

THE BROWNIAN LIMIT OF SEPARABLE PERMUTATIONS¹

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We study uniform random permutations in an important class of pattern-avoiding permutations: the separable permutations. We describe the asymptotics of the number of occurrences of any fixed given pattern in such a random permutation in terms of the Brownian excursion. In the recent terminology of permutons, our work can be interpreted as the convergence of uniform random separable permutations towards a “Brownian separable permuton”.

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1. Introduction. The aim of this article is to study the asymptotic properties of an important class of pattern-avoiding permutations: the separable permutations. Our main result is the description of the asymptotics in n of the number of occurrences of any fixed given pattern in a uniform separable permutation of n elements.

1.1. *Pattern-avoiding permutations.* We first give some definitions. For any n , the set of permutations of $[n] := \{1, 2, \dots, n\}$ is denoted by \mathfrak{S}_n . We write permutations of \mathfrak{S}_n in one-line notation as $\sigma = \sigma_1\sigma_2 \dots \sigma_n$. For a permutation σ in \mathfrak{S}_n , the size n of σ is denoted by $|\sigma|$. For $\sigma \in \mathfrak{S}_n$, and $I \subset [n]$ of cardinality k , let $\text{pat}_I(\sigma)$ be the permutation of \mathfrak{S}_k induced by $\{\sigma_i, i \in I\}$. For example, for $\sigma = 65831247$ and $I = \{2, 5, 7\}$ we have

$$\text{pat}_{\{2,5,7\}}(\overline{65831247}) = 312$$

since the values in the subsequence $\sigma_2\sigma_5\sigma_7 = 514$ are in the same relative order as in the permutation 312. A permutation $\pi = \text{pat}_I(\sigma)$ is a *pattern* involved in σ , and the subsequence $(\sigma_i)_{i \in I}$ is an *occurrence* of π in σ .

All along this paper, we use letter σ for a (large) permutation of size n , and letter π for a pattern of size $k \leq n$. We denote by $\widetilde{\text{occ}}(\pi, \sigma)$ the proportion of occurrences of a pattern π in σ . More formally,

$$\widetilde{\text{occ}}(\pi, \sigma) = \frac{1}{\binom{n}{k}} \text{card}\{I \subset [n] \text{ of cardinality } k \text{ such that } \text{pat}_I(\sigma) = \pi\}.$$

Equivalently, $\widetilde{\text{occ}}(\pi, \sigma)$ is the probability to have $\text{pat}_I(\sigma) = \pi$ if I is randomly and uniformly chosen among the $\binom{n}{k}$ subsets of $[n]$ with k elements. If $|\pi| > |\sigma|$, we set conveniently $\widetilde{\text{occ}}(\pi, \sigma) = 0$.

We say that σ *avoids* τ if there is no occurrence of τ in σ , that is, $\widetilde{\text{occ}}(\tau, \sigma) = 0$. For any (finite or infinite) set of patterns τ_1, τ_2, \dots , we denote by $\text{Av}_n(\tau_1, \tau_2, \dots)$ the set of permutations of size n that avoid all the τ_i 's. Then the infinite union $\text{Av}(\tau_1, \tau_2, \dots) = \bigcup_n \text{Av}_n(\tau_1, \tau_2, \dots)$ is called a *class* of (pattern-avoiding) permutations. Equivalently,³ a class of permutations is a set \mathcal{C} of permutations such that, for any $\sigma \in \mathcal{C}$ and any pattern π of σ , it holds that $\pi \in \mathcal{C}$.

³This statement is folklore in the literature on permutation patterns. A proof can be found in [12], paragraph 5.1.2 for instance. See also [12], paragraph 7.2.3 for a proof that an infinite set of excluded patterns is sometimes necessary to describe a class.

Classes of permutations have been intensively studied for their combinatorial and algorithmic properties over the last 50 years. An account of the past and current research on these classes can be found in [12, 28, 40]. Finding the enumeration of specific classes, defined by the avoidance of a small number of small patterns, has been one of the first problems studied in this field. It started with the proof that $Av(\tau)$ is counted by the Catalan numbers, for any $\tau \in \mathfrak{S}_3$, and the research on this topic still continues, as witnessed by the summary [42]. The combinatorics of classes of permutations has however expanded in several other directions, including a general approach to the study of classes of permutations based on various notions of structure, like the *substitution decomposition* ([2], Proposition 2 or [40], Section 3.2) to which we shall return later in this introduction.

The probabilistic study of classes of permutations is much more recent and, just like their combinatorial study at its beginning, it focuses on the study of specific classes with small excluded patterns. More precisely, the probabilistic counterpart of the study of specific classes is centered on the following interesting question: Given a fixed pattern τ , what can we say about a typical σ in $Av_n(\tau)$ (for large n)? Recently, many authors have considered this problem for different choices of small patterns τ . We mention a few of them:

- The question was initiated in a paper of Madras and Liu [32] in relation with a Monte-Carlo algorithm to approximate growth rates of permutation classes. In subsequent papers, Atapour and Madras [4] and Madras and Pehlivan [33] started the study of uniform permutations in $Av_n(\tau)$ for small patterns τ .
- In parallel, Miner and Pak [34] described very precisely the asymptotic shape of a uniform element in $Av_n(\tau)$ for the 6 patterns τ in \mathfrak{S}_3 . Dokos and Pak [16] have then obtained similar results for random doubly alternating Baxter permutations. Note also that Miner and Pak discuss at the end of their paper a possible connection with the Brownian excursion.
- Such a connection between $Av_n(\tau)$ for $\tau \in \mathfrak{S}_3$ and the Brownian excursion is explained by Hoffman, Rizzolo and Slivken [21]. Many combinatorial consequences are given, in particular a precise description of fixed points of such permutations [20].
- In a parallel line of research, Bóna [10, 11] investigates the behaviour of $\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma)]$ for σ uniform in $Av_n(132)$, and several fixed π 's. Similar results for other permutation classes and various patterns π have then been obtained by Homberger [22], Chang, Eu and Fu [14] and Rudolf [38].
- The question of finding limiting distributions for $\widetilde{\text{occ}}(\pi, \sigma)$ for σ uniform in $Av_n(\tau)$, rather than studying only its expectation, was raised by Janson, Nakamura and Zeilberger in [25]: the authors gave some algorithms to find limits of moments for small π and τ . A bit later, Janson [24] has given for every π the asymptotic behaviour of the random variable $\widetilde{\text{occ}}(\pi, \sigma)$ for σ uniform in $Av_n(132)$. For instance, he expresses in terms of the Brownian excursion area the asymptotic behaviour of $\widetilde{\text{occ}}(12, \sigma)$.

- In his recent thesis, Bevan describes the limit shape of permutations in so-called *connected monotone grid classes* [7], Chapter 6. This result is the first that deals with an infinite family of permutation classes.
- Even if it does not involve strictly speaking *pattern-avoiding permutations*, we mention the recent work of Kenyon, Kral', Radin and Winkler [27]. They prove a large deviation theorem for permutations seen as probability measures on the square. Their result yields limit shapes of random permutations with fixed densities of some fixed pattern π_1, \dots, π_r . This parallels similar results on graphons, which are well-studied objects in random graph theory.

In the current paper, we are specifically interested in the class of *separable* permutations.

DEFINITION 1.1. A permutation σ is *separable* if σ avoids both 2413 and 3142.

We obtain results similar to those of Janson [24] for $\text{Av}(132)$, namely we study occurrences of any pattern π in uniform random separable permutations. There is however an important difference between our work and all classes studied so far: random permutations in any of these previously studied classes have a limit, which is *deterministic* at first order, whereas random separable permutations have a limit, which is *nondeterministic* at first order. The limit of random separable permutations will be discussed in Section 1.3, and the proof that it is not deterministic will be given in Section 9.2. This is also visible on Figure 1, which shows two large typical separable permutations obtained using a Boltzmann random sampler.

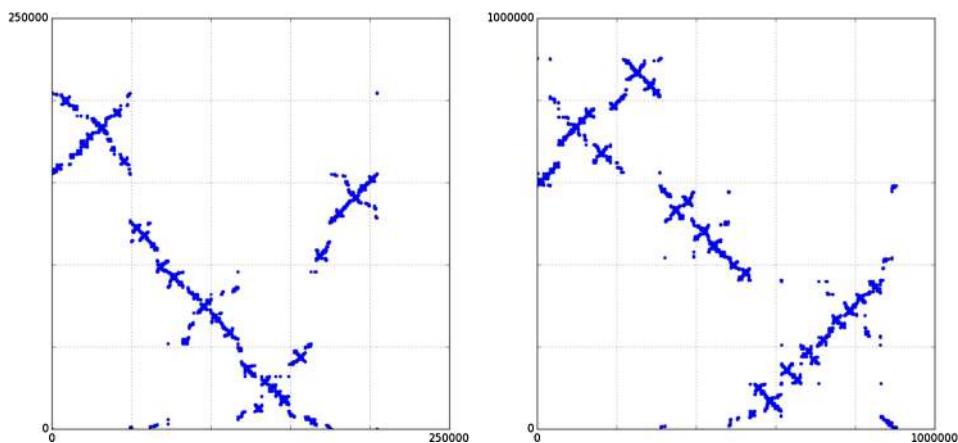


FIG. 1. Two uniform random separable permutations of sizes respectively $n = 204,523$ and $n = 903,073$ [a permutation σ is represented here with its diagram: for every $i \leq n$, there is a dot at coordinates (i, σ_i)].

There are several reasons that motivate our choice of studying the class of separable permutations, in addition to it being one of the most studied classes after $\text{Av}(\tau)$ for τ of size 3. Separable permutations have a very nice and robust combinatorial structure: they can be completely decomposed using direct sums and skew sums and, therefore, can be represented as signed Schröder trees (see Section 2.2). This encoding with trees is essential in proving a variety of results about separable permutations in different fields, for instance:

- the algorithmic problem of PERMUTATION PATTERN MATCHING is NP-hard in general, but polynomial on separable permutations [13];
- from an enumerative combinatorics point of view, in addition to being simple to count, separable permutations display remarkable *equipopularity* properties, see [1].

Besides, separable permutations appear naturally in several problems, at first sight independent from permutation pattern theory:

- as the permutations sortable by certain sorting devices (pop-stacks in series) [5];
- as space-filling permutation matrices in bootstrap percolation [39];
- as possible polynomial interchanges (i.e., studying in which possible ways the relative order of the values of a family of polynomials can be modified when crossing a common zero) [18].

Finally, the class of separable permutations is the simplest case of a nontrivial *substitution-closed* (also called wreath-closed [2]) class, and we believe that the results obtained here might be extended to any substitution-closed class; see the discussion on universality in Section 1.4.

1.2. *Overview of our results.* Throughout this paper, let us denote by σ_n a uniform separable permutation of size n . Our goal is to describe the limit of σ_n when n goes to infinity. Our main result gives, for any π , the asymptotics of the distribution of $\widetilde{\text{occ}}(\pi, \sigma_n)$ when n tends to infinity. We will see in Section 1.3 an equivalent formulation in terms of weak convergence of probability measures on the square.

Our main theorem is the following.

THEOREM 1.2. *Let σ_n be a uniform separable permutation of size n . There exists a collection of random variables (Λ_π) , π ranging over all permutations, defined on the same probability space, such that for all π , $0 \leq \Lambda_\pi \leq 1$ and:*

(i) *If π is a separable permutation of size at least 2, Λ_π is a nondeterministic random variable.*

(If π is the permutation of size 1, $\Lambda_\pi = 1$. If π is not separable, $\Lambda_\pi = 0$.)

(ii) *For all π , when $n \rightarrow +\infty$,*

$$\widetilde{\text{occ}}(\pi, \sigma_n) \xrightarrow{d} \Lambda_\pi,$$

where \xrightarrow{d} denotes the convergence in distribution.

(iii) Moreover, the convergence holds jointly, that is, for any finite sequence of permutations (π_1, \dots, π_r) ,

$$(\widetilde{\text{occ}}(\pi_1, \sigma_n), \dots, \widetilde{\text{occ}}(\pi_r, \sigma_n)) \xrightarrow{d} (\Lambda_{\pi_1}, \dots, \Lambda_{\pi_r}).$$

(On the right-hand side, the Λ_{π_i} 's are not independent.)

Theorem 1.2 is not just an existential result: in this paper we give for any pattern π a construction of Λ_π (Definition 3.1, page 2155) that can be briefly explained as follows. There is a natural way (reviewed in Section 2.4) to extract a (signed) tree with $|\pi|$ leaves from a realization of the (signed) Brownian excursion. The variable Λ_π is the probability that this tree is one of the *separation trees* of π (see Section 2.2 for the definition).

Statement (i) of the theorem is proved in Section 9.2, while Statement (iii) [Statement (ii) being just a special case] is proved in Sections 4 to 7, following the proof schema detailed in Section 3.4.

Theorem 1.2 shows in particular that, for every separable pattern π of size k , the number of occurrences of π in σ_n must be renormalized by n^k to have a nontrivial limit in distribution. This is in contrast with the result of Janson [24], Theorem 2.1 for σ uniform in $\text{Av}_n(132)$. Indeed, in his result, the numbers of occurrences of different patterns of the same size are normalized by different powers of n to have nontrivial limits in distribution.

In addition to the convergence in distribution, we also prove the convergence of all joint moments (in fact, we first prove the convergence of joint moments, and then deduce the convergence in distribution). This is especially interesting since the joint moments in the limit can be computed explicitly.

More precisely, all joint moments can be expressed in the limit from expectations of Λ_π 's (see Proposition 9.3), and the expectation of Λ_π can be expressed in terms of the number N_π of signed binary trees associated with the permutation π (these are also sometimes called *separation trees* of π ; see Section 2.2 for their definition). The latter is proved in Proposition 9.1, which reads as follows.

THEOREM 1.3. *For any permutation π of size k ,*

$$\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)] \xrightarrow{n \rightarrow +\infty} \frac{N_\pi}{2^{k-1} \text{Cat}_{k-1}},$$

where we denote by $\text{Cat}_k := \frac{1}{k+1} \binom{2k}{k}$ the k th Catalan number.

REMARK 1.4. Our proof of Theorem 1.3 involves the random variables Λ_π , which are constructed from the Brownian excursion. We do not know if there exists a proof using only discrete arguments.

In addition to being defined combinatorially, the numbers N_π are easy to compute; see Observation 9.2. This gives, for example,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{E}[\widetilde{\text{occ}}(12, \sigma_n)] &= \lim_{n \rightarrow +\infty} \mathbb{E}[\widetilde{\text{occ}}(21, \sigma_n)] = \frac{1}{2}; \\ \lim_{n \rightarrow +\infty} \mathbb{E}[\widetilde{\text{occ}}(123, \sigma_n)] &= \lim_{n \rightarrow +\infty} \mathbb{E}[\widetilde{\text{occ}}(321, \sigma_n)] = \frac{1}{4}; \\ \lim_{n \rightarrow +\infty} \mathbb{E}[\widetilde{\text{occ}}(132, \sigma_n)] &= \lim_{n \rightarrow +\infty} \mathbb{E}[\widetilde{\text{occ}}(213, \sigma_n)] = \frac{1}{8}; \\ \lim_{n \rightarrow +\infty} \mathbb{E}[\widetilde{\text{occ}}(231, \sigma_n)] &= \lim_{n \rightarrow +\infty} \mathbb{E}[\widetilde{\text{occ}}(312, \sigma_n)] = \frac{1}{8}. \end{aligned}$$

Limits of higher (joint) moments can also be computed explicitly, as shown in Proposition 9.3. For example, we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} \text{Var}[\widetilde{\text{occ}}(12, \sigma_n)] &= \frac{1}{30}; \\ \lim_{n \rightarrow +\infty} \text{Var}[\widetilde{\text{occ}}(132, \sigma_n)] &= \frac{3}{560}; \\ \lim_{n \rightarrow +\infty} \mathbb{E}[\widetilde{\text{occ}}(12, \sigma_n) \cdot \widetilde{\text{occ}}(123, \sigma_n)] &= \frac{43}{280}. \end{aligned}$$

These values have been computed automatically with a Sage program [15] that the authors can make available on request. We refer to Section 9.1 for a discussion on the computation of joint moments, both from the theoretical and algorithmic points of view.

For the curious reader, we give the first few moments of Λ_{12} [i.e., the limits of the first few moments of $\widetilde{\text{occ}}(12, \sigma_n)$]:

$$\begin{aligned} \mathbb{E}[\Lambda_{12}] &= \frac{1}{2}; & \mathbb{E}[\Lambda_{12}^2] &= \frac{17}{60}; & \mathbb{E}[\Lambda_{12}^3] &= \frac{7}{40}; \\ \mathbb{E}[\Lambda_{12}^4] &= \frac{6361}{55,440}; & \mathbb{E}[\Lambda_{12}^5] &= \frac{1741}{22,176}. \end{aligned}$$

We did not recognize the first moments of any “usual” distribution on $[0, 1]$.

1.3. *Interpretation of our main result in terms of permutons.* We recall the notion of *permutons* introduced in [23]. Note that, in [23], permutons are called *limit permutons* and have two equivalent definitions (see [23], Section 2.3); the name *permuton* was proposed in [19] in analogy with the graph analogue *graphon*.

DEFINITION 1.5. A *permuton* is a probability measure μ on the square $[0, 1]^2$ with uniform marginals, that is, for any a and b with $0 \leq a \leq b \leq 1$,

$$\mu([a, b] \times [0, 1]) = \mu([0, 1] \times [a, b]) = b - a.$$

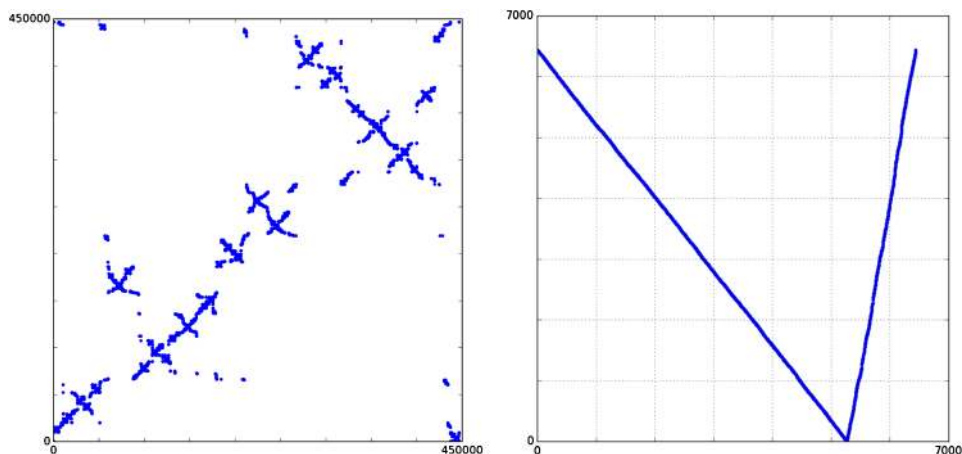


FIG. 2. On the left, a typical permutation of size 446,699 in the substitution-closed class whose simple permutations are 2413, 3142 and 24153. On the right, a typical permutation of size 6441 in the nonsubstitution-closed class $\text{Av}(2413, 1243, 2341, 531642, 41352)$.

One can associate a permuton μ_σ to a permutation σ of size n . We define μ_σ as having density n on each square $[(i-1)/n, i/n] \times [(\sigma(i)-1)/n, \sigma(i)/n]$ (for $1 \leq i \leq n$) and density 0 elsewhere; that is, each of these squares has total weight $1/n$, uniformly distributed in it. This is a natural way to encode and rescale a permutation, very close to the graphical representation that we use on Figures 1 and 2.

Since permutons are measures, it is natural to speak about weak convergence of permutons. We will see in Section 8 that our main result, combined with previous results on permutons, implies the following.

THEOREM 1.6. *Let σ_n be a uniform random separable permutation of size n . There exists a random permuton μ such that μ_{σ_n} tends to μ in distribution in the weak convergence topology. Moreover, μ is not deterministic.*

The distribution of this permuton μ is unique, since it is defined as a limit in distribution. The proof of the existence of μ is not constructive, but involves the variables Λ_π , which are themselves built using a realization of the Brownian excursion. We therefore call μ the *Brownian separable permuton*.

REMARK 1.7. There are many examples of convergence of large combinatorial structures towards continuum objects built from Brownian motion (or related processes: the Brownian bridge and excursion). One can mention:

- Random mappings and Brownian bridges;
- Random trees and Brownian trees;
- Random graphs and Brownian motion.

A very good reference for these topics is [36]. Theorem 1.6 fits naturally in this body of literature.

Note that the fact that separable permutations have a Brownian limit in some sense should not come as a surprise. Indeed, separable permutations of size n can be encoded by (signed) *Schröder trees* with n leaves (see Section 2.2). Like for many families of trees, the limit of Schröder trees with a fixed number of leaves (leaving signs aside) is related to the Brownian excursion: more precisely, Pitman and Rizzolo [37] and Kortchemski [29] proved that the *contour* of a uniform Schröder tree with n leaves tends to the Brownian excursion. This result is essential in our approach.

1.4. *Perspectives.* We think that the Brownian separable permutation μ is an interesting object and is worth being studied. In particular, we would like to address the following questions.

Construction of μ At the moment, μ is defined in an indirect way, as limit of discrete objects. Is there a way to define/construct the random measure μ directly in the continuum, for example, as a function of the (signed) Brownian excursion?

Properties of μ It would be interesting to find some almost-sure properties of μ . Is it absolutely continuous or singular with respect to Lebesgue measure on the square? (Note that, since its marginals are uniform, it cannot have atoms.) One also expects that μ is fractal in some sense, because of the link with the Brownian excursion and the visual aspect of the simulations in Figure 1. This raises the following question: what is the Hausdorff dimension of its support?

Universality of μ We believe that μ is the limit (in the sense of permutations) of many other classes of random pattern-avoiding permutations. Recall indeed that the class of separable permutations is the first nontrivial case of a substitution-closed class of permutations. Such classes are those whose structure is well understood using the substitution decomposition mentioned earlier in this introduction, even more so when they contain a finite number of so-called *simple* permutations (see [40], Section 3.2). For any substitution-closed class, the permutations it contains may be represented by trees, called (*substitution*) *decomposition trees*. These generalize signed Schröder trees by introducing other types of vertices, labeled by the simple permutations in the class. As the encoding of separable permutations by Schröder trees is crucial in this work, our results might extend to all substitution-closed classes containing finitely many simple permutations. Figure 2 shows two typical permutations: one in a substitution-closed class, one in a class that is not closed under substitution. On these examples, it seems clear that the first one looks similar to the separable case, whereas the second one does not.

Note added after the peer-review process: after the submission of the present article, subsequent results have been achieved on the topic. A construction of the

Brownian separable permuton has been given by Maazoun in [31]. One can prove from this construction that the support of μ is completely disconnected and has Hausdorff dimension 1, with probability 1. Moreover, the authors of this paper, together with Maazoun, have proved the following universality result: under a simple sufficient condition, a uniform random permutation in a substitution-closed class does indeed converge towards the Brownian separable permuton (or a trivial one-parameter deformation of it); see [6].

1.5. *Outline of the paper.* Our paper is organized as follows:

- Section 2 gives all the preliminaries needed to define the limit random variables Λ_π and records a lot of easy useful facts about permutations and trees.
- Section 3 defines the variables Λ_π and presents the structure of the proof of Theorem 1.2(iii).
- Sections 4 to 6 go through the several steps of this proof (the outline of the proof itself is given in Section 3.4).
- We gather all the arguments and conclude the proof of Theorem 1.2(iii) in Section 7.
- Section 8 contains the proof of the permuton interpretation of our main result: Theorem 1.6.
- Section 9 studies some properties of Λ_π : combinatorial formulas for the moments in Section 9.1 and proof of Theorem 1.2(i) (Λ_π is not deterministic) in Section 9.2.
- We collect several useful properties of the Brownian excursion in the [Appendix](#).

2. Permutations, trees and excursions.

2.1. Basics on trees.

DEFINITION 2.1. A *Schröder tree* is either a leaf, or consists of a root vertex with an ordered k -tuple of subtrees attached to the root ($k \geq 2$), which are themselves Schröder trees.

Nonleaf vertices of a tree are called *internal vertices*.

We consider only *finite* Schröder trees, that is, those having finitely many leaves and internal vertices.

In our context, the *size* of a tree t is its number of leaves. It is denoted $|t|$, whereas $\#t$ denotes the number of vertices of t (including both leaves and internal vertices).

Because every internal vertex of a Schröder tree has at least 2 children, it follows immediately (by induction) the following.

OBSERVATION 2.2. For every Schröder tree t , $2|t| \geq \#t + 1$.

A *binary tree* is a Schröder tree where there are exactly 2 subtrees attached to every internal vertex.

In this article, we consider unlabeled trees. Nevertheless, since we work with *plane trees* (i.e., trees in which the subtrees attached to a vertex are ordered), there is a canonical way to label the *leaves* of a tree (from left to right). Then a subset of the set of leaves of a tree t is canonically represented by a subset I of $[|t|]$.

DEFINITION 2.3. Let t be a Schröder tree and u and v be two vertices of t . Denote by r the root of t . The (*first*) *common ancestor* of u and v is the vertex furthest away from r that appears on both paths from r to u and from r to v in t .

DEFINITION 2.4. Let t be a Schröder tree. Any subset I of the leaves of t induces a subtree t_I of t , which is also a Schröder tree, defined as follows:

- the leaves of t_I are the elements of I ;
- the internal vertices of t_I are the vertices of t that are common ancestors of two leaves in I ;
- the ancestor-descendant relation in t_I is inherited from the one in t ;
- the order between the children of an internal vertex of t_I is inherited from t .

Note that if t is a binary tree, then so is t_I .

DEFINITION 2.5. A *signed Schröder tree* is a pair (t, ε) , where t is a Schröder tree and ε a function from the set of internal vertices of t to $\{+, -\}$.

DEFINITION 2.6. Let (t, ε) be a signed Schröder tree. Any subset I of the leaves of t induces a signed subtree (t_I, ε_I) of (t, ε) , where t_I is as in Definition 2.4 and ε_I is the restriction of ε on the set of internal vertices of t_I .

EXAMPLE 2.7. Consider the signed Schröder tree on the left-hand side of Figure 3. To ease the presentation, we have indicated the canonical labeling of its leaves from left to right. We take the subset of leaves $I = \{2, 4, 7\}$ (circled on the picture). Then the signed subtree (t_I, ε_I) is represented on the right-hand side of Figure 3. (Again labels on the leaves are here to simplify the presentation, our objects are in essence not labeled.)

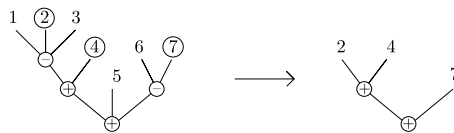


FIG. 3. Subtree of a signed Schröder tree. The labels on the leaves are here only to witness the embedding of the right tree in the left one.

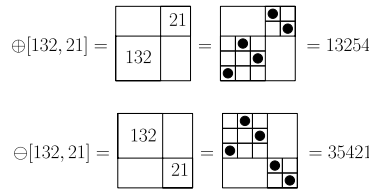


FIG. 4. Direct sum and skew sum of permutations on their diagrams. Recall that the diagram of a permutation σ of size n is the set of points at coordinates (i, σ_i) , for $1 \leq i \leq n$.

2.2. Separable permutations and Schröder trees.

DEFINITION 2.8. Let π and σ be two permutations of respective sizes k and ℓ . Their direct sum and skew sum are the permutations of size $k + \ell$ defined by

$$\begin{aligned} \oplus[\pi, \sigma] &= \pi \oplus \sigma = \pi_1 \cdots \pi_k(\sigma_1 + k) \cdots (\sigma_\ell + k) \quad \text{and} \\ \ominus[\pi, \sigma] &= \pi \ominus \sigma = (\pi_1 + \ell) \cdots (\pi_k + \ell)\sigma_1 \cdots \sigma_\ell. \end{aligned}$$

The operators \oplus and \ominus being associative, direct sums and skew sums with $r \geq 2$ components are defined in the obvious way. We will use the notation $\oplus[\pi^1, \dots, \pi^r]$ instead of $\pi^1 \oplus \cdots \oplus \pi^r$ (and similarly for \ominus). Examples with $k = 3, \ell = 2$ are provided in Figure 4, which also illustrates the graphical interpretation of these operations on permutation diagrams.

DEFINITION 2.9. Let (t, ε) be a signed Schröder tree. The permutation associated with (t, ε) , denoted $\text{perm}(t, \varepsilon)$, is inductively defined by:

- if t is a leaf, then $\text{perm}(t, \varepsilon) = 1$;
- otherwise, denoting by t_1, \dots, t_r the children of the root of t from left to right, and ε_i the restriction of ε to the vertices of t_i ,

$$\text{perm}(t, \varepsilon) = \begin{cases} \oplus[\text{perm}(t_1, \varepsilon_1), \dots, \text{perm}(t_r, \varepsilon_r)] & \text{if the root of } t \text{ has sign } +; \\ \ominus[\text{perm}(t_1, \varepsilon_1), \dots, \text{perm}(t_r, \varepsilon_r)] & \text{if the root of } t \text{ has sign } -. \end{cases}$$

If $\text{perm}(t, \varepsilon) = \sigma$, we say that (t, ε) is a signed tree of σ .

Note that the i th leaf of (t, ε) (from left to right) corresponds to the i th element of $\text{perm}(t, \varepsilon)$. Binary signed trees of a permutation σ are sometimes called separation trees of σ .

For example, let (t, ε) be the tree on the left-hand side of Figure 3 of Example 2.7. Then

$$\begin{aligned} \text{perm}(t, \varepsilon) &= \oplus[\oplus[\ominus[1, 1, 1], 1], 1, \ominus[1, 1]] = \oplus[\oplus[321, 1], 1, 21] \\ &= \oplus[3214, 1, 21] = 3214576. \end{aligned}$$

On the other hand, if we consider the tree (t_I, ε_I) on the right-hand side of Figure 3, we have

$$\text{perm}(t_I, \varepsilon_I) = \oplus[\oplus[1, 1], 1] = 123.$$

In other words, (t, ε) and (t_I, ε_I) are signed trees of 3214576 and 123, respectively.

OBSERVATION 2.10. There is an important consequence of the definition of the signed tree (t, ε) of a permutation σ , which is easily seen on our figures. For $i < j$, then $\sigma_i < \sigma_j$ (resp., $\sigma_i > \sigma_j$) if and only if the common ancestor of leaves i, j has sign + (resp., -).

A consequence of this observation is that the map perm is compatible with taking substructures.

OBSERVATION 2.11. Let (t, ε) be a signed Schröder tree. Let I be a subset of $[|t|]$ and (t_I, ε_I) be the signed subtree of (t, ε) induced by I . Then it holds that $\text{pat}_I(\text{perm}(t, \varepsilon)) = \text{perm}(t_I, \varepsilon_I)$.

We continue Example 2.7. Taking $I = \{2, 4, 7\}$, we have seen in previous examples that $\text{perm}(t, \varepsilon) = 3214576$ and $\text{perm}(t_I, \varepsilon_I) = 123$. According to Observation 2.11, we should have $\text{pat}_{\{2,4,7\}}(3214576) = 123$, which is indeed the case. This is illustrated by Figure 5.

In the **Introduction**, separable permutations have been defined as avoiding specific patterns. In fact, they can also be characterized using signed Schröder trees.

PROPOSITION 2.12. *Separable permutations are exactly those that can be obtained as $\text{perm}(t, \varepsilon)$ for a signed binary tree (t, ε) , or equivalently, those that can be obtained as $\text{perm}(t, \varepsilon)$ for a signed Schröder tree (t, ε) .*

PROOF. The first part (with binary trees) has been established in [13].

Assume now that a permutation σ can be obtained as $\text{perm}(t, \varepsilon)$ for a signed Schröder tree (t, ε) ; we need to prove that σ can also be obtained as $\text{perm}(t', \varepsilon')$ for a signed binary tree (t', ε') . To obtain (t', ε') from (t, ε) , it is enough to replace every vertex with a label δ and $k > 2$ subtrees t_1, \dots, t_k by a binary tree with k leaves on which t_1, \dots, t_k are pending and with all internal vertices labeled δ .

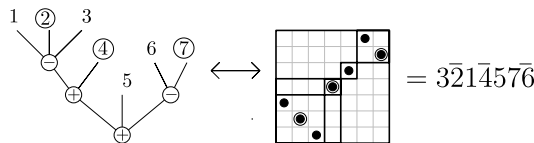


FIG. 5. One of the signed trees associated with $\sigma = 3214576$. The elements of the pattern $\text{pat}_{\{2,4,7\}}(\sigma)$ are circled/overlined.

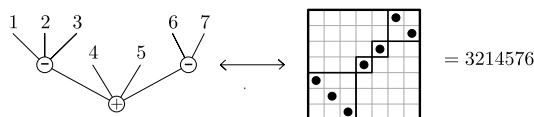


FIG. 6. The decomposition tree of a separable permutation.

Figure 12 (page 2180) shows all binary trees that can be obtained by the above construction from a specific signed Schröder tree. □

The advantage of Schröder trees over binary trees is that the correspondence tree-permutation can be made one-to-one in more natural way.

PROPOSITION 2.13. *The map perm is a size-preserving bijection between separable permutations and signed Schröder trees in which the signs alternate [i.e., all internal vertices at even (resp., odd) distance from the root have the same sign as the root (resp., the opposite sign)].*

Proposition 2.13 is a consequence of the more general substitution decomposition theorem for permutations [2], Proposition 2. In this context, the unique signed Schröder tree (t, ε) in which the signs alternate such that $\text{perm}(t, \varepsilon) = \pi$ is called the (substitution) decomposition tree of π .

Let us return to Example 2.7. The permutation 3214576 which is induced by the tree on the left-hand side of Figure 3 is separable and the Schröder tree in which signs alternate that corresponds to it is shown in Figure 6.

Recall that σ_n denotes a uniform random separable permutation of size n . Also, throughout the paper, we denote by T_n a uniform random Schröder tree with n leaves. From Proposition 2.13, we have the following.

COROLLARY 2.14. *Let B be a balanced Bernoulli variable with values in $\{+, -\}$, independent from T_n . Let \mathcal{E}_n be the sign function on the internal vertices of T_n , such that the signs alternate and the root of T_n has sign B . Then σ_n has the same distribution as $\text{perm}(T_n, \mathcal{E}_n)$.*

Pitman and Rizzolo observed (see [37], Theorem 1) that T_n behaves like a Galton–Watson tree conditioned to have n leaves (we refer to [41], Section 3 for basics of Galton–Watson trees).

PROPOSITION 2.15. *Let ν be the probability distribution defined by*

$$\nu(0) = 2 - \sqrt{2}, \quad \nu(1) = 0, \quad \nu(i) = \left(1 - \frac{\sqrt{2}}{2}\right)^{i-1} \quad (\text{for all } i \geq 2).$$

Then:

- ν has mean 1 and variance $4(\sqrt{2} - 1)$;

- T_n has the same distribution as a Galton–Watson tree with offspring distribution ν , conditioned to have n leaves.

Note that there are actually infinitely many probability distributions ν such that T_n has the same distribution as a Galton–Watson tree with offspring distribution ν , conditioned to have n leaves. The one chosen in Proposition 2.15 is such that ν has mean 1. The Galton–Watson tree model is then *critical*, and most convergence results in the literature are established in this case (see, e.g., Propositions 3.5 and 2.23 below).

2.3. *Contours and excursions.*

DEFINITION 2.16. An *excursion* is a continuous function $f : [0, 1] \rightarrow [0, +\infty)$ with $f(0) = f(1) = 0$.

Note that with our convention we allow $f(t) = 0$ for $t \notin \{0, 1\}$. We associate canonically with a tree an excursion, called its *contour*.

DEFINITION 2.17 (Contour C_t). Let t be a (binary or Schröder) tree. Recall that $\#t$ denotes the total number of vertices of t , and denote by V the set of vertices of t .

Consider the function $\text{dfs}_t : \{0, 1, \dots, 2\#t - 2\} \rightarrow V$ (dfs stands for *depth first search*) defined by:

- $\text{dfs}_t(0)$ is the root of t ;
- if $\text{dfs}_t(i) = v$, then $\text{dfs}_t(i + 1)$ is the leftmost child of v that has not yet been visited, if it exists, and the parent of v otherwise.

The *contour* of t is the function $C_t : [0, 2\#t - 2] \rightarrow [0; +\infty)$ such that for all integers i in $[0, 2\#t - 2]$, $C_t(i)$ is the distance from the root of t to the vertex $\text{dfs}_t(i)$, and C_t is linear between those points.

EXAMPLE 2.18. An example of a tree t and its contour C_t is given on Figure 7 (the reader should disregard the signs for the moment). The tiny numbers beside each vertex v indicate the times i in $[0, 2\#t - 2]$ such that $\text{dfs}_t(i) = v$.

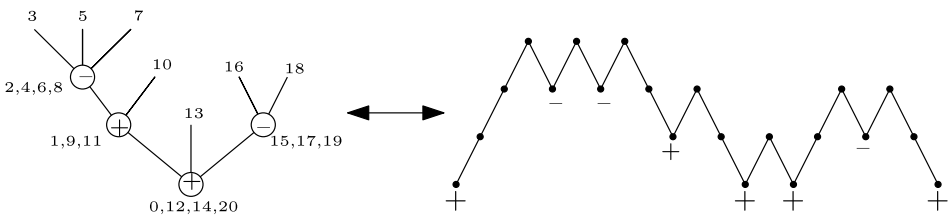


FIG. 7. A (signed) tree and its (signed) contour.

DEFINITION 2.19 (Normalized contour \widetilde{C}_t). Let t be a Schröder tree. The *normalized contour* of t is the function $\widetilde{C}_t : [0, 1] \rightarrow [0, +\infty)$ defined as follows:

$$\text{for all } u \in [0, 1], \quad \widetilde{C}_t(u) = \frac{1}{\sqrt{|t|}} C_t((2\#t - 2)u).$$

By definition, both C_t and \widetilde{C}_t are continuous, so that \widetilde{C}_t is an excursion.

Some properties of the local maxima and local minima of \widetilde{C}_t follow from its definition, and will be essential for our purpose.

OBSERVATION 2.20. Let t be a tree with $|t| > 1$. If there is a *local maximum* of \widetilde{C}_t at u , then $u = \frac{i}{(2\#t-2)}$ for some integer i and $\text{dfs}_t(i)$ is a *leaf* of t . This defines a bijection between the leaves of t and the local maxima of \widetilde{C}_t .

This allows us to identify the leaves of t with the x -coordinates of the local maxima of \widetilde{C}_t . We will often do so in the sequel, and we introduce the following notation.

DEFINITION 2.21 (ℓ_t). For any tree t with $|t| > 1$, we define $\ell_1 < \dots < \ell_{|t|}$ to be the x -coordinates of the local maxima of \widetilde{C}_t (which correspond to the leaves of t), and we set $\ell_t = \{\ell_1, \dots, \ell_{|t|}\} \subset [0, 1]$.

OBSERVATION 2.22. Let t be a tree with $|t| > 1$. If there is a *local minimum* of \widetilde{C}_t in u , then $u = \frac{i}{(2\#t-2)}$ for some integer i and $\text{dfs}_t(i)$ is an *internal vertex* of t . Note however that a single internal vertex can correspond to several local minima of \widetilde{C}_t .

Using the description of Schröder trees as Galton–Watson trees (see Proposition 2.15), Pitman and Rizzolo [37] and Kortchemski [29] have proved that the normalized contour of a uniform Schröder tree converges in distribution to a multiple of the Brownian excursion.⁴ Throughout the paper, we denote by e the Brownian excursion.

PROPOSITION 2.23. *The following convergence holds in distribution in the space $\mathcal{C}[0, 1]$ of real-valued continuous functions on $[0, 1]$:*

$$(\widetilde{C}_{T_n}(u))_{u \in [0, 1]} \xrightarrow{n \rightarrow +\infty} (\lambda \cdot e(u))_{u \in [0, 1]},$$

where $\lambda = \sqrt{2 + 3/\sqrt{2}}$.

⁴For the readers who need background on the Brownian excursion, we have collected in the [Appendix](#) (one of) its definition(s), together with some useful properties, and bibliographical references.

We now define signed analogues of contours and excursions.

DEFINITION 2.24. A *signed excursion* is a pair (f, s) where f is an excursion and s a function from the set of the local minima of f to $\{+, -\}$.

From Observation 2.22, we may define the signed contour of a signed tree as follows.

DEFINITION 2.25. Let (t, ε) be a signed Schröder tree. The *signed contour* of (t, ε) is the pair (\widetilde{C}_t, s) , where s associates to every local minimum of \widetilde{C}_t reached in $u = \frac{i}{(2\#t-2)}$ the sign of the internal vertex $\text{dfs}_t(i)$ of t .

EXAMPLE 2.26. The reader is now invited to look again at Figure 7, taking the signs into consideration.

We may also define the signed Brownian excursion by the following.

DEFINITION 2.27. The *signed Brownian excursion* is the pair (e, S) where e is the Brownian excursion and S is the function assigning balanced independent signs on the local minima of e .

The probability space and the σ -algebra on which this object is constructed will be introduced in Section 2.5.

REMARK 2.28. Some readers may have seen the term “*signed Brownian excursion*” refer to an excursion of Brownian motion whose length has been normalized to 1, but that may be either positive on $(0, 1)$ or negative on $(0, 1)$. The meaning of this expression in our paper is different.

REMARK 2.29. Consider a uniform random Schröder tree T_n of size n and a random sign function \mathcal{E}_n on its internal vertices defined as in Corollary 2.14. If we use Definition 2.25 on (T_n, \mathcal{E}_n) , we get a random signed excursion $(\widetilde{C}_{T_n}, S_n)$. Since \widetilde{C}_{T_n} converges towards a multiple of the Brownian excursion e , a natural question is the convergence of $(\widetilde{C}_{T_n}, S_n)$ towards the signed Brownian excursion. A first difficulty in proving such a result would be to find a good topology on signed excursions. Another, maybe deeper, obstacle is that the signs assigned to the local minima in $(\widetilde{C}_{T_n}, S_n)$ are far from independent: recall that on the corresponding tree, signs on internal vertices alternate. Our work shows the convergence of $(\widetilde{C}_{T_n}, S_n)$ to (e, S) in a weaker sense. Indeed, the next subsection explains how to extract a permutation of any fixed size from a signed excursion, and Theorem 1.2(iii) proves the joint convergence in distribution of the permutations extracted from $(\widetilde{C}_{T_n}, S_n)$ to those extracted from (e, S) .

2.4. *Extracting trees and permutations from excursions.* Given k points in an excursion, there is a natural way to associate a tree with this data. We now explain this construction. This is classical in the random tree literature and can, for example, be found in Le Gall’s survey [30], Section 2.5 (except that he considers geometric trees, that is, with edge-lengths, while we are only interested in combinatorial trees).

DEFINITION 2.30 (Tree). Let f be an excursion and $\mathbf{x} = \{x_1, \dots, x_k\}$ be a set of k points in $[0, 1]$. Without loss of generality, we assume $x_1 < \dots < x_k$. For all $1 \leq i \leq k - 1$, let m_i be the minimum value of f on $[x_i, x_{i+1}]$.

We associate to (f, \mathbf{x}) a Schröder tree $\text{Tree}(f, \mathbf{x})$, defined recursively as follows. If $k = 1$, then $\text{Tree}(f, \mathbf{x})$ is a leaf.

Otherwise, let $m = \min_i \{m_i\}$ and denote by $i_1 < i_2 < \dots < i_p$ all the indices i_j such that $m_{i_j} = m$. For notational convention, let $i_0 = 0$ and $i_{p+1} = k$. For all $0 \leq j \leq p$, define $\mathbf{x}^{(j)} = \{x_{i_j+1}, \dots, x_{i_{j+1}}\}$. Then $\text{Tree}(f, \mathbf{x})$ is the tree whose root has arity $p + 1$ and whose children are $\text{Tree}(f, \mathbf{x}^{(0)}), \dots, \text{Tree}(f, \mathbf{x}^{(p)})$.

OBSERVATION 2.31. If all m_i are distinct, then $\text{Tree}(f, \mathbf{x})$ is a binary tree.

EXAMPLE 2.32. An example of the construction presented in Definition 2.30 is given on Figure 8 (the reader should disregard the signs for the moment).

OBSERVATION 2.33. By construction, the leaves of $\text{Tree}(f, \mathbf{x})$ are in one-to-one correspondence with the x_i ’s. Moreover, to each m_i corresponds an internal vertex of $\text{Tree}(f, \mathbf{x})$; note however that several m_i ’s may correspond to the same vertex of $\text{Tree}(f, \mathbf{x})$ (this happens exactly for m_i and m_j having the same value such that no m_k for $i < k < j$ is smaller). In addition, the common ancestor of the leaves corresponding to x_i and x_{i+1} in $\text{Tree}(f, \mathbf{x})$ is the internal vertex that corresponds to m_i . More generally, the common ancestor of the leaves corresponding to x_i and x_j , with $i < j$, is the internal vertex that corresponds to $\min_{i \leq h < j} \{m_h\}$.

An interesting special case consists in considering the normalized contour \tilde{C}_t of a tree t , and choosing \mathbf{x} to be a subset of ℓ_t (defined in Definition 2.21).

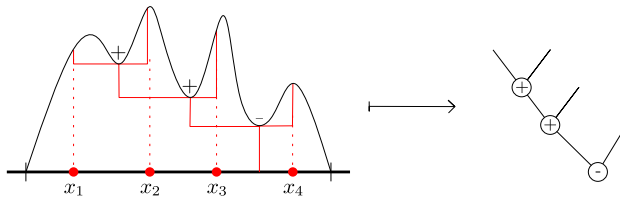


FIG. 8. *Extracting a (signed) tree from a (signed) excursion.*

OBSERVATION 2.34. Let t be a Schröder tree, and I be a subset of its set of leaves. Let \mathbf{x} be the subset of ℓ_t corresponding to the x -coordinates of these leaves. Then $\text{Tree}(\widehat{C}_t, \mathbf{x}) = t_I$, that is to say the tree extracted from \mathbf{x} in the contour of t is the same as the subtree of t induced by I .

A similar statement holds when extracting trees in excursions that are not necessarily contours; this observation (or rather its signed analogue) will be useful in Section 4.

OBSERVATION 2.35. Let f be an excursion and $\mathbf{x} = \{x_1, \dots, x_k\}$ be a set of k points in $[0, 1]$. Let $t = \text{Tree}(f, \mathbf{x})$. Let \mathbf{y} be a subset of \mathbf{x} , and I be the corresponding subset of the set of leaves of t . Then the tree extracted from \mathbf{y} in f is the subtree of t induced by I , that is, $\text{Tree}(f, \mathbf{y}) = t_I$.

We now discuss the signed analogue of the extraction of trees from excursions.

DEFINITION 2.36 (Tree_\pm). Let (f, s) be a signed excursion, and let also $\mathbf{x} = \{x_1, \dots, x_k\}$ be a set of k points in $[0, 1]$, with $x_1 < \dots < x_k$. As in Definition 2.30, for all $1 \leq i \leq k - 1$, denote by m_i the minimum value of f on $[x_i, x_{i+1}]$.

We assume that the following condition holds:

$$(C) \quad \left\{ \begin{array}{l} \bullet m_i \text{ is reached in the interior of } [x_i, x_{i+1}], \text{ that is,} \\ \text{for all } i, \text{ there exists } y \in (x_i, x_{i+1}) \text{ such that } f(y) = m_i. \\ \bullet \text{ For all } i, \text{ all the local minima with value } m_i \text{ on } (x_i, x_{i+1}) \\ \text{are given the same sign by } s, \text{ abusively denoted } s(m_i). \\ \bullet \text{ If } m_i = m_j \text{ and there is no } i < k < j \text{ such that } m_k < m_i, \\ \text{then } s(m_i) = s(m_j). \end{array} \right.$$

Then $\text{Tree}_\pm(f, s, \mathbf{x})$ is defined like $\text{Tree}(f, \mathbf{x})$, except that, at each stage of the construction, we associate with the root the sign of the corresponding local minimum (or minima) of f . Doing so, at the end of the construction, every internal vertex of $\text{Tree}_\pm(f, s, \mathbf{x})$ has a sign.

Condition (C) ensures that the signs are well defined. Indeed, the first condition guarantees that f has a minimum on (x_i, x_{i+1}) , which is therefore a local minimum of f (of value m_i), and consequently has a sign. The second condition implies that, for all i , all minima of f on (x_i, x_{i+1}) have the same sign, and the third one ensures that (with the notation of Definition 2.30) all m_i have the same sign, and recursively so.

Because a separable permutation is associated with each signed Schröder tree (see Definition 2.9 and Proposition 2.12), we can extract a separable permutation from a signed excursion in which k points are chosen.

DEFINITION 2.37. Let (f, s) be a signed excursion and $\mathbf{x} = \{x_1, \dots, x_k\}$ be a set of k points in $[0, 1]$. Assume (f, s, \mathbf{x}) satisfies Condition (\mathcal{C}) . The separable permutation associated with (f, s, \mathbf{x}) is

$$\text{Perm}(f, s, \mathbf{x}) := \text{perm}(\text{Tree}_\pm(f, s, \mathbf{x})).$$

EXAMPLE 2.38. The reader is now invited to look again at Figure 8, taking the signs into consideration. The permutation associated to the signed tree on the right is 2341, so that in this case, $\text{Perm}(f, s, \mathbf{x}) = 2341$.

OBSERVATION 2.39. When considering a signed tree (t, ε) and its signed contour (\widetilde{C}_t, s) , and choosing \mathbf{x} as a subset of ℓ_t , then Condition (\mathcal{C}) is always satisfied. Indeed, all local minima of \widetilde{C}_t that are required to have the same sign correspond to the same internal vertex of t .

Moreover, we have the signed equivalents of Observations 2.34 and 2.35.

OBSERVATION 2.40. Let (t, ε) be a signed Schröder tree and (\widetilde{C}_t, s) be its signed contour. Let I be a subset of the set of leaves of t , and \mathbf{x} be the corresponding subset of ℓ_t . Then $\text{Tree}_\pm(\widetilde{C}_t, s, \mathbf{x}) = (t_I, \varepsilon_I)$, that is, the signed tree extracted from \mathbf{x} in the signed contour of (t, ε) is defined and is the same as the signed subtree of (t, ε) induced by I .

OBSERVATION 2.41. Let (f, s) be a signed excursion and $\mathbf{x} = \{x_1, \dots, x_k\}$ be a set of k points in $[0, 1]$. Assume that $\text{Tree}_\pm(f, s, \mathbf{x})$ is defined and let us denote $(t, \varepsilon) = \text{Tree}_\pm(f, s, \mathbf{x})$. Let \mathbf{y} be a subset of \mathbf{x} , and let I be the corresponding subset of the set of leaves of (t, ε) . Then the signed tree extracted from \mathbf{y} in (f, s) is defined and is the subtree of (t, ε) induced by I , that is, $\text{Tree}_\pm(f, s, \mathbf{y}) = (t_I, \varepsilon_I)$.

EXAMPLE 2.42. For (f, s, \mathbf{x}) as in Figure 8 and $I = \{1, 3\}$, then the resulting tree is $\text{Tree}_\pm(f, s, \{x_1, x_3\}) = (t_I, \varepsilon_I) = \bigvee_{\oplus}$.

Getting to permutations, we can combine Observations 2.11 and 2.40 to obtain the following.

OBSERVATION 2.43. Let (t, ε) be a signed Schröder tree, and (f, s) be its signed contour. Consider a subset I of the set of leaves of t , and denote by \mathbf{x} the corresponding subset of ℓ_t . We have

$$\text{Perm}(f, s, \mathbf{x}) = \text{pat}_I(\text{perm}(t, \varepsilon)).$$

EXAMPLE 2.44. We consider the signed Schröder tree (t, ε) from Example 2.7 and the set $I = \{2, 4, 7\}$. We have in this case $\text{pat}_I(\text{perm}(t, \varepsilon)) = 123$. On the other hand, the construction of $\text{Perm}(f, s, \mathbf{x})$ is illustrated on Figure 9 and also yields the permutation 123.

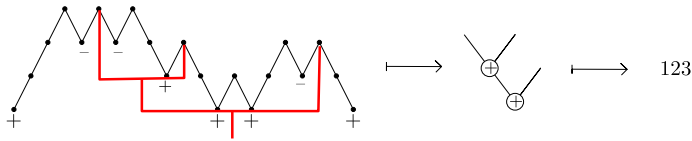


FIG. 9. Extracting a signed tree from the signed contour of a bigger tree and a subset of its leaves.

Another situation where we will extract a permutation from a signed excursion is the following.

OBSERVATION 2.45. Let (f, s) be a realization of the signed Brownian excursion (e, S) and \mathbf{x} be a set of k points taken uniformly and independently at random in $[0, 1]$. Then Condition (C) is satisfied with probability 1 for (e, S, \mathbf{x}) , and the extracted tree is binary (see Lemmas A.1 and A.2 in the Appendix).

2.5. *Measurability issues.* In this section, we construct a suitable probability space and σ -algebra for the signed Brownian excursion. A possibility would be to take a Brownian excursion e and a family of i.i.d. signs $(s_x)_{x \in [0,1]}$, independent from the excursion (and then consider only those indexed by minima of e). With this construction, the function $(f, s, \mathbf{x}) \mapsto \text{Tree}_{\pm}(f, s, \mathbf{x})$ would however not be measurable. We therefore give a slightly more subtle construction.

We fix an enumeration of the rational intervals in $[0, 1]$, that is, a bijection

$$\Psi : \mathbb{Z}_{>0} \rightarrow \{[p, q] \in \mathbb{Q}^2, 0 \leq p < q \leq 1\}.$$

Denote $\Psi(i) = [p_i, q_i]$. Let b be the position of a local minimum in an excursion f . We say that b is associated with $[p_i, q_i]$ if $b = \operatorname{argmin}_{[p_i, q_i]} f$ and if i is minimal with this property. Clearly, such an i always exists.

To construct the signed Brownian excursion (e, S) , we now take a Brownian excursion e and an i.i.d. sequence of balanced signs $(S_i)_{i \geq 1}$, independent from e . Consider a (strict) local minimum b of e and let i be such that b is associated with $[p_i, q_i]$. We then think as S_i as being the sign of b .

A given interval $[p_i, q_i]$ can be associated with at most one b (the position of the minimum of e on $[p_i, q_i]$; by Lemma A.2, almost surely, this minimum is unique), which guarantees the independence of the signs of the different local minima. Note that some interval $[p_i, q_i]$ may be associated with no local minimum b (when the position of the minimum of e on $[p_i, q_i]$ is associated with $[p_j, q_j]$, for some $j < i$), which does not create any problem.

We claim that, on this probability space, the function $(f, s, \mathbf{x}) \mapsto \text{Tree}_{\pm}(f, s, \mathbf{x})$ is measurable. The unsigned version $(f, \mathbf{x}) \mapsto \text{Tree}(f, \mathbf{x})$ is a classical object in the literature [30], Section 2.5 and it is easy to prove its measurability. The difficulty comes from the sign, and hence we focus on the case $|\mathbf{x}| = 2$, the general one

following easily. We have

$$\left\{ (f, s, x_1, x_2) : \text{Tree}_{\pm}(f, s, x_1, x_2) = \bigvee_{\oplus} \right\}$$

$$= \bigcup_{i \geq 1} \{ (f, s, x_1, x_2) : \text{argmin}_{[x_1; x_2]} f \text{ is associated with } [p_i, q_i] \text{ and } s_i = + \}.$$

It is easy to check that this is a countable union of measurable sets, hence a measurable set itself. Therefore, our measurability claim is proved.

The function Perm is then also measurable since Perm = perm ◦ Tree_±. These are the only two functions of the signed Brownian excursion of interest for this paper.

3. The variables Λ_π: Definition, properties and proof schema of the main result.

3.1. *Description of the limit variables Λ_π.* Recall from the Introduction that $\widetilde{\text{occ}}(\pi, \sigma_n)$ denotes the proportion of occurrences of π in a uniform random separable permutation of size n. Our main result [see Theorem 1.2(ii) and (iii), page 2138] is the (joint) convergence in distribution of $\widetilde{\text{occ}}(\pi, \sigma_n)$ towards a random variable Λ_π that we now define.

We denote by 1[A] the characteristic function of an event A.

DEFINITION 3.1. Let π be a pattern and (e, S) be the signed Brownian excursion. Let also k = |π|. Let X₁, X₂, . . . , X_k be k uniform and independent points in [0, 1], independent from (e, S) and set **X** = {X₁, X₂, . . . , X_k}. The random variable Λ_π is defined by

$$(1) \quad \Lambda_{\pi} = \mathbb{P}(\text{Perm}(e, S, \mathbf{X}) = \pi \mid e, S).$$

Note that this may be rephrased as $\Lambda_{\pi} = \mathbb{E}(\mathbf{1}[\text{Perm}(e, S, \mathbf{X}) = \pi] \mid e, S)$.

More precisely, we define the infinite-dimensional random vector (Λ_π)_π, indexed by all patterns π of all sizes as follows: for each π, let **X**^(π) be a set of |π| independent uniform random variables in [0, 1], taken independently for different patterns π, then we set

$$(2) \quad (\Lambda_{\pi})_{\pi} = \mathbb{E}((\mathbf{1}[\text{Perm}(e, S, \mathbf{X}^{(\pi)}) = \pi])_{\pi} \mid e, S).$$

In this definition, and throughout the paper, the event “Perm(e, S, **X**) = π” should be understood as “Perm(e, S, **X**) is defined and is equal to π”.

Since the finite-dimensional marginals of (Λ_π)_π appear in our main theorem, rather than the whole vector, let us write down explicitly the definition of these finite-dimensional laws: for any patterns π₁, . . . , π_r we have

$$(3) \quad (\Lambda_{\pi_1}, \dots, \Lambda_{\pi_r}) = \mathbb{E}((\mathbf{1}[\text{Perm}(e, S, \mathbf{X}^{(\pi_1)}) = \pi_1], \dots, \mathbf{1}[\text{Perm}(e, S, \mathbf{X}^{(\pi_r)}) = \pi_r]) \mid e, S).$$

Note that in the definition of Λ_π , we condition on the *random variable* (e, S) , so that Λ_π is a random variable itself. Moreover, in (2) and (3), all coordinates are defined by conditioning on the *same realization* (e, S) . Thus, the variables Λ_π corresponding to different patterns π are *not independent*.

The reader less familiar with probability theory might be more comfortable with the following equivalent description of Λ_π . For simplicity, we only discuss below the definition of the one-dimensional random variable Λ_π (for a fixed pattern π), and not of the full infinite-dimensional vector $(\Lambda_\pi)_\pi$.

Let us define a function Ψ_π on signed excursions as follows:

$$\Psi_\pi(f, s) = \mathbb{P}(\text{Perm}(f, s, \mathbf{X}) = \pi),$$

where \mathbf{X} is a set of $|\pi|$ uniform and independent points in $[0, 1]$. Then we set $\Lambda_\pi := \Psi_\pi(e, S)$ to be the image of the signed Brownian excursion by Ψ_π .

In other words, in (1), the probability is taken with respect to \mathbf{X} , while we consider a realization of (e, S) . In such situations, we will sometimes use a superscript on \mathbb{P} to record the source of randomness: namely, we write (1) as

$$\Lambda_\pi = \mathbb{P}^{\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}) = \pi).$$

Similarly, we use superscripts on expectation symbols \mathbb{E} to indicate the source of randomness.

OBSERVATION 3.2. With this notation, we have the obvious compatibility relation

$$\mathbb{E}^{e,S}[\mathbb{E}^{\mathbf{X}}(g(e, S, \mathbf{X}))] = \mathbb{E}^{e,S,\mathbf{X}}[g(e, S, \mathbf{X})],$$

for any function g such that these quantities are defined. If g is the indicator function of an event A , this can be rewritten as

$$\mathbb{E}^{e,S}[\mathbb{P}^{\mathbf{X}}(A)] = \mathbb{P}^{e,S,\mathbf{X}}(A).$$

To finish this section, we discuss trivial cases of our main theorem, when π is not separable and when π is the permutation of size 1.

REMARK 3.3. Observe that if π is not a separable permutation, from Proposition 2.12 it cannot be obtained as $\text{Perm}(e, S, \mathbf{X})$, and thus Λ_π is identically equal to 0 in this case. Clearly, $\widetilde{\text{occ}}(\pi, \sigma_n)$ is also identically equal to 0 in this case since a separable permutation cannot have a nonseparable pattern (permutation classes are by definition downward-closed for the pattern relation).

REMARK 3.4. If $\pi = 1$ is the only permutation of size 1, then regardless of $\mathbf{X} = (X_1)$ we have that $\text{Perm}(e, S, \mathbf{X})$ is the permutation of size 1. Consequently, Λ_π is identically equal to 1 in this case. Similarly, $\widetilde{\text{occ}}(\pi, \sigma_n)$ is also identically equal to 1, since every element of a permutation is an occurrence of the pattern $\pi = 1$.

Thus, our main theorem is vacuous in the special cases π not separable or $\pi = 1$.

3.2. *The leaf distribution function of a tree.* Before going further, we need a detour through distribution functions, to encode the positions of the leaves in the renormalized contour of a tree t .

A distribution function F is a right-continuous nondecreasing function from \mathbb{R} to $[0, 1]$ with $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow +\infty} F(x) = 1$. A real-valued random variable X has distribution function F if, for all x , one has $F(x) = P(X \leq x)$. In the sequel, we only consider distribution functions F such that $F(0) = 0$ and $F(1) = 1$ (equivalently, the associated random variables have values in $[0, 1]$); and we identify F with its restriction on the domain $[0, 1]$.

For such a distribution function F , the pseudo-inverse F^* of F is defined as follows: for u in $[0, 1]$, we set $F^*(u) = \inf\{x \in [0, 1] : F(x) \geq u\}$. One can check that for all $u \in [0, 1]$ and $\varepsilon > 0$,

$$F(F^*(u)) \geq u \geq F(F^*(u) - \varepsilon).$$

Moreover, if U is a uniform random variable on $[0, 1]$, then $F^*(U)$ has distribution function F .

The following distribution functions will be of particular interest in this work:

- For the uniform distribution on $[0, 1]$, we have $F_U(x) = \min(x, 1)\mathbf{1}[x \geq 0]$.
- With any tree t with set of leaves $\ell_t = \{\ell_1, \dots, \ell_{|t|}\}$ ($|t| > 1$), we associate its *leaf distribution function* F_t defined by

$$(4) \quad F_t(x) = \frac{1}{|t|} \sum_{i=1}^{|t|} \mathbf{1}[\ell_i \leq x].$$

By definition, taking a random variable with distribution F_t corresponds to choosing a leaf of t uniformly at random (more precisely the x -coordinate ℓ_i of the corresponding leaf in the normalized contour of t). Figure 10 shows the leaf distribution function associated with the (unsigned) tree of Figure 7, page 2148.

The following statement was essentially proved by Pitman and Rizzolo in [37].

PROPOSITION 3.5. *Let T_n be a uniform random Schröder tree with n leaves. Defining the random distribution function F_{T_n} as above, we have the following convergence in probability:*

$$\|F_{T_n} - F_U\|_\infty \rightarrow 0.$$

Informally, choosing uniformly at random a leaf in T_n (or rather the corresponding x -coordinate in the normalized contour \widetilde{C}_{T_n}) amounts in the limit to choosing a uniform point in $[0, 1]$.

PROOF OF PROPOSITION 3.5. Recall from Proposition 2.15, that T_n can be seen as a Galton–Watson tree with a specific offspring distribution, conditioned to have n leaves.

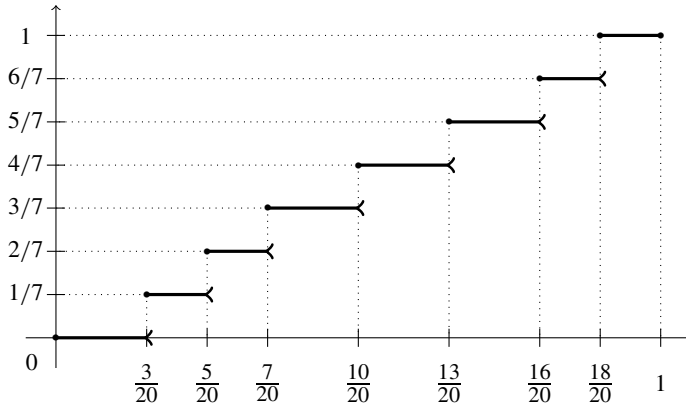


FIG. 10. The leaf distribution function F_t for the tree t shown in Figure 7. Note that F_t is piecewise constant and all gaps are of height $\frac{1}{|t|}$ (here, $\frac{1}{7}$), but pieces may have different widths (here, $\frac{2}{20}$ or $\frac{3}{20}$). Informally, Proposition 3.5 states that these widths are asymptotically close to each other.

In the proof of [37], Theorem 8, Pitman and Rizzolo established for such models the convergence of the empirical distribution of the location of leaves (that they denote ν^n) to the Lebesgue measure on $[0, 1]$. This convergence holds in probability for the weak topology.

Weak convergence is equivalent to the convergence of distribution functions at continuity points of the limit, so that their statement correspond to the convergence of F_{T_n} to F_U in probability for the pointwise convergence topology. Moreover, since distribution functions are nondecreasing and since the limit F_U is continuous, it is well known and easy to see that pointwise convergence implies uniform convergence, so that the proposition is proved. \square

We finish by a simple lemma, that is used in Sections 5 and 6.

LEMMA 3.6. If F is a distribution function, then $\|F^* - F_U^*\|_\infty \leq \|F - F_U\|_\infty$.

Note that on $[0, 1]$, $F_U^* = F_U = \text{id}_{[0,1]}$.

PROOF OF LEMMA 3.6. Choose $u \in [0, 1]$. If $u \geq F^*(u)$ then

$$\begin{aligned} |F^*(u) - F_U(u)| &= u - F^*(u) \leq F(F^*(u)) - F^*(u) \\ &\leq \sup_{t \in [0,1]} |F(t) - t| = \|F - F_U\|_\infty. \end{aligned}$$

If on the contrary $u < F^*(u)$, then write for a small $\varepsilon > 0$

$$|F^*(u) - F_U(u)| = F^*(u) - u \leq F^*(u) - \varepsilon - F(F^*(u) - \varepsilon) + \varepsilon$$

$$\begin{aligned} &\leq \sup_{t \in [-\varepsilon, 1-\varepsilon]} |t - F(t)| + \varepsilon \\ &\leq \varepsilon + \sup_{t \in [0, 1-\varepsilon]} |t - F(t)| + \varepsilon \leq \|F - F_U\|_\infty + 2\varepsilon, \end{aligned}$$

and let $\varepsilon \rightarrow 0$. \square

3.3. *Reformulation of the main theorem.* To prove the convergence of the quantity $\widetilde{\text{occ}}(\pi, \sigma_n)$ towards Λ_π , it is useful to describe these two random variables in a similar manner. More precisely, we define a function Pr^{Perm} in such a way that both $\widetilde{\text{occ}}(\pi, \sigma_n)$ and Λ_π have a natural expression using Pr^{Perm} . Along the way, we also define a tree analogue Pr^{Tree} of Pr^{Perm} , which we shall use in the proof of Theorem 1.2(iii).

DEFINITION 3.7 (Pr^{Tree}). Let t_0 be a tree with k leaves, f be an excursion and F be a distribution function with $F(0) = 0$ and $F(1) = 1$. Let X_1, \dots, X_k be independent random variables with distribution function F and \mathbf{X} be the set $\{X_1, \dots, X_k\}$. Then $\text{Pr}^{\text{Tree}}(t_0; f, F) = \mathbb{P}^{\mathbf{X}}(\text{Tree}(f, \mathbf{X}) = t_0)$.

DEFINITION 3.8 (Pr^{Perm}). Let π be a permutation of size k , (f, s) be a signed excursion and F be a distribution function with $F(0) = 0$ and $F(1) = 1$. Let X_1, \dots, X_k be independent random variables with distribution function F and \mathbf{X} be the set $\{X_1, \dots, X_k\}$. Then $\text{Pr}^{\text{Perm}}(\pi; f, s, F) = \mathbb{P}^{\mathbf{X}}(\text{Perm}(f, s, \mathbf{X}) = \pi)$.

Note that in the above two definitions, if the event $\text{Tree}(f, \mathbf{X}) = t_0$ [resp., $\text{Perm}(f, s, \mathbf{X}) = \pi$] is realized, then there is no repetition in X_1, \dots, X_k .

OBSERVATION 3.9. By definition, it follows that $\Lambda_\pi = \text{Pr}^{\text{Perm}}(\pi; e, S, F_U)$.

LEMMA 3.10. Let (t, ε) be a signed tree and $\sigma = \text{perm}(t, \varepsilon)$ be the corresponding permutation. Let (\widetilde{C}_t, s) be the signed contour of (t, ε) and F_t be the distribution function associated with t . It holds that

$$\text{Pr}^{\text{Perm}}(\pi; \widetilde{C}_t, s, F_t) = \frac{(|t|)_k}{|t|^k} \widetilde{\text{occ}}(\pi, \sigma),$$

where $(x)_k$ denotes the falling factorial $x(x - 1) \cdots (x - k + 1)$.

PROOF. Let X_1, \dots, X_k be independent random variables with distribution function F_t [in this case, $\text{Perm}(\widetilde{C}_t, s, \mathbf{X})$ is always defined—see Observation 2.39]. In other terms, we take k leaves of t independently uniformly at random. The probability that they are different is clearly $(|t|)_k / |t|^k$. Conditioned to that event, $\mathbf{X} = \{X_1, \dots, X_k\}$ is a uniform subset of k leaves of t . From Observation 2.43, we have

$$\text{Perm}(\widetilde{C}_t, s, \mathbf{X}) = \text{pat}_I(\sigma),$$

where I is a uniform random k -element subset of $[[t]]$. The probability that the right-hand side is equal to π is by definition $\widetilde{\text{occ}}(\pi, \sigma)$. Thus,

$$\mathbb{P}(\text{Perm}(\widetilde{C}_t, s, \mathbf{X}) = \pi \mid \mathbf{X} \text{ is repetition-free}) = \widetilde{\text{occ}}(\pi, \sigma),$$

which completes the proof. \square

COROLLARY 3.11. *Recall that T_n denotes a uniform random Schröder tree with n leaves, and \mathcal{E}_n the sign function on the internal vertices of T_n , such that the signs alternate and the root of T_n has a balanced sign. Denote by $(\widetilde{C}_{T_n}, S_n)$ the signed contour of (T_n, \mathcal{E}_n) . Then*

$$\widetilde{\text{occ}}(\pi, \sigma_n) = \text{Pr}^{\text{Perm}}(\pi; \widetilde{C}_{T_n}, S_n, F_{T_n}) \left(1 + \mathcal{O}\left(\frac{1}{n}\right) \right).$$

Using Observation 3.9 and Corollary 3.11, our main result, that is, the joint convergence in distribution of $\widetilde{\text{occ}}(\pi, \sigma_n)$ to Λ_π [see Theorem 1.2(iii)] can now be written as the joint convergence (for any finite family of patterns π):

$$(5) \quad \text{Pr}^{\text{Perm}}(\pi; \widetilde{C}_{T_n}, S_n, F_{T_n}) \xrightarrow{d} \text{Pr}^{\text{Perm}}(\pi; e, S, F_U),$$

3.4. Outline of the proof. The proof of (5) consists in several steps, as follows. Recall that the normalized contour \widetilde{C}_{T_n} and the leaf distribution function F_{T_n} of T_n converge to the Brownian excursion e and the uniform distribution function F_U , respectively.

A natural way to proceed would be to prove the convergence of $(\widetilde{C}_{T_n}, S_n)$ to the signed Brownian excursion, and the continuity of $\text{Pr}^{\text{Perm}}(\pi; \cdot)$. As discussed in Remark 2.29, a major difficulty when trying to prove that $(\widetilde{C}_{T_n}, S_n)$ converges to (e, S) is that the signs on the local minima of \widetilde{C}_{T_n} are far from independent. Instead of attacking this difficulty head-on, we have developed a proof along a different line. Recall that our goal is to prove the convergence in distribution of $\widetilde{\text{occ}}(\pi, \sigma_n)$ to Λ_π .

The main part of our proof is to show the convergence of $\widetilde{\text{occ}}(\pi, \sigma_n)$ to Λ_π *in expectation*. In Section 4, we prove that this is enough to have convergence in distribution. Indeed we shall see that all moments of Λ_π are determined by expectations of Λ_ρ , for larger permutations ρ . Moreover, we shall see that a similar statement holds in the limit for $\widetilde{\text{occ}}(\pi, \sigma_n)$. Therefore, convergence of expectations implies convergence of all (joint) moments. And since the random variables are bounded by 1, convergence of (joint) moments implies (multi-dimensional) convergence in distribution.

The goal of Section 5 is to prove the convergence in distribution of the *unsigned* trees extracted from the normalized contour of a uniform Schröder tree to the *unsigned* trees extracted from the Brownian excursion. This is achieved by proving

the continuity of $\Pr^{\text{Tree}}(t_0; \cdot)$, and using the convergence of \widetilde{C}_{T_n} to e and the convergence of F_{T_n} to F_U . Since this step does not involve signs, there is no major difficulty here.

In Section 6, we return to signed objects, proving that the signs on extracted trees are balanced and independent. More precisely, we examine the signs of the internal vertices of a signed tree (t_0, ε) extracted from $(\widetilde{C}_{T_n}, S_n)$, and we prove that, when n goes to infinity, these signs are balanced and independent. Note that a similar statement for (e, S) is obvious, because the signs are chosen balanced and independent in (e, S) . The proof for $(\widetilde{C}_{T_n}, S_n)$ involves the fact that the relative distances between the points of \widetilde{C}_{T_n} corresponding to the vertices of t_0 tend to infinity, and a subtree exchangeability argument.

Finally, we put all these results together in Section 7, to conclude the proof of Theorem 1.2(iii). More precisely, using Sections 5 and 6 we show the convergence of $\widetilde{\text{occ}}(\pi, \sigma_n)$ to Λ_π in expectation, which implies Theorem 1.2(iii) using Section 4.

4. Expectations determine joint moments. Recall that, for each n , σ_n denotes a uniform random separable permutation of size n , and that we want to prove the convergence in distribution of the random variables $\widetilde{\text{occ}}(\pi, \sigma_n)$ to Λ_π . Since the random variables $\widetilde{\text{occ}}(\pi, \sigma_n)$ and the candidate limit Λ_π are bounded by 1, (multi-dimensional) convergence in distribution is equivalent to convergence of (joint) moments. (This can be seen, e.g., as a consequence of the Stone–Weierstrass theorem, which ensures that polynomials are dense in the set of continuous functions from $[0, 1]^r$ to \mathbb{R} .) Concretely, Theorem 1.2(iii) (page 2138) is equivalent to the following statement.

THEOREM 4.1. *For any list of patterns π_1, \dots, π_r (possibly with repetitions),*

$$\mathbb{E} \left[\prod_{i=1}^r \widetilde{\text{occ}}(\pi_i, \sigma_n) \right] \longrightarrow \mathbb{E} \left[\prod_{i=1}^r \Lambda_{\pi_i} \right].$$

Instead of proving a convergence in distribution of random variables, we can therefore limit ourselves to proving a convergence of real numbers, which is a lot more tractable. To make our task even simpler, we show in Corollary 4.6 below that Theorem 4.1 follows if we prove, for all π , the convergence of $\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)]$ to $\mathbb{E}[\Lambda_\pi]$. The key point is that $\prod_{i=1}^r \widetilde{\text{occ}}(\pi_i, \sigma)$ can be expressed combinatorially as a linear combination of $\widetilde{\text{occ}}(\rho, \sigma)$ for larger patterns ρ , and that the same relation holds between $\prod_{i=1}^r \Lambda_{\pi_i}$ and the Λ_ρ 's.

DEFINITION 4.2. Let ρ be a permutation of size K and π_1, \dots, π_r be permutations such that $\sum_{i=1}^r |\pi_i| = K$. An ordered set-partition $(I_i)_{1 \leq i \leq r}$ of $\{1, \dots, K\}$ is *compatible* with $\rho, \pi_1, \dots, \pi_r$ if for all i , $\text{pat}_{I_i}(\rho) = \pi_i$.

The number of ordered set-partitions compatible with $\rho, \pi_1, \dots, \pi_r$ is denoted by $d_{\pi_1, \dots, \pi_r}^\rho$. We also consider the proportion

$$c_{\pi_1, \dots, \pi_r}^\rho = \frac{d_{\pi_1, \dots, \pi_r}^\rho}{\binom{K}{|\pi_1|, \dots, |\pi_r|}}$$

of ordered set-partitions of $\{1, \dots, K\}$ which are compatible with $\rho, \pi_1, \dots, \pi_r$.

EXAMPLE 4.3. Let $\rho = 1342, \pi_1 = 21$ and $\pi_2 = 12$. There are 6 ordered set-partitions of $\{1, \dots, 4\}$, which are $(\{1, 2\}, \{3, 4\}), (\{1, 3\}, \{2, 4\}), (\{1, 4\}, \{2, 3\}), (\{2, 3\}, \{1, 4\}), (\{2, 4\}, \{1, 3\}), (\{3, 4\}, \{1, 2\})$. Let (I_1, I_2) be one of the first four ordered pairs, then $\text{pat}_{I_1}(\rho) = 12 \neq \pi_1$ thus (I_1, I_2) is not compatible with ρ, π_1, π_2 . On the contrary, the last two pairs are compatible with ρ, π_1, π_2 . Then $d_{21, 12}^{1342} = 2$ and $c_{21, 12}^{1342} = 1/3$.

PROPOSITION 4.4. Fix a list of patterns π_1, \dots, π_r , and denote by K the sum of their sizes. Then

$$\prod_{i=1}^r \Lambda_{\pi_i} = \sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \Lambda_\rho.$$

PROOF. We fix a realization (e, S) of the Brownian excursion all along the proof. The main idea to prove the equality in Proposition 4.4 is to show that both parts represent the probability of the same event. First, we describe the event.

We fix a list of patterns π_1, \dots, π_r and we denote by K the sum of their sizes. Let us take K independent uniform random variables in $[0, 1]$. We denote the $|\pi_1|$ first ones (in the order of sampling) by $X_1^1, \dots, X_{|\pi_1|}^1$, the following ones $X_1^2, \dots, X_{|\pi_2|}^2$, and so on until the $|\pi_r|$ last ones which are denoted $X_1^r, \dots, X_{|\pi_r|}^r$. Let $\mathbf{X}^{(i)} = \{X_j^i \mid 1 \leq j \leq |\pi_i|\}$ for all i and $\mathbf{X} = \bigcup_{i=1}^r \mathbf{X}^{(i)}$. The event we are considering is then: for each i , $\text{Perm}(e, S, \mathbf{X}^{(i)}) = \pi_i$. Since (e, S) is a fixed realization of the Brownian excursion, all the probabilities below should be understood with respect to the random variables \mathbf{X} , as indicated by the notation $\mathbb{P}^{\mathbf{X}}$.

Let $\mathcal{P}^{\pi_1, \dots, \pi_r} = \mathbb{P}^{\mathbf{X}}(\forall i, \text{Perm}(e, S, \mathbf{X}^{(i)}) = \pi_i)$. Since the X_j^i are independent, we have

$$\mathcal{P}^{\pi_1, \dots, \pi_r} = \prod_{i=1}^r \mathbb{P}^{\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}^{(i)}) = \pi_i) = \prod_{i=1}^r \Lambda_{\pi_i}.$$

It remains to prove that $\mathcal{P}^{\pi_1, \dots, \pi_r} = \sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \Lambda_\rho$. We compute this probability by conditioning on the value of $\text{Perm}(e, S, \mathbf{X})$. Since (e, S, \mathbf{X}) satisfies Condition (C) (page 2152) with probability 1 (see Observation 2.45, page 2154), the permutation $\text{Perm}(e, S, \mathbf{X})$ is almost surely defined and has size K . Thus,

$$(6) \quad \mathcal{P}^{\pi_1, \dots, \pi_r} = \sum_{\rho \in \mathfrak{S}_K} \mathcal{P}_\rho \times \mathbb{P}^{\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}) = \rho)$$

where $\mathcal{P}_\rho = \mathbb{P}^{\mathbf{X}}(\forall i, \text{Perm}(e, S, \mathbf{X}^{(i)}) = \pi_i \mid \text{Perm}(e, S, \mathbf{X}) = \rho)$.

By definition, $\mathbb{P}^{\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}) = \rho) = \Lambda_\rho$, so we just need to prove that $\mathcal{P}_\rho = c_{\pi_1, \dots, \pi_r}^\rho$ to complete the proof.

Consider a realization of the random variables \mathbf{X} . We say that X_j^i has rank k if it is the k th smallest value among all variables of \mathbf{X} . Then we can associate with \mathbf{X} an ordered set-partition of $\{1, \dots, K\}$ that we denote abusively $\text{Part}(\mathbf{X})$ and that is defined as follows: the i th subset of $\text{Part}(\mathbf{X})$ is obtained from $\mathbf{X}^{(i)}$ by replacing each X_j^i by its rank. For example,

$$\text{Part}(\{0.7, 0.9, 0.2\}, \{0.5, 0.8\}, \{0.3\}) = (\{1, 4, 6\}, \{3, 5\}, \{2\})$$

since $0.2 < 0.3 < 0.5 < 0.7 < 0.8 < 0.9$.

Let \mathcal{I} be the set of all ordered set-partitions of $\{1, \dots, K\}$ in r subsets such that the i th one has $|\pi_i|$ elements. Then, by conditioning on the value of $\text{Part}(\mathbf{X})$, we have

$$\begin{aligned} \mathcal{P}_\rho &= \sum_{(I_i) \in \mathcal{I}} \mathbb{P}^{\mathbf{X}}(\text{Part}(\mathbf{X}) = (I_i)) \\ &\quad \times \mathbb{P}^{\mathbf{X}}(\forall i, \text{Perm}(e, S, \mathbf{X}^{(i)}) = \pi_i \mid \text{Perm}(e, S, \mathbf{X}) = \rho \text{ and } \text{Part}(\mathbf{X}) = (I_i)). \end{aligned}$$

Recall that, by construction, $\text{Perm} = \text{perm} \circ \text{Tree}_\pm$. From Observations 2.41 (page 2153) and 2.11 (page 2146),

$$\begin{aligned} &\mathbb{P}^{\mathbf{X}}(\forall i, \text{Perm}(e, S, \mathbf{X}^{(i)}) = \pi_i \mid \text{Perm}(e, S, \mathbf{X}) = \rho \text{ and } \text{Part}(\mathbf{X}) = (I_i)) \\ &= \begin{cases} 1 & \text{if } (I_i) \text{ is compatible with } \rho, \pi_1, \dots, \pi_r; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In addition, we claim that $\text{Part}(\mathbf{X})$ is uniformly distributed in \mathcal{I} , that is, for each (I_i) in \mathcal{I} , we have $\mathbb{P}^{\mathbf{X}}(\text{Part}(\mathbf{X}) = (I_i)) = 1/\binom{K}{|\pi_1|, \dots, |\pi_r|}$. Indeed each possible relative order of the (X_j^i) occurs with the same probability. Moreover, the block sizes of ordered set-partitions in \mathcal{I} are prescribed: $\text{card}(I_i) = |\pi_i|$. Therefore, each ordered set-partition in \mathcal{I} corresponds to the same number of relative orders of the variables (X_j^i) ; and this proves our claim.

Finally, we obtain that $\mathcal{P}_\rho = d_{\pi_1, \dots, \pi_r}^\rho / \binom{K}{|\pi_1|, \dots, |\pi_r|} = c_{\pi_1, \dots, \pi_r}^\rho$. Going back to (6), we get $\mathcal{P}^{\pi_1, \dots, \pi_r} = \sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \Lambda_\rho$, which concludes the proof. \square

PROPOSITION 4.5. *Fix a list of patterns π_1, \dots, π_r , and denote by K the sum of their sizes. Then for any permutation σ of size n ,*

$$\prod_{i=1}^r \widetilde{\text{occ}}(\pi_i, \sigma) = \sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \widetilde{\text{occ}}(\rho, \sigma) + \mathcal{O}\left(\frac{1}{n}\right).$$

The constant hidden in the \mathcal{O} symbol does depend on π_1, \dots, π_r , but is uniform on σ .

PROOF. Let k_i be the size of π_i . We denote by S_i the set of k_i -element subsets, I , of $[n]$ such that $\text{pat}_I(\sigma) = \pi_i$. Then

$$\prod_{i=1}^r \widetilde{\text{occ}}(\pi_i, \sigma) = \prod_{i=1}^r \frac{\text{card}(S_i)}{\binom{n}{k_i}} = \frac{\prod_{i=1}^r \text{card}(S_i)}{\prod_{i=1}^r \binom{n}{k_i}}.$$

We set $S_\rho = \{(s_1, \dots, s_r) \in S_1 \times S_2 \times \dots \times S_r \text{ such that } \text{pat}_I(\sigma) = \rho \text{ for } I = \bigcup_{i=1}^r s_i\}$. Then

$$\begin{aligned} \prod_{i=1}^r \text{card}(S_i) &= \text{card}(S_1 \times S_2 \times \dots \times S_r) \\ &= \text{card}\left(\bigcup_{\rho \in \mathfrak{S}_K} S_\rho\right) + \text{card}\left(S_1 \times S_2 \times \dots \times S_r \setminus \bigcup_{\rho \in \mathfrak{S}_K} S_\rho\right). \end{aligned}$$

Note that the sets S_ρ are disjoint: indeed $S_\rho \cap S_{\rho'} \neq \emptyset \Rightarrow \rho = \text{pat}_I(\sigma) = \rho'$. Therefore, denoting $R = S_1 \times S_2 \times \dots \times S_r \setminus \bigcup_{\rho \in \mathfrak{S}_K} S_\rho$, we have

$$\prod_{i=1}^r \text{card}(S_i) = \sum_{\rho \in \mathfrak{S}_K} \text{card}(S_\rho) + \text{card}(R).$$

We first study R . Let $(s_1, \dots, s_r) \in S_1 \times S_2 \times \dots \times S_r$, and define $I = \bigcup_{i=1}^r s_i$. Note that $\text{card}(I) \leq K$, and that the inequality is strict if and only if $(s_1, \dots, s_r) \in R$. Consequently, it holds that

$$R = \left\{ (s_1, \dots, s_r) \in S_1 \times S_2 \times \dots \times S_r \text{ such that } \text{card}\left(\bigcup_{i=1}^r s_i\right) < K \right\}.$$

It follows that $\text{card}(R) = \mathcal{O}(n^{K-1})$, where the constant hidden in the \mathcal{O} symbol depends on the k_i 's but not on n . Since $\prod_{i=1}^r \binom{n}{k_i} \sim \text{cst} n^K$ for some constant cst that depends on the k_i 's only, we obtain that $\frac{\text{card}(R)}{\prod_{i=1}^r \binom{n}{k_i}} = \mathcal{O}\left(\frac{1}{n}\right)$, where the constant hidden in the \mathcal{O} symbol depends on the k_i 's only.

We now study S_ρ . Each element of S_ρ corresponds to an occurrence of ρ in σ . Conversely, a given occurrence of ρ in σ may lead to one, several or no element(s) of S_ρ : this depends on the number of ways to partition the set of indices of σ corresponding to the occurrence of ρ into an ordered sequence (s_1, \dots, s_r) of r sets such that each s_i induces the pattern π_i . The number of ways to do this ordered partition does not depend on the occurrence of ρ that is considered, and there are by definition $d_{\pi_1, \dots, \pi_r}^\rho$ ways to do this partition. Thus,

$$\begin{aligned} \text{card}(S_\rho) &= d_{\pi_1, \dots, \pi_r}^\rho \times \text{number of occurrences of } \rho \text{ in } \sigma \\ &= \binom{K}{k_1, \dots, k_r} c_{\pi_1, \dots, \pi_r}^\rho \times \binom{n}{K} \widetilde{\text{occ}}(\rho, \sigma). \end{aligned}$$

Putting things together,

$$\begin{aligned} \prod_{i=1}^r \widetilde{\text{occ}}(\pi_i, \sigma) &= \sum_{\rho \in \mathfrak{S}_K} \frac{\text{card}(S_\rho)}{\prod_{i=1}^r \binom{n}{k_i}} + \frac{\text{card}(R)}{\prod_{i=1}^r \binom{n}{k_i}} \\ &= \sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \widetilde{\text{occ}}(\rho, \sigma) \frac{\binom{K}{k_1, \dots, k_r} \binom{n}{K}}{\prod_{i=1}^r \binom{n}{k_i}} + \mathcal{O}\left(\frac{1}{n}\right). \end{aligned}$$

By simple computations, we have

$$\frac{\binom{K}{k_1, \dots, k_r} \binom{n}{K}}{\prod_{i=1}^r \binom{n}{k_i}} = \frac{n!}{K!(n-K)!} \frac{K!}{k_1! \cdots k_r!} \frac{k_1!(n-k_1)!}{n!} \cdots \frac{k_r!(n-k_r)!}{n!} = 1 + \mathcal{O}\left(\frac{1}{n}\right),$$

where again the constant hidden in the \mathcal{O} symbol depends on the k_i 's only. To conclude, we need to remark that the sum $\sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \widetilde{\text{occ}}(\rho, \sigma)$ is bounded independently of n ; indeed, each term $c_{\pi_1, \dots, \pi_r}^\rho \widetilde{\text{occ}}(\rho, \sigma)$ is bounded by 1, and there are $K!$ terms. It then follows that

$$\prod_{i=1}^r \widetilde{\text{occ}}(\pi_i, \sigma) = \sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \widetilde{\text{occ}}(\rho, \sigma) + \mathcal{O}\left(\frac{1}{n}\right),$$

where the constant hidden in the \mathcal{O} symbol does depend on π_1, \dots, π_r , but is uniform on σ . \square

COROLLARY 4.6. *Theorem 4.1 [and therefore Theorem 1.2(iii)] is equivalent to the following statement:*

$$(7) \quad \text{For any pattern } \pi, \quad \mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)] \longrightarrow \mathbb{E}[\Lambda_\pi].$$

PROOF. We assume that equation (7) holds. Let π_1, \dots, π_r be a list of patterns, and denote by K the sum of their sizes. Then, from Propositions 4.4 and 4.5, we have

$$\begin{aligned} \mathbb{E}\left[\prod_{i=1}^r \widetilde{\text{occ}}(\pi_i, \sigma_n)\right] &= \sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \mathbb{E}[\widetilde{\text{occ}}(\rho, \sigma_n)] + o(1) \\ &= \sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \mathbb{E}[\Lambda_\rho] + o(1) = \mathbb{E}\left[\prod_{i=1}^r \Lambda_{\pi_i}\right] + o(1). \end{aligned}$$

Note that to get the second line, we used that $\sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho$ does not depend on n . The above computations show that equation (7) implies Theorem 4.1. \square

The next three sections focus on proving equation (7); the proof will be completed in Theorem 7.3.

5. Continuity of Pr^{Tree} . In this section, we prove the convergence in distribution of the trees extracted from the normalized contour of a uniform Schröder tree to the trees extracted from the Brownian excursion (in the unsigned case). This result is stated in Corollary 5.4. It follows easily from earlier results and a continuity property of Pr^{Tree} (defined in Definition 3.7, page 2159) that we establish in Lemma 5.2 below. Let us first set up some notation.

Fix a tree t_0 with k leaves. We consider the map $(f, F) \mapsto \text{Pr}^{\text{Tree}}(t_0; f, F)$. We use the uniform topology (i.e., the topology induced by the supremum norm) both on the set of excursions and on the set of distribution functions.

Throughout the section, if f is an excursion and $x \leq y$ in $[0, 1]$, we denote $m(f; x, y) = \min_{[x,y]} f$. Let $X_1 \leq X_2 \cdots \leq X_k$ be the reordering of k uniform i.i.d. random variables in $[0, 1]$. We say that an excursion f has the *distinct minima property* if $m(f, X_1, X_2), \dots, m(f, X_{k-1}, X_k)$ are distinct with probability 1 (the probability here is taken with respect to X_1, \dots, X_k).

LEMMA 5.1. *Let e be the Brownian excursion, then a.s. e has the distinct minima property.*

PROOF. This can be seen as a consequence of Lemma A.4, but since this lemma uses a deep result of [30], let us give a more elementary proof.

If e is the Brownian excursion, and $X_1 < \cdots < X_k$ is the reordering of k i.i.d. uniform random variables in $[0, 1]$, then from Lemma A.1 with probability 1 no X_i is a one-sided minimum. When this is the case, for each i , $m(f, X_i, X_{i+1})$ cannot be reached on the extremities of the interval $[X_i, X_{i+1}]$ and is therefore a local minimum. Consequently, using Lemma A.2, the values $m(f, X_1, X_2), \dots, m(f, X_{k-1}, X_k)$ are distinct, almost surely. \square

LEMMA 5.2. *Let t_0 be a fixed tree with k leaves. If f is an excursion with the distinct minima property, then $\text{Pr}^{\text{Tree}}(t_0; \dots)$ is continuous in (f, F_U) with respect to the uniform topology.*

PROOF. We prove that $\text{Pr}^{\text{Tree}}(t_0; g, G) \rightarrow \text{Pr}^{\text{Tree}}(t_0; f, F_U)$ as soon as $g \rightarrow f$ and $G \rightarrow F_U$. Fix $\varepsilon, \delta > 0$ and consider an excursion g and a distribution function G with $\|f - g\|_\infty \leq \varepsilon$ and $\|F_U - G\|_\infty \leq \delta$.

Recall from Lemma 3.6 (page 2158) that $\|F_U^* - G^*\|_\infty \leq \|F_U - G\|_\infty \leq \delta$ and that F_U^* is the identity on $[0, 1]$. Let $X_1 < \cdots < X_k$ be the reordering of k independent uniform random variables, and let $\mathbf{x} = (X_1, \dots, X_k)$. Furthermore, set $Y_i = G^*(X_i)$. Note that G^* is nondecreasing, so that $Y_1 \leq \cdots \leq Y_k$. We write $\mathbf{y} = (Y_1, \dots, Y_k)$. By construction, we have that:

- \mathbf{y} has the distribution of the reordering of k i.i.d. random variables of distribution G ;
- and, for each i , one has $|Y_i - X_i| \leq \delta$.

It follows from $\|f - g\|_\infty \leq \varepsilon$ that $|m(g, Y_i, Y_{i+1}) - m(f, Y_i, Y_{i+1})| \leq \varepsilon$ for $i \leq k - 1$. This implies that, for each $i \leq k - 1$,

$$(8) \quad |m(g, Y_i, Y_{i+1}) - m(f, X_i, X_{i+1})| \leq \varepsilon + \omega(f, \delta),$$

where ω is the modulus of continuity of f defined by

$$(9) \quad \omega(f, \delta) = \sup_{|r-s| \leq \delta} |f(r) - f(s)|.$$

Recall that $\text{Tree}(f, \mathbf{x})$ is the tree extracted from the set of points \mathbf{x} on the excursion f . We have

$$(10) \quad \begin{aligned} & |\text{Pr}^{\text{Tree}}(t_0; f, F_U) - \text{Pr}^{\text{Tree}}(t_0; g, G)| \\ &= |\mathbb{P}^{\mathbf{x}}(\text{Tree}(f, \mathbf{x}) = t_0) - \mathbb{P}^{\mathbf{x}}(\text{Tree}(g, \mathbf{y}) = t_0)| \\ &\leq \mathbb{P}^{\mathbf{x}}(\text{Tree}(f, \mathbf{x}) \neq \text{Tree}(g, \mathbf{y})). \end{aligned}$$

Above and in what follows, the randomness is only given by \mathbf{x} , the variables \mathbf{y} are functions of the \mathbf{x} 's, and hence random as well, while f is nonrandom. We emphasize this by using the notation $\mathbb{P}^{\mathbf{x}}$.

By construction (see Definition 2.30, page 2151), the tree $\text{Tree}(f, \mathbf{x})$ only depends on the relative order of $m(f, X_1, X_2), \dots, m(f, X_{k-1}, X_k)$. We denote $\text{gap}(f, \mathbf{x})$ the minimal difference between any two of these values. Since f is assumed to have the distinct minima property, $\text{gap}(f, \mathbf{x})$ is nonzero with probability 1.

From (8), we see that the numbers $m(g, Y_1, Y_2), \dots, m(g, Y_{k-1}, Y_k)$ are in the same relative order as $m(f, X_1, X_2), \dots, m(f, X_{k-1}, X_k)$ as soon as we have $\varepsilon + \omega(f, \delta) < \text{gap}(f, \mathbf{x})/2$. If this is the case, then $\text{Tree}(f, \mathbf{x}) = \text{Tree}(g, \mathbf{y})$. Therefore,

$$\begin{aligned} \mathbb{P}^{\mathbf{x}}(\text{Tree}(f, \mathbf{x}) \neq \text{Tree}(g, \mathbf{y})) &\leq \mathbb{P}^{\mathbf{x}}(\text{gap}(f, \mathbf{x})/2 \leq \varepsilon + \omega(f, \delta)) \\ &\xrightarrow{\varepsilon, \delta \rightarrow 0} \mathbb{P}^{\mathbf{x}}(\text{gap}(f, \mathbf{x}) = 0) = 0. \end{aligned}$$

With (10), this completes the proof. \square

By definition, the (substitution) decomposition tree of σ is the unique signed Schröder tree (t, ε) in which the signs alternate such that $\text{perm}(t, \varepsilon) = \sigma$. By Corollary 2.14 (page 2147), in the case where σ_n is a uniform random separable permutation of size n , then its (unsigned) decomposition tree is a uniform Schröder tree with n leaves, denoted T_n all along the article.

PROPOSITION 5.3. *Let σ_n be a uniform random separable permutation of size n and (\widetilde{C}_{T_n}) be the normalized contour of its decomposition tree. We also define F_{T_n} as usual [see (4), page 2157]. Let e be the Brownian excursion and define $\lambda = \sqrt{2 + 3/\sqrt{2}}$. Then we have*

$$(\widetilde{C}_{T_n}, F_{T_n}) \xrightarrow{d} (\lambda \cdot e, F_U)$$

in distribution, with respect to the uniform topology.

PROOF. From Propositions 2.23 (page 2149) and 3.5 (page 2157), we know that $\widetilde{C}_{T_n} \xrightarrow{d} \lambda \cdot e$ and $F_{T_n} \xrightarrow{d} F_U$. It remains to prove the joint convergence. Note that the limit F_U of F_{T_n} is deterministic.

Thus, we want to use a theorem of Billingsley ([8], Theorem 3.9) that asserts that if $X'_n \xrightarrow{d} X'$ and $X''_n \xrightarrow{d} a''$ (where X'_n and X' are random variables with values in a metric space S' , X''_n are random variables with values in a metric space S'' and a'' is a deterministic element of S''), then $(X'_n, X''_n) \xrightarrow{d} (x', a'')$, provided that $T = S' \times S''$ is separable.

The hypothesis that T is separable is in fact only needed to ensure that the Borel σ -algebra on T is the product of the Borel σ -algebras on S' and S'' . For this to be the case, it is sufficient that either S' or S'' is separable; see [9], Lemma 6.4.2.

In our case, the functions \widetilde{C}_{T_n} and the limit $\lambda \cdot e$ are random elements in the space $S' = \mathcal{C}[0, 1]$ of continuous functions on $[0, 1]$, which is known to be separable (see [8], Example 1.3). We can therefore use Billingsley's theorem and the proposition is proved. \square

COROLLARY 5.4. *With the above notation, for any fixed t_0 , we have the convergence in distribution:*

$$\Pr^{\text{Tree}}(t_0; \widetilde{C}_{T_n}, F_{T_n}) \xrightarrow{d} \Pr^{\text{Tree}}(t_0; e, F_U).$$

In particular, since these random variables are bounded, we have

$$\mathbb{E}^{\sigma_n}[\Pr^{\text{Tree}}(t_0; \widetilde{C}_{T_n}, F_{T_n})] \rightarrow \mathbb{E}^e[\Pr^{\text{Tree}}(t_0; e, F_U)].$$

PROOF. This corollary is a simple application of the mapping theorem [8], Theorem 2.7 because of the following facts:

- $(\widetilde{C}_{T_n}, F_{T_n}) \xrightarrow{d} (\lambda \cdot e, F_U)$, which is proved in Proposition 5.3;
- $\Pr^{\text{Tree}}(t_0; \dots)$ is invariant when the first argument is multiplied by a constant and is continuous at (e, F_U) for almost every e , which follows from Lemmas 5.1 and 5.2. \square

6. Signs are asymptotically balanced and independent. We now return to the study of signed trees. The purpose of this section is to justify that asymptotically (when n goes large) the $k - 1$ signs in the tree $\text{Tree}_{\pm}(\widetilde{C}_{T_n}, S_n, \ell)$ are independent and balanced, when ℓ is a k -element subset of the set of leaves of T_n chosen uniformly at random.

PROPOSITION 6.1. *Let σ_n be a uniform random separable permutation of size n and (t_0, ε) be a signed binary tree of size k . As usual, let $(\widetilde{C}_{T_n}, S_n)$ be the signed contour of the decomposition tree of σ_n . We also consider a uniform random k -element subset ℓ of leaves of T_n . Then*

$$(11) \quad \mathbb{P}^{\sigma_n, \ell}(\text{Tree}_{\pm}(\widetilde{C}_{T_n}, S_n, \ell) = (t_0, \varepsilon)) = \frac{1}{2^{k-1}} \mathbb{P}^{\sigma_n, \ell}(\text{Tree}(\widetilde{C}_{T_n}, \ell) = t_0) + o(1).$$

The proof is given at the end of the present section. The core of this proof is Lemma 6.2 below, regarding heights of branching points of marked leaves in uniform random Schröder trees. We first introduce notation.

Recall that T_n is a uniform Schröder tree of size n . We take a set of k leaves of T_n uniformly at random. As in Definition 2.21 (page 2149), denote their x -coordinates by $\ell_1 < \dots < \ell_k$, and $\boldsymbol{\ell} = (\ell_1, \dots, \ell_k)$ and let H_i be the height of ℓ_i in T_n . For $i = 1, \dots, k - 1$, denote by M_i the height of the common ancestor of the leaves ℓ_i and ℓ_{i+1} (defining the height as the distance from the root).

LEMMA 6.2. *When $n \rightarrow +\infty$, the parities of M_1, \dots, M_{k-1} are asymptotically balanced, independent from each other and from the subtree t_0 induced by the k leaves.*

More formally, we fix a binary tree t_0 with k leaves and we condition on the event that the subtree of T_n induced by $\boldsymbol{\ell}$ is t_0 . For all $\delta_1, \dots, \delta_{k-1} \in \{0, 1\}$,

$$\mathbb{P}(M_1 \equiv \delta_1, \dots, M_{k-1} \equiv \delta_{k-1} \mid \text{Tree}(\widetilde{C}_{T_n}, \boldsymbol{\ell}) = t_0) \xrightarrow{n \rightarrow +\infty} \frac{1}{2^{k-1}},$$

where $a \equiv b$ means $a = b \pmod 2$.

PROOF. The proof essentially relies on a subtree exchangeability argument. In order to give intuition, we start by the case $k = 2$.

In this case, there is a unique binary tree with two leaves, so the conditioning is not relevant. We want to prove that the common ancestor of two leaves ℓ_1 and ℓ_2 , chosen uniformly at random in T_n among all 2-element subsets of leaves, has an even height with probability asymptotically $1/2$.

Fix two integers $n, h \geq 1$. We consider the set \mathcal{T}_n^h of Schröder trees t with n leaves, with two marked leaves ℓ_1 and ℓ_2 ($\ell_1 < \ell_2$), such that ℓ_1 has height h . Such a tree t can be canonically described as follows:

- Take a chain $v_0, \dots, v_h = \ell_1$ going from the root of t to the first marked leaf.
- Then for each $x < h$, glue a tree T_x^l and a tree T_x^r , respectively, on the left and on the right of v_x .
- One of the trees T_x^r , say $T_{x_0}^r$, contains the second marked leaf ℓ_2 . Then the vertex v_{x_0} is the common ancestor of ℓ_1 and ℓ_2 and has height $M_1 = x_0$.

For every permutation α of $\{0, 1, \dots, h - 1\}$, replacing each pair (T_x^l, T_x^r) by $(T_{\alpha(x)}^l, T_{\alpha(x)}^r)$ in the above decomposition provides a bijection from \mathcal{T}_n^h to itself. This bijection maps a tree with $M_1 = x_0$ onto a tree with $M_1 = \alpha(x_0)$.

Applying this construction to our random tree T_n with its two marked leaves, we get that for each h, x with $0 \leq x \leq h - 1$

$$\mathbb{P}(M_1 = x \mid H_1 = h) = \frac{1}{h}.$$

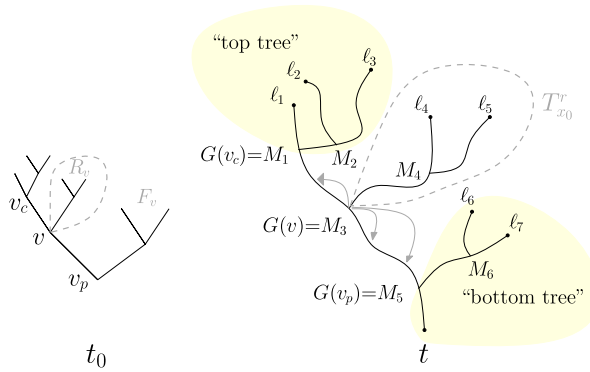


FIG. 11. The subtree exchangeability argument.

Therefore, we obtain

$$\begin{aligned} \mathbb{P}(M_1 \text{ is odd}) &= \sum_{h \geq 1} \mathbb{P}(M_1 \text{ is odd} \mid H_1 = h) \mathbb{P}(H_1 = h) \\ &= \sum_{h \geq 1} \frac{\lfloor h/2 \rfloor}{h} \mathbb{P}(H_1 = h) = \frac{1}{2} - \mathcal{O}\left(\mathbb{E}\left(\frac{1}{H_1}\right)\right), \end{aligned}$$

which goes to $1/2$, as soon as H_1 goes to infinity in probability. This will be proved in Lemma 6.3 below.

We now prove Lemma 6.2 in the general case $k \geq 2$. Since the proof uses a lot of notation, the reader is invited to look regularly at Figure 11, which illustrates the main definitions.

We fix $k \geq 2$ and a binary tree t_0 with k leaves. We denote by $\mathcal{T}_n(t_0)$ the set of Schröder trees of size n with k marked leaves $\ell_1 < \dots < \ell_k$, which induce the subtree t_0 . For such a tree t , each vertex v in t_0 corresponds to a vertex $G(v)$ in t : $G(v)$ is either one of the marked leaves of t (if v is a leaf of t_0) or a common ancestor of two marked leaves of t (if v is an internal vertex of t_0). We denote by $h_t(v)$ the height of $G(v)$ in t .

We fix an internal vertex v of t_0 . Define v_c as the left child of v in t_0 and v_p as the parent of v . For $t \in \mathcal{T}_n(t_0)$, the distance between $G(v_c)$ and $G(v_p)$ is $d_t(v) := h_t(v_c) - h_t(v_p)$. [If v is the root, then v_p is not defined and we shall think of $G(v_p)$ as a virtual new root of t that is an ancestor of every vertex in t and has height -1 ; in particular, we take $h_t(v_p) = -1$.]

The vertices of t_0 (both leaves and internal vertices, v excluded) are partitioned in two sets: the right subtree R_v of v and its complement set F_v . For a sequence of heights $\mathbf{h} = (h_u)_{u \in F_v}$, we define

$$\mathcal{T}_n^{v, \mathbf{h}}(t_0) = \{t \in \mathcal{T}_n(t_0), \text{ for all } u \in F_v, h_t(u) = h_u\}.$$

Note that, for every t in this set, $d_t(v) = h_{v_c} - h_{v_p}$; we denote this common value by $d_{\mathbf{h}, v}$. (If v is the root of t_0 , we set $h_{v_p} = -1$.)

Similar to the decomposition in the case $k = 2$, each tree in $\mathcal{T}_n^{v, \mathbf{h}}(t_0)$ can be decomposed as follows:

- A chain $w_0, \dots, w_{d_{\mathbf{h},v}}$ going from $G(v_p)$ to $G(v_c)$.
- For each $0 < x < d_{\mathbf{h},v}$, a tree T_x^l and a tree T_x^r , respectively, on the left and on the right of w_x .
- One of the trees T_x^r , say $T_{x_0}^r$, contains $G(R_v)$. Then the vertex w_{x_0} is $G(v)$ and has height $h_{v_p} + x_0$.
- A “top tree” which consists of the offspring of $G(v_c)$.
- All others vertices of t form a connected subtree called the “bottom tree”. (If v is the root of t_0 , there is no bottom tree.)

For every permutation α of $\{1, \dots, d_{\mathbf{h},v} - 1\}$, replacing each pair (T_x^l, T_x^r) by $(T_{\alpha(x)}^l, T_{\alpha(x)}^r)$ in the above decomposition provides a bijection from $\mathcal{T}_n^{v, \mathbf{h}}(t_0)$ to itself. Clearly, this bijection maps a tree with $h_t(v) = h_{v_p} + x_0$ onto a tree with $h_t(v) = h_{v_p} + \alpha(x_0)$.

We now work with a uniform Schröder tree T_n of size n with k marked leaves, and we condition on the fact that these leaves induce the subtree t_0 . We use the notation $\mathbb{P}_{t_0}(T_n \in A) := \mathbb{P}(T_n \in A \mid T_n \in \mathcal{T}_n(t_0))$. The above discussion implies that for every $h_{v_p} < x < h_{v_c}$ we have

$$\mathbb{P}_{t_0}(h_{T_n}(v) = x \mid T_n \in \mathcal{T}_n^{v, \mathbf{h}}(t_0)) = \frac{1}{h_{v_c} - h_{v_p} - 1} = \frac{1}{d_{\mathbf{h},v} - 1} = \frac{1}{d_{T_n}(v) - 1}.$$

Therefore,

$$\left| \mathbb{P}_{t_0}(h_{T_n}(v) \text{ is even} \mid T_n \in \mathcal{T}_n^{v, \mathbf{h}}(t_0)) - \frac{1}{2} \right| \leq \frac{1}{d_{T_n}(v) - 1}.$$

We introduce a subset E_n^v of $\mathcal{T}_n(t_0)$ of *well-behaved* trees:

$$E_n^v := \{t \in \mathcal{T}_n(t_0), d_t(v) \geq n^{1/4}\}.$$

We will prove in Lemma 6.3 that for all v , $\mathbb{P}_{t_0}(E_n^v) \rightarrow 1$.

Note that if \mathbf{h} is such that $\mathcal{T}_n^{v, \mathbf{h}}(t_0) \subset E_n^v$, then

$$\left| \mathbb{P}_{t_0}(h_{T_n}(v) \text{ is even} \mid T_n \in \mathcal{T}_n^{v, \mathbf{h}}(t_0)) - \frac{1}{2} \right| \leq \frac{1}{n^{1/4} - 1}.$$

As a consequence, we have

$$(12) \quad \mathbb{P}_{t_0}(h_{T_n}(v) \text{ is even} \mid T_n \in \mathcal{T}_n^{v, \mathbf{h}}(t_0)) = \frac{1}{2} + o(1),$$

where the error term $o(1)$ is bounded independently of \mathbf{h} , provided that we have $\mathcal{T}_n^{v, \mathbf{h}}(t_0) \subset E_n^v$.

Internal vertices of t_0 can be canonically indexed by $\{1, \dots, k - 1\}$: v_i is the common ancestor of ℓ_i, ℓ_{i+1} . By definition, $M_i = h_t(v_i)$ and we denote $E_n^i := E_n^{v_i}$.

Fix $i \geq 1$. For $j < i$, the vertex v_j lies in F_{v_i} , so that the fact that T_n belongs to $\mathcal{T}_n^{v_i, \mathbf{h}}(t_0)$ determines the random variables $(M_j)_{j < i}$. Clearly, this also determines whether T_n belongs to E_n^i or not. Consequently, for all $i \leq k - 1$, the event

$$\{M_1 \equiv \delta_1, \dots, M_{i-1} \equiv \delta_{i-1}, T_n \in E_n^i\}$$

can be written as a disjoint union (on \mathbf{h}) of events of the form $T_n \in \mathcal{T}_n^{v_i, \mathbf{h}}(t_0)$. (12) yields

$$\mathbb{P}_{t_0}(M_i \equiv \delta_i \mid M_1 \equiv \delta_1, \dots, M_{i-1} \equiv \delta_{i-1}, T_n \in E_n^i) = \frac{1}{2} + o(1).$$

Furthermore, for any event A , we have $\mathbb{P}_{t_0}(A) = \mathbb{P}_{t_0}(A, T_n \in E_n^{k-1}) + o(1)$ because $\mathbb{P}_{t_0}(T_n \in E_n^{k-1}) \rightarrow 1$ (Lemma 6.3). Using this twice, we obtain

$$\begin{aligned} \mathbb{P}_{t_0}(M_1 \equiv \delta_1, \dots, M_{k-1} \equiv \delta_{k-1}) &= \mathbb{P}_{t_0}(M_1 \equiv \delta_1, \dots, M_{k-1} \equiv \delta_{k-1}, T_n \in E_n^{k-1}) + o(1) \\ &= \mathbb{P}_{t_0}(M_{k-1} \equiv \delta_{k-1} \mid M_1 \equiv \delta_1, \dots, M_{k-2} \equiv \delta_{k-2}, T_n \in E_n^{k-1}) \\ &\quad \times (\mathbb{P}_{t_0}(M_1 \equiv \delta_1, \dots, M_{k-2} \equiv \delta_{k-2}) + o(1)) + o(1) \\ &= \frac{1}{2} \mathbb{P}_{t_0}(M_1 \equiv \delta_1, \dots, M_{k-2} \equiv \delta_{k-2}) + o(1). \end{aligned}$$

Repeating this argument $k - 2$ times concludes the proof. \square

It remains to prove that with high probability a uniform Schröder tree T_n is *well behaved*: the heights H_i of marked leaves and M_j of their common ancestors and the distances $|M_i - M_j|$ and $|M_i - H_j|$ are all larger than $n^{1/4}$ (in fact, larger than $\varepsilon_n n^{1/2}$ for all $\varepsilon_n \rightarrow 0$).

LEMMA 6.3. *We re-use the notation of the proof of Lemma 6.2. In particular, E_n^v denotes the set of trees t in $\mathcal{T}_n(t_0)$ such that $d_t(v) \geq n^{1/4}$. For all internal vertices v of t_0 ,*

$$\mathbb{P}(T_n \notin E_n^v \mid T_n \in \mathcal{T}_n(t_0)) \rightarrow 0.$$

In the special case where $t_0 = \vee$ (i.e., $k = 2$), every pair (ℓ_1, ℓ_2) of marked leaves of a Schröder tree with $n \geq 2$ leaves induces t_0 , and the height of ℓ_1 goes to infinity in probability when $n \rightarrow +\infty$.

PROOF. First, Corollary 5.4 gives the probability of the conditioning event in the limit

$$\lim_{n \rightarrow +\infty} \mathbb{P}(T_n \in \mathcal{T}_n(t_0)) = \mathbb{P}(\text{Tree}(e, \mathbf{U}) = t_0),$$

where \mathbf{U} is a set of k i.i.d. uniform random variables in $[0, 1]$. From Lemma A.3, we know that this quantity is strictly larger than 0. Therefore, it is enough to prove that $\mathbb{P}(T_n \notin E_n^v) \rightarrow 0$ without the conditioning.

By definition, for each internal vertex v in t_0 , $d_t(v)$ is one of the following:

- the height H_i of a marked leaf;
- the height M_i of the common ancestor of two marked leaves;
- the difference of heights $|M_i - M_j|$ between two common ancestors;
- the difference of heights $|M_i - H_j|$ between a leaf and a common ancestor.

Thus, the first statement of the lemma will follow if we show that

$$\mathbb{P} \left(\begin{array}{l} \text{There are } i \leq k, j \leq k - 1 \text{ such that} \\ H_i \leq n^{1/4} \text{ or } M_j \leq n^{1/4} \text{ or } |H_i - M_j| \leq n^{1/4} \\ \text{or } |M_i - M_j| \leq n^{1/4} (i \neq j) \end{array} \right) \rightarrow 0.$$

Actually, we prove the stronger statement that for any two i, j , all sequences of random variables $\frac{1}{\sqrt{n}}H_i, \frac{1}{\sqrt{n}}M_j, \frac{1}{\sqrt{n}}|M_i - M_j|, \frac{1}{\sqrt{n}}|H_i - M_j|$ converge in distribution to positive random variables.

We will only write the details for the sequence $(\frac{1}{\sqrt{n}}|M_i - M_j|)$, the proof being identical in the other cases. Recall that the x -coordinate of ℓ_i has the same distribution as $F_{T_n}^*(U_i)$, where:

- U_i is the i th smallest value among k i.i.d. uniform random variables in $[0, 1]$.
- $F_{T_n}^*$ is the pseudo-inverse of F_{T_n} defined in (4) (page 2157); combining Proposition 3.5, Lemma 3.6 (page 2157), we know that $F_{T_n}^*(x)$ tends to x when n tends to infinity (uniformly in x).

By construction, we have (in what follows \widetilde{C}_{T_n} is the normalized contour function of T_n , and the U_i 's are independent from \widetilde{C}_{T_n})

$$M_i = \min_{[F_{T_n}^*(U_i), F_{T_n}^*(U_{i+1})]} \sqrt{n} \times \widetilde{C}_{T_n}.$$

Since $(\widetilde{C}_{T_n})_n$ converges to $\lambda \cdot e$ (Proposition 2.23, page 2149) and $F_{T_n}^*(U_i)$ tends to U_i , we have

$$\frac{1}{\sqrt{n}}|M_i - M_j| \xrightarrow{(d)} \lambda \left| \min_{[U_i, U_{i+1}]} e - \min_{[U_j, U_{j+1}]} e \right|.$$

This completes the proof of the first statement of Lemma 6.3.

In particular, we have proved that H_1 goes to infinity with n in probability. The second statement of the lemma is just a rephrasing of this, in the particular case where $t_0 = \vee$. \square

REMARK 6.4. The joint convergence of the $\frac{1}{\sqrt{n}}H_i$ and $\frac{1}{\sqrt{n}}M_i$ is observed by Aldous in the proof of [3], Theorem 20, with the slight difference that he works

with uniform random *vertices* in the tree, while we consider uniform random *leaves*.

PROOF OF PROPOSITION 6.1. Let σ_n be a uniform separable permutation of size n . From Corollary 2.14 (page 2147), it has the same distribution as $\text{perm}(T_n, \mathcal{E}_n)$ where T_n is a uniform Schröder tree with n leaves and \mathcal{E}_n is the sign function on the internal vertices of T_n , such that the signs alternate and such that the root r of T_n has a balanced sign: $\mathcal{E}_n(r) = +$ with probability $1/2$. Recall that \widetilde{C}_{T_n} [resp., $(\widetilde{C}_{T_n}, S_n)$] denotes the normalized contour of T_n [resp., the signed contour of (T_n, \mathcal{E}_n)].

As in the statement of the proposition, consider a uniform random k -element subset ℓ of leaves of T_n . Recall from Observations 2.34 and 2.40 that $\text{Tree}(\widetilde{C}_{T_n}, \ell)$ [resp., $\text{Tree}_\pm(\widetilde{C}_{T_n}, S_n, \ell)$] is the subtree of T_n [resp., the signed subtree of (T_n, \mathcal{E}_n)] induced by these leaves. We condition on the fact that $\text{Tree}(\widetilde{C}_{T_n}, \ell) = t_0$. It is enough to prove that the probability that the signs in $\text{Tree}_\pm(\widetilde{C}_{T_n}, S_n, \ell)$ coincide with a fixed sign function ε on the internal vertices of t_0 is $1/2^{k-1} + o(1)$. Recall that the signs in $\text{Tree}_\pm(\widetilde{C}_{T_n}, S_n, \ell)$ correspond to signs of the common ancestors of the marked leaves in (T_n, \mathcal{E}_n) .

We further condition on the fact that the root of T_n has sign $+$. Then a vertex in T_n has sign $+$ if it has even height and sign $-$ if it has odd height. Therefore, the probability that the common ancestors have given signs correspond to the probability that their heights have given parities. This probability is $1/2^{k-1} + o(1)$ by Lemma 6.2.

The same holds if we had conditioned on the fact that the root of T_n has sign $-$. We can therefore conclude, that, conditioning only on the event $\text{Tree}(\widetilde{C}_{T_n}, \ell) = t_0$, the probability that the common ancestors have given signs is $1/2^{k-1} + o(1)$, which completes the proof of the proposition. \square

7. Conclusion of the proof. In this section, we prove that the convergence of the expectation for Pr^{Tree} implies the one of the expectation for Pr^{Perm} . In order to do that, the expectation for Pr^{Perm} is expressed in terms of the one for Pr^{Tree} (as a linear combination) in the continuous and discrete cases. This is possible since the signs are balanced and independent as it has been proven in Proposition 6.1.

LEMMA 7.1. *Let π be a pattern of size k and (e, S) be the signed Brownian excursion*

$$\mathbb{E}^{e, S}[\text{Pr}^{\text{Perm}}(\pi; e, S, F_U)] = \frac{1}{2^{k-1}} \sum_{(t_0, \varepsilon)} \mathbb{E}^e[\text{Pr}^{\text{Tree}}(t_0; e, F_U)],$$

where the sum runs over all signed binary trees (t_0, ε) of π .

PROOF. By definition,

$$\text{Pr}^{\text{Perm}}(\pi; e, S, F_U) = \mathbb{P}^{\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}) = \pi),$$

where \mathbf{X} is a set of k independent uniform variables in $[0, 1]$. Thus, with Observation 3.2 (page 2156),

$$\mathbb{E}^{e,S}[\text{Pr}^{\text{Perm}}(\pi; e, S, F_U)] = \mathbb{P}^{e,S,\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}) = \pi).$$

Since $\text{Perm}(e, S, \mathbf{X}) = \text{perm}(\text{Tree}_{\pm}(e, S, \mathbf{X}))$ this is exactly the probability that $\text{Tree}_{\pm}(e, S, \mathbf{X})$ is equal to one of the signed trees that are pre-images of π by perm . In other terms,

$$\mathbb{E}^{e,S}[\text{Pr}^{\text{Perm}}(\pi; e, S, F_U)] = \sum_{(t_0, \varepsilon)} \mathbb{P}^{e,S,\mathbf{X}}(\text{Tree}_{\pm}(e, S, \mathbf{X}) = (t_0, \varepsilon)),$$

where the sum runs over all signed trees of π . From Observation 2.31 (page 2151), if t_0 is not binary, then $\text{Tree}(e, S, \mathbf{X}) = t_0$ has probability 0, since from Lemma A.2 a.s. all minima of e have distinct values. Consequently, in this case $\text{Tree}_{\pm}(e, S, \mathbf{X}) = (t_0, \varepsilon)$ has also probability 0, and the sum above can be restricted to the signed *binary* trees of π . For each such signed binary tree (t_0, ε) , we have

$$\mathbb{P}^{e,S,\mathbf{X}}(\text{Tree}_{\pm}(e, S, \mathbf{X}) = (t_0, \varepsilon)) = \frac{1}{2^{k-1}} \mathbb{P}^{e,\mathbf{X}}(\text{Tree}(e, \mathbf{X}) = t_0).$$

Indeed, signs are taken independently on local minima of e and thus on vertices of t_0 . Finally, with Observation 3.2 again, the probability on the right-hand side is

$$(13) \quad \mathbb{P}^{e,\mathbf{X}}(\text{Tree}(e, \mathbf{X}) = t_0) = \mathbb{E}^e[\mathbb{P}^{\mathbf{X}}(\text{Tree}(e, \mathbf{X}) = t_0)] = \mathbb{E}^e[\text{Pr}^{\text{Tree}}(t_0; e, F_U)],$$

which completes the proof. \square

Using Proposition 6.1, we obtain a discrete analogue of this lemma.

LEMMA 7.2. *Let π be a pattern of size k and σ_n be a uniform random separable permutation of size n . As usual, let $(\widetilde{C}_{T_n}, S_n)$ be the signed contour of the decomposition tree of σ_n ,*

$$\mathbb{E}^{\sigma_n}[\text{Pr}^{\text{Perm}}(\pi; \widetilde{C}_{T_n}, S_n, F_{T_n})] = \frac{1}{2^{k-1}} \sum_{(t_0, \varepsilon)} \mathbb{E}^{\sigma_n}[\text{Pr}^{\text{Tree}}(t_0; \widetilde{C}_{T_n}, F_{T_n})] + o(1),$$

where the sum runs over all signed binary trees (t_0, ε) of π .

PROOF. Let \mathbf{X} be a set of k independent variables taken with distribution F_{T_n} . Remember that this amounts to choosing k leaves of T_n independently and uniformly at random. Using an argument similar to that in the previous proof, we have

$$(14) \quad \mathbb{E}^{\sigma_n}[\text{Pr}^{\text{Perm}}(\pi; \widetilde{C}_{T_n}, S_n, F_{T_n})] = \sum_{(t_0, \varepsilon)} \mathbb{P}^{\sigma_n, \mathbf{X}}(\text{Tree}_{\pm}(\widetilde{C}_{T_n}, S_n, \mathbf{X}) = (t_0, \varepsilon)),$$

where the sum runs over all signed trees (t_0, ε) of π .

It holds that

$$\mathbb{P}^{\sigma_n, \mathbf{X}}(\text{Tree}_{\pm}(\widetilde{C}_{T_n}, S_n, \mathbf{X}) = (t_0, \varepsilon)) \leq \mathbb{P}^{\sigma_n, \mathbf{X}}(\text{Tree}(\widetilde{C}_{T_n}, \mathbf{X}) = t_0)$$

and

$$\begin{aligned} \mathbb{P}^{\sigma_n, \mathbf{X}}(\text{Tree}(\widetilde{C}_{T_n}, \mathbf{X}) = t_0) &= \mathbb{E}^{\sigma_n}[\mathbb{P}^{\mathbf{X}}(\text{Tree}(\widetilde{C}_{T_n}, \mathbf{X}) = t_0)] \\ &= \mathbb{E}^{\sigma_n}[\text{Pr}^{\text{Tree}}(t_0; \widetilde{C}_{T_n}, F_{T_n})]. \end{aligned}$$

If t_0 is not binary, using Corollary 5.4 (page 2168), we further have

$$\mathbb{E}^{\sigma_n}[\text{Pr}^{\text{Tree}}(t_0; \widetilde{C}_{T_n}, F_{T_n})] \rightarrow \mathbb{E}^e[\text{Pr}^{\text{Tree}}(t_0; e, F_U)] = 0.$$

Recall indeed that, since a.s. all minima of e have distinct values from Lemma A.2, the trees extracted from e at uniformly distributed points are binary with probability 1.

Therefore, the sum in (14) can be replaced by a sum over signed *binary* trees (t_0, ε) of π , making only an error of $o(1)$. For signed binary trees (t_0, ε) , we use Proposition 6.1 (page 2168) to derive that

$$\begin{aligned} \mathbb{P}^{\sigma_n, \mathbf{X}}(\text{Tree}_{\pm}(\widetilde{C}_{T_n}, S_n, \mathbf{X}) = (t_0, \varepsilon)) &= \mathbb{P}^{\sigma_n, \mathbf{X}}(\text{Tree}_{\pm}(\widetilde{C}_{T_n}, S_n, \mathbf{X}) = (t_0, \varepsilon) \mid \text{no repetition in } \mathbf{X}) + o(1) \\ &= \frac{1}{2^{k-1}} \mathbb{P}^{\sigma_n, \mathbf{X}}(\text{Tree}(\widetilde{C}_{T_n}, \mathbf{X}) = t_0 \mid \text{no repetition in } \mathbf{X}) + o(1) \\ &= \frac{1}{2^{k-1}} \mathbb{P}^{\sigma_n, \mathbf{X}}(\text{Tree}(\widetilde{C}_{T_n}, \mathbf{X}) = t_0) + o(1). \end{aligned}$$

Finally, since

$$\begin{aligned} \mathbb{P}^{\sigma_n, \mathbf{X}}(\text{Tree}(\widetilde{C}_{T_n}, \mathbf{X}) = t_0) &= \mathbb{E}^{\sigma_n}[\mathbb{P}^{\mathbf{X}}(\text{Tree}(\widetilde{C}_{T_n}, \mathbf{X}) = t_0)] \\ &= \mathbb{E}^{\sigma_n}[\text{Pr}^{\text{Tree}}(t_0; \widetilde{C}_{T_n}, F_{T_n})], \end{aligned}$$

this completes the proof. \square

Using these expressions of the expectation for Pr^{Perm} in terms of the one for Pr^{Tree} in the continuous and discrete cases, we can now prove our main result.

THEOREM 7.3. *For any pattern π ,*

$$\mathbb{E}[\text{occ}(\pi, \sigma_n)] \longrightarrow \mathbb{E}[\Lambda_{\pi}].$$

PROOF. From Corollary 5.4, we have that, for each tree t_0 ,

$$\mathbb{E}^{\sigma_n}[\text{Pr}^{\text{Tree}}(t_0; \widetilde{C}_{T_n}, F_{T_n})] \rightarrow \mathbb{E}^e[\text{Pr}^{\text{Tree}}(t_0; e, F_U)].$$

Combining this with Lemmas 7.1 and 7.2, we get that for each pattern π ,

$$\mathbb{E}^{\sigma_n}[\text{Pr}^{\text{Perm}}(\pi; \widetilde{C}_{T_n}, S_n, F_{T_n})] \rightarrow \mathbb{E}^{e, S}[\text{Pr}^{\text{Perm}}(\pi; e, S, F_U)].$$

From Corollary 3.11 (page 2160), $\widetilde{\text{occ}}(\pi, \sigma_n) = \text{Pr}^{\text{Perm}}(\pi; \widetilde{C}_{T_n}, S_n, F_{T_n})(1 + \mathcal{O}(\frac{1}{n}))$. Together with Observation 3.9 (stating that $\Lambda_\pi = \text{Pr}^{\text{Perm}}(\pi; e, S, F_U)$), this completes the proof of Theorem 7.3. \square

As shown by Corollary 4.6 (page 2165), Theorems 4.1 and 1.2(iii) follow from Theorem 7.3 above.

8. Permuton interpretation of our main result. The goal of this section is to prove Theorem 1.6 (page 2141). We first need additional material on permutons. Recall from Definition 1.5 that permutons, which were introduced in [23], are probability measures on $[0, 1]^2$ with uniform marginals. Given a (deterministic) permuton μ and an integer k , there is a natural way to define a random permutation Π_k^μ of size k .

DEFINITION 8.1. Let μ be a permuton and k be an integer. Take k points in $[0, 1]^2$ independently according to μ . A.s. these k points have distinct x -coordinates and distinct y -coordinates (since μ has uniform marginals). Therefore, we can order these points $(X_1, Y_1), \dots, (X_k, Y_k)$ such that $Y_1 < Y_2 < \dots < Y_k$. Then Π_k^μ is the unique permutation such that $X_{\Pi_k^\mu(1)} < X_{\Pi_k^\mu(2)} < \dots < X_{\Pi_k^\mu(k)}$.

For a permutation π of size k , following the notation of [23], we then define $t(\pi, \mu)$ as the probability that $\Pi_k^\mu = \pi$.

We say that a (deterministic) sequence of permutations (σ_n) with sizes going to infinity converges to the permuton μ if, for all π , $\widetilde{\text{occ}}(\pi, \sigma_n)$ tends to $t(\pi, \mu)$.

It is noticed in [23], equation (49), that the convergence of (σ_n) to a permuton μ is equivalent to the weak convergence of the associated permutons (μ_{σ_n}) to μ . In particular, this implies the uniqueness of limits of sequences of permutations. Moreover, from [23], Theorem 1.6(i), if (σ_n) is a deterministic sequence of permutations such that $\widetilde{\text{occ}}(\pi, \sigma_n)$ has a limit for all π , then there exists a (necessarily unique) permuton μ such that (σ_n) tends towards μ .

We can now proceed to the proof of Theorem 1.6.

PROOF OF THEOREM 1.6. Item (iii) in Theorem 1.2 asserts that the finite-dimensional laws of $(\widetilde{\text{occ}}(\pi, \sigma_n))_\pi$ converge to those of $(\Lambda_\pi)_\pi$ (here, vectors are indexed by all permutations).

This is equivalent to the convergence in distribution of the infinite-dimensional vector $(\widetilde{\text{occ}}(\pi, \sigma_n))_\pi$ towards $(\Lambda_\pi)_\pi$ in the product topology (from the definitions of the convergence in distribution and of the product topology). From Skorohod’s representation theorem [26], Theorem 3.30, there exists a probability space Ω , random variables $(\mathbf{O}_n)_{n \geq 1}$ and $\mathbf{\Lambda}'$ such that:

- (i) for each $n \geq 1$, \mathbf{O}_n is a random vector $(O_{n,\pi})_\pi$ indexed by all permutations that has the same law as $(\widetilde{\text{occ}}(\pi, \sigma_n))_\pi$;

(ii) Λ' is a random vector $(\Lambda'_\pi)_\pi$ indexed by all permutations that has the same law as $(\Lambda_\pi)_\pi$;

(iii) we have sure convergence (in the product topology) of O_n to Λ' when n tends to infinity, that is, for any $\omega \in \Omega$, we have

$$\text{for any permutation } \pi, \quad O_{n,\pi}(\omega) \rightarrow \Lambda'(\omega)_\pi.$$

We now aim at constructing random variables σ'_n such that

$$(O_{n,\pi})_\pi = (\widetilde{\text{occ}}(\pi, \sigma'_n))_\pi,$$

so that we can apply Theorem 1.6(i) of [23] in each $\omega \in \Omega$.

Fix $n \geq 1$. Since $(O_{n,\pi})_\pi$ and $(\widetilde{\text{occ}}(\pi, \sigma_n))_\pi$ have the same distribution and since this distribution is supported on a finite set, one can assume that they have the same image set. Thus, for any ω in Ω , $(O_{n,\pi}(\omega))_\pi$ is equal to $(\widetilde{\text{occ}}(\pi, \sigma))_\pi$ for some permutation σ of size n that we denote $\sigma'_n(\omega)$. This defines a sequence of random permutations σ'_n defined on the probability space Ω with the property that $(O_{n,\pi})_\pi = (\widetilde{\text{occ}}(\pi, \sigma'_n))_\pi$. Note that σ'_n is measurable since, for a permutation τ of size n , we have $\{\omega : \sigma'_n(\omega) = \tau\} = \{\omega : O_{n,\tau}(\omega) = 1\}$.

Next, we claim that for any fixed n , σ'_n has the same distribution as σ_n . Indeed, n being fixed, considering the restriction of $(\widetilde{\text{occ}}(\pi, \sigma_n))_\pi$ [resp., $(\widetilde{\text{occ}}(\pi, \sigma'_n))_\pi$] to patterns π of size n gives the distribution of σ_n (resp., σ'_n). The claim then follows since $(\widetilde{\text{occ}}(\pi, \sigma_n))_\pi$ and $(\widetilde{\text{occ}}(\pi, \sigma'_n))_\pi$ have the same distribution (both the same as that of $(O_{n,\pi})_\pi$).

From Item (iii) above, and by definition of σ'_n , for any $\omega \in \Omega$, the following holds:

$$\text{for any permutation } \pi, \quad \widetilde{\text{occ}}(\pi, \sigma'_n(\omega)) \rightarrow \Lambda'(\omega)_\pi.$$

Using [23], Theorem 1.6(i), this implies that, for any $\omega \in \Omega$, there exists a (unique) permuton $\mu(\omega)$ such that $\sigma'_n(\omega)$ tends to $\mu(\omega)$ in the sense of Definition 8.1, or equivalently $\mu_{\sigma'_n(\omega)}$ converges weakly to $\mu(\omega)$. Since $\omega \mapsto \mu(\omega)$ is the pointwise limit of the measurable functions $\omega \mapsto \mu_{\sigma'_n(\omega)}$ in the metrizable weak topology, we know that μ is measurable.

Using that pointwise convergence implies convergence in distribution and that σ'_n and σ_n have the same distribution, we deduce that μ_{σ_n} tends in distribution to μ in the weak convergence topology, as claimed.

Finally, it should be noted that, for any π , $\Lambda_\pi \stackrel{(d)}{=} t(\pi, \mu)$, using the notation t from Definition 8.1, since both are the limit in distribution of $\widetilde{\text{occ}}(\pi, \sigma_n)$. From Theorem 1.2(i) Λ_{12} is not deterministic, so neither is μ . \square

9. Some properties of Λ_π . In this section we give some additional results concerning the limit variables Λ_π . These results are not needed in the proof of our main theorem [Theorem 1.2(iii)]. However, the more information we have on Λ_π , the more interesting our theorem is.

The first result, given in Section 9.1, is a way of computing (joint) moments of the variables Λ_π . This has been used in the Introduction to give explicit values for the limit of (joint) moments of $\widetilde{\text{occ}}(\pi, \sigma_n)$. Our second result presented in Section 9.2 is the fact that the variables Λ_π are not deterministic when π is separable. This corresponds to Item (i) in Theorem 1.2.

9.1. *Computing expectation and other moments of Λ_π .* Recall that N_π denotes the number of signed binary trees associated with the permutation π .

PROPOSITION 9.1. *For any permutation π of size k ,*

$$\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)] \longrightarrow \mathbb{E}[\Lambda_\pi] = \frac{N_\pi}{2^{k-1} \cdot \text{Cat}_{k-1}}.$$

PROOF. For any π , the convergence of $\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)]$ to $\mathbb{E}[\Lambda_\pi]$ is given by Theorem 7.3 (page 2176). Let π be a permutation of size k and let us prove that

$$\mathbb{E}[\Lambda_\pi] = \frac{N_\pi}{2^{k-1} \cdot \text{Cat}_{k-1}}.$$

We consider the signed Brownian excursion (e, S) . We take k points X_1, \dots, X_k in $[0, 1]$ uniformly at random, independently from each other and from (e, S) , and we let $\mathbf{X} = (X_1, \dots, X_k)$. From Lemma 7.1 (see also (13) in its proof), we have

$$\mathbb{E}^{e,S}[\text{Pr}^{\text{Perm}}(\pi; e, S, F_U)] = \frac{1}{2^{k-1}} \sum_{(t_0, \varepsilon)} \mathbb{P}^{e, \mathbf{X}}(\text{Tree}(e, \mathbf{X}) = t_0),$$

where the sum runs over all signed binary trees (t_0, ε) of π .

Lemma A.3, a deep result on Brownian excursion, states that $\text{Tree}(e, \mathbf{X})$ is a uniform binary tree with k leaves, that is, for any binary tree t_0 with k leaves,

$$\mathbb{P}^{e, \mathbf{X}}(\text{Tree}(e, \mathbf{X}) = t_0) = \frac{1}{\text{Cat}_{k-1}}.$$

The proposition then follows immediately. \square

In addition, N_π , the number of signed binary trees of a given permutation π , can be expressed combinatorially from the decomposition tree of π . More precisely, we have the following.

OBSERVATION 9.2. For any permutation π , denoting t_π its (unique) decomposition tree, it holds that

$$N_\pi := \text{card}\{(t_0, \varepsilon), t_0 \text{ binary} : \text{perm}(t_0, \varepsilon) = \pi\} = \prod_{v \text{ int. vertex of } t_\pi} \text{Cat}_{\text{deg}(v)-1},$$

where $\text{deg}(v)$ is the number of children of v .

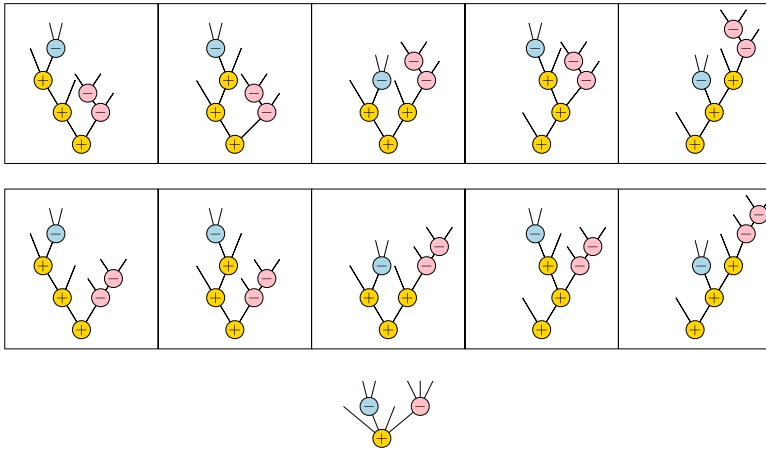


FIG. 12. The decomposition tree of 1324765 and its $\text{Cat}_3 \times \text{Cat}_2 \times \text{Cat}_1 = 10$ signed binary trees.

PROOF. Given a signed tree of π , and a vertex v with sign ε in this tree, the following transformation produces a tree which is still a signed tree of π : assuming that k subtrees T_1, \dots, T_k are attached to v , replace v by a binary tree with k leaves and all internal vertices labeled ε and where the i th leaf is replaced by T_i . Conversely, each signed binary tree of π can be obtained from the decomposition tree t_π of π applying this transformation on all internal vertices v of t_π . An example is given in Figure 12. \square

PROPOSITION 9.3. Let π_1, \dots, π_r be a list of patterns, and let $K = \sum_{i=1}^r |\pi_i|$. We have

$$\mathbb{E} \left[\prod_{i=1}^r \widetilde{\text{occ}}(\pi_i, \sigma_n) \right] \longrightarrow \mathbb{E} \left[\prod_{i=1}^r \Lambda_{\pi_i} \right] = \sum_{\rho \in \mathfrak{S}_K} c_{\pi_1, \dots, \pi_r}^\rho \mathbb{E}[\Lambda_\rho],$$

where for any $\rho \in \mathfrak{S}_K$, $c_{\pi_1, \dots, \pi_r}^\rho$ denotes the proportion of ordered set-partitions of $\{1, \dots, K\}$ which are compatible with $\rho, \pi_1, \dots, \pi_r$; see Definition 4.2, page 2161.

PROOF. See the proof of Corollary 4.6 for the convergence result, and Proposition 4.4 for the expression of $\mathbb{E}[\prod_{i=1}^r \Lambda_{\pi_i}]$. \square

The results of this section enable the automatic computation of joint moments of Λ_π . As mentioned in the Introduction, we have implemented this in Sage. Let us discuss quickly algorithmic questions behind this implementation:

- From Proposition 9.1, computing the expectation essentially amounts to computing N_π . Finding the degree of the root of the decomposition tree of π is easy. To simplify the discussion (but without loss of generality), suppose that this root

has sign $+$. Then its degree is the number of integers $i \leq k$ such that

$$\{\pi(1), \dots, \pi(i)\} = \{1, \dots, i\}, \quad \text{or equivalently} \quad \max_{j \leq i} \pi(j) = i.$$

Thanks to the second formulation, this can be computed in linear time reading the permutation from left-to-right, keeping only in memory the maximum of the values already read.

Degrees of all vertices can then be computed recursively, and we find N_π using Observation 9.2, in quadratic time.

- Higher moments or joint moments are more complex to compute. Fix a list of patterns π_1, \dots, π_r of respective sizes k_1, \dots, k_r and set $K = k_1 + \dots + k_r$. While convenient in proofs, the definition of the coefficients $c_{\pi_1, \dots, \pi_r}^\rho$ (see Definition 4.2, page 2161) is not optimal from a practical point of view. It is however possible to generate directly the multi-set of permutations of \mathfrak{S}_K , where each ρ appears with multiplicity $d_{\pi_1, \dots, \pi_r}^\rho$. If $\mathbf{Pos} = (\text{Pos}_1, \dots, \text{Pos}_r)$ and $\mathbf{Val} = (\text{Val}_1, \dots, \text{Val}_r)$ are ordered set-partitions of $\{1, \dots, K\}$ such that $\text{card}(\text{Pos}_i) = \text{card}(\text{Val}_i) = k_i$, we can define a permutation $\rho(\mathbf{Pos}, \mathbf{Val})$ as follows: $\rho(\mathbf{Pos}, \mathbf{Val})$ assigns the $\pi_i(j)$ th smallest value in Val_i to the j th smallest value in Pos_i . It is easy to check that any given ρ is constructed this way from exactly $d_{\pi_1, \dots, \pi_r}^\rho$ pairs $(\mathbf{Pos}, \mathbf{Val})$. We have used this construction in our implementation of the computation of joint moments.

Even if this is more efficient than a naive implementation of Definition 4.2, the complexity still grows very quickly.

Indeed, the number of pairs $(\mathbf{Pos}, \mathbf{Val})$ as above is equal to $\binom{K}{k_1, \dots, k_r}^2$. In practice, we have only been able to compute the first four moments of Λ_{12} with this algorithm (the fifth moment given in the Introduction has been inferred from the first four using the symmetry $\Lambda_{12} \stackrel{(d)}{=} 1 - \Lambda_{12}$). We believe that modifying slightly the program to enable parallel computing may allow to compute the fifth moment with the above algorithm in a reasonable time but that the sixth moment, which involves more than 5×10^{13} pairs $(\mathbf{Pos}, \mathbf{Val})$, will remain out of reach.

9.2. Λ_π is not deterministic. Finally, we will prove Theorem 1.2(i) We have already seen in Remark 3.3 (page 2156) that, when π is not separable, Λ_π is identically 0 (in particular, deterministic). We have also seen in Remark 3.4 (page 2156) that for the permutation π of size 1, Λ_π is identically 1 (and thus, again deterministic). On the contrary, we now prove that if π is separable of size at least 2, then Λ_π is a true *random* variable and not a deterministic constant.

When π is separable of size at least 2, to prove that Λ_π is not deterministic, it would be sufficient to check that $\text{Var}(\Lambda_\pi) > 0$. Although Proposition 9.3 provides an expression of the variance of Λ_π , this expression does not allow us to prove directly that $\text{Var}(\Lambda_\pi) > 0$. Instead, we prove in Proposition 9.5 that $\Lambda_\pi \neq \mathbb{E}[\Lambda_\pi]$ with positive probability. From the expression of $\mathbb{E}[\Lambda_\pi]$ given in Proposition 9.1,

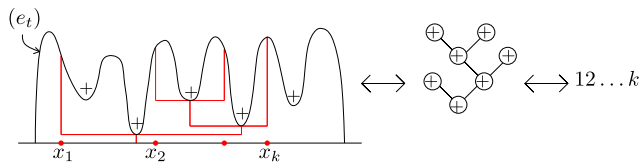


FIG. 13. A realization of (e, S) for which $\text{Perm}(e, S, \mathbf{X}) = 123 \dots k$ with high probability.

it follows immediately that $\mathbb{E}[\Lambda_\pi] > 0$ if π is separable. So the proof of Theorem 1.2(i) will be completed as soon as we prove that Λ_π takes values as close to 0 as wanted with positive probability. This is done in Lemma 9.4.

LEMMA 9.4. Let π be a separable permutation of size $k \geq 2$, (e, S) be the signed Brownian excursion and N be a positive integer. There exists an event $E_N = E_N(e, S)$ such that:

- E_N occurs with positive probability;
- If (e, S) is such that E_N occurs, then $\mathbb{P}^{\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}) = \pi) \leq \frac{k^2}{N}$, where $\mathbf{X} = (X_1, \dots, X_k)$ and the X_i 's are uniform independent points in $[0, 1]$.

PROOF. Assume first that $\pi \neq 123 \dots k$. The key idea of the proof is that for some realizations of (e, S) (see a sketch in Figure 13), $\text{Perm}(e, S, \mathbf{X}) = 123 \dots k$ with high probability.

We set $\eta = \frac{1}{4N^2}$ and define intervals as follows:

- for integers ℓ between 0 and N , we set $J_\ell = [\frac{\ell}{N} - \eta; \frac{\ell}{N} + \eta] \cap [0, 1]$;
- for integers ℓ between 1 and N , we set $I_\ell = [\frac{\ell-1}{N} + \eta; \frac{\ell}{N} - \eta]$.

These intervals are represented on Figure 14. We have $\text{Length}(I_\ell) = 1/N - 2\eta$ for each ℓ . Moreover, by construction $\text{Length}(J_\ell) = 2\eta$ for $\ell \neq 0$ and $\ell \neq N$, while $\text{Length}(J_0) = \text{Length}(J_N) = \eta$.

Let $E_N = E_N(e, S)$ be the event defined as follows:

- (i) For each ℓ , $\min_{J_\ell} e \leq 1/10$.
- (ii) $\min_{I_1 \cup I_2 \cup \dots \cup I_N} e \geq 2/10$.
- (iii) The signs of the $N - 1$ local minima of e on J_1, \dots, J_{N-1} are all $+$.

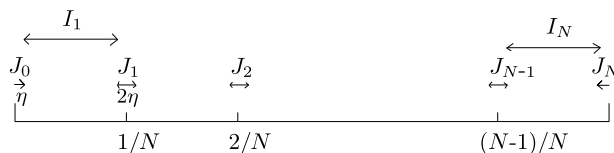


FIG. 14. Intervals I_ℓ and J_ℓ .

Taking f a piecewise linear excursion such that $f(\frac{\ell}{N}) = 0$ for integers ℓ between 0 and N and $f(x) = 3/10$ for $x \in I_1 \cup I_2 \cup \dots \cup I_N$, and with $\delta = 1/10$, it follows from Lemma A.5 that

$$\mathbb{P}^{e,S}(E_N) = \frac{1}{2^{N-1}} \mathbb{P}^e(e \text{ satisfies Items (i) and (ii)}) > 0.$$

Consider now a realization (e, S) of the signed Brownian excursion such that E_N is realized, and let $\mathbf{X} = (X_1, \dots, X_k)$ be k uniform independent points in $[0, 1]$. The construction of E_N ensures that $\text{Perm}(e, S, \mathbf{X}) = 123 \dots k$ as soon as each X_i is in an interval I_ℓ such that I_ℓ does not contain any other X_j . Equivalently, if $\text{Perm}(e, S, \mathbf{X}) \neq 123 \dots k$ then one of the two following events occurs:

- For some i , X_i belongs to an interval J_ℓ .
- For some $i \neq j$, X_i and X_j belong to the same interval I_ℓ .

Therefore, we get

$$\begin{aligned} \mathbb{P}^{\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}) = \pi) &\leq \mathbb{P}^{\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}) \neq 123 \dots k) \\ &\leq k \sum_{\ell=0}^N \mathbb{P}^{\mathbf{X}}(X_1 \in J_\ell) + \binom{k}{2} \sum_{\ell=1}^N \mathbb{P}^{\mathbf{X}}(X_1, X_2 \in I_\ell) \\ &\leq k2N\eta + \binom{k}{2} N \left(\frac{1}{N} - 2\eta\right)^2 \\ &\leq \frac{k}{2N} + \binom{k}{2} \frac{1}{N} = \frac{k^2}{N} \quad \left(\text{recall that } \eta = \frac{1}{4N^2}\right). \end{aligned}$$

This completes the proof of Lemma 9.4 when $\pi \neq 123 \dots k$. By symmetry, the same result holds for $\pi \neq k \dots 321$. Therefore, the statement of Lemma 9.4 holds for all separable permutations of size at least 2. \square

We can now establish the announced proposition.

PROPOSITION 9.5. *For any separable pattern π of size at least 2,*

$$\mathbb{P}^{e,S}(\Lambda_\pi < \mathbb{E}[\Lambda_\pi]) > 0.$$

In particular, Λ_π is not almost surely constant.

PROOF. Let $k \geq 2$ be the size of π . Since $\mathbb{E}[\Lambda_\pi] > 0$ (from Proposition 9.1) and k is fixed, one can choose N big enough such that $k^2/N < \mathbb{E}[\Lambda_\pi]$. For such a value of N , let E_N be the event given by Lemma 9.4. If (e, S) is such that E_N is realized, then (with $\mathbf{X} = (X_1, \dots, X_k)$ a tuple of k uniform independent points in $[0, 1]$)

$$\Lambda_\pi = \mathbb{P}^{\mathbf{X}}(\text{Perm}(e, S, \mathbf{X}) = \pi) \leq \frac{k^2}{N} < \mathbb{E}[\Lambda_\pi].$$

Since E_N has positive probability, the event $\Lambda_\pi < \mathbb{E}[\Lambda_\pi]$ also occurs with positive probability. \square

APPENDIX: USEFUL FACTS ON THE BROWNIAN EXCURSION

For the convenience of the reader, we now record several useful properties of a typical realization of the Brownian excursion e .

There are several ways to define the Brownian excursion, the most convenient for us is to draw a realization of e from a realization of the Brownian motion $(B_t)_{t \geq 0}$, as follows (see [36], Section 0.2). Consider

$$(15) \quad a = \sup\{t \leq 1 : B_t = 0\}, \quad b = \inf\{t \geq 1 : B_t = 0\}$$

(almost surely $a < 1 < b$), and set

$$(e(s))_{0 \leq s \leq 1} := \left(\frac{1}{\sqrt{b-a}} |B_{a+s(b-a)}| \right)_{0 \leq s \leq 1}.$$

Thus, the Brownian excursion e is seen as a dilatation of a piece of B . It follows that some almost-sure properties of the set of local extrema of B remain true for e .

Recall that by definition, x is a one-sided local minimum for f if, for some $\varepsilon > 0$,

$$f(x) = \min_{[x-\varepsilon, x]} f \quad \text{or} \quad f(x) = \min_{[x, x+\varepsilon]} f.$$

LEMMA A.1. *The set of one-sided local minima of the Brownian excursion e has Lebesgue measure 0, almost surely.*

PROOF. We first prove an analogous statement for the Brownian motion $(B_t)_{t \geq 0}$. We denote $\text{Omin}(B)$ the set of one-sided local minima of B , and $\text{Leb}(\text{Omin}(B))$ its Lebesgue measure. We have

$$\text{Leb}(\text{Omin}(B)) = \int_0^{+\infty} \mathbf{1}_{u \in \text{Omin}(B)} du,$$

so that, taking the expectation with respect to B ,

$$\mathbb{E}[\text{Leb}(\text{Omin}(B))] = \int_0^{+\infty} \mathbb{P}(u \in \text{Omin}(B)) du = 0.$$

In the last equality, we used that $\mathbb{P}(u \in \text{Omin}(B)) = 0$ for every fixed u : Theorem 1.27 in [35] gives a similar result for local minima (i.e., two-sided minima), but the proof is easily adapted to the case of one-sided minima. Thus, $\text{Leb}(\text{Omin}(B))$ is a nonnegative random variable with expectation 0 and, therefore, is equal to 0 almost surely.

The statement then follows for $\text{Leb}(\text{Omin}(e))$ by dilatation, since the dilatation of a set of measure zero has measure zero as well. \square

We now discuss values of local minima.

LEMMA A.2. *With probability one, the Brownian excursion e has no two local minima with the same value.*

PROOF. For the Brownian motion, it is the statement of [26], Lemma 11.15. This is also true for e since it is a dilatation of the Brownian motion. \square

An important consequence of Lemma A.2 in the present paper is that, for all set of k distinct points \mathbf{x} of $[0, 1]$ and for almost all realizations of e , the tree $\text{Tree}(e, \mathbf{x})$ obtained in Section 2.4 is a binary tree (because of Observation 2.31).

A remarkable fact is that if \mathbf{x} is uniformly distributed this random binary tree is uniform (see [30], Section 2.6).

LEMMA A.3. *Fix $k \geq 2$ and a binary tree t_0 with k leaves. Let U_1, \dots, U_k be k uniform and independent random variables in $[0, 1]$, independent from e . Then*

$$\mathbb{P}(\text{Tree}(e, \{U_1, \dots, U_k\}) = t_0) = \frac{1}{\text{Cat}_{k-1}}.$$

It is in fact even possible to describe the law of the *geometric tree* extracted from e and U_1, \dots, U_k , that is, a tree with edge-lengths that are nonnegative real numbers (see [30], Theorem 2.11). In this paper, we use a rather weak consequence of this result.

LEMMA A.4. *Take k i.i.d. uniform random variables in $[0, 1]$ independently from e and call them $U_1 < \dots < U_k$. Set $M_i = \min_{[U_i, U_{i+1}]} e$. Then the random vector*

$$\mathbf{v} = (e(U_1), \dots, e(U_k), M_1, \dots, M_{k-1})$$

has distinct coordinates with probability 1.

PROOF. As said above, the law of the geometric tree extracted from e and U_1, \dots, U_k is given in [30], Theorem 2.11; this law has a density with respect to the uniform distribution on geometric trees.

Conditioning on the fact that $\text{Tree}(e, \{U_1, \dots, U_k\})$ is a given t_0 with k leaves, the coordinates of \mathbf{v} are sums of edge-lengths of the geometric tree. Hence, the vector \mathbf{v} has a density with respect to the Lebesgue measure on \mathbb{R}^{2k-1} . Without conditioning, \mathbf{v} has also a density, which is simply the mean of the conditional densities. This implies the lemma. \square

Finally, we need the fact that the Brownian excursion e is arbitrary close to any fixed Lipschitz excursion with positive probability. (A Lipschitz excursion is simply an excursion that is also a Lipschitz function, that is, there exists $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all x, y in $[0, 1]$.)

LEMMA A.5. For every Lipschitz excursion f and $\delta > 0$,

$$\mathbb{P}\left(\sup_{0 \leq s \leq 1} |e(s) - f(s)| \leq \delta\right) > 0.$$

PROOF. The proof relies on a similar result for Brownian motion [17], Section 1.4. Let us give some details.

Fix a Lipschitz excursion f and $\delta > 0$ as in the statement of the lemma. Without loss of generality, assume that $\delta < 1/2$. Define $g_\delta(t) = \min(\delta, t, 1 - t)$, so that $\|g_\delta\|_\infty = \delta$. Then

$$|e(s) - f(s) - g_{\delta/2}(s)| \leq \delta/2 \quad \Rightarrow \quad |e(s) - f(s)| \leq \delta.$$

Therefore, replacing if necessary f by $f + g_{\delta/2}$ and δ by $\delta/2$, we can assume without loss of generality that the following holds:

(16) for any $\eta \leq \delta/2$ and $s \in [\eta, 1 - \eta]$, one has $f(s) \geq \eta$.

We extend f to the interval $[-\delta, 1 + \delta]$ by setting $f(t) = t$ if $t \leq 0$ and $f(t) = 1 - t$ if $t \geq 1$. Let $(B_t)_{t \geq 0}$ be a realization of the Brownian motion. Let $\eta \in (0, \delta/2]$. Using the results of [17], Section 1.4, we know that with positive probability we have

(17)
$$\sup_{s \in [1/2 - \delta, 3/2 + \delta]} \left| B_s - f\left(s - \frac{1}{2}\right) \right| < \eta.$$

Together with (16), this implies in particular that

$$\begin{cases} B_s > 0 & \text{if } s \in \left[\frac{1}{2} + \eta, \frac{3}{2} - \eta\right]; \\ B_s < 0 & \text{if } s \in \left[\frac{1}{2} - \delta, \frac{1}{2} - \eta\right] \cup \left[\frac{3}{2} + \eta, \frac{3}{2} + \delta\right]. \end{cases}$$

Therefore, if we define a and b as in (15), we have

(18)
$$\frac{1}{2} - \eta < a < \frac{1}{2} + \eta, \quad \frac{3}{2} - \eta < b < \frac{3}{2} + \eta.$$

In particular, $|b - a - 1| < 2\eta$. For s in $[0, 1]$, we can write

$$\begin{aligned} |e(s) - f(s)| &= \left| \frac{1}{\sqrt{b-a}} B_{a+s(b-a)} - f(s) \right| \\ &\leq \frac{1}{\sqrt{b-a}} \left| B_{a+s(b-a)} - f\left(a - \frac{1}{2} + s(b-a)\right) \right| \\ &\quad + \left| \frac{1}{\sqrt{b-a}} - 1 \right| \cdot \left| f\left(a - \frac{1}{2} + s(b-a)\right) \right| \\ &\quad + \left| f\left(a - \frac{1}{2} + s(b-a)\right) - f(s) \right|. \end{aligned}$$

Using (17) and (18), the inequality $|b - a - 1| < 2\eta$ and the fact that f is a bounded Lipschitz function, it is not hard to see that this upper bound is smaller than $C\eta$ for some constant C . Since η can be chosen as small as wanted, we may assume $C\eta \leq \delta$ and we get $|e(s) - f(s)| \leq \delta$ (for all s in $[0, 1]$).

In summary, for η sufficiently small, (17) implies $\sup_{0 \leq s \leq 1} |e(s) - f(s)| \leq \delta$, so that the latter occurs with positive probability, as wanted. \square

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