is constantly 0 on $\Lambda_t \cap \mathbb{R}$.

The proof is similar to the proof in [2] of similar results about the map H_t.

Proof. Continuity of J_t on $\overline{\Delta}_t$ follows from [2, Lemma 3], which shows continuity of G_{x_0} on $\overline{\Delta}_t$. To show injectivity of J_t, suppose, toward a contradiction, that $J_t(z_1) = J_t(z_2)$, for some $z_1 \neq z_2$ in $\overline{\Delta}_t$. Then, using the definition (5.12) of J_t, we have

$$t(G_\mu(z_2) - G_\mu(z_1)) = z_2 - z_1.$$

This shows

$$t \int_{\mathbb{R}} \frac{z_1 - z_2}{(z_1 - x)(z_2 - x)}\, d\mu(x) = z_2 - z_1.$$

Since we are assuming that z_1 and z_2 are distinct, we can divide by $z_1 - z_2$ to obtain

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(z_1 - x)(z_2 - x)} = -\frac{1}{t}. \tag{5.15}$$

Since $z_1, z_2 \in \overline{\Delta}_t$, we have $T(z_1) \leq 1/t$ and $T(z_2) \leq 1/t$. Thus, by the Cauchy–Schwarz inequality,

$$\left| \int_{\mathbb{R}} \frac{d\mu(x)}{(z_1 - x)(z_2 - x)} \right|^2 \leq \int_{\mathbb{R}} \frac{d\mu(x)}{|z_1 - x|^2} \int_{\mathbb{R}} \frac{d\mu(x)}{|z_2 - x|^2} \leq \frac{1}{t^2}.$$

By (5.15), we have equality in the above Cauchy–Schwarz inequality. Therefore, there exists an $\alpha \in \mathbb{C}$ such that the relation

$$\frac{1}{\bar{z}_2 - x} = \frac{\alpha}{z_1 - x}, \tag{5.16}$$

or, equivalently,

$$(\alpha - 1)x = \alpha \bar{z}_2 - z_1 \tag{5.17}$$

holds for μ-almost every x. Since μ is assumed not to be a δ-measure, we must have $\alpha = 1$, or else x would equal the constant value $(\alpha\bar{z}_2 - z_1)/(\alpha - 1)$ for μ-almost every x. With $\alpha = 1$, we find that $z_1 = \bar{z}_2$. But now if we substitute $z_1 = \bar{z}_2$ into (5.15), we obtain

$$\int_{\mathbb{R}} \frac{d\mu(x)}{|z_1 - x|^2} = -\frac{1}{t}$$

which is impossible. This shows z_1 and z_2 cannot be distinct and Point 1 is established.

For Point 2, fix a_0 with $v_t(a_0) > 0$. We compute that

$$G'_{x_0}(\lambda_0) = -\int_{\mathbb{R}} \frac{d\mu(x)}{(\lambda_0 - x)^2}$$

so that

$$\left| G'_{x_0}(a_0 + iv_t(a_0)) \right| \leq \int_{\mathbb{R}} \frac{d\mu(x)}{(a_0 - x)^2 + v_t(a_0)^2} = \frac{1}{t}. \tag{5.18}$$

We claim that this inequality must be strict. Otherwise, we would have equality in the "putting the absolute value inside the integral" inequality. This would mean, by the proof of Theorem 1.33 of [31], that

$$\frac{1}{(\lambda_0 - x)^2}$$

would have the same phase for μ-almost every x. But since λ_0 is in the upper half plane, the phase of $\lambda_0 - x$ increases from 0 to π as x increases from $-\infty$ to ∞. Thus, the phase of $1/(\lambda_0 - x)^2$ decreases from 2π to 0 as x increases from $-\infty$ to

∞. Therefore, $1/(\lambda_0 - x)^2$ cannot have the same phase μ-almost every x unless μ is a δ-measure.

Now,

$$\frac{d}{da_0}G_{x_0}(a_0 + iv_t(a_0)) = G'_{x_0}(a_0 + iv_t(a_0))\left(1 + i\frac{dv_t(a_0)}{da_0}\right).$$

Since (5.18) is a strict inequality,

$$\left|\frac{d}{da_0}G_{x_0}(a_0 + iv_t(a_0))\right|^2 < \frac{1}{t^2}\left(1 + \left(\frac{dv_t(a_0)}{da_0}\right)^2\right). \tag{5.19}$$

Since

$$\mathrm{Im}[G_{x_0}(a_0 + iv_t(a_0))] = -v_t(a_0)\int_{\mathbb{R}} \frac{d\mu(x)}{(a_0 - x)^2 + v_t(a_0)^2} = -\frac{v_t(a_0)}{t},$$

we have

$$\left(\frac{d}{da_0}\,\mathrm{Im}[G_{x_0}(a_0 + iv_t(a_0))]\right)^2 = \frac{1}{t^2}\frac{dv_t(a_0)}{a_0}$$

and (5.19) becomes

$$\left(\frac{d}{da_0}\,\mathrm{Re}[G_{x_0}(a_0 + iv_t(a_0))]\right)^2 < \frac{1}{t^2}.$$

This shows, using the definition $a_t(a_0) = \mathrm{Re}[J_t(a_0 + iv_t(a_0))]$,

$$a_t'(a_0) = 1 - t\frac{d}{da_0}\,\mathrm{Re}[G_{x_0}(a_0 + iv_t(a_0))] \in (0, 2),$$

as claimed.

We now turn to Point 3. To show that the function $a_t(a_0)$ is strictly increasing with a_0, we use two observations. First, by Point 2, a_t is increasing at any point a_0 where $v_t(a_0) > 0$. Second, when $v_t(a_0) = 0$, we have

$$a_t(a_0) = a_0 - t\int_{\mathbb{R}} \frac{1}{a_0 - x}\, d\mu(x). \tag{5.20}$$

We claim that the right-hand side of (5.20) is an increasing function of a_0. Suppose that $a_0 < a_1$ and $v_t(a_0) = v_t(a_1) = 0$. We compute

$$a_t(a_1) - a_t(a_0) = (a_1 - a_0)\left(1 + t\int \frac{1}{(a_1 - x)(a_0 - x)}\, d\mu(x)\right). \tag{5.21}$$

By Cauchy–Schwarz inequality and (5.4),

$$\left|\int \frac{1}{(a_1 - x)(a_0 - x)}\, d\mu(x)\right|^2 \leq \int \frac{d\mu(x)}{(a_1 - x)^2}\int \frac{d\mu(x)}{(a_0 - x)^2} \leq \frac{1}{t^2};$$

the Cauchy–Schwarz inequality is indeed strict by the reasoning leading to (5.16) and (5.17). This proves that the right-hand side of (5.21) is positive and we have established our claim.

Consider any two points a_0 and a_1 with $a_0 < a_1$; we wish to show that $a_t(a_0) < a_t(a_1)$.

We consider four cases, corresponding to whether $v_t(a_0)$ and $v_t(a_1)$ are zero or positive. If $v_t(a_0)$ and $v_t(a_1)$ are both zero, we use (5.20) and immediately conclude that $a_t(a_0) < a_t(a_1)$. If $v_t(a_0) = 0$ but $v_t(a_1) > 0$, then let α be the infimum of the interval I around a_1 on which v_t is positive, so that $v_t(\alpha) = 0$ and $a_0 \leq \alpha$.

Then $a_t(a_0) \leq a_t(\alpha)$ by (5.20) and $a_t(\alpha) < a_t(a_1)$ by the positivity of a_t' on I. The remaining cases are similar; the case where both $v_t(a_0)$ and $v_t(a_1)$ are positive can be subdivided into two cases depending on whether or not a_0 and a_1 are in the same interval of positivity of v_t.

Finally, we show that a_t maps $\mathbb{R}$ onto $\mathbb{R}$. Since x_0 is assumed to be bounded, the law μ of x_0 is compactly supported. It then follows easily from the condition (5.4) for v_t to be zero that $v_t(a_0) = 0$ whenever $|a_0|$ is large enough. Thus, for $|a_0|$ large, the formula (5.20) applies, and we can easily see that $\lim_{a_0 \to -\infty} a_t(a_0) = -\infty$ and $\lim_{a_0 \to +\infty} a_t(a_0) = +\infty$.

For Point 4, it follows easily from Point 3 and the definition (5.13) of a_t that J_t maps the graph of v_t to the graph of a function. When then note that (5.14) holds when $v_t(a_0) = 0$—both sides are zero. To establish (5.14) when $v_t(a_0) > 0$, we compute that

$$\begin{aligned}
\operatorname{Im}[J_t(a_0 + iv_t(a_0))] &= v_t(a_0) - t \operatorname{Im} \int_{\mathbb{R}} \frac{1}{a_0 + iv_t(a_0) - x} \, d\mu(x) \\
&= v_t(a_0) + tv_t(a_0) \int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + v_t(a_0)^2} \, d\mu(x) \\
&= 2v_t(a_0),
\end{aligned}$$

by the defining property (5.3) of v_t.

For Point 5, we note that the graph of v_t, together with the point at infinity, forms a Jordan curve in the Riemann sphere, with the region above the graph as the interior of the disk—and similarly with v_t replaced by b_t. Since $J_t(\lambda_0)$ tends to infinity as λ_0 tends to infinity, J_t defines a continuous map of the closed disk bounded by $\operatorname{graph}(v_t) \cup \{\infty\}$ to the closed disk bounded by $\operatorname{graph}(b_t) \cup \{\infty\}$, and this map is a homeomorphism on the boundary. By an elementary topological argument, J_t must map the closed disk *onto* the closed disk.

Finally, for Point 6, we use the description of Λ_t in Proposition 5.2 as the region bounded by the graphs of v_t and $-v_t$. The complement of Λ_t thus consists of the region on or above the graph of v_t or on or below the graph of $-v_t$. By Point 5 and the fact that J_t commutes the complex conjugation, J_t will map the complement of Λ_t to the region on or above the graph of b_t or on or below the graph of $-b_t$. Thus, from Definition 5.4, Ω_t will be the region bounded by the graphs of b_t and $-b_t$. □

5.4. **The method of Janik, Nowak, Papp, Wambach, and Zahed.** We now discuss the work of Janik, Nowak, Papp, Wambach, and Zahed [23] which gives a nonrigorous but illuminating method of computing the support of the Brown measure of $x_0 + i\sigma_t$. (See especially Section V of [23].) This method does not say anything about the Brown measure besides what its support should be. Furthermore, it is independent of the method used by Jarosz and Nowak in [24, 25] and discussed in Section 9.

Recall the definition of the Cauchy transform of an operator A in (5.8). We note that if λ is outside the spectrum of A, then we may safely put $\varepsilon = 0$ in the function

$$S_A(\lambda, \varepsilon) := \tau[\log((A - \lambda)^*(A - \lambda) + \varepsilon)].$$

Using the formula (3.20) for the derivative of the trace of the logarithm, we can easily compute that

$$\frac{\partial}{\partial\lambda}S_A(\lambda,0)=\tau[(\lambda-A)^{-1}]=G_A(\lambda).$$

But since $G_A(\lambda)$ depends holomorphically on λ, we find that

$$\begin{aligned}\Delta_\lambda S_A(\lambda,0)&=4\frac{\partial}{\partial\bar\lambda}\frac{\partial}{\partial\lambda}S_A(\lambda,0)\\&=4\frac{\partial}{\partial\bar\lambda}G_A(\lambda)\\&=0,\end{aligned}$$

so that the Brown measure is zero. This argument shows that the Brown measure is zero outside the spectrum of A.

Now, in the case $A=x_0+i\sigma_t$, the authors of [23] attempt to determine the maximum set on which the function

$$\frac{\partial}{\partial\lambda}S(t,\lambda,0)$$

remains holomorphic. We start with Biane's subordination function identity (5.10), which we rewrite as follows. Let σ be a fixed semicircular element, so that the law of σ_t is the same as that of $\sqrt{t}\sigma$. Then set $u=\sqrt{t}$, so that (5.10) reads as

$$\tau[\{\lambda+u^2G_{x_0}(\lambda)-(x_0+u\sigma)\}^{-1}]=G_{x_0}(\lambda).$$

We then formally analytically continue to $u=i\sqrt{t}$, giving

$$\tau[\{\lambda-tG_{x_0}(\lambda)-(x_0+i\sqrt{t}\sigma)\}^{-1}]=G_{x_0}(\lambda).$$

Thus,

$$G_{x_0+\sigma_t}(\lambda+tG_{x_0}(\lambda))=G_{x_0}(\lambda)=G_{x_0+i\sigma_t}(\lambda-tG_{x_0}(\lambda)).$$

In terms of the maps H_t and J_t defined in (5.9) and (5.12), respectively, we then have

$$G_{x_0+i\sigma_t}(J_t(\lambda))=G_{x_0+\sigma_t}(H_t(\lambda))$$

or

$$G_{x_0+i\sigma_t}(J_t(H_t^{-1}(\lambda)))=G_{x_0+\sigma_t}(\lambda).\tag{5.22}$$

We also note that from the definitions (5.9) and (5.12), we have $J_t(\lambda)=2\lambda-H_t(\lambda)$, so that

$$J_t(H_t^{-1}(z))=2H_t^{-1}(z)-z.\tag{5.23}$$

Then since the right-hand side of (5.22) is holomorphic on the whole upper half-plane, the authors of [23] argue that the identity (5.22) actually holds on the whole upper half-plane. If that claim actually holds, we will have the identity

$$G_{x_0+i\sigma_t}(z)=G_{x_0+\sigma_t}(H_t(J_t^{-1}(z)))\tag{5.24}$$

for all z in the range of $J_t\circ H_t^{-1}$, namely for all z (in the upper half-plane) outside the closure of Ω_t. An exactly parallel argument then applies in the lower half-plane. The authors thus wish to conclude that $G_{x_0+i\sigma_t}$ is defined and holomorphic on the complement of $\overline{\Omega}_t$, which would show that the Brown measure of $x_0+i\sigma_t$ is zero there.

We emphasize that the argument for (5.22) is rigorous for all sufficiently large λ, simply because the quantity $\tau[(\lambda-A)^{-1}]$ depends holomorphically on both the complex number λ and the operator A. But just because the right-hand side of the identity extends holomorphically to the upper half-plane does not by itself mean that the identity continues to hold on the whole upper half-plane. Thus, the argument in [23] is not entirely rigorous. Nevertheless, it certainly gives a natural explanation of how the domain Ω_t arises.

The identities (5.22) and (5.24) already indicate a close relationship between the operators $x_0+i\sigma_t$ and $x_0+\sigma_t$. In Section 8, we will find an even closer relationship: The push-forward of the Brown measure of $x_0+i\sigma_t$ under a certain map $Q_t:\overline{\Omega}_t\to\mathbb{R}$ is precisely the law of $x_0+\sigma_t$. The map Q_t is constructed as follows: It is the unique map of $\overline{\Omega}_t$ to $\mathbb{R}$ that agrees with $H_t\circ J_t^{-1}$ on $\partial\Omega_t$ and maps vertical segments in $\overline{\Omega}_t$ to points in $\mathbb{R}$.

6. Outside the domain

In this section, we show that the Brown measure of $x_0+i\sigma_t$ is zero in the complement of the closure of the domain Ω_t in Definition 5.4. We outline our strategy in Section 6.1 and then give a rigorous argument in Section 6.2.

6.1. **Outline.**

Our goal is to compute the Laplacian with respect to λ of the function

$$s_t(\lambda)=\lim_{\varepsilon\to 0^+}S(t,\lambda,\varepsilon).$$

We use the Hamilton–Jacobi method of Proposition 4.2, which gives us a formula for $S(t,\lambda(t),\varepsilon(t))$. Since (Proposition 4.4) $\varepsilon(t)=\varepsilon_0(1-p_0t)^2$, we can attempt to make $\varepsilon(t)$ approach 0 by letting ε_0 approach zero. This strategy, however, can only succeed if the lifetime of the path remains at least t in the limit as $\varepsilon_0\to 0$. Thus, we must take λ_0 for which $T(\lambda_0)\geq t$, where T is as in Definition 4.6. We therefore consider λ_0 in $\overline{\Lambda}_t^c$, where Λ_t is as in Definition 5.1.

If we formally put $\varepsilon_0=0$, then $\varepsilon(t)=0$, and, by Proposition 4.7, we have

$$\lambda(t)=J_t(\lambda_0). \tag{6.1}$$

Now, by Proposition 5.5, J_t maps Λ_t^c injectively onto Ω_t^c. Thus, for any $\lambda\in\overline{\Omega}_t^c$, we may choose $\lambda_0=J_t^{-1}(\lambda)$. Then, if we *formally* apply the Hamilton–Jacobi formula (4.5) with $\varepsilon_0=0$, we get

$$S(t,\lambda,0)=\int_{\mathbb{R}}\log(|J_t^{-1}(\lambda)-x|^2)\,d\mu(x)-\frac{t}{4}(p_{a,0}^2-p_{b,0}^2),$$

where, with $\varepsilon_0=0$, the initial momenta in (4.4) may be computed as

$$\begin{aligned}
p_{a,0}&=2\int_{\mathbb{R}}\frac{(a_0-x)}{(a_0-x)^2+b_0^2}\,d\mu(x)=2\operatorname{Re}\int_{\mathbb{R}}\frac{1}{\lambda_0-x}\,d\mu(x)\\
p_{b,0}&=2\int_{\mathbb{R}}\frac{b_0}{(a_0-x)^2+b_0^2}\,d\mu(x)=-2\operatorname{Im}\int_{\mathbb{R}}\frac{1}{\lambda_0-x}\,d\mu(x).
\end{aligned}$$

Thus,

$$S(t,\lambda,0)=\int_{\mathbb{R}}\log(|J_t^{-1}(\lambda)-x|^2)\,d\mu(x)-t\operatorname{Re}[G_{x_0}(J_t^{-1}(\lambda_0))^2], \tag{6.2}$$

where

$$G_{x_0}(\lambda)=\tau((\lambda-x_0)^{-1})=\int_{\mathbb{R}}\frac{1}{\lambda-x}\,d\mu(x).$$

The right-hand side of (6.2) is the composition of a harmonic function and a holomorphic function and is therefore harmonic. We thus wish to conclude that Brown measure of $x_0 + i\sigma_t$ is zero outside $\overline{\Omega}_t$.

The difficulty with the preceding argument is that the function $S(t, \lambda, \varepsilon)$ is only known ahead of time to be defined for $\varepsilon > 0$. Thus, the PDE in Theorem 3.1 is only known to hold when $\varepsilon > 0$ and the Hamilton–Jacobi formula is only valid when $\varepsilon(t)$ remains positive. We are therefore not allowed to set $\varepsilon_0 = 0$ in the Hamilton–Jacobi formula (4.5). Now, if λ is outside the spectrum of $x_0 + i\sigma_t$, then we can see that $S(t, \lambda, \varepsilon)$ continues to make sense for $\varepsilon = 0$ and even for ε slightly negative, and the PDE and Hamilton–Jacobi formula presumably apply. But of course we do not know that every point in $\overline{\Omega}_t$ is outside the spectrum of $x_0 + i\sigma_t$; if we did, an elementary property of the Brown measure would already tell us that the Brown measure is zero there.

If, instead, we let ε_0 *approach* zero from above, we find that

$$\lim_{\varepsilon_0 \to 0^+} S(t, \lambda(t), \varepsilon(t)) = \int_{\mathbb{R}} \log(|J_t^{-1}(\lambda) - x|^2)\, d\mu(x) - t\mathrm{Re}\,[G_{x_0}(J_t^{-1}(\lambda))^2]. \quad (6.3)$$

Now, as $\varepsilon_0 \to 0^+$, we can see that $\lambda(t)$ approaches λ and $\varepsilon(t)$ approaches 0. But there is still a difficulty, because the function $s_t(\lambda)$ is defined as the limit of $S(t, \lambda, \varepsilon)$ as ε tends to zero *with* λ *fixed*. But on the left-hand side of (6.3), $\lambda = \lambda(t)$ is not fixed, because it depends on ε_0. To overcome this difficulty, we will use the inverse function theorem to show that, for each $t > 0$, the function S has an extension to a neighborhood of $(t, \lambda, 0)$ that is continuous in the λ- and ε-variables. Thus, the limit of S along *any* path approaching $(t, \lambda, 0)$ is the same as the limit with λ fixed and ε tending to zero.

6.2. **Rigorous treatment.** In this section, we establish the following rigorous version of (6.2), which shows that the support of the Brown measure of $x_0 + i\sigma_t$ is contained in $\overline{\Omega}_t$. Recall that $s_t(\lambda)$ is the limit of $S(t, \lambda, \varepsilon)$ as ε approaches zero from above with λ fixed.

Theorem 6.1. *If* λ *is not in* $\overline{\Omega}_t$*, we have*

$$s_t(\lambda) = \int_{\mathbb{R}} \log(|J_t^{-1}(\lambda) - x|^2)\, d\mu(x) - t\mathrm{Re}\,[G_{x_0}(J_t^{-1}(\lambda))^2] \quad (6.4)$$

and $\Delta s_t(\lambda) = 0$.

The theorem will follow from the argument in Section 6.1, once the following regularity result is established.

Proposition 6.2. *Fix a time* $t > 0$ *and a point* $\lambda^* \in \overline{\Omega}_t^c$*. Then the function* $(\lambda, \varepsilon) \mapsto S(t, \lambda, \varepsilon)$ *extends to a real analytic function defined in a neighborhood of* $(\lambda^*, 0)$ *inside* $\mathbb{C} \times \mathbb{R}$.

We will need the following preparatory result.

Lemma 6.3. *If* λ_0 *is not in* $\overline{\Lambda}_t$*, there is a neighborhood of* λ_0 *in* $\overline{\Lambda}_t^c$ *that does not intersect* $\mathrm{supp}(\mu)$.

The result of this lemma does not hold if we replace $\overline{\Lambda}_t^c$ by Λ_t^c. As a counterexample, if $\mu = 3x^2\, dx$ on $[0, 1]$, then using the criterion (5.6) for $\Lambda_t \cap \mathbb{R}$, we find that $0 \in \Lambda_t^c$ for small enough t, but $0 \in \mathrm{supp}(\mu)$.

Proof. It is clear that the statement of this lemma holds unless $\lambda_0 \in \mathbb{R}$ since x_0 is self-adjoint.

Consider, then, a point $\lambda_0 \in \overline{\Lambda}_t^c \cap \mathbb{R}$. Choose an interval (α, β) around λ_0 contained in $\overline{\Lambda}_t^c \cap \mathbb{R}$. We claim that $\mu((\alpha, \beta))$ must be zero. To see, note that since the points in (α, β) are outside Λ_t, we have (Definition 5.1)

$$\int_{\mathbb{R}} \frac{1}{(a_0 - x)^2}\, d\mu(x) \le \frac{1}{t}$$

for all $a_0 \in (\alpha, \beta)$. If we integrate the above integral with respect to the Lebesgue measure in a_0, we have

$$\int_\alpha^\beta \int_{\mathbb{R}} \frac{1}{(a_0 - x)^2}\, d\mu(x)\, da_0 < \infty.$$

We may then reverse the order of integration and restrict the integral with respect to μ to (α, β) to get

$$\int_\alpha^\beta \int_\alpha^\beta \frac{1}{(a_0 - x)^2}\, da_0\, d\mu(x) < \infty. \tag{6.5}$$

But

$$\int_\alpha^\beta \frac{1}{(a_0 - x)^2}\, da_0 = \infty$$

for all $x \in (\alpha, \beta)$. Thus, the only way (6.5) can hold is if $\mu((\alpha, \beta) = 0$. □

We now work toward the proof of Proposition 6.2. In light of the formulas for the solution path in Proposition 4.4, we consider the map V_t given by $V_t(a_0, b_0, \varepsilon_0) = (a_t, b_t, \varepsilon_t)$, where

$$\begin{aligned} a_t(a_0, b_0, \varepsilon_0) &= a_0 - \frac{t}{2} p_{a,0} \\ b_t(a_0, b_0, \varepsilon_0) &= b_0 + \frac{t}{2} p_{b,0} \\ \varepsilon_t(a_0, b_0, \varepsilon_0) &= \varepsilon_0 (1 - p_0 t)^2. \end{aligned}$$

This map is initially defined for $\varepsilon_0 > 0$, which guarantees that the integrals (4.4) defining $p_{a,0}$ and $p_{b,0}$ are convergent, even if $b_0 = 0$. But if λ_0 is in $\overline{\Lambda}_t^c$, Lemma 6.3 guarantees that λ_0 is outside the closed support of μ, so that the integrals are convergent when $\varepsilon_0 = 0$ and even when ε_0 is slightly negative. Thus, for any $\lambda_0 \in \overline{\Lambda}_t^c$, we can extend V_t to a neighborhood of $(\lambda_0, 0)$ using the same formula. We note that when $\varepsilon_0 = 0$, we have

$$a_t(a_0, b_0, 0) + i b_t(a_0, b_0, 0) = J_t(a_0 + i b_0), \tag{6.6}$$

as in (6.1).

Lemma 6.4. *If λ_0 is not in $\overline{\Lambda}_t$, the Jacobian matrix of V_t at $(\lambda_0, 0)$ is invertible.*

Proof. If we vary a_0 or b_0 with ε_0 held equal to 0, then ε remains equal to zero, so that

$$\frac{\partial \varepsilon_t}{\partial a_0}(\lambda_0, 0) = \frac{\partial \varepsilon_t}{\partial b_0}(\lambda_0, 0) = 0.$$

Meanwhile, from the formula for ε_t, we obtain

$$\frac{\partial \varepsilon_t}{\partial \varepsilon_0}(\lambda_0, 0) = (1 - t p_0)^2.$$

Thus, using (6.6), we find that the Jacobian matrix of V at $(a_0, b_0, 0)$ has the form

$$\begin{pmatrix} K & * \\ 0 & (1 - tp_0)^2 \end{pmatrix},$$

where K is the 2×2 Jacobian matrix of the map J_t.

Since $\lambda_0 \in \overline{\Lambda}_t^c$, we have $T(\lambda_0) = 1/p_0 > 1/t$, so that $1 - tp_0 > 0$. Furthermore, since J_t is injective on $\overline{\Lambda}_t^c$, its complex derivative must be nonzero at λ_0, so that K is invertible. We can then see that the Jacobian matrix of V_t at $(t, \lambda_0, 0)$ has nonzero determinant. □

We are now ready for the proof of our regularity result.

Proof of Propositon 6.2. Define a function HJ by the right-hand side of the first Hamilton–Jacobi formula (4.5), namely,

$$\mathrm{HJ}(a_0, b_0, \varepsilon_0, t) = S(0, \lambda_0, \varepsilon_0) - t\left[\frac{1}{4}(p_{a,0}^2 - p_{b,0}^2) - \varepsilon_0 p_0^2\right]. \tag{6.7}$$

Now take $\lambda^* \in \overline{\Omega}_t^c$ and let $\lambda_0^* = J_t^{-1}(\lambda^*)$, so that $\lambda_0^* \in \overline{\Lambda}_t^c$. By Lemma 6.4 and the inverse function theorem, V_t has an analytic inverse in a neighborhood U of $(\lambda^*, 0)$. By shrinking U if necessary, we can assume that the λ_0-component of $V^{-1}(\lambda, \varepsilon)$ lies in $\overline{\Lambda}_t^c$ for all (λ, ε) in U. We now claim that for each fixed $t > 0$, the map

$$(\lambda, \varepsilon) \mapsto \mathrm{HJ} \circ V_t^{-1}(\lambda, \varepsilon) \tag{6.8}$$

gives the desired analytic extension of $S(t, \cdot, \cdot)$ to a neighborhood of $(\lambda^*, 0)$.

We first note that $\mathrm{HJ} \circ V_t^{-1}$ is smooth, where we use Lemma 6.3 to guarantee that the momenta in the definition of HJ are well defined. We then argue that for all (λ, ε) in U with $\varepsilon > 0$, the value of $\mathrm{HJ} \circ V_t^{-1}(\lambda, \varepsilon)$ agrees with $S(t, \lambda, \varepsilon)$. To see this, note first that if $(\lambda, \varepsilon) \in U$ has $\varepsilon > 0$, then the ε_0-component of $V_t^{-1}(\lambda, \varepsilon)$ must be positive, as is clear from the formula for $\varepsilon_t(a_0, b_0, \varepsilon_0)$. Since, also, the λ_0-component of $V_t^{-1}(\lambda, \varepsilon)$ is in $\overline{\Lambda}_t^c$, the small-ε_0 lifetime of the path is at least t, so that when $\varepsilon_0 > 0$, the lifetime is greater than t. Thus, for (λ, ε) in U with $\varepsilon > 0$, the first Hamilton–Jacobi formula (4.5) tells us that, indeed, $S(t, \lambda, \varepsilon) = \mathrm{HJ}(V_t^{-1}(\lambda, \varepsilon))$. □

We now come to the proof of our main result.

Proof of Theorem 6.1. Once we know that (6.8) gives an analytic extension of $S(t, \cdot, \cdot)$, we conclude that the function s_t defined as

$$s_t(\lambda) = \lim_{\varepsilon \to 0} S(t, \lambda, \varepsilon)$$

can be computed as

$$s_t(\lambda) = \mathrm{HJ} \circ V_t^{-1}(\lambda, 0). \tag{6.9}$$

The point of this observation is that because $\mathrm{HJ} \circ V_t^{-1}$ is analytic (in particular, continuous), we can compute $s_t(\lambda)$ by taking the limit of $\mathrm{HJ} \circ V_t^{-1}(\delta, \varepsilon)$ along *any* path ending at $(\lambda, 0)$, rather than having to fix λ and let ε tend to zero.

Fix a point λ in $\overline{\Omega}_t^c$ and let $\lambda_0 = J_t^{-1}(\lambda)$, so that $T(\lambda_0) \geq t$. Then for any $\varepsilon_0 > 0$, the lifetime of the path with initial conditions $(\lambda_0, \varepsilon_0)$ will be greater than t and the first Hamilton–Jacobi formula (4.5) tells us that

$$S(t, \lambda(t), \varepsilon(t)) = S(0, \lambda_0, \varepsilon_0) - t\left[\frac{1}{4}(p_{a,0}^2 - p_{b,0}^2) - \varepsilon_0 p_0^2\right].$$

As $\varepsilon_0 \to 0$, we find that $\lambda(t) \to J_t(\lambda_0) = \lambda$ and $\varepsilon(t) \to 0$. Thus, by (6.9) and the continuity of $\mathrm{HJ} \circ V_t^{-1}$, we have

$$s_t(\lambda) = S(0, \lambda_0, 0) - t \lim_{\varepsilon_0 \to 0} \left[\frac{1}{4}(p_{a,0}^2 - p_{b,0}^2) - \varepsilon_0 p_0^2 \right],$$

which gives the claimed expression (6.4).

Now, if λ is outside of $\overline{\Omega}_t$, then $J_t^{-1}(\lambda)$ is outside of $\bar{\Lambda}_t$, which means (Lemma 6.3) that $J_t^{-1}(\lambda)$ is outside the support of the measure μ. It is then easy to see that s_t is a composition of a harmonic function and a holomorphic function, which is harmonic. ☐

7. Inside the domain

7.1. **Outline.** In Section 6, we computed the Brown measure in the complement of $\overline{\Omega}_t$ and found that it is zero there. Our strategy was to apply the Hamilton–Jacobi formulas with λ_0 in the complement of $\overline{\Lambda}_t$ and ε_0 chosen to be very small, so that $\lambda(t)$ is in the complement of $\overline{\Omega}_t$ and $\varepsilon(t)$ is also very small. If, on the other hand, we take λ_0 inside Λ_t, then (by definition) $T(\lambda_0) < t$, meaning that the small-ε_0 lifetime of the path is less than t. Thus, for $\lambda_0 \in \Lambda_t$ and ε_0 small, the Hamilton–Jacobi formulas are not applicable at time t.

In this section, then, we will use a different strategy. We recall from Proposition 4.4 that $\varepsilon(t) = \varepsilon_0(1 - p_0 t)^2$. Thus, an alternative way to make $\varepsilon(t)$ small is to take $\varepsilon_0 > 0$ and arrange for p_0 to be close to $1/t$. Thus, for each point λ in Ω_t, we will try to find $\lambda_0 \in \Lambda_t$ and $\varepsilon_0 > 0$ so that $p_0 = 1/t$ and $\lambda(t) = \lambda$. (If $p_0 = 1/t$ then the solution to the system of ODEs blows up at time t, so that technically we are not allowed to apply the Hamilton–Jacobi formulas at time t. But we will gloss over this point for now and return to it in Section 7.1.3.)

Once we have understood how to choose λ_0 and ε_0 as functions of $\lambda \in \Omega_t$, we will then apply the Hamilton–Jacobi method to compute the Brown measure inside Ω_t. Specifically, we will use the second Hamilton–Jacobi formula (4.6) to compute the first derivatives of $S(t, \lambda, 0)$ with respect to a and b. We then compute the second derivatives to get the density of the Brown measure.

7.1.1. *Mapping onto Ω_t.* We first describe how to choose λ_0 and $\varepsilon_0 > 0$ as functions of $\lambda \in \Omega_t$ so that $\lambda(t) = \lambda$ and $\varepsilon(t) = 0$. If $a_0 + ib_0 \in \Lambda_t$, then $|b_0| < v_t(a_0)$. Then from the defining property (5.3) of the function v_t, we see that if we take

$$\varepsilon_0 = \varepsilon_0^t(a_0) := v_t(a_0)^2 - b_0^2, \tag{7.1}$$

then ε_0 is positive and plugging this value of ε_0 into the formula (4.4) for p_0 gives

$$p_0 = \int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + v_t(a_0)^2}\, d\mu(x) = \frac{1}{t},$$

as desired.

It remains to see how to choose λ_0 so that (with ε_0 given by (7.1)) we will have $\lambda(t) = \lambda$. Since $p_0 = 1/t$, Proposition 4.5 applies:

$$a(t) = t \int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + v_t(a_0)^2}\, d\mu(x) \tag{7.2}$$

$$b(t) = 2b_0. \tag{7.3}$$

If we want $\lambda(t)$ to equal $\lambda = a + ib$, then (7.3) immediately tells us that we should choose $b_0 = b/2$. We will show in Section 7.2 that (7.2) can be solved for a_0 as a function of a and t; we use the notation $a_0^t(a)$ for the solution.

Summary 7.1. *For all $\lambda = a + ib \in \Omega_t$, the following procedure shows how to choose $\lambda_0 = a_0 + ib_0 \in \Lambda_t$ and $\varepsilon_0 > 0$ so that, with these initial conditions, we will have $\lambda(t) = \lambda$ and $\varepsilon(t) = 0$. First, we use the condition*

$$\int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + v_t(a_0)^2}\, d\mu(x) = \frac{1}{t}$$

to determine v_t as a function of a_0. Second, use the condition

$$\int_{\mathbb{R}} \frac{x}{(a_0 - x)^2 + v_t(a_0)^2}\, d\mu(x) = \frac{a}{t}$$

to determine a_0 as a function a_0^t of a. Then we take

$$\begin{aligned} b_0 &= b/2 \\ \varepsilon_0 &= v_t(a_0^t(a))^2 - b_0^2. \end{aligned}$$

7.1.2. *Computing the Brown measure.* Using the choices for λ_0 and ε_0 in Summary 7.1, we then apply the second Hamilton–Jacobi formula (4.6). Since $\lambda(t) = \lambda$ and $\varepsilon(t) = 0$ and p_b is a constant of motion,

$$\frac{\partial S}{\partial b}(t, \lambda, 0) = p_b(t) = p_{b,0}.$$

But since, by (4.7), $p_{b,0} = 2b_0p_0$, we obtain

$$\frac{\partial S}{\partial b}(t, \lambda, 0) = 2b_0p_0 = \frac{b}{t},$$

since we are assuming that $p_0 = 1/t$.

Similarly,

$$\begin{aligned} \frac{\partial S}{\partial a}(t, \lambda, 0) &= p_a(t) \\ &= p_{a,0} \\ &= 2a_0p_0 - 2p_1. \\ &= \frac{2a_0^t(a)}{t} - \frac{2a}{t}, \end{aligned}$$

where we have used (7.2) and the formula (4.8) for $p_{a,0}$.

Conclusion 7.2. *The preceding argument suggests that for $\lambda = a + ib \in \Omega_t$, we should have*

$$\begin{aligned} \frac{\partial s_t}{\partial a} &= \frac{2}{t}(a_0^t(a) - a) \\ \frac{\partial s_t}{\partial b} &= \frac{b}{t}. \end{aligned}$$

If this is correct, then the density of the Brown measure in Ω_t is readily computed as

$$\frac{1}{4\pi}\left(\frac{\partial^2 S}{\partial a^2} + \frac{\partial^2 S}{\partial b^2}\right)(t, \lambda, 0) = \frac{1}{2\pi t}\left(\frac{da_0^t(a)}{da} - \frac{1}{2}\right),$$

as claimed in Theorem 1.2. In particular, the density of the Brown measure in Λ_t would be independent of $b = \operatorname{Im}\lambda$.

7.1.3. *Technical issues.* The preceding argument is not rigorous, since the Hamilton–Jacobi formulas are only known to hold as long as $\varepsilon(s)$ remains positive for all $0 \leq s \leq t$. That is to say, if $\varepsilon(t) = 0$ then we are not allowed to use the formulas at time t. We can try to work around this point by letting ε_0 *approach* the value $\varepsilon_0^t(\lambda_0) := v_t(a_0)^2 - b_0^2$ in (7.1) from above. Then we have a situation similar to the one in (6.3), namely

$$\lim_{\varepsilon_0 \to \varepsilon_0^t(\lambda_0)^+} \frac{\partial S}{\partial a}(t, \lambda(t), \varepsilon(t)) = \frac{2}{t}(a_0^t(a) - a) \tag{7.4}$$

$$\lim_{\varepsilon_0 \to \varepsilon_0^t(\lambda_0)^+} \frac{\partial S}{\partial b}(t, \lambda(t), \varepsilon(t)) = \frac{b}{t}, \tag{7.5}$$

where

$$\lim_{\varepsilon_0 \to \varepsilon_0^t(\lambda_0)} \lambda(t) = \lambda; \qquad \lim_{\varepsilon_0 \to \varepsilon_0^t(\lambda_0)} \varepsilon(t) = 0.$$

But the Brown measure is computed by first evaluating the limit

$$s_t(\lambda) := \lim_{\varepsilon \to 0^+} S(t, \lambda, \varepsilon),$$

where the limit is taken as $\varepsilon \to 0$ *with* λ *fixed*, and then taking the distributional Laplacian with respect to λ. Since $\lambda(t)$ is not fixed in (7.4) and (7.5), it is not clear that these limits are actually computing $\partial s_t/\partial a$ and $\partial s_t/\partial b$. The main technical challenge of this section is, therefore, to establish enough regularity of S near $(t, \lambda, \varepsilon)$ to verify that $\partial s_t/\partial a$ and $\partial s_t/\partial b$ are actually given by the right-hand sides of (7.4) and (7.5).

7.2. **Surjectivity.** In this section, we show that the procedure in Summary 7.1 actually gives a continuous map of Λ_t onto Ω_t. Given any $\lambda_0 \in \Lambda_t$, choose $\varepsilon_0 = \varepsilon_0^t(\lambda_0)$ as in (7.1), so that

$$\lim_{s \to t} \varepsilon(s) = 0.$$

Then define

$$U_t(\lambda_0) = \lim_{s \to t} \lambda(s).$$

By Proposition 4.5, we have

$$U_t(a_0 + ib_0) = a_t(a_0) + 2ib_0$$

where

$$a_t(a_0) = t \int_{\mathbb{R}} \frac{1}{(a_0 - x)^2 + v_t(a_0)^2} \, d\mu(x). \tag{7.6}$$

Since we assume $\lambda_0 \in \Lambda_t$, we have $v_t(a_0) > 0$ and we therefore have an alternative formula:

$$a_t(a_0) = \mathrm{Re}[J_t(a_0 + iv_t(a_0))]. \tag{7.7}$$

It is a straightforward computation to check that the right-hand sides of (7.6) and (7.7) agree, using that the identity (5.3) holds when $v_t(a_0) > 0$.

The main result in this section is stated in the following theorem. We remind the reader of the definition (5.12) of the map J_t.

Theorem 7.3. *The following results hold.*

(1) *The map U_t extends continuously to $\overline{\Lambda}_t$. This extension is the unique continuous map of $\overline{\Lambda}_t$ into $\overline{\Omega}_t$ that (a) agrees with J_t on $\partial\Lambda_t$ and (b) maps each vertical segment in $\overline{\Lambda}_t$ linearly to a vertical segment in $\overline{\Omega}_t$.*
(2) *The map U_t is a homeomorphism from Λ_t onto Ω_t.*

Most of what we need to prove the theorem is already in Proposition 5.5.

Proof. Proposition 5.5 showed that the right-hand side of (7.7) is continuous for all $a_0 \in \mathbb{R}$. Using this formula for a_t, we see that U_t actually extends continuously to the whole complex plane. It is then a simple computation to check that

$$\operatorname{Im} J_t(a_0 \pm iv_t(a_0)) = \pm 2v_t(a_0).$$

This formula, together with (7.7), shows that U_t agrees with J_t for all points in $\partial\Lambda_t$ having nonzero imaginary parts. Then points in $\partial\Lambda_t$ on the real axis are limits of points in $\partial\Lambda_t$ with nonzero imaginary parts. Thus, U_t indeed agrees with J_t on $\partial\Lambda_t$. Also U_t is linear on each vertical segment. Since Λ_t is bounded by the graphs of v_t and $-v_t$, it is easy to see that U_t is the *unique* map with these two properties.

By Proposition 5.5, Ω_t is bounded by the graphs b_t and $-b_t$, where the graph of b_t is the image of the graph of v_t under J_t. From this result, it follows easily that U_t is a homeomorphism. □

We conclude this section by giving bounds on the real parts of points in Ω_t, in terms of the law μ of x_0.

Proposition 7.4. *Let*

$$M = \sup\ \operatorname{supp}(\mu), \quad m = \inf\ \operatorname{supp}(\mu).$$

Then

$$m < \inf(\Omega_t \cap \mathbb{R}) \quad \text{and} \quad \sup(\Omega_t \cap \mathbb{R}) < M.$$

In particular, every point λ in $\overline{\Omega}_t$ has $m < \operatorname{Re}\lambda < M$.

Proof. Let $\tilde{a}_0 = \sup(\Lambda_t \cap \mathbb{R})$. Then $v_t(\tilde{a}_0) = 0$, which means (Proposition 5.2) that

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(\tilde{a}_0 - x)^2} \leq \frac{1}{t}.$$

Then

$$\sup(\Omega_t \cap \mathbb{R}) = a_t(\tilde{a}_0) = t\int_{\mathbb{R}} \frac{x\,d\mu(x)}{(\tilde{a}_0 - x)^2} \leq M.$$

Because of our standing assumption that μ is not a δ-measure, this inequality is strict. The inequality for $\inf(\Omega_t \cap \mathbb{R})$ can be proved similarly. □

7.3. **Regularity.** Define a function $\tilde{S}$ by

$$\tilde{S}(t, \lambda, z) = S(t, \lambda, z^2)$$

for $z > 0$.

Proposition 7.5. *Fix a time $t > 0$ and a point $\lambda^* \in \Omega_t$. Then the function $(\lambda, z) \mapsto \tilde{S}(t, \lambda, z)$ extends to a real analytic function defined in a neighborhood of $(\lambda^*, 0)$ inside $\mathbb{C} \times \mathbb{R}$.*

Once the proposition is established, the function $s_t(\lambda) := \lim_{\varepsilon \to 0^+} S(t, \lambda, \varepsilon)$ can be computed as $s_t(\lambda) = \tilde{S}(t, \lambda, 0)$. Since $\tilde{S}(t, \lambda, z)$ is smooth in λ and z, we can compute s_t (or any of its derivatives) at λ^* by evaluating $\tilde{S}(t, \lambda, z)$ (or any of its derivatives) along any path where $\lambda \to \lambda^*$ and $z \to 0$. Thus, Proposition 7.5 will allow us to make rigorous the argument leading to 7.2. Specifically, we will be able to conclude that the left-hand sides of (7.4) and (7.5) are actually equal to $\partial s_t/\partial a$ and $\partial s_t/\partial b$, respectively.

Remark 7.6. *The function S itself does not have a smooth extension of the same sort that $\tilde{S}$ does. Indeed, since $\sqrt{\varepsilon}p_\varepsilon$ is a constant of motion, the second Hamilton–Jacobi formula (4.6) tells us that $\partial S/\partial\varepsilon$ must blow up like $1/\sqrt{\varepsilon}$ as we approach $(t,\lambda^*,0)$ along a solution of the system (4.2). The same reasoning tells us that the extended $\tilde{S}$ does* not *satisfy $\tilde{S}(t,\lambda,z)=S(t,\lambda,z^2)$ for $z<0$. Indeed, since $\sqrt{\varepsilon}p_\varepsilon$ is a constant of motion, $\frac{\partial\tilde{S}}{\partial z}(t,\lambda,z)=2\sqrt{\varepsilon}\frac{\partial S}{\partial\varepsilon}(t,\lambda,z^2)$ has a nonzero limit as $z\to 0$. Thus, $\tilde{S}$ cannot have a smooth extension that is even in z.*

To prove Proposition 7.5, we will use a strategy similar to the one in Section 6.2. For each $t>0$, we define a map

$$W_t(a_0,b_0,\varepsilon_0)=(a_t,b_t,z_t)$$

by

$$\begin{aligned}a_t&=a(t,a_0,b_0,\varepsilon_0)\\ b_t&=b(t,a_0,b_0,\varepsilon_0)\\ z_t&=\sqrt{\varepsilon(t,a_0,b_0,\varepsilon_0)}\end{aligned}$$

where a, b, ε are defined as in Proposition 4.4. The last component z_t can be expressed explicitly as

$$z_t=\sqrt{\varepsilon_0}\,(1-tp_0)\,.$$

The map W_t is initially defined only for

$$\varepsilon_0>\varepsilon_0^t(\lambda_0):=v_t(a_0)^2-b_0^2.$$

This condition guarantees that $p_0<1/t$, so that the lifetime of the path is greater than t. But for each $t>0$ and $\lambda_0\in\Lambda_t$, we can extend W_t to a neighborhood of $(a_0,b_0,\varepsilon_0^t(\lambda_0))$, simply by using the same formulas. We note that if $\varepsilon_0>\varepsilon_0^t(\lambda_0)$, then $p_0<1/t$ so that $z_t>0$; and if $\varepsilon_0<\varepsilon_0^t(\lambda_0)$ then $p_0>1/t$ so that $z_t<0$.

Lemma 7.7. *For all $t>0$ and $\lambda_0=a_0+ib_0\in\Lambda_t$, the Jacobian of W_t at $(a_0,b_0,\varepsilon_0^t(\lambda_0))$ is invertible, where $\varepsilon_0^t(\lambda_0)=v_t(a_0)^2-b_0^2$.*

Proof. We introduce the notations

$$\begin{aligned}q_0&=\int_{\mathbb{R}}\frac{d\mu(x)}{((a_0-x)^2+v_t(a_0)^2)^2}\\ q_1&=\int_{\mathbb{R}}\frac{(a_0-x)\,d\mu(x)}{((a_0-x)^2+v_t(a_0)^2)^2}\\ q_2&=\int_{\mathbb{R}}\frac{(a_0-x)^2\,d\mu(x)}{((a_0-x)^2+v_t(a_0)^2)^2}.\end{aligned}$$

Note that $q_0>0$ and $q_2>0$. When $\varepsilon_0=\varepsilon_0^t(\lambda_0)$, we can write p_0 in terms of q_0 and q_2 as

$$p_0=q_2+v_t(a_0)^2q_0. \tag{7.8}$$

We now compute the Jacobian matrix of W_t at the point $(\lambda_0,\varepsilon_0^t(\lambda_0))$, with $\lambda_0\in\Lambda_t$. Using the formulas (4.4) for $p_{a,0}$ and $p_{b,0}$, we can compute that

$$\begin{array}{lll}\frac{\partial p_{a,0}}{\partial a_0}=2(-q_2+v_t(a_0)^2q_0) & \frac{\partial p_{a,0}}{\partial b_0}=-4b_0q_1 & \frac{\partial p_{a,0}}{\partial\varepsilon_0}=-2q_1\\ \frac{\partial p_0}{\partial a_0}=-2q_1 & \frac{\partial p_0}{\partial b_0}=-2b_0q_0 & \frac{\partial p_0}{\varepsilon_0}=-q_0\end{array}.$$

The Jacobian of W_t at $(a_0, b_0, \varepsilon_0^t(\lambda_0))$ then has the following form,

$$DW_t = \begin{pmatrix} t(\frac{1}{t} + q_2 - v_t(a_0)^2 q_0) & 2tb_0q_1 & tq_1 \\ -2tb_0q_1 & t(\frac{1}{t} + p_0 - 2b_0^2 q_0) & -tb_0q_0 \\ 2t\sqrt{\varepsilon_0}q_1 & 2t\sqrt{\varepsilon_0}b_0q_0 & \frac{1-tp_0}{2\sqrt{\varepsilon_0}} + t\sqrt{\varepsilon_0}q_0 \end{pmatrix}.$$

Since $\varepsilon_0 = \varepsilon_0^t(\lambda_0)$, we have $1/t = p_0$, and by (7.8), the $(1,1)$-entry can be simplified to $2tq_2$. The Jacobian matrix DW_t then simplifies to

$$DW_t = 2t \begin{pmatrix} q_2 & b_0q_1 & \frac{1}{2}q_1 \\ -b_0q_1 & (q_2 + (v_t(a_0)^2 - b_0^2)q_0) & -\frac{1}{2}b_0q_0 \\ \sqrt{\varepsilon_0}q_1 & \sqrt{\varepsilon_0}b_0q_0 & \frac{1}{2}\sqrt{\varepsilon_0}q_0 \end{pmatrix}.$$

We compute the determinant of DW_t by first adding $-2b_0$ times the third column to the second column and then using a cofactor expansion along the second column. The result is

$$\det DW_t = 4t^3\sqrt{\varepsilon_0}(q_2 + v_t(a_0)^2 q_0)(q_2q_0 - q_1^2)$$

Now, by the Cauchy–Schwarz inequality,

$$q_0q_2 - q_1^2 \geq 0$$

and it cannot be an equality unless μ is a δ-measure. Therefore, we conclude that $\det DW_t$ is positive, establishing the proposition. □

Proof of Proposition 7.5. The proof is extremely similar to the proof of Proposition 6.2; the desired extension is given by the map

$$(\lambda, z) \mapsto \mathrm{HJ}(W_t^{-1}(\lambda, z)),$$

where HJ is the Hamilton–Jacobi function in (6.7). Take $z > 0$ and let $(\mu_0, \delta_0) = W_t^{-1}(\lambda, z)$. Then we must have $\delta_0 > \varepsilon_0^t(\mu_0)$ or else $z_t(\mu_0, \delta_0) = z$ would be negative. Thus, the lifetime of the path will be greater than t and the Hamilton–Jacobi formula will apply. Thus, the Hamilton–Jacobi formula (4.5) shows that $\mathrm{HJ}(W_t^{-1}(\lambda, z))$ agrees with $\tilde{S}(t, \lambda, z)$ for $z > 0$. □

7.4. **Computing the Brown measure.** Using Proposition 7.5, we can show that the left-hand sides of (7.4) and (7.5) are actually equal to $\partial s_t/\partial a$ and $\partial s_t/\partial b$.

Corollary 7.8. *For any $\lambda = a + ib$ in Ω_t we have*

$$\frac{\partial s_t}{\partial a} = \frac{2}{t}(a_0^t(a) - a), \quad \frac{\partial s_t}{\partial b} = \frac{b}{t},$$

so that

$$\Delta s_t(\lambda) = \frac{2}{t}\left(\frac{da_0^t(a)}{da} - \frac{1}{2}\right).$$

Proof. Fix $t > 0$ and $\lambda^* \in \Omega_t$. By Proposition 7.5, the function

$$s_t(\lambda) := \lim_{\varepsilon \to 0^+} S(t, \lambda, \varepsilon)$$

may be computed, for λ in a neighborhood of λ^*, as

$$s_t(\lambda) = \tilde{S}(t, \lambda, 0).$$

Since $\tilde{S}$ is smooth, we can evaluate $\tilde{S}$ or any of its derivatives at $(t, \lambda, 0)$ by taking limits along any path we choose with t fixed. Thus, the left-hand sides of (7.4) and (7.5) are actually equal to $\partial s_t/\partial a$ and $\partial s_t/\partial b$. The formula for Δs_t then follows by taking second derivatives with respect to a and to b and simplifying. □

We now establish our main result, a formula for the Brown measure of $x_0 + i\sigma_t$.

Theorem 7.9. *The open set Ω_t is a set of full measure for the Brown measure of $x_0 + i\sigma_t$. Inside Ω_t, the Brown measure is absolutely continuous with a strictly positive density w_t given by*

$$w_t(\lambda) = \frac{1}{2\pi t}\left(\frac{da_0^t(a)}{da} - \frac{1}{2}\right), \quad \lambda = a + ib. \tag{7.9}$$

Since $w_t(\lambda)$ is independent of b, we see that w_t is constant along the vertical segments inside Ω_t.

Proof. Corollary 7.8 shows that in Ω_t, the Brown measure of $x_0 + i\sigma_t$ has a density given by (7.9). It then follows from Point 2 of Proposition 5.5 that

$$\frac{da_0^t(a)}{da} > \frac{1}{2},$$

showing that w_t is positive in Ω_t. It remains to show that Ω_t is a set of full Brown measure. Since the Brown measure is zero outside $\overline{\Omega}_t$, we see that Ω_t will have full measure provided that the boundary of Ω_t has measure zero. While it may be possible to prove this directly using the strategy in Section 7.4 of [12], we instead use the approach used in [22].

In Theorem 8.2, we will consider a probability measure ρ_t on Λ_t. We will then show that the push-forward of ρ_t under the map $U_t : \Lambda_t \to \Omega_t$ agrees with $\mathrm{Brown}(x_0 + i\sigma_t)$ on Ω_t. Since the preimage of Ω_t under U_t is Λ_t and $\rho_t(\Lambda_t) = 1$, we see that the $\mathrm{Brown}(x_0 + i\sigma_t)$ assigns full measure to Ω_t. □

8. Two results about push-forwards of the Brown measure

In this section, we show how $\mathrm{Brown}(x_0 + i\sigma_t)$ is related to two other measure by means of pushing forward under appropriate maps. To motivate one of our results, let us consider the case that $x_0 = \tilde{\sigma}_s$, a semicircular element of variance s freely independent of σ_t. It is known (see [4, Example 5.3] and Section 10.1) that in this case, the Brown measure of $\tilde{\sigma}_s + i\sigma_t$ is uniformly distributed on an ellipse. It follows that the distribution of $\mathrm{Re}\,\lambda$ with respect to $\mathrm{Brown}(\tilde{\sigma}_s + i\sigma_t)$ is semicircular—which is the same (up to scaling by a constant) as the distribution of $\tilde{\sigma}_s + \sigma_t$. (See Figure 7.)

Point 2 of Theorem 8.2 generalizes the preceding result to the case of arbitrary x_0, in which the map $\lambda \mapsto \mathit{const.}\,\mathrm{Re}\,\lambda$ is replaced by a certain map $Q_t : \overline{\Omega}_t \to \mathbb{R}$. When the distribution of x_0 is semicircular, $Q_t(\lambda)$ is a multiple of the real part of λ, as in (8.1).

Recall that c_t denotes the circular Brownian motion. We will make use of the map $U_t : \overline{\Lambda}_t \to \overline{\Omega}_t$ described in Section 7.2, and another map $Q_t : \overline{\Omega}_t \to \mathbb{R}$ which we now define. Recall from Sections 5.2 and 5.3 that the inverse of the map J_t takes $\partial\Omega_t$ to $\partial\Lambda_t$ and that the map H_t takes $\partial\Lambda_t$ to $\mathbb{R}$, so that $H_t \circ J_t^{-1}$ takes $\partial\Omega_t$ to $\mathbb{R}$.

Definition 8.1. *Let $Q_t : \overline{\Omega}_t \to \mathbb{R}$ be the unique map that agrees with $H_t \circ J_t^{-1}$ on $\partial\Omega_t$ and maps vertical segments in $\overline{\Omega}_t$ to points in $\mathbb{R}$.*

The map Q_t is visualized in Figure 4. In the case that x_0 is semicircular with variance s, one can easily use the computations in Section 10.1 to show that

$$Q_t(a + ib) = \frac{s + t}{s}a. \tag{8.1}$$

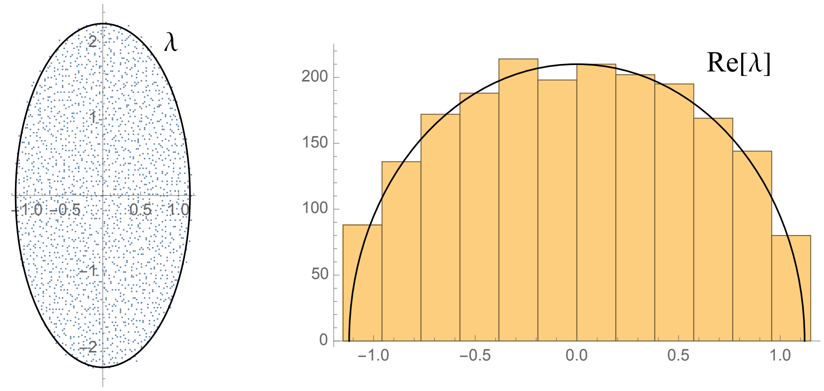

FIGURE 7. A random matrix approximation to the Brown measure of $x_0 + i\sigma_t$ when x_0 is semicircular (left) and the distribution of the real parts of the eigenvalues (right).

In general, we may compute Q_t more explicitly as follows. We first map $a + ib$ to the point $a + ib_t(a)$ on $\partial\Omega_t$. Next, we compute

$$J_t^{-1}(a + ib_t(a)) = a_0^t(a) + iv_t(a_0^t(a)).$$

Next, we use the identity $H(J_t^{-1}(z)) = 2J_t^{-1}(z) - z$ in (5.23). Finally, we recall that $H_t \circ J_t^{-1}$ is real-valued on $\partial\Omega_t$. Thus,

$$\begin{aligned} Q_t(a + ib) &= \mathrm{Re}\{2J_t^{-1}[a + ib_t(a)] - (a + ib_t(a))\} \\ &= 2a_0^t(a) - a. \end{aligned}$$

Theorem 8.2. *The following results hold.*

1. *The push-forward of the Brown measure of $x_0 + c_t$ under the map U_t in Theorem 7.3 is the Brown measure of $x_0 + i\sigma_t$.*
2. *The push-forward of the Brown measure of $x_0 + i\sigma_t$ under the map Q_t is the law of $x_0 + \sigma_t$.*

Proof. By Theorem 3.9 in [22], the Brown measure ρ_t of $x_0 + c_t$ can be written as

$$\begin{aligned} d\rho_t &= \frac{1}{\pi t}\left(1 - \frac{t}{2}\frac{d}{da_0}\int_{\mathbb{R}} \frac{x\,d\mu(x)}{(a_0 - x)^2 + v_t(a_0)^2}\right)\,da_0\,db_0 \\ &= \frac{1}{\pi t}\left(1 - \frac{1}{2}\frac{da_t}{da_0}\right)\,da_0\,db_0 \end{aligned}$$

for $a_0 + ib_0 \in \Lambda_t$. Now, under the map U_t, we have $a = a_t(a_0)$ and $b = 2b_0$. Thus,

$$\begin{aligned} d\rho_t &= \frac{1}{\pi t}\left(1 - \frac{1}{2}\frac{da_t}{da_0}\right)\,\frac{da_0^t}{da}da\,\frac{db}{2} \\ &= \frac{1}{2\pi t}\left(\frac{da_0^t}{da} - \frac{1}{2}\right)\,da\;db \end{aligned}$$

for $a + ib \in \Omega_t$. This last expression in the formula for the restriction of the Brown measure to Ω_t.

Since ρ_t is a probability measure on Λ_t, we find that the Brown measure of $x_0 + i\sigma_t$ assigns mass 1 to Ω_t, as noted in the proof of Theorem 7.9. Thus, there is no mass of $\mathrm{Brown}(x_0 + i\sigma_t)$ anywhere else and the pushforward of ρ_t under U_t is precisely $\mathrm{Brown}(x_0 + i\sigma_t)$.

To prove Point 2, we consider the unique map $\Psi_t : \overline{\Lambda}_t \to \mathbb{R}$ that agrees with H_t on $\partial\Lambda_t$ and is constant along vertical segments in Λ_t. Then $Q_t = \Psi_t \circ U_t^{-1}$. (Both Q_t and $H_t \circ U_t^{-1}$ agree with $\Psi_t \circ J_t^{-1}$ on $\partial\Omega_t$ and are constant along vertical segments inside $\overline{\Omega}_t$.) By Point 1, the push-forward of the Brown measure of $x_0 + i\sigma_t$ under U_t^{-1} is the Brown measure ρ_t of $x_0 + c_t$. By Theorem 3.13 of [22], the push-forward of the Brown measure ρ_t by Ψ_t of $x_0 + c_t$ is the law of $x_0 + \sigma_t$ and Point 2 follows. □

9. The method of Jarosz and Nowak

9.1. **The formula for the Brown measure.** We now describe a different approach to computing the Brown measure of $x_0 + i\sigma_t$, developed by Jarosz and Nowak in the physics literature [24, 25]. As discussed in the introduction, the method is not rigorous as written, but could conceivably be made rigorous using the general framework developed by Belinschi, Mai, and Speicher in [6]. (See also related results in [7].) We emphasize, however, that (so far as we know) no explicit computation of the case of $x_0 + i\sigma_t$ has been made using the framework in [6].

Jarosz and Nowak work with an operator of the form $H_1 + iH_2$, where H_1 and H_2 are arbitrary freely independent elements. Then on p. 10118 of [25], they present an algorithm by which the "nonholomorphic Green's function" of $H_1 + iH_2$ may be computed. In the notation of this paper, the nonholomorphic Green's function is the function $\partial s_t/\partial\lambda$, so that the Brown measure may be computed by taking the $\bar{\lambda}$-derivative:

$$\frac{1}{\pi}\frac{\partial}{\partial\bar{\lambda}}\frac{\partial s_t}{\partial\lambda} = \frac{1}{4\pi}\Delta_\lambda s_t(\lambda).$$

Examples are presented in Section 6.1 of [24] in which H_2 is semicircular and H_2 has various different distributions. We now work out their algorithm in detail in the case that $H_1 = x_0$ is an *arbitrary* self-adjoint element and $H_2 = \sigma_t$.

We refer to [24, 25] for the framework used in the algorithm, involving "quaternionic Green's functions." We present only the final algorithm for computation, described in Eqs. (75)–(79) of [25], and we specialize to the case $H_1 = x_0$ and $H_2 = \sigma_t$. The algorithm, adapted to our notation, is as follows. We fix a complex number $\lambda = a + ib$. Then we introduce three unknown quantities, complex numbers g and g' and a real number m. These are supposed to satisfy three equations:

$$B_{x_0}(g) = a + \frac{m}{g} \tag{9.1}$$

$$B_{\sigma_t}(g') = b + \frac{1-m}{g'} \tag{9.2}$$

$$|g| = |g'|, \tag{9.3}$$

where B_{x_0} and B_{σ_t} are the "Blue's functions," that is, the inverse functions of the Cauchy transforms of x_0 and σ_t, respectively. We are supposed to solve these equations for g and g' as functions of a and b. Once this is done, we have

$$\frac{\partial s_t}{\partial\lambda}(a,b) = \operatorname{Re} g - i \operatorname{Re} g'. \tag{9.4}$$

Since $\partial/\partial\lambda = (\partial/\partial a - i\ \partial/\partial b)/2$, (9.4) may be written equivalently as

$$\frac{\partial s_t}{\partial a}(a,b) = 2\operatorname{Re} g; \tag{9.5}$$

$$\frac{\partial s_t}{\partial b} = 2\operatorname{Re} g'. \tag{9.6}$$

That is to say, the real parts of g and g' determine the derivatives of s_t with respect to a and b, respectively.

Proposition 9.1. *The Jarosz–Nowak method when applied to $x_0 + i\sigma_t$ gives the following result. We try to solve the equation*

$$g = G_{x_0}(a + t\bar{g})$$

for g as a function of a and t, with the solution denoted $g_t(a)$. Then, inside the support of the Brown measure, its density ρ_t is a function of a and t only, namely

$$\rho_t(a) = \frac{1}{4\pi}\left(\frac{1}{t} + 2\frac{d}{da}\operatorname{Re} g_t(a)\right). \tag{9.7}$$

Proof. It is known that the Blue's function of σ_t is given by $B_{\sigma_t}(g') = 1/g' + tg'$. (This statement is equivalent to saying that the R-transform of σ_t is given by $R(z) = tz$, as in [38, Example 3.4.4].) Plugging this expression into (9.2) and simplifying, we obtain a quadratic equation:

$$t(g')^2 - bg' + m = 0,$$

whose roots are

$$g' = \frac{b \pm \sqrt{b^2 - 4mt}}{2t}. \tag{9.8}$$

Assuming (as Jarosz and Nowak implicitly do) that these roots are complex, we find that

$$\operatorname{Re}(g') = \frac{b}{2t}. \tag{9.9}$$

Thus, without even using (9.1) or (9.3), we find by (9.6) that

$$\frac{\partial s_t}{\partial b} = \frac{b}{t},$$

which agrees with what we found in Corollary 7.8.

Meanwhile, assuming still that the roots in (9.8) are complex, we find that

$$\begin{aligned}
|g'|^2 &= \frac{1}{4t^2}(b^2 + 4mt - b^2) \\
&= \frac{m}{t}.
\end{aligned}$$

Then (9.3) says that $|g|^2 = |g'|^2 = m/t$, so that $m = t\,|g|^2$. Thus, after replacing m by $t\,|g|^2$, (9.1) becomes

$$B_{x_0}(g) = a + t\bar{g}.$$

Since B_{x_0} is the inverse function to G_{x_0}, this equation may be rewritten as

$$g = G_{x_0}(a + t\bar{g}). \tag{9.10}$$

We hope that this equation will implicitly determine the complex number g as a function of a (and t). We therefore write g as $g_t(a)$.

We then substitute the expression for g into (9.5), giving

$$\frac{\partial s_t}{\partial a} = 2\operatorname{Re} g_t(a).$$

The density ρ_t of the Brown measure is then computed as

$$\begin{aligned}\rho_t(a,b) &= \frac{1}{4\pi}\left(\frac{\partial^2 s_t}{\partial b^2} + \frac{\partial^2 s_t}{\partial a^2}\right) \\ &= \frac{1}{4\pi}\left(\frac{1}{t} + 2\frac{d}{da}\operatorname{Re} g_t(a)\right),\end{aligned}$$

as claimed. □

Proposition 9.2. *In the Jarosz–Nowak method, the quantity* $\operatorname{Re} g_t(a)$ *may be computed as*

$$\operatorname{Re} g_t(a) = \frac{1}{t}(a_0^t(a) - a),$$

where the function a_0^t *is as in Summary 7.1. Thus, the formula (9.7) for the Brown measure in the Jarosz–Nowak method agrees with what we found in Theorem 7.9.*

Proof. The imaginary part of the equation (9.10) for g says that

$$\begin{aligned}\operatorname{Im} g &= \operatorname{Im} \int_{\mathbb{R}} \frac{d\mu(x)}{a + t\bar{g} - x} \\ &= t \operatorname{Im} g \int_{\mathbb{R}} \frac{d\mu(x)}{(a + t\operatorname{Re} g - x)^2 + t^2(\operatorname{Im} g)^2}.\end{aligned}$$

Thus, at least when $\operatorname{Im} g \neq 0$, we get

$$\int \frac{d\mu(x)}{(a + t\operatorname{Re} g - x)^2 + t^2(\operatorname{Im} g)^2} = \frac{1}{t}. \tag{9.11}$$

We may now apply the equation (5.3) that defines the function v_t with a replaced by $a + t\operatorname{Re} g$, giving

$$t\operatorname{Im} g = \pm v_t(a + t\operatorname{Re} g). \tag{9.12}$$

We now look at the real part of (9.10):

$$\begin{aligned}\operatorname{Re} g &= \operatorname{Re} \int_{\mathbb{R}} \frac{d\mu(x)}{a + t\bar{g} - x} \\ &= (a + t\operatorname{Re} g)\int_{\mathbb{R}} \frac{1}{(a + t\operatorname{Re} g - x)^2 + t^2(\operatorname{Im} g)^2}\ d\mu(x) \\ &\quad - \int_{\mathbb{R}} \frac{x}{(a + t\operatorname{Re} g - x)^2 + t^2(\operatorname{Im} g)^2}\ d\mu(x).\end{aligned}$$

Using (9.11) and (9.12) this equation simplifies to

$$a = t\int_{\mathbb{R}} \frac{x}{(a + t\operatorname{Re} g - x)^2 + v_t(a + t\operatorname{Re} g)^2}\ d\mu(x). \tag{9.13}$$

Now, if we let

$$a_0 = a + t\operatorname{Re} g, \tag{9.14}$$

then (9.13) is just the equation for a in terms of a_0 that we found in our Hamilton–Jacobi analysis (Proposition 4.5). Thus,

$$\operatorname{Re} g = \frac{1}{t}(a_0 - a),$$

as claimed. □

9.2. **The support of the Brown measure.** We now examine the condition for the boundary of the support of the Brown measure, as given in Eq. (80) of [25]:

$$(\operatorname{Re} g)^2 + (\operatorname{Re} g')^2 = |g|^2 . \tag{9.15}$$

Proposition 9.3. *The condition for a point $a + ib$ to be on the boundary in the Jarosz–Nowak method is that*

$$b = 2v_t(a_0^t(a)).$$

Such points are precisely the boundary points of our domain Ω_t.

Proof. We cancel $(\operatorname{Re} g)^2$ from both sides of (9.15), leaving us with

$$(\operatorname{Re} g')^2 = (\operatorname{Im} g)^2.$$

Now, we have found in (9.9) that $\operatorname{Re} g' = b/(2t)$ and in (9.12) that $\operatorname{Im} g = \pm v_t(a + t \operatorname{Re} g)/t$. But in (9.14), we have identified $a + t \operatorname{Re} g$ with $a_0^t(a)$. Thus, the condition (9.15) for the boundary reads

$$\frac{b}{2t} = \pm \frac{v_t(a_0^t(a))}{t},$$

or $b = 2v_t(a_0^t(a))$, which is the condition for the boundary of Ω_t (Point 4 of Proposition 5.5). □

10. Examples

In this section, we compute three examples, in which the law of x_0 is semicircular, Bernoulli, or uniform. Additional examples, computed by a different method, were previously worked out by Jarosz and Nowak in [24, Section 6.1].

We also mention that we can take x_0 to have the form $x_0 = y_0 + \tilde{\sigma}_s$, where $\tilde{\sigma}_s$ is another semicircular Brownian motion and y_0, $\tilde{\sigma}_s$, and σ_t are all freely independent. Thus, our results allow one to determine the Brown measure for the sum of the elliptic element $\tilde{\sigma}_s + i\sigma_t$ and the freely independent self-adjoint element y_0. The details of this analysis will appear elsewhere.

10.1. **The elliptic law.** In our first example, the law μ of x_0 is a semicircular distribution with variance s. Then $x_0 + i\sigma_t$ has the form of an **elliptic element** $\tilde{\sigma}_s + i\sigma_t$, where $\tilde{\sigma}.$ and $\sigma.$ are two freely independent semicircular Brownian motions. The associated "elliptical law," in various forms, has been studied extensively going back to the work of Girko [15]. The elliptical case was worked out by Jarosz and Nowak in [25, Section 3.6]. The Brown measure of $\tilde{\sigma}_s + i\sigma_t$ was also computed by Biane and Lehner [4, Example 5.3] by a different method. We include this example as a simple demonstration of the effectiveness of our method.

Theorem 10.1. *The Brown measure of $\tilde{\sigma}_s + i\sigma_t$ is supported in the closure of the ellipse centered at the origin with semi-axes $2s/\sqrt{s+t}$ and $2t/\sqrt{s+t}$. The density of the Brown measure is constant*

$$\frac{1}{4\pi}\left(\frac{1}{s} + \frac{1}{t}\right)$$

in the domain.

We apply our result in this paper with $x_0 = \tilde{\sigma}_s$. In the next proposition we compute Λ_t.

Proposition 10.2. *We can parametrize the upper boundary curve of* Λ_t *by*

$$a_0 + iv_t(a_0) = \frac{(2s+t)q + it\sqrt{4(s+t)-q^2}}{2(s+t)}, \quad q \in [-2\sqrt{s+t}, 2\sqrt{s+t}]; \tag{10.1}$$

therefore, Λ_t *is the ellipse centered at the origin with semi-axes* $\frac{2s+t}{\sqrt{s+t}}$ *and* $\frac{t}{\sqrt{s+t}}$.

Proof. Recall that Δ_t denotes the region in the upper half plane above the graph of v_t, so that $\overline{\Delta}_t$ is the region on or above the graph of v_t. Recall also that Biane [2] has shown that the function H_t in (5.9) maps $\overline{\Delta}_t$ injectively onto the closed upper half plane.

In the case at hand, the Cauchy transform of $\tilde{\sigma}_s$ is $G_{\tilde{\sigma}_s}(z) = \left(z - \sqrt{z^2-4s}\right)/(2s)$. We can then compute the function H_t in (5.9) as

$$H_t(z) = z + tG_{\sigma_s}(z) = z + t\left(\frac{z - \sqrt{z^2-4s}}{2s}\right), \quad z \in \overline{\Delta}_t.$$

The inverse map H_t^{-1} is then easily computed as

$$H_t^{-1}(z) = \frac{(2s+t)z + t\sqrt{z^2-4(s+t)}}{2(s+t)}, \quad \operatorname{Im} z \geq 0. \tag{10.2}$$

The part of the graph of v_t where $v_t > 0$ comes from the values of H_t^{-1} on the real axis having nonzero imaginary part, that is, for real numbers q with $|q| < 2\sqrt{s+t}$. Plugging these numbers into (10.2) gives the claimed form (10.1). □

Proposition 10.3. *The boundary curve of* Ω_t *can be parametrized by*

$$a + ib_t(a) = \frac{sq + it\sqrt{4(s+t)-q^2}}{s+t}, \quad q \in [-2\sqrt{s+t}, 2\sqrt{s+t}].$$

Consequently, Ω_t *is an ellipse centered at the origin with semi-axes* $\frac{2s}{\sqrt{s+t}}$ *and* $\frac{2t}{\sqrt{s+t}}$.

Proof. By Proposition 10.2, the upper boundary of Λ_t can be parametrized by the curve in (10.1). By Definition 5.4, we find the boundary curve of Ω_t by applying the map J_t in (5.12), which satisfies $J_t(z) = 2z - H_t(z)$. Thus,

$$\begin{aligned} a + ib_t(a) &= J_t(a_0 + iv_t(a_0)) \\ &= \frac{sq + it\sqrt{4(s+t)-q^2}}{s+t}. \end{aligned}$$

which traces an ellipse centered at the origin with semi-axes $\frac{2s}{s+t}$ and $\frac{2t}{\sqrt{s+t}}$. □

Proof of Theorem 10.1. The domain Ω_t is computed in Proposition 10.3. By Proposition 10.3,

$$\begin{aligned} a &= \frac{sq}{s+t} \\ a_0^t(a) &= \frac{(2s+t)q}{2(s+t)} = \frac{(2s+t)}{2s}a. \end{aligned}$$

It follows that the density of the Brown measure is

$$\begin{aligned} \frac{1}{2\pi t}\left(\frac{da_0^t(a)}{da} - \frac{1}{2}\right) &= \frac{1}{2\pi t}\left(\frac{2s+t}{2s} - \frac{1}{2}\right) \\ &= \frac{1}{4\pi}\left(\frac{1}{s} + \frac{1}{t}\right), \end{aligned}$$

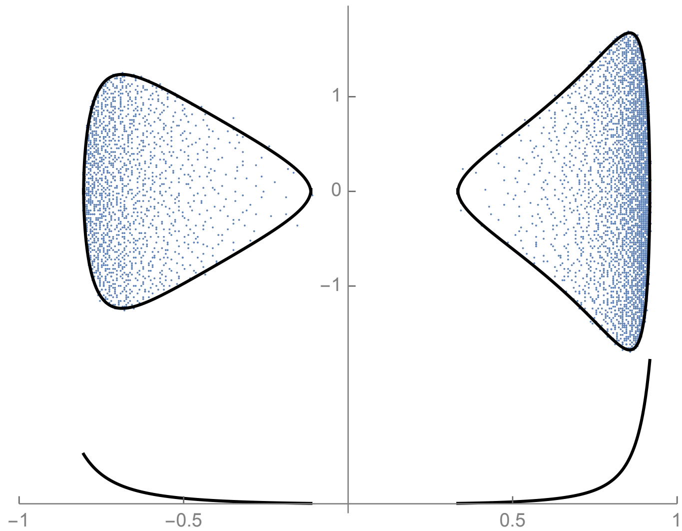

FIGURE 8. The domain Ω_t in the Bernoulli case with a simulation of the eigenvalues (top), plotted with the density of the Brown measure (in Ω_t) as a function of a (bottom). Shown for $\alpha = 2/3$ and $t = 1$.

as claimed. □

10.2. **Bernoulli case.** In our second example, the law μ of x_0 is Bernoulli distributed, with mass α at 1 and mass $\beta = 1 - \alpha$ at -1, for $0 < \alpha < 1$. The case $\alpha = 1/2$ was previously analyzed in the paper of Stephanov [32] and also in Section V of [24] by different methods.

Denote by $Q(a)$ the quartic polynomial

$$-4a^4 + 4t(\alpha - \beta)a^3 - (t^2 + 4t - 8)a^2 + 2t(t-2)(\alpha - \beta)a - (\alpha - \beta)^2 t^2 + 4t - 4. \quad (10.3)$$

Then the domain Ω_t and the density of the Brown measure in this example are computed in the following proposition.

Proposition 10.4. *Any $\lambda \in \Omega_t$ satisfies $|\mathrm{Re}\,\lambda| < 1$. The domain Ω_t is given by*

$$\Omega_t = \left\{ a + ib \in \mathbb{C} \,\middle|\, b^2 < \frac{Q(a)}{(1-a^2)^2} \right\}$$

so that

$$\Omega_t \cap \mathbb{R} = \{a \in \mathbb{R} | Q(a) > 0\}.$$

The density of the Brown measure in this Bernoulli case is given by

$$w_t(\lambda) = \frac{1}{4\pi}\left(-\frac{1}{t} + \frac{\beta}{(a-1)^2} + \frac{\alpha}{(a+1)^2}\right).$$

See Figure 8.

Proof. Recall the functions $a_0^t(a)$ and $b_t(a)$ defined by the four equations (1.4)–(1.7). We now compute these functions for the Bernoulli case. The equations (1.4) and (1.5) take the following form in the Bernoulli case:

$$\frac{\alpha}{(a_0-1)^2+v^2}+\frac{\beta}{(a_0+1)^2+v^2}=\frac{1}{t}$$
$$\frac{\alpha}{(a_0-1)^2+v^2}-\frac{\beta}{(a_0+1)^2+v^2}=\frac{a}{t}.$$

We then introduce the new variables

$$A=(a_0-1)^2+v^2$$
$$B=(a_0+1)^2+v^2$$

so that

$$\frac{\alpha}{A}+\frac{\beta}{B}=\frac{1}{t}$$
$$\frac{\alpha}{A}-\frac{\beta}{B}=\frac{a}{t}.$$

Then we can solve for A and B as

$$A=\frac{2\alpha t}{1+a},\quad B=\frac{2\beta t}{1-a}.$$

We can recover a_0 and v^2 as

$$a_0=\frac{1}{4}(B-A)=\frac{t}{2}\frac{a+\beta-\alpha}{1-a^2}$$

and

$$\begin{aligned}v^2&=\frac{A+B}{2}-a_0^2-1\\&=\frac{\alpha t(1-a)+\beta t(1+a)}{1-a^2}-\left(\frac{t}{2}\frac{a+\beta-\alpha}{1-a^2}\right)^2-1\\&=\frac{Q(a)}{4(1-a^2)^2}\end{aligned}$$

where Q is defined in (10.3). Recalling that $b_t(a)=2v$, we find that

$$a_0^t(a)=\frac{t}{2}\frac{a+\beta-\alpha}{1-a^2} \tag{10.4}$$

and

$$b_t(a)^2=\frac{Q(a)}{(1-a^2)^2}. \tag{10.5}$$

Equation (10.5) gives the claimed form of the domain Ω_t. The density of the Brown measure is then computed from (10.4) using the formula in Theorem 7.9. □

10.3. **Uniform case.** In our third and final example, μ is uniformly distributed on $[-1,1]$. The case in which μ is uniformly distributed on any interval can be reduced to this case as follows. First, by shifting x_0 by a constant, we can assume that the interval has the form $[-A,A]$. Once this is the case, we write

$$x_0+i\sigma_t=A\left(x_0/A+i\sigma_t/A\right),$$

where the law of x_0/A is uniform on $[-1,1]$ and σ_t/A has the same $*$-distribution as σ_{t/A^2}. Thus, to compute the Brown measure in this case, we use the formulas below with t replaced by t/A^2 and then scale the entire Brown measure by a factor of A.

When μ is uniform on $[-1,1]$—in particular, symmetric about 0—the Brown measure is symmetric about the imaginary axis. The key is again solving the equations (1.4) and (1.5) that define the functions $a_0^t(a)$ and $b_t(a)$.

Proposition 10.5. *Let $v_{\max}$ be the smallest positive real number v such that*

$$\frac{1}{v} = \tan\left(\frac{v}{t}\right).$$

Also let $A_0^t(v)$ be given by

$$A_0^t(v) = \sqrt{2v\cot\left(\frac{2v}{t}\right) + 1 - v^2}.$$

for $|v| \leq v_{\max}$. (When $v=0$, we understand the above formula as $A_0^t(0) = \sqrt{t+1}$.) Then the following results hold.

(1) *The domain Ω_t has only one connected component, and is given by*

$$\Omega_t = \{\pm A_t(b/2) + iy|\ \ |y| < b,\ \ |b| \leq 2v_{\max}\} \tag{10.6}$$

where

$$A_t(v) = A_0^t(v) + \frac{t}{4}\log\left(1 - \frac{4A_0^t(v)}{(A_0^t(v)+1)^2 + v^2}\right).$$

That is, $\partial\Omega_t$ consists of the two curves

$$\pm A_t(b/2) + ib, \quad |b| \leq 2v_{\max}.$$

In particular, the function b_t defined in (5.14) is unimodal with a peak at 0.

(2) *The domain Ω_t satisfies*

$$\Omega_t \cap \mathbb{R} = \left\{a \in \mathbb{R} \,\middle|\, |a| < \sqrt{t+1} - \frac{t}{2}\log\left(\frac{\sqrt{t+1}+1}{\sqrt{t+1}-1}\right)\right\}.$$

(3) *The density of the Brown measure in Ω_t is (as always) a function of a and t only, and the graph of this function is traced out by the curve*

$$(A_t(v), W_t(v)), \quad |v| < v_{\max}, \tag{10.7}$$

where

$$W_t(v) = \frac{t^2 + 4(t+2)v^2 - t(t+4v^2)\cos(4v/t) - 4tv\sin(4v/t)}{4\pi t\left(-t^2 + 8v^2 + t^2\cos(4v/t)\right)}.$$

To the extent that we can compute the height b_t of Ω_t as a function of a, we can then compute the density of the Brown measure as a function of a by replacing v by $b_t(a)/2$ in the above expression. That is, the density $w_t(\lambda)$ is given by

$$w_t(\lambda) = W_t(b_t(a)/2). \tag{10.8}$$

See Figure 9.

Before we prove Proposition 10.5, we need some computations about v_t and Λ_t from the following proposition.

Proposition 10.6. *The following results about v_t and Λ_t hold.*

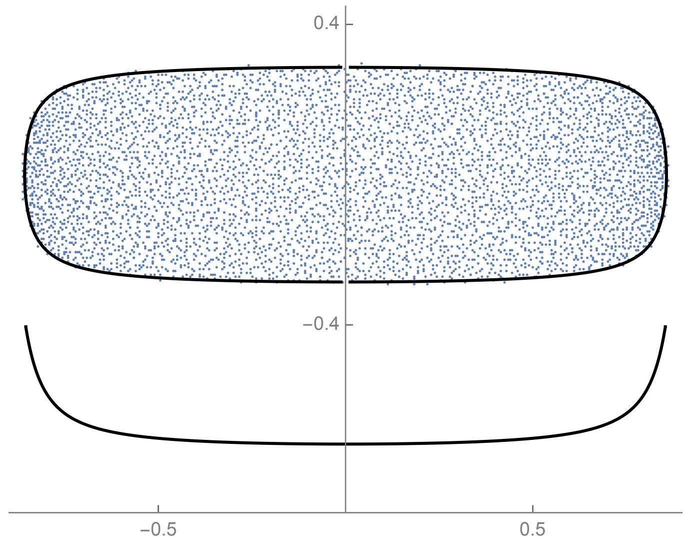

FIGURE 9. The domain Ω_t in the uniform case with a simulation of the eigenvalues (top), plotted with the density of the Brown measure (in Ω_t) as a function of a (bottom). Shown for $t = 0.1$.

(1) *The function v_t is unimodal, with a peak at $a_0 = 0$. The maximum $v_{\max} = v_t(0)$ is the smallest positive real number v such that*

$$\frac{1}{v} = \tan\left(\frac{v}{t}\right).$$

In particular, $v_t(0) < \frac{\pi t}{2}$.

(2) *The domain Λ_t has only one connected component. Its boundary can be described by the two curves*

$$\pm A_0^t(v) + iv, \quad |v| \le v_{\max}$$

where

$$A_0^t(v) = \sqrt{2v\cot\left(\frac{2v}{t}\right) + 1 - v^2}. \tag{10.9}$$

(3) *The domain Λ_t satisfies*

$$\Lambda_t \cap \mathbb{R} = (-\sqrt{t+1}, \sqrt{t+1}). \tag{10.10}$$

Proof. In the uniform case, (1.4) takes the form

$$\int_{\mathbb{R}} \frac{d\mu(x)}{(a_0 - x)^2 + v^2} = \frac{\arctan\left(\frac{1-a_0}{v}\right) + \arctan\left(\frac{1+a_0}{v}\right)}{2v} = \frac{1}{t}. \tag{10.11}$$

Using the addition law of inverse tangent,

$$\arctan(A) + \arctan(B) = \arctan\left(\frac{A+B}{1-AB}\right),$$

we can easily solve for a_0^2 as a function of v:

$$a_0^2 = 2v\cot\left(\frac{2v}{t}\right) + 1 - v^2. \tag{10.12}$$

Restricted to $a_0 \geq 0$, (10.12) defines $a_0 = A_0^t$ as a function of v as in (10.9).

The function $v_t(a_0)$ cannot be represented as an elementary function of a_0; we, however, have proved in the preceding paragraph that v_t restricted to $a_0 \geq 0$ in Λ_t has an inverse A_0^t. The function v_t then must be strictly decreasing from 0 to $\sup(\Lambda_t \cap \mathbb{R})$; by symmetry, it is strictly increasing from $\inf(\Lambda_t \cap \mathbb{R})$ to 0. In particular, v_t is unimodal with global maximum at $a_0 = 0$. Putting $a_0 = 0$ in (10.11), the maximum $v_{\max} = v_t(0)$ is the smallest positive real number v such that

$$\frac{1}{v} = \tan\left(\frac{v}{t}\right).$$

Thus, $v_t(0) = v_{\max} < \frac{\pi t}{2}$. This proves Point 1.

Since v_t is unimodal, the domain Λ_t has only one connected component. By Definition 5.1, Λ_t is symmetric about the real axis. In our case, Λ_t is also symmetric about the imaginary axis and the right hand side of (10.9) defines an even function of v. Thus, the boundary of Λ_t can be described by the curves

$$\partial\Lambda_t = \{(\pm A_0^t(v), v)|\ \ |v| \leq v_{\max}\},$$

which is Point 2. Since $a_0(v)^2 \to t+1$ as $v \to 0$, (10.10) holds, which proves Point 3. □

Proof of Proposition 10.5. In the uniform case, the equation (1.5) takes the form

$$\begin{aligned} a &= t\int_{\mathbb{R}} \frac{x}{(a_0 - x)^2 + v_t(a_0)^2}\ d\mu(x) \\ &= a_0 + \frac{t}{4}\log\left(1 - \frac{4a_0}{(a_0+1)^2 + v_t(a_0)^2}\right). \end{aligned}$$

For $a \geq 0$ in Ω_t, we can express a as a function $a = A_t(v)$ of $v = v_t(a_0)$ using (10.9) as

$$A_t(v) = A_0^t(v) + \frac{t}{4}\log\left(1 - \frac{4A_0^t(v)}{(A_0^t(v)+1)^2 + v^2}\right) \tag{10.13}$$

for $0 < v \leq v_{\max}$.

Using the definition of b_t in (5.14), $b_t(a) = 2v_t(a_0^t(a)) = 2v$. Thus, b_t is unimodal on $\Omega_t \cap \mathbb{R}$ with a peak at 0. The domain Ω_t has only one connected component whose boundary can be described by the two curves

$$\pm A_t(b/2) + ib, \quad |b| \leq 2v_{\max}.$$

Thus, (10.6) follows. This proves Point 1.

Using (10.10), we can compute the limit as $v \to 0$ in (10.13), from which the claimed from of $\Omega_t \cap \mathbb{R}$ follows, establishing Point 2.

Now, since both a_0 and a are functions of v when a_0 and a are nonnegative, we can compute

$$\frac{da_0^t(a)}{da} = \frac{dA_0^t(v)/dv}{dA_t(v)/dv}. \tag{10.14}$$

This result also holds for negative a because Λ_t and Ω_t are symmetric about the imaginary axis. The density of the Brown measure is then computed using Theorem 7.9. Using (10.14), the density can be expressed in terms of v as

$$\frac{1}{2\pi t}\left(\frac{da_0^t(a)}{da} - \frac{1}{2}\right) = \frac{t^2 + 4(t+2)v^2 - t(t+4v^2)\cos\left(\frac{4v}{t}\right) - 4tv\sin\left(\frac{4v}{t}\right)}{4\pi t\left(-t^2 + 8v^2 + t^2\cos\left(\frac{4v}{t}\right)\right)},$$

establishing (10.7). Since $b_t(a) = 2v$, we obtain (10.8), completing the proof. □

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References

[1] Z. D. Bai, Circular law, *Ann. Probab.* **25** (1997), 494–529.
[2] P. Biane, On the free convolution with a semi-circular distribution, *Indiana Univ. Math. J.* **46** (1997), 705–718.
[3] P. Biane, Processes with free increments, *Math. Z.* **227** (1998), 143–174.
[4] P. Biane and F. Lehner, Computation of some examples of Brown's spectral measure in free probability, *Colloq. Math.* **90** (2001), 181–211.
[5] P. Biane and R. Speicher, Stochastic calculus with respect to free Brownian motion and analysis on Wigner space, *Probab. Theory Related Fields* **112** (1998), 373–409.
[6] S. T. Belinschi, T. Mai, and R. Speicher, Analytic subordination theory of operator-valued free additive convolution and the solution of a general random matrix problem, *J. Reine Angew. Math.* **732** (2017), 21–53.
[7] S. T. Belinschi, P. Śniady, and R. Speicher, Eigenvalues of non-Hermitian random matrices and Brown measure of non-normal operators: Hermitian reduction and linearization method, *Linear Algebra Appl.* **537** (2018), 48–83.
[8] L. G. Brown, Lidskiĭ's theorem in the type II case. *In* Geometric methods in operator algebras (Kyoto, 1983), 1–35, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, 1986.
[9] Z. Burda, J. Grela, M. A. Nowak, W. Tarnowski, and P. Warchoł, Dysonian dynamics of the Ginibre ensemble, *Phys. Rev. Letters* **113** (2014), article 104102.
[10] Z. Burda, J. Grela, M. A. Nowak, W. Tarnowski, and P. Warchoł, Unveiling the significance of eigenvectors in diffusing non-Hermitian matrices by identifying the underlying Burgers dynamics, *Nucl. Phys. B* **897** (2015), 421–447.
[11] N. Demni and T. Hamdi, Support of the Brown measure of the product of a free unitary Brownian motion by a free self-adjoint projection, *J. Funct. Anal.*, **282** (2022), Article 109362, 36 pp.
[12] B. K. Driver, B. C. Hall, and T. Kemp, The Brown measure of the free multiplicative Brownian motion, arXiv:1903.11015 [math.PR].
[13] L. C. Evans, Partial differential equations. Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010. xxii+749 pp.
[14] V. L. Girko, The circular law, *Theory Probab. Appl.* **29** (1984), 694–706.
[15] V. L. Girko, Elliptic law, *Theory Probab. Appl.* **30** (1985), 677–690.
[16] J. Grela, M. A. Nowak, and W. Tarnowski, Eikonal formulation of large dynamical random matrix models, *Phys. Rev. E* **104** (2021), Article 054111, 12 pp.
[17] A. Guionnet, M. Krishnapur, and O. Zeitouni, The single ring theorem, *Ann. of Math.* (2) **174** (2011), 1189–1217.
[18] B. C. Hall, PDE methods in random matrix theory. In Harmonic analysis and applications (M. T. Rassias, Ed.), 77-124, Springer 2021.
[19] B. C. Hall and C.-W. Ho, The Brown measure of a family of free multiplicative Brownian motions, arXiv:2104.07859 [math.PR]
[20] C.-W. Ho, The Brown measure of the sum of a self-adjoint element and an elliptic element, arXiv:2007.06100 [math.OA]
[21] C.-W. Ho, The Brown measure of unbounded variables with free semicircular imaginary part, arXiv:2011.14222 [math.OA]
[22] C.-W. Ho and P. Zhong, Brown Measures of free circular and multiplicative Brownian motions with self-adjoint and unitary initial conditions, *J. Europ. Math. Soc.*, to appear.
[23] R. Janik, M. A. Nowak, G. Papp, J. Wambach, and I. Zahed, Non-Hermitian random matrix models: free random variable approach, *Phys. Rev. E* **55** (1997), 4100–4106.
[24] A. Jarosz and M. A. Nowak, A novel approach to non-Hermitian random ratrix models, preprint arXiv:math-ph/0402057.
[25] A Jarosz and M. A. Nowak, Random Hermitian versus random non-Hermitian operators—unexpected links, *J. Phys. A* **39** (2006), 10107–10122.
[26] V. Kargin, Subordination for the sum of two random matrices, *Ann. Probab.* **43** (2015), 2119–2150.

[27] T. Kemp, The large-N limits of Brownian motions on $\mathbb{GL}_N$, *Int. Math. Res. Not.*, (2016), 4012–4057.
[28] J. A. Mingo and R. Speicher, Free probability and random matrices. Fields Institute Monographs, 35. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2017.
[29] E. A. Nikitopoulos, Itô's formula for noncommutative C^2 functions of free Itô processes with respect to circular Brownian motion, arXiv:2011.08493 [math.OA].
[30] L. Pastur and V. Vasilchuk, On the law of addition of random matrices, *Comm. Math. Phys.* **214** (2000), 249–286.
[31] W. Rudin, Real and complex analysis, *McGraw-Hill.* (1987).
[32] M. A. Stephanov, Random matrix model of QCD at finite density and the nature of the quenched limit, *Phys. Rev. Letters* **76** (1996), 4472-4475.
[33] T. Tao and V. Vu, Random matrices: universality of ESDs and the circular law. With an appendix by Manjunath Krishnapur, *Ann. Probab.* **38** (2010), 2023–2065.
[34] D. V. Voiculescu, Symmetries of some reduced free product C^*-algebras. *In* "Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)," 556–588, Lecture Notes in Math., 1132, Springer, Berlin, 1985.
[35] D. V. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* **104** (1991), 201–220.
[36] D. V. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory. I. Comm. Math. Phys. 155, 1 (1993), 71–92.
[37] D. V. Voiculescu, The coalgebra of the free difference quotient and free probability, *Int. Math. Res. Not.* **2000** (2000), 79–106.
[38] D. V. Voiculescu, K. J. Dykema, and A. Nica, Free random variables. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups. CRM Monograph Series, 1. American Mathematical Society, Providence, RI, 1992. vi+70.
[39] P. Zhong, Brown measure of the sum of an elliptic operator and a free random variable in a finite von Neumann algebra, arXiv:2108.09844 [math.OA]

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