The Brownian map is the scaling limit of uniform random plane quadrangulations

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Abstract

We prove that uniform random quadrangulations of the sphere with n faces, endowed with the usual graph distance and renormalized by $n^{-1/4}$, converge as $n \to \infty$ in distribution for the Gromov-Hausdorff topology to a limiting metric space. We validate a conjecture by Le Gall, by showing that the limit is (up to a scale constant) the so-called *Brownian map*, which was introduced by Marckert & Mokkadem and Le Gall as the most natural candidate for the scaling limit of many models of random plane maps. The proof relies strongly on the concept of *geodesic stars* in the map, which are configurations made of several geodesics that only share a common endpoint and do not meet elsewhere.

Contents

1 Introduction

Random plane maps and their scaling limits are a sort of two-dimensional analog of random walks and Brownian motion, in which one wants to approximate a random continuous surface using large random graphs drawn on the 2-sphere [1]. A *plane map* is a proper embedding, without edge-crossings, of a finite connected graph in the two-dimensional sphere. Loops or multiple edges are allowed. We say that a map is *rooted* if one of its oriented edges, called the root, is distinguished. Two (rooted) maps $\mathbf{m}, \mathbf{m}'$ are considered equivalent if there exists a direct homeomorphism of the sphere that maps $\mathbf{m}$ onto $\mathbf{m}'$ (and maps the root of $\mathbf{m}$ to that of $\mathbf{m}'$ with preserved orientations). Equivalent maps will systematically be identified in the sequel, so that the set of maps is a countable set with this convention.

From a combinatorial and probabilistic perspective, the maps called *quadrangulations*, which will be the central object of study in this work, are among the simplest to manipulate. Recall that the *faces* of a map are the connected component of the complement of the embedding. A map is called a quadrangulation if all its faces have degree 4, where the degree of a face is the number of edges that are incident to it (an edge which is incident to only one face has to be counted twice in the computation of the degree of this face). Let $\mathbf{Q}$ be the set of plane, rooted quadrangulations, and $\mathbf{Q}_n$ be the set of elements of $\mathbf{Q}$ with n faces. Let Q_n be uniformly distributed in $\mathbf{Q}_n$. We identify Q_n with the finite metric space $(V(Q_n), d_{Q_n})$, where $V(Q_n)$ is the set of vertices of Q_n, and d_{Q_n} is the usual graph distance on $V(Q_n)$. We see Q_n as a random variable with values in the space $\mathbb{M}$ of compact metric spaces considered up to isometry. The space $\mathbb{M}$ is endowed with the Gromov-Hausdorff topology [6]: The distance between two elements $(X, d), (X', d')$ in $\mathbb{M}$ is given by

$$\mathtt{d}_{\mathrm{GH}}((X,d),(X',d')) = \frac{1}{2}\inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R})\,,$$

where the infimum is taken over all *correspondences* $\mathcal{R}$ between X and X', i.e. subsets of $X \times X'$ whose canonical projections $\mathcal{R} \to X, X'$ are onto, and $\mathrm{dis}(\mathcal{R})$ is the *distortion* of

$\mathcal{R}$, defined by

$$\mathrm{dis}(\mathcal{R}) = \sup_{(x,x'),(y,y')\in\mathcal{R}} |d(x,y) - d'(x',y')| \,.$$

The space $(\mathbb{M}, \mathtt{d}_{\mathrm{GH}})$ is then a separable and complete metric space [11].

It turns out that typical graph distances in Q_n are of order $n^{1/4}$ as $n \to \infty$, as was shown in a seminal paper by Chassaing & Schaeffer [8]. Since then, much attention has been drawn by the problem of studying the scaling limit of Q_n, i.e. to study the asymptotic properties of the space $n^{-1/4}Q_n := (V(Q_n), n^{-1/4}d_{Q_n})$ as $n \to \infty$. Le Gall [14] obtained a compactness result, namely that the laws of $(n^{-1/4}Q_n, n \geq 1)$, form a relatively compact family of probability distributions on $\mathbb{M}$. This means that along suitable subsequences, the sequence $n^{-1/4}Q_n$ converges in distribution in $\mathbb{M}$ to some limiting space. Such limiting spaces are called "Brownian maps", so as to recall the fact that Brownian motion arises as the scaling limit of discrete random walks. Many properties satisfied by *any* Brownian map (i.e. by any limit in distribution of $n^{-1/4}Q_n$ along some subsequence) are known. In particular, Le Gall showed that their topology is independent of the choice of a subsequence [14]. Then Le Gall & Paulin identified this topology with the topology of the 2-dimensional sphere [18, 24]. Besides, the convergence of *two-point functions* and *three-point functions*, that is, of the joint laws of rescaled distances between 2 or 3 randomly chosen vertices in $V(Q_n)$, was also established respectively by Chassaing & Schaeffer [8] and Bouttier & Guitter [5]. These convergences occur without having to extract subsequences. See [17] for a recent survey of the field.

It is thus natural to conjecture that all Brownian maps should in fact have the same distribution, and that the use of subsequences in the approximation by quadrangulations is superfluous. A candidate for a limiting space (which is sometimes also called the Brownian map, although is was not proved that this space arises as the limit of $n^{-1/4}Q_n$ along some subsequence) was described in equivalent, but slightly different forms, by Marckert and Mokkadem [21] and Le Gall [14].

The main goal of this work is to prove these conjectures, namely, that $n^{-1/4}Q_n$ converges in distribution as $n \to \infty$ to the conjectured limit of [21, 14]. This unifies the several existing definitions of Brownian map we just described, and lifts the ambiguity that there could have been more than one limiting law along a subsequence for $n^{-1/4}Q_n$.

In order to state our main result, let us describe the limiting Brownian map. This space can be described from a pair of random processes $(\mathbf{e}, Z)$. Here, $\mathbf{e} = (\mathbf{e}_t, 0 \leq t \leq 1)$ is the so-called *normalized Brownian excursion*. It can be seen as a positive excursion of standard Brownian motion conditioned to have duration 1. The process $Z = (Z_t, 0 \leq t \leq 1)$ is the so-called *head of the Brownian snake* driven by $\mathbf{e}$: Conditionally given $\mathbf{e}$, Z is a centered Gaussian process with continuous trajectories, satisfying

$$E[|Z_s - Z_t|^2 \,|\, \mathbf{e}] = d_{\mathbf{e}}(s,t) \qquad s,t \in [0,1] \,,$$

where

$$d_{\mathbf{e}}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2 \inf_{s\wedge t \leq u \leq s \vee t} \mathbf{e}_u \,.$$

The function $d_{\mathbf{e}}$ defines a pseudo-distance on $[0,1]$: This means that $d_{\mathbf{e}}$ satisfies the properties of a distance, excepting separation, so that it can hold that $d_{\mathbf{e}}(s,t) = 0$ with

$s \neq t$. We let $\mathcal{T}_{\mathbf{e}} = [0,1]/\{d_{\mathbf{e}} = 0\}$, and denote the canonical projection by $p_{\mathbf{e}} : [0,1] \to \mathcal{T}_{\mathbf{e}}$. It is obvious that $d_{\mathbf{e}}$ passes to the quotient to a distance function on $\mathcal{T}_{\mathbf{e}}$, still called $d_{\mathbf{e}}$. The space $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is called Aldous' *continuum random tree*. The definition of Z implies that a.s., for every s, t such that $d_{\mathbf{e}}(s,t) = 0$, one has $Z_s = Z_t$. Therefore, it is convenient to view Z as a function $(Z_a, a \in \mathcal{T}_{\mathbf{e}})$ indexed by $\mathcal{T}_{\mathbf{e}}$. Roughly speaking, Z can be viewed as the Brownian motion indexed by the Brownian tree.

Next, we set

$$D^\circ(s,t) = Z_s + Z_t - 2\max\Big(\inf_{s\le u\le t} Z_u, \inf_{t\le u\le s} Z_u\Big), \qquad s,t \in [0,1],$$

where $s \le u \le t$ means $u \in [s,1] \cup [0,t]$ when $t < s$. Then let, for $a, b \in \mathcal{T}_{\mathbf{e}}$,

$$D^\circ(a,b) = \inf\{D^\circ(s,t) : s,t \in [0,1], p_{\mathbf{e}}(s) = a, p_{\mathbf{e}}(t) = b\}.$$

The function D° on $[0,1]^2$ is a pseudo-distance, but D° on $\mathcal{T}_{\mathbf{e}}^2$ does not satisfy the triangle inequality. This motivates writing, for $a, b \in \mathcal{T}_{\mathbf{e}}$,

$$D^*(a,b) = \inf\Big\{\sum_{i=1}^{k-1} D^\circ(a_i, a_{i+1}) : k \ge 1, a = a_1, a_2, \ldots, a_{k-1}, a_k = b\Big\}.$$

The function D^* is now a pseudo-distance on $\mathcal{T}_{\mathbf{e}}$, and we finally define

$$S = \mathcal{T}_{\mathbf{e}}/\{D^* = 0\},$$

which we endow with the quotient distance, still denoted by D^*. Alternatively, letting, for $s, t \in [0,1]$,

$$D^*(s,t) = \inf\Big\{\sum_{i=1}^{k} D^\circ(s_i, t_i) : k \ge 1, s = s_1, t = t_k, d_{\mathbf{e}}(t_i, s_{i+1}) = 0, 1 \le i \le k-1\Big\},$$

one can note that S can also be defined as the quotient metric space $[0,1]/\{D^* = 0\}$. The space (S, D^*) is a *geodesic metric space*, meaning that for every two points $x, y \in S$, there exists an isometry $\gamma : [0, D^*(x,y)] \to S$ such that $\gamma(0) = x, \gamma(D^*(x,y)) = y$. The function γ is called a *geodesic* from x to y.

The main result of this paper is the following.

Theorem 1 *As $n \to \infty$, the metric space $(V(Q_n), (8n/9)^{-1/4} d_{Q_n})$ converges in distribution for the Gromov-Hausdorff topology on $\mathbb{M}$ to the space (S, D^*).*

The strategy of the proof is to obtain properties of the geodesic paths that have to be satisfied in any distributional limit of $n^{-1/4}Q_n$, and which are sufficient to give "formulas" for distances in these limiting spaces that do not depend on the choice of subsequences. An important object of study is the occurence of certain types of *geodesic stars* in the Brownian map, i.e. of points from which radiate several disjoint geodesic *arms*. We hope that the present study will pave the way to a deeper understanding of geodesic stars in the Brownian map.

The rest of the paper is organised as follows. The next section recalls an important construction from [14] of the potential limits in distributions of the spaces $n^{-1/4}Q_n$, which will allow to reformulate slightly Theorem 1 into the alternative Theorem 3. Then we show how the proof of Theorem 3 can be obtained as a consequence of two intermediate statements, Propositions 6 and 8. Section 3 is devoted to the proof of the first proposition, while Section 4 reduces the proof of Proposition 8 to two key "elementary" statements, Lemmas 18 and 19, which deal with certain properties of geodesic stars in the Brownian map, with two or three arms. The proofs of these lemmas contains most of the novel ideas and techniques of the paper. Using a generalization of the Cori-Vauquelin-Schaeffer bijection found in [25], we will be able to translate the key statements in terms of certain probabilities for families of labeled maps, which have a simple enough structure so that we are able to derive their scaling limits. These discrete and continuous structures will be described in Sections 5 and 6, while Section 7 is finally devoted to the proof of the two key lemmas.

Let us end this Introduction by mentioning that, simultaneously to the elaboration of this work, Jean-François Le Gall independently found a proof of Theorem 1. His method is different from ours, and we believe that both approaches present specific interests.

Notation conventions. In this paper, we let $V(\mathbf{m}), E(\mathbf{m})$ and $F(\mathbf{m})$ be the sets of vertices, edges and faces of the map $\mathbf{m}$. Also, we let $\vec{E}(\mathbf{m})$ be the set of oriented edges of $\mathbf{m}$, so that $\#\vec{E}(\mathbf{m}) = 2\#E(\mathbf{m})$. If $e \in \vec{E}(\mathbf{m})$, we let $e_- e_+ \in V(\mathbf{m})$ be the origin and target of e. The reversal of $e \in \vec{E}(\mathfrak{s})$ is denoted by $\overline{e}$.

If $e \in \vec{E}(\mathbf{m}), f \in F(\mathbf{m}), v \in V(\mathbf{m})$, we say that f and e are incident if f lies *to the left of* e when following the orientation of e. We say that v and e are incident if $v = e_-$. If $e \in E(\mathbf{m})$ is not oriented, we say that f and e are incident if f is incident to e or $\overline{e}$. A similar definition holds for incidence between v and e. Finally, we say that f and v are incident if v and f are incident to a common $e \in E(\mathbf{m})$. The sets of vertices, edges and oriented edges incident to a face f are denoted by $V(f), E(f), \vec{E}(f)$. We will also use the notation $V(f \cap f'), E(f \cap f')$ for vertices and edges simultaneously incident to the two faces $f, f' \in F(\mathbf{m})$.

If $\mathbf{m}$ is a map and v, v' are vertices of $\mathbf{m}$, a *chain* from v to v' is a finite sequence $(e^{(1)}, e^{(2)}, \ldots, e^{(k)})$ of oriented edges, such that $e^{(1)}_- = v, e^{(k)}_+ = v'$, and $e^{(i)}_+ = e^{(i+1)}_-$ for every $i \in \{1, 2, \ldots, k-1\}$. The integer k is called the length of the chain, and we allow also the chain of length 0 from v to itself. The *graph distance* $d_{\mathbf{m}}(v, v')$ in $\mathbf{m}$ described above is then the minimal k such that there exists a chain with length k from v to v'. A chain with minimal length is called a *geodesic chain*.

In this paper, we will often let C denote a positive, finite constant, whose value may vary from line to line. Unless specified otherwise, the random variables considered in this paper are supposed to be defined on a probability space $(\Omega, \mathcal{F}, P)$.

2 Preliminaries

2.1 Extracting distributional limits from large quadrangulations

As mentioned in the introduction, it is known that the laws of the rescaled quadrangulations $(8n/9)^{-1/4}Q_n$ form a compact sequence of distributions on $\mathbb{M}$. Therefore, from every subsequence, it is possible to further extract a subsequence along which $(8n/9)^{-1/4}Q_n$ converges in distribution to a random variable (S', D') with values in $\mathbb{M}$. Theorem 1 then simply boils down to showing that this limit has the same distribution as (S, D^*), independently on the choices of subsequences.

In order to be able to compare efficiently the spaces (S', D') and (S, D^*), we perform a particular construction, due to [14], for which the spaces are the same quotient space, i.e. $S = S'$. This is not restrictive, in the sense that this construction can always be performed up to (yet) further extraction. We recall some of its important aspects.

Recall that a quadrangulation $\mathbf{q} \in \mathbf{Q}_n$, together with a distinguished vertex v_* can be encoded by a labeled tree with n edges *via* the so-called Cori-Vauquelin-Schaeffer bijection, which was introduced by Cori and Vauquelin [9], and considerably developed by Schaeffer, starting with his thesis [32]. If $(\mathbf{t}, \mathbf{l})$ is the resulting labeled tree, then the vertices of $\mathbf{q}$ distinct from v_* are exactly the vertices of $\mathbf{t}$, and $\mathbf{l}$ is, up to a shift by its global minimum over $\mathbf{t}$, the function giving the graph distances to v_* in $\mathbf{q}$. In turn, the tree $(\mathbf{t}, \mathbf{l})$ can be conveniently encoded by *contour and label functions*: Heuristically, this function returns the height (distance to the root of $\mathbf{t}$) and label of the vertex visited at time $0 \le k \le 2n$ when going around the tree clockwise. These functions are extended by linear interpolation to continuous functions on $[0, 2n]$.

If $\mathbf{q} = Q_n$ is a uniform random variable in $\mathbf{Q}_n$ and v_* is uniform among the $n + 2$ vertices of Q_n, then the resulting labeled tree (T_n, ℓ_n) has a contour and label function (C_n, L_n) such that C_n is a simple random walk in $\mathbb{Z}$, starting at 0 and ending at 0 at time $2n$, and conditioned to stay non-negative. Letting u_i^n be the vertex of $\mathbf{t}_n$ visited at step i of the contour, we let

$$D_n(i/2n, j/2n) = \left(\frac{9}{8n}\right)^{1/4} d_{Q_n}(u_i^n, u_j^n)\,, \qquad 0 \le i, j \le 2n\,,$$

and extend D_n to a continuous function on $[0, 1]^2$ by interpolation, see [14] for details. Then, the distributions of the triples of processes

$$\left(\left(\frac{C_n(2ns)}{\sqrt{2n}}\right)_{0\le s\le 1}, \left(\frac{L_n(2ns)}{(8n/9)^{1/4}}\right)_{0\le s\le 1}, (D_n(s,t))_{0\le s,t\le 1}\right), \qquad n \ge 1$$

form a relatively compact family of probability distributions. Therefore, from every subsequence, we can further extract a certain subsequence (n_k), along which the above triples converge in distribution to a limit $(\mathbf{e}, Z, D)$, where D is a random pseudo-distance on $[0, 1]$, and $(\mathbf{e}, Z)$ is the head of the Brownian snake process described in the Introduction. We may and will assume that this convergence holds a.s., by using the Skorokhod representation theorem, and up to changing the underlying probability space. Implicitly, until Section 4, all the integers n and limits $n \to \infty$ that are considered will be along the subsequence (n_k), or some further extraction.

The function D is a class function for $\{d_{\mathbf{e}} = 0\}$, which induces a pseudo-distance on $\mathcal{T}_{\mathbf{e}}/\{d_{\mathbf{e}} = 0\}$, still denoted by D for simplicity. Viewing successively D as a pseudo-distance on $[0,1]$ and $\mathcal{T}_{\mathbf{e}}$, we can let

$$S' = [0,1]/\{D = 0\} = \mathcal{T}_{\mathbf{e}}/\{D = 0\},$$

and endow it with the distance induced by D, still written D by a similar abuse of notation as above for simplicity. The space (S', D) is then a random geodesic metric space.

On the other hand, we can define D° and D^* out of $(\mathbf{e}, Z)$ as in the Introduction, and let $S = \mathcal{T}_{\mathbf{e}}/\{D^* = 0\}$. The main result of [14] is the following.

Proposition 2 (Theorem 3.4 in [14]) (i) *The three subsets of* $\mathcal{T}_{\mathbf{e}} \times \mathcal{T}_{\mathbf{e}}$

$$\{D = 0\}, \quad \{D^\circ = 0\}, \quad \{D^* = 0\}$$

are a.s. the same. In particular, the quotient sets S' *and* S *are a.s. equal, and* (S, D), (S, D^*) *are homeomorphic.*
(ii) *Along the subsequence* (n_k), $(Q_n, (9/8n)^{1/4} d_{Q_n})$ *converges a.s. in the Gromov-Hausdorff sense to* (S, D).

Using the last statement, Theorem 1 is now a consequence of the following statement, which is the result that we are going to prove in the remainder of this paper.

Theorem 3 *Almost-surely, it holds that* $D = D^*$.

2.2 A short review of results on S

A word on notation. Since we are considering several metrics D, D^* on the same set S, a little care is needed when we consider balls or geodesics, as we must mention to which metric we are referring to. For $x \in S$ and $r \geq 0$, we let

$$B_D(x,r) = \{y \in S : D(x,y) < r\}, \qquad B_{D^*}(x,r) = \{y \in S : D^*(x,y) < r\},$$

and we call them respectively the (open) D-ball and the D^*-ball with center r and radius r. Similarly, a continuous path γ in S will be called a D-geodesic, resp. a D^*-geodesic, if it is a geodesic path in (S, D) resp. (S, D^*). Note that since (S, D) and (S, D^*) are a.s. homeomorphic, a path in S is continuous for the metric D if and only if it is continuous for the metric D^*. When it is unambiguous from the context which metric we are dealing with, we sometimes omit the mention of D or D^* when considering balls or geodesics.

Basic properties of D, D°, D^*. Recall that $p_{\mathbf{e}} : [0,1] \to \mathcal{T}_{\mathbf{e}}$ is the canonical projection, we will also let $\mathbf{p}_Z : \mathcal{T}_{\mathbf{e}} \to S$ be the canonical projection, and $\mathbf{p} = \mathbf{p}_Z \circ p_{\mathbf{e}}$. The function D° was defined on $[0,1]^2$ and $(\mathcal{T}_{\mathbf{e}})^2$, it also induces a function on S^2 by letting

$$D^\circ(x,y) = \inf\{D^\circ(a,b) : a,b \in \mathcal{T}_{\mathbf{e}}, \mathbf{p}_Z(a) = x, \mathbf{p}_Z(b) = y\}\,.$$

Again, D° does not satisfy the triangle inequality. However, it holds that

$$D(x,y) \le D^*(x,y) \le D^\circ(x,y)\,, \qquad x,y \in S\,.$$

One of the difficulties in handling S is its definition using two successive quotients, so we will always mention whether we are considering D, D^*, D° on $[0,1], \mathcal{T}_{\mathbf{e}}$ or S.

We define the *uniform measure* λ on S to be the push-forward of the Lebesgue measure on $[0,1]$ by $\mathbf{p}$. This measure will be an important tool to sample points randomly on S.

Furthermore, by [19], the process Z attains its overall minimum a.s. at a single point $s_* \in [0,1]$, and the class $\rho = \mathbf{p}(s_*)$ is called the *root* of the space (S,D). One has, by [14],

$$D(\rho,x) = D^\circ(\rho,x) = D^*(\rho,x) = Z_x - \inf Z\,, \qquad \text{for every } x \in S\,. \tag{1}$$

Here we viewed Z as a function on S, which is licit because Z is a class function for $\{D = 0\}$, coming from the fact that $D(s,t) \ge |Z_s - Z_t|$ for every $s,t \in [0,1]$. The latter property can be easily deduced by passing to the limit from the discrete counterpart $D_n(i/2n, j/2n) \ge (9/8n)^{1/4}|L_n(i) - L_n(j)|$, which is a consequence of standard properties of the Cori-Vauquelin-Schaeffer bijection.

Geodesics from the root in S. Note that the definition of the function D° on $[0,1]^2$ is analogous to that of $d_{\mathbf{e}}$, using Z instead of $\mathbf{e}$. Similarly to $\mathcal{T}_{\mathbf{e}}$, we can consider yet another quotient $\mathcal{T}_Z = [0,1]/\{D^\circ = 0\}$, and endow it with the distance induced by D°. The resulting space is a random $\mathbb{R}$-*tree*, that is a geodesic metric space in which any two points are joined by a unique continuous injective path up to reparameterization. This comes from general results on encodings of $\mathbb{R}$-trees by continuous functions, see [10] for instance. The class of s_*, that we still call ρ, is distinguished as the root of $\mathcal{T}_Z$, and any point in this space (say encoded by the time $s \in [0,1]$) is joined to ρ by a unique geodesic. A formulation of the main result of [15] is that this path projects into (S,D) as a geodesic $\gamma^{(s)}$ from ρ to $\mathbf{p}(s)$, and that *any* D-geodesic from ρ can be described in this way. In particular, this implies that $D^\circ(x,y) = D(x,y) = Z_y - Z_x$ whenever x lies on a D-geodesic from ρ to y.

This implies the following improvement of (1). In a metric space (X,d), we say that (x,y,z) are aligned if $d(x,y) + d(y,z) = d(x,z)$: Note that this notion of alignment depends on the order in which x,y,z are listed (in fact, of the middle term only). In a geodesic metric space, this is equivalent to saying that y lies on a geodesic from x to z.

Lemma 4 *Almost surely, for every $x,y \in S$, it holds that (ρ,x,y) are aligned in (S,D) if and only if they are aligned in (S,D^*), and in this case it holds that*

$$D(x,y) = D^\circ(x,y) = D^*(x,y)$$

To prove this lemma, assume that (ρ, x, y) are aligned in (S, D). We already saw that this implies that $D(x,y) = D^\circ(x,y)$, so that necessarily $D^*(x,y) = D(x,y)$ as well, since $D \leq D^* \leq D^\circ$. By (1) this implies that (ρ, x, y) are aligned in (S, D^*), as well as the last conclusion of the lemma. Conversely, if (ρ, x, y) are aligned in (S, D^*), then by (1), the triangle inequality, and the fact that $D \leq D^*$,

$$D^*(\rho, y) = D(\rho, y) \leq D(\rho, x) + D(x, y) \leq D^*(\rho, x) + D^*(x, y) = D^*(\rho, y)\,,$$

meaning that there must be equality throughout. Hence (ρ, x, y) are aligned in (S, D), and we conclude as before.

Another important consequence of this description of geodesics is that the geodesics $\gamma^{(s)}, \gamma^{(t)} \in [0,1]$ are bound to merge into a single path in a neighborhood of ρ, a phenomenon called *confluence of geodesics*. This particularizes to the following statement.

Lemma 5 *Let $s, t \in [0,1]$. Then the images of $\gamma^{(s)}, \gamma^{(t)}$ coincide in the complement of $B_D(\mathbf{p}(s), D^\circ(s,t))$ (or of $B_D(\mathbf{p}(t), D^\circ(s,t))$).*

To prove this lemma, it suffices to note that $D^\circ(s,t)$ is the length of the path obtained by following $\gamma^{(s)}$ back from its endpoint $\mathbf{p}(s)$ until it coalesces with $\gamma^{(t)}$ in the tree $\mathcal{T}_Z$, and then following $\gamma^{(t)}$ up to $\mathbf{p}(t)$. Note that the same is true with D^* instead of D because of Lemma 4.

2.3 Plan of the proof

In this section, we decompose the proof of Theorem 3 into several intermediate statements. The two main ones (Propositions 6 and 8) will be proved in Sections 3 and 4.

The first step is to show that the distances D and D^* are almost equivalent distances.

Proposition 6 *Let $\alpha \in (0,1)$ be fixed. Then almost-surely, there exists a (random) $\varepsilon_1 > 0$ such that for every $x, y \in S$ with $D(x,y) \leq \varepsilon_1$, one has $D^*(x,y) \leq D(x,y)^\alpha$.*

The second step is based on a study of particular points of the space (S, D), from which emanate *stars* made of several disjoint geodesic paths, which we also call *arms* by analogy with the so-called "arm events" of percolation.

Definition 7 *Let (X, d) be a geodesic metric space, and $x_1, \ldots, x_k, x$ be $k+1$ points in X. We say that x is a k-star point with respect to $x_1, \ldots, x_k$ if for every geodesic paths (arms) $\gamma_1, \ldots, \gamma_k$ from x to $x_1, \ldots, x_k$ respectively, it holds that for every $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, the paths γ_i, γ_j intersect only at x. We let $\mathcal{G}(X; x_1, \ldots, x_k)$ be the set of points $x \in X$ that are k-star points with respect to $x_1, \ldots, x_k$.*

Conditionally on (S, D), let x_1, x_2 be random points of S with distribution λ. These points can be constructed by considering two independent uniform random variables U_1, U_2 in $[0,1]$, independent of $(\mathbf{e}, Z, D)$, and then setting $x_1 = \mathbf{p}(U_1)$ and $x_2 = \mathbf{p}(U_2)$. These random variables always exist up to enlarging the underlying probability space if necessary.

Then by [15, Corollary 8.3 (i)] (see also [25]), with probability 1, there is a unique D-geodesic from x_1 to x_2, which we call γ. By the same result, the geodesics from ρ to x_1 and x_2 are also unique, we call them γ_1 and γ_2. Moreover, ρ is a.s. not on γ, because γ_1, γ_2 share a common initial segment (this is the confluence property mentioned earlier). So trivially $D(x_1, \rho) + D(x_2, \rho) > D(x_1, x_2)$ meaning that (x_1, ρ, x_2) are not aligned.

We let

$$\Gamma = \gamma([0, D(x_1, x_2)]) \cap \mathcal{G}(S; x_1, x_2, \rho)\,.$$

Equivalently, with probability 1, it holds that $y \in \Gamma$ if and only if any geodesic from y to ρ does not intersect γ except at y itself: Note that the a.s. unique geodesic from y to x_1 is the segment of γ that lies between y and x_1, for otherwise, there would be several distinct geodesics from x_1 to x_2.

Proposition 8 *There exists $\delta \in (0, 1)$ for which the following property is satisfied almost-surely: There exists a (random) $\varepsilon_2 > 0$ such that for every $\varepsilon \in (0, \varepsilon_2)$, the set Γ can be covered with at most $\varepsilon^{-(1-\delta)}$ balls of radius ε in (S, D).*

Note that this implies a (quite weak) property of the Hausdorff dimension of Γ.

Corollary 9 *The Hausdorff dimension of Γ in (S, D) is a.s. strictly less than 1.*

We do not know the exact value of this dimension. The largest constant δ that we can obtain following the approach of this paper is not much larger than 0.00025, giving an upper bound of 0.99975 for the Hausdorff dimension of Γ.

With Propositions 6 and 8 at hand, proving Theorem 3 takes only a couple of elementary steps, which we now perform.

Lemma 10 *Let (s, t) be a non-empty subinterval of $[0, D(x_1, x_2)]$ such that $\gamma(v) \notin \Gamma$ for every $v \in (s, t)$. Then, there exists a unique $u \in [s, t]$ such that*

- *$(\rho, \gamma(s), \gamma(u))$ are aligned, and*
- *$(\rho, \gamma(t), \gamma(u))$ are aligned.*

Proof. Fix $v \in (s, t)$. Since $\gamma(v) \notin \Gamma$, there exists a geodesic from $\gamma(v)$ to ρ that intersects the image of γ at some point $\gamma(v')$, with $v' \neq v$. In particular, the points $(\rho, \gamma(v'), \gamma(v))$ are aligned. Let us assume first that $v' < v$, and let

$$w = \inf\{v'' \in [s, v] : (\rho, \gamma(v''), \gamma(v)) \text{ are aligned}\}\,.$$

Then $w \in [s, v)$ since v' is in the above set. Let us show that $w = s$. If if was true that $w > s$, then $\gamma(w)$ would not be an element of Γ, and some geodesic from $\gamma(w)$ to ρ would thus have to intersect the image of γ at some point $\gamma(w')$ with $w' \neq w$. But we cannot have $w' < w$ by the minimality property of w, and we cannot have $w' > w$ since otherwise, both $(\rho, \gamma(w), \gamma(w'))$ on the one hand and, $(\rho, \gamma(w'), \gamma(w))$ on the other hand, would be aligned. This shows that $w = s$, and this implies that $(\rho, \gamma(s), \gamma(v))$ are aligned. By a similar reasoning, if it holds that $v' > v$, then $(\rho, \gamma(t), \gamma(v))$ are aligned.

At this point, we can thus conclude that for every $v \in (s, t)$, either $(\rho, \gamma(s), \gamma(v))$ are aligned or $(\rho, \gamma(t), \gamma(v))$ are aligned. The conclusion follows by defining

$$u = \sup\{u' \in [s, t] : (\rho, \gamma(s), \gamma(u')) \text{ are aligned}\},$$

as can be easily checked. □

Let δ be as in Proposition 8. Let $\varepsilon > 0$ be chosen small enough so that Γ is covered by balls $B_D(x_{(i)}, \varepsilon), 1 \le i \le K$, for some $x_{(1)}, x_{(2)}, \dots, x_{(K)}$ with $K = \lfloor \varepsilon^{-(1-\delta)} \rfloor$. Without loss of generality, we can assume that $\{x_1, x_2\} \subset \{x_{(1)} \dots, x_{(K)}\}$, up to increasing K by 2, or leaving K unchanged at the cost of taking smaller δ and ε. We can also assume that all the balls $B_D(x_{(i)}, \varepsilon)$ have a non-empty intersection with the image of γ, by discarding all the balls for which this property does not hold. Let

$$r_i = \inf\{t \ge 0 : \gamma(t) \in B_D(x_{(i)}, \varepsilon)\}, \qquad r'_i = \sup\{t \le D(x_1, x_2) : \gamma(t) \in B_D(x_{(i)}, \varepsilon)\},$$

so that $r_i < r'_i$ for $i \in \{1, 2, \dots, K\}$, since $B_D(x_{(i)}, \varepsilon)$ is open, and let

$$A = \bigcup_{i=1}^{K} [r_i, r'_i], \tag{2}$$

so that $\Gamma \subset \gamma(A)$.

Lemma 11 *For every $i \in \{1, 2, \dots, K\}$ and $r \in [r_i, r'_i]$, one has $D(\gamma(r), x_{(i)}) \le 2\varepsilon$.*

Proof. If this was not the case, then since γ is a D-geodesic path that passes through $\gamma(r_i), \gamma(r)$ and $\gamma(r'_i)$ in this order, we would have $D(\gamma(r_i), \gamma(r'_i)) = D(\gamma(r_i), \gamma(r)) + D(\gamma(r), \gamma(r'_i)) > 2\varepsilon$. But on the other hand,

$$D(\gamma(r_i), \gamma(r'_i)) \le D(\gamma(r_i), x_{(i)}) + D(x_{(i)}, \gamma(r'_i)) < 2\varepsilon,$$

a contradiction. □

Let I be the set of indices $j \in \{1, 2, \dots, K\}$ such that $[r_j, r'_j]$ is maximal for the inclusion order in the family $\{[r_j, r'_j], 1 \le j \le K\}$. Note that the set A of (2) is also equal to $A = \bigcup_{i \in I} [r_i, r'_i]$. This set can be uniquely rewritten in the form $A = \bigcup_{i=0}^{K'-1} [t_i, s_{i+1}]$ where $K' \le \#I$, and

$$0 = t_0 < s_1 < t_1 < s_2 < \dots < s_{K'-1} < t_{K'-1} < s_{K'} = D(x_1, x_2).$$

Here, the fact that $t_0 = 0$ and $s_{K'} = D(x_1, x_2)$ comes from the assumption that x_1, x_2 are in the set $\{x_{(i)}, 1 \le i \le K\}$. We let $x_{(i)} = \gamma(s_i)$ for $1 \le i \le K'$ and $y_{(i)} = \gamma(t_i)$ for $0 \le i \le K' - 1$.

Lemma 12 *Almost-surely, it holds that for every ε small enough,*

$$\sum_{i=0}^{K'-1} D^*(y_{(i)}, x_{(i+1)}) \leq 4K\varepsilon^{(2-\delta)/2} .$$

Proof. Consider one of the connected components $[t_i, s_{i+1}]$ of A. Let

$$J_i = \{j \in I : [r_j, r'_j] \subset [t_i, s_{i+1}]\} ,$$

so that $\sum_{0\leq i\leq K'-1} \#J_i = \#I \leq K$. The lemma will thus follow if we can show that $D^*(y_{(i)}, x_{(i+1)}) \leq 2\#J_i\varepsilon^{(2-\delta)/2}$ for every ε small enough. By the definition of I, it holds that if $[r_j, r'_j]$ and $[r_k, r'_k]$ with $j, k \in I$ have non-empty intersection, then these two intervals necessarily overlap, meaning that $r_j \leq r_k \leq r'_j \leq r'_k$ or vice-versa. Let us re-order the $r_j, j \in J_i$ as $r_{j_k}, 1 \leq k \leq \#J_i$ in non-decreasing order. Then $\gamma(r_{j_1}) = y_{(i)}$ and $\gamma(r'_{j_{\#J_i}}) = x_{(i+1)}$, and

$$D^*(y_{(i)}, x_{(i+1)}) \leq \sum_{k=1}^{\#J_i-1} D^*(\gamma(r_{j_k}), \gamma(r_{j_{k+1}})) + D^*(\gamma(r_{j_{\#J_i}}), x_{(i+1)}) . \tag{3}$$

By Lemma 11 and the overlapping property of the intervals $[r_j, r'_j]$, we have that

$$D(\gamma(r_{j_k}), \gamma(r_{j_{k+1}})) \leq 4\varepsilon , \qquad D(\gamma(r_{j_{\#J_i}}), x_{(i+1)}) \leq 4\varepsilon .$$

Now apply Proposition 6 with $\alpha = (2-\delta)/2$ to obtain from (3) that a.s. for every ε small enough,

$$D^*(y_{(i)}, x_{(i+1)}) \leq \#J_i(4\varepsilon)^{(2-\delta)/2} \leq 4\#J_i\varepsilon^{(2-\delta)/2} ,$$

which concludes the proof. □

We can now finish the proof of Theorem 3. The complement of the set A in $[0, D(x_1, x_2)]$ is the union of the intervals (s_i, t_i) for $1 \leq i \leq K'-1$. Now, for every such i, the image of the interval (s_i, t_i) by γ does not intersect Γ, since $\Gamma \subset \gamma(A)$ and γ is injective. By Lemma 10, for every $i \in \{1, 2, \ldots, K'-1\}$, we can find $u_i \in [s_i, t_i]$ such that $(\rho, \gamma(s_i), \gamma(u_i))$ are aligned, as well as $(\rho, \gamma(t_i), \gamma(u_i))$. Letting $x_{(i)} = \gamma(s_i), y_{(i)} = \gamma(t_i)$ and $z_{(i)} = \gamma(u_i)$, by Lemma 4,

$$D^*(x_{(i)}, z_{(i)}) = D(x_{(i)}, z_{(i)}) , \qquad D^*(y_{(i)}, z_{(i)}) = D(y_{(i)}, z_{(i)}) , \qquad 1 \leq i \leq K'-1 .$$

By the triangle inequality, and the fact that γ is a D-geodesic, we have a.s.

$$\begin{aligned} D^*(x_1, x_2) &\leq \sum_{i=1}^{K'-1} (D^*(x_{(i)}, z_{(i)}) + D^*(z_{(i)}, y_{(i)})) + \sum_{i=0}^{K'-1} D^*(y_{(i)}, x_{(i+1)}) \\ &= \sum_{i=1}^{K'-1} (D(x_{(i)}, z_{(i)}) + D(z_{(i)}, y_{(i)})) + \sum_{i=0}^{K'-1} D^*(y_{(i)}, x_{(i+1)}) \\ &\leq D(x_1, x_2) + \sum_{i=0}^{K'-1} D^*(y_{(i)}, x_{(i+1)}) \\ &\leq D(x_1, x_2) + 4K\varepsilon^{(2-\delta)/2} , \end{aligned}$$

where we used Lemma 12 at the last step, assuming ε is small enough. Since $K \le \varepsilon^{-(1-\delta)}$, this is enough to get $D^*(x_1, x_2) \le D(x_1, x_2)$ by letting $\varepsilon \to 0$. Since $D \le D^*$ we get $D(x_1, x_2) = D^*(x_1, x_2)$.

Note that the previous statement holds a.s. for $\lambda \otimes \lambda$-almost every $x_1, x_2 \in S$. This means that if $x_1, x_2, \dots$ is an i.i.d. sequence of λ-distributed random variables (this can be achieved by taking an i.i.d. sequence of uniform random variables on $[0,1]$, independent of the Brownian map, and taking their images under $\mathbf{p}$), then almost-surely one has $D^*(x_i, x_j) = D(x_i, x_j)$ for every $i, j \ge 1$. But since the set $\{x_i, i \ge 1\}$ is a.s. dense in (S, D) (or in (S, D^*)), the measure λ being of full support, we obtain by a density argument that a.s., for every $x, y \in S$, $D^*(x, y) = D(x, y)$. This ends the proof of Theorem 3.

3 Rough comparison between D and D^*

The goal of this Section is to prove Proposition 6. We start with an elementary statement in metric spaces.

Lemma 13 *Let (X, d) be an arcwise connected metric space, and let x, y be two distinct points in X. Let γ be a continuous path with extremities x and y. Then for every $\eta > 0$, there exist at least $K = \lfloor d(x, y)/(2\eta) \rfloor + 1$ points $y_1, \dots, y_K$ in the image of γ such that $d(y_i, y_j) \ge 2\eta$ for every $i, j \in \{1, 2, \dots, K\}$ with $i \ne j$.*

Proof. Let us assume without loss of generality that γ is parameterized by $[0, 1]$ and that $\gamma(0) = x, \gamma(1) = y$. Also, assume that $d(x, y) \ge 2\eta$, since the statement is trivial otherwise. In this case we have $K \ge 2$.

Set $s_0 = 0$, and by induction, let

$$s_{i+1} = \sup\{t \le 1 : d(\gamma(t), \gamma(s_i)) < 2\eta\}, \qquad i \ge 0.$$

The sequence $(s_i, i \ge 0)$ is non-decreasing, and it holds that $d(\gamma(s_i), \gamma(s_{i+1})) \le 2\eta$ for every $i \ge 0$. Let $i \in \{0, 1, \dots, K-2\}$. Then, since $x = \gamma(0) = \gamma(s_0)$,

$$d(x, \gamma(s_i)) \le \sum_{j=0}^{i-1} d(\gamma(s_j), \gamma(s_{j+1})) \le 2\eta i \le 2\eta(K-2) \le d(x, y) - 2\eta,$$

which implies, by the triangle inequality,

$$d(\gamma(s_i), y) \ge d(x, y) - d(x, \gamma(s_i)) \ge d(x, y) - (d(x, y) - 2\eta) = 2\eta.$$

From this, and the definition of $(s_i, i \ge 0)$ it follows that $d(\gamma(s_i), \gamma(s_j)) \ge 2\eta$ for every $i \in \{0, 1, \dots, K-2\}$ and $j > i$. This yields the wanted result, setting $y_i = \gamma(s_{i-1})$ for $1 \le i \le K$. □

Our next tool is a uniform estimate for the volume of D-balls in S.

Lemma 14 *Let $\eta \in (0,1)$ be fixed. Then almost-surely, there exists a (random) $C \in (0,\infty)$ such that for every $r \geq 0$ and every $x \in S$, one has*

$$\lambda(B_D(x,r)) \leq Cr^{4-\eta}\,.$$

This is an immediate consequence of a result by Le Gall [15, Corollary 6.2], who proves the stronger fact that the optimal random "constant" C of the statement has moments of all orders. In the remainder of this section, we will use let C, c for such almost-surely finite positive random variables. As long as no extra property besides the a.s. finiteness of these random variables is required, we keep on calling them C, c even though they might differ from statement to statement or from line to line, just as if they were universal constants.

The following statement is a rough uniform lower estimate for D^*-balls.

Lemma 15 *Let $\eta \in (0,1)$ be fixed. Then almost-surely, there exists a (random) $c \in (0,\infty)$ and $r_0 > 0$ such that for every $r \in [0, r_0]$ and every $x \in S$, one has*

$$\lambda(B_{D^*}(x,r)) \geq cr^{4+\eta}\,.$$

Proof. We use the fact that $B_{D^\circ}(x,r) \subseteq B_{D^*}(x,r)$ for every $x \in S$ and $r \geq 0$, where $B_{D^\circ}(x,r) = \{y \in S : D^\circ(x,y) < r\}$. We recall that D° does not satisfy the triangle inequality, which requires a little extra care when manipulating "balls" of the form $B_{D^\circ}(x,r)$.

By definition, $D^\circ(x,y) = \inf_{s,t} D^\circ(s,t)$ where the infimum is over $s,t \in [0,1]$ such that $\mathbf{p}(s) = x, \mathbf{p}(t) = y$. Consequently, for every $s \in [0,1]$ such that $\mathbf{p}(s) = x$,

$$\mathbf{p}(\{t \in [0,1] : D^\circ(s,t) < r\}) \subseteq B_{D^\circ}(x,r)\,,$$

which implies, by definition of λ,

$$\lambda(B_{D^*}(x,r)) \geq \lambda(B_{D^\circ}(x,r)) \geq \mathrm{Leb}(\{t \in [0,1] : D^\circ(s,t) < r\})\,.$$

Consequently, for every $r > 0$,

$$\lambda(B_{D^*}(x,r)) \geq \mathrm{Leb}(\{t \in [0,1] : D^\circ(s,t) \leq r/2\})\,. \tag{4}$$

We use the fact that Z is a.s. Hölder-continuous with exponent $1/(4+\eta)$, which implies that a.s. there exists a random $C \in (0,\infty)$, such that for every $h \geq 0$, $\omega(Z,h) \leq Ch^{1/(4+\eta)}$, where

$$\omega(Z,h) = \sup\{|Z_t - Z_s| : s,t \in [0,1], |t-s| \leq h\}$$

is the oscillation of Z. Since

$$D^\circ(s,t) \leq Z_s + Z_t - 2\min_{u \in [s\wedge t, s\vee t]} Z_u \leq 2\omega(Z, |t-s|)\,,$$

we obtain that for every $s \in [0,1]$, for every $h > 0$ and $t \in [(s-h)\vee 0, (s+h)\wedge 1]$,

$$D^\circ(s,t) \leq 2Ch^{1/(4+\eta)}\,.$$

Letting $h = (r/(4C))^{4+\eta}$ yields

$$\mathrm{Leb}(\{t \in [0,1] : D^\circ(s,t) \le r/2\}) \ge (2h) \wedge 1\,.$$

This holds for every $s \in [0,1], r \ge 0$ so that, by (4),

$$\lambda(B_{D^*}(x,r)) \ge \Big(\frac{2}{(4C)^{4+\eta}} r^{4+\eta}\Big) \wedge 1\,.$$

This yields the wanted result with $r_0 = 4C/2^{1/(4+\eta)}$ and $c = 2/(4C)^{4+\eta}$. □

We are now able to prove Proposition 6, and we argue by contradiction. Let $\alpha \in (0,1)$ be fixed, and assume that the statement of the proposition does not hold. This implies that with positive probability, one can find two sequences $(x_n, n \ge 0)$ and $(y_n, n \ge 0)$ of points in S such that $D(x_n,y_n)$ converges to 0, and $D^*(x_n,y_n) > D(x_n,y_n)^\alpha$ for every $n \ge 0$. From now on until the end of the proof, we restrict ourselves to this event of positive probability, and almost-sure statements will implicitly be in restriction to this event.

Let γ_n be a D-geodesic path from x_n to y_n. Let $V^D_\beta(\gamma_n)$ be the D,β-thikening of the image of γ_n:

$$V^D_\beta(\gamma_n) = \{x \in S : \exists t \in [0, D(x_n,y_n)], D(\gamma_n(t),x) < \beta\}\,.$$

Then $V^D_\beta(\gamma_n)$ is contained in a union of at most $\lfloor D(x_n,y_n)/(2\beta)\rfloor + 1$ D-balls of radius 2β: Simply take centers y_n and $\gamma_n(2\beta k)$ for $1 \le k \le \lfloor D(x_n,y_n)/(2\beta)\rfloor$. Consequently, for every $\beta > 0$,

$$\lambda(V^D_\beta(\gamma_n)) \le \Big(\frac{D(x_n,y_n)}{2\beta} + 1\Big) \sup_{x\in S} \lambda(B_D(x,2\beta))\,.$$

By applying Lemma 14, for any $\eta \in (0,1)$, whose value will be fixed later on, we obtain a.s. the existence of $C \in (0,\infty)$ such that for every $n \ge 0$ and $\beta \in [0, D(x_n,y_n)]$,

$$\lambda(V^D_\beta(\gamma_n)) \le C\beta^{3-\eta} D(x_n,y_n)\,. \tag{5}$$

Let $V^{D^*}_\beta(\gamma_n)$ be the D^*,β-thikening of γ_n, defined as $V^D_\beta(\gamma_n)$ above but with D^* instead of D. The spaces (S,D^*) and (S,D) being homeomorphic, we obtain that (S,D^*) is arcwise connected and γ_n is a continuous path in this space. Therefore Lemma 3 applies: For every $\beta > 0$ we can find points $y_1,\ldots,y_K$, $K = \lfloor D^*(x_n,y_n)/(2\beta)\rfloor + 1$, such that $D^*(y_i,y_j) \ge 2\beta$ for every $i,j \in \{1,2,\ldots,K\}$ with $i \ne j$. From this, it follows that the balls $B_{D^*}(y_i,\beta), 1 \le i \le K$ are pairwise disjoint, and they are all included in $V^{D^*}_\beta(\gamma_n)$. Hence,

$$\lambda(V^{D^*}_\beta(\gamma_n)) \ge \sum_{i=1}^K \lambda(B_{D^*}(y_i,\beta)) \ge K \inf_{x\in S} \lambda(B_{D^*}(x,\beta)) \ge \frac{D^*(x_n,y_n)}{2\beta} \inf_{x\in S} \lambda(B_{D^*}(x,\beta))\,.$$

For the same $\eta \in (0,1)$ as before, by using Lemma 15 and the definition of x_n, y_n, we conclude that a.s. there exists $c \in (0,\infty)$ and $r_0 > 0$ such that for every $\beta \in [0, r_0]$,

$$\lambda(V_\beta^{D^*}(\gamma_n)) \geq c\beta^{3+\eta} D^*(x_n, y_n) \geq c\beta^{3+\eta} D(x_n, y_n)^\alpha . \tag{6}$$

But since $D \leq D^*$, we have that $V_\beta^{D^*}(\gamma_n) \subseteq V_\beta^D(\gamma_n)$, so that (5) and (6) entail that for every $\beta \in [0, D(x_n, y_n) \wedge r_0]$,

$$\beta^{2\eta} \leq CD(x_n, y_n)^{1-\alpha} ,$$

for some a.s. finite $C > 0$. But taking $\eta = (1-\alpha)/4$ and then $\beta = D(x_n, y_n) \wedge r_0$, we obtain since $D(x_n, y_n) \to 0$ that $D(x_n, y_n)^{(1-\alpha)/2} = O(D(x_n, y_n)^{1-\alpha})$, which is a contradiction. This ends the proof of Proposition 6. □

4 Covering 3-star points on typical geodesics

We now embark in our main task, which is to prove Proposition 8.

4.1 Entropy number estimates

We use the same notation as in Section 2.3. In this section, we are going to fix two small parameters $\delta, \beta \in (0, 1/2)$, which will be tuned later on: The final value of δ will be the one that appears in Proposition 8.

We want to estimate the number of D-balls of radius ε needed to cover the set Γ. Since we are interested in bounding this number by $\varepsilon^{-(1-\delta)}$ we can consider only the points of Γ that lie at distance at least $8\varepsilon^{1-\beta}$ from x_1, x_2 and ρ. Since Γ is included in the image of γ, the remaining part of Γ can certainly be covered with at most $32\varepsilon^{-\beta}$ balls of radius ε. Since we chose $\beta, \delta < 1/2$, we have $\beta < 1 - \delta$, so this extra number of balls will be negligible compared to $\varepsilon^{-(1-\delta)}$.

So for $\varepsilon > 0$, let $\mathcal{N}_\Gamma(\varepsilon)$ be the minimal $n \geq 1$ such that there exist n points $x_{(1)}, \dots, x_{(n)} \in S$ such that

$$\Gamma \setminus (B_D(x_1, 8\varepsilon^{1-\beta}) \cup B_D(x_2, 8\varepsilon^{1-\beta}) \cup B_D(\rho, 8\varepsilon^{1-\beta})) \subset \bigcup_{i=1}^n B_D(x_{(i)}, \varepsilon) .$$

We call Γ_ε the set on the left-hand side. We first need a simple control on D°.

Lemma 16 *Fix $\delta > 0$. Almost surely, there exists a (random) $\varepsilon_3(\delta) \in (0,1)$ such that for every $t \in [0,1]$, $\varepsilon \in (0, \varepsilon_3(\delta))$ and $s \in [(t - \varepsilon^{4+\delta}) \vee 0, (t + \varepsilon^{4+\delta}) \wedge 1]$, it holds that $D^\circ(s,t) \leq \varepsilon/2$.*

Proof. This is an elementary consequence of the fact that Z is a.s. Hölder-continuous with any exponent $\alpha \in (0, 1/4)$, and the definition of D°. From this, we get that $D^\circ(s,t) \leq 2K_\alpha |t-s|^\alpha$, where K_α is the α-Hölder norm of Z. If $|t-s| \leq \varepsilon^{4+\delta}$, then

$D^\circ(s,t) \leq 2K_\alpha \varepsilon^{(4+\delta)\alpha}$, so it suffices to choose $\alpha \in (1/(4+\delta), 1/4)$, and then $\varepsilon_3(\delta) = (4K_\alpha)^{-((4+\delta)\alpha-1)^{-1}} \wedge (1/2)$. $\square$

For every $y \in S$, let $t \in [0,1]$ such that $\mathbf{p}(t) = y$. We let

$$F(y,\varepsilon) = \mathbf{p}([(t-\varepsilon^{4+\delta}) \vee 0, (t+\varepsilon^{4+\delta}) \wedge 1]), \qquad \varepsilon > 0.$$

Note that in general, $F(y,\varepsilon)$ does depend on the choice of t, so we let this choice be arbitrary, for instance, t can be the smallest possible in $[0,1]$. Lemma 16 and the fact that $D \leq D^\circ$ entails that $F(y,\varepsilon)$ is included in $B_D(y,\varepsilon/2)$ for every $\varepsilon \in (0,\varepsilon_3(\delta))$, but $F(y,\varepsilon)$ is not necessarily a neighborhood of y in S. We can imagine it as having the shape of a fan with apex at y.

Lemma 17 *Let $\delta > 0$ and $\varepsilon \in (0,\varepsilon_3(\delta))$ be as in Lemma 16. Let $t_{(1)}, t_{(2)}, \ldots, t_{(N)}$ be elements in $[0,1]$ such that the intervals $[(t_{(i)} - \varepsilon^{4+\delta}/2) \vee 0, (t_{(i)} + \varepsilon^{4+\delta}/2) \wedge 1], 1 \leq i \leq N$ cover $[0,1]$. Then, letting $x_{(i)} = \mathbf{p}(t_{(i)})$,*

$$\mathcal{N}_\Gamma(\varepsilon) \leq \sum_{i=1}^N \mathbf{1}_{\{x_{(i)} \in \bigcup_{y \in \Gamma_\varepsilon} F(y,\varepsilon)\}}.$$

Proof. Let $y \in \Gamma_\varepsilon$ be given. By assumption, for every $t \in [0,1]$, the interval $[(t-\varepsilon^{4+\delta}) \vee 0, (t+\varepsilon^{4+\delta}) \wedge 1]$ contains at least one point $t_{(i)}$. By definition, this means that $F(y,\varepsilon)$ contains at least one of the points $x_{(i)}, 1 \leq i \leq N$. By the assumption on ε, we obtain that $y \in B_D(x_{(i)}, \varepsilon/2)$. Therefore, the union of balls $B_D(x_{(i)}, \varepsilon)$, where $x_{(i)}$ is contained in $F(y,\varepsilon)$ for some $y \in \Gamma_\varepsilon$, cover Γ_ε. $\square$

Let $(U_0^{(i)}, i \geq 1)$ be an i.i.d. sequence of uniform random variables on $[0,1]$, independent of $(\mathbf{e}, Z, D)$ and of U_1, U_2. We let $x_0^{(i)} = \mathbf{p}(U_0^{(i)})$. For $\varepsilon > 0$ let $N_\varepsilon = \lfloor \varepsilon^{-4-2\delta} \rfloor$. The probability of the event $\mathcal{B}_\varepsilon$ that the intervals $[(U_0^{(i)} - \varepsilon^{4+\delta}/2) \vee 0, (U_0^{(i)} + \varepsilon^{4+\delta}/2) \wedge 1], 1 \leq i \leq N_\varepsilon$ do not cover $[0,1]$ is less than the probability that there exists a $j \leq 2\varepsilon^{-4-\delta}$ such that $[j\varepsilon^{4+\delta}/2, ((j+1)\varepsilon^{4+\delta}/2) \wedge 1]$ does not contain any of the $U_0^{(i)}, 1 \leq i \leq N_\varepsilon$. This has a probability at most $2\varepsilon^{-4-\delta} e^{-\varepsilon^{-\delta}/2}$, which decays faster than any power of ε as $\varepsilon \to 0$. Using Lemma 17, we get the existence of a finite constant C such that for every $\varepsilon > 0$,

$$\begin{aligned} & P(\mathcal{N}_\Gamma(\varepsilon) \geq \varepsilon^{-(1-\delta)}, \varepsilon \leq \varepsilon_3(\delta)) \\ \leq \; & P(\mathcal{B}_\varepsilon) + P\left(\sum_{i=1}^{N_\varepsilon} \mathbf{1}_{\{x_0^{(i)} \in \bigcup_{y \in \Gamma} F(y,\varepsilon)\}} \geq \varepsilon^{-(1-\delta)}, \varepsilon \leq \varepsilon_3(\delta)\right) \\ \leq \; & C\varepsilon^4 + \varepsilon^{1-\delta}\varepsilon^{-4-2\delta} P\left(x_0 \in \bigcup_{y \in \Gamma_\varepsilon} F(y,\varepsilon), \varepsilon \leq \varepsilon_3(\delta)\right), \end{aligned}$$

where we used the Markov inequality in the second step, and let $x_0 = x_0^{(1)}$.

The uniqueness properties of geodesics in the Brownian map already mentioned before imply that a.s. there is a unique geodesic γ_0 from x_0 to ρ, and a unique geodesic between

x_i and x_j for $i,j \in \{0,1,2\}$. We let γ_{01}, γ_{02} be the geodesics from x_0 to x_1 and x_2, and recall that γ is the unique geodesic between x_1 and x_2.

Since $F(y,\varepsilon) \subseteq B_D(y,\varepsilon/2)$ whenever $\varepsilon \le \varepsilon_3(\delta)$, and $\Gamma_\varepsilon \subset \gamma$, we get that if $x_0 \in \bigcup_{y\in\Gamma_\varepsilon} F(y,\varepsilon)$ then $B_D(x_0,\varepsilon/2)$ intersects γ. Moreover, recall from Lemma 5 that D° is a measure of how quickly two geodesics in the Brownian map coalesce. By definition, we have $D^\circ(x_0,y) \le \varepsilon/2$ whenever $x_0 \in F(y,\varepsilon)$, so that outside of $B_D(x_0,\varepsilon/2)$, the image of γ_0 is included in some geodesic γ' going from y to ρ. Since $y \in \Gamma$, by definition, γ' does not intersect γ_{12} except at the point y itself.

We conclude that the event $\mathcal{A}_0(\varepsilon) = \{x_0 \in \bigcup_{y\in\Gamma_\varepsilon} F(y,\varepsilon) \text{ and } \varepsilon \in (0,\varepsilon_3(\delta))\}$ implies that either of the events $\mathcal{A}_1(\varepsilon,\beta)$ or $\mathcal{A}_2(2\varepsilon^{1-\beta})$ occurs, where

- $\mathcal{A}_1(\varepsilon,\beta)$ is the event that $D(x_0,x_1)\wedge D(x_0,x_2) \ge 7\varepsilon^{1-\beta}$, that γ intersects $B_D(x_0,\varepsilon/2)$, and there exists a point of $\gamma_{01}\cup\gamma_{02}$ out of $B_D(x_0,2\varepsilon^{(1-\beta)})$ not belonging to γ,
- $\mathcal{A}_2(\varepsilon)$ is the event that $D(x_0,x_1)\wedge D(x_0,x_2)\wedge D(x_0,\rho) \ge 7\varepsilon/2$, that γ intersects $B_D(x_0,\varepsilon)$, and that the geodesics $\gamma_0,\gamma_{01},\gamma_{02}$ do not intersect outside of $B_D(x_0,\varepsilon)$.

Indeed, on $\mathcal{A}_0(\varepsilon)$, we first note that $D(x_0,\Gamma_\varepsilon) < \varepsilon$, and since

$$D(x_1,\Gamma_\varepsilon)\wedge D(x_2,\Gamma_\varepsilon)\wedge D(\rho,\Gamma_\varepsilon) \ge 8\varepsilon^{1-\beta}$$

we obtain

$$D(x_0,x_1)\wedge D(x_0,x_2)\wedge D(x_0,\rho) \ge 7\varepsilon^{1-\beta} = (7/2)\cdot 2\varepsilon^{1-\beta}\,.$$

Then on $\mathcal{A}_0(\varepsilon)\setminus\mathcal{A}_1(\varepsilon,\beta)$, outside of $B_D(x_0,2\varepsilon^{1-\beta})$, the geodesics γ_{01} and γ_{02} are included in γ, and γ_0 is included in the geodesic γ' discussed above, so $\mathcal{A}_2(2\varepsilon^{1-\beta})$ occurs.

The key lemmas give an estimation of the probabilities for these events.

Lemma 18 *For every $\beta \in (0,1)$, there exists $C \in (0,\infty)$ such that for every $\varepsilon > 0$,*

$$P(\mathcal{A}_1(\varepsilon,\beta)) \le C\varepsilon^{3+\beta}\,.$$

Lemma 19 *There exist $\chi \in (0,1)$ and $C \in (0,\infty)$ such that for every $\varepsilon > 0$,*

$$P(\mathcal{A}_2(\varepsilon)) \le C\varepsilon^{3+\chi}\,.$$

Taking these lemmas for granted, we can now conclude the proof of Proposition 8. The constant χ of Lemma 19 will finally allow us to tune the parameters δ,β, and we choose first β so that $(1-\beta)(3+\chi) > 3$, and then $\delta > 0$ so that

$$\delta < \frac{\beta}{3} \wedge \frac{(1-\beta)(3+\chi)-3}{3}\,.$$

From our discussion, we have, for $\varepsilon_3 > 0$ fixed and $\varepsilon \in (0,\varepsilon_3)$

$$\begin{aligned} & P(\mathcal{N}_\Gamma(\varepsilon) \ge \varepsilon^{-(1-\delta)}, \varepsilon_3(\delta) \ge \varepsilon_3) \\ \le\ & C\varepsilon^4 + \varepsilon^{-3-3\delta}(P(\mathcal{A}_1(\varepsilon,\beta)) + P(\mathcal{A}_2(2\varepsilon^{1-\beta}))) \\ \le\ & C(\varepsilon^4 + \varepsilon^{\beta-3\delta} + \varepsilon^{(1-\beta)(3+\chi)-3-3\delta})\,. \end{aligned} \tag{7}$$

By our choice of δ, β, the exponents in (7) are strictly positive. This gives the existence of some $\psi > 0$ such that for every $\varepsilon_3 > 0$, there exists a $C > 0$ such that $\varepsilon \le \varepsilon_3$ implies

$$P(\mathcal{N}_\Gamma(\varepsilon) \ge \varepsilon^{-(1-\delta)}, \varepsilon_3(\delta) \ge \varepsilon_3) \le C\varepsilon^\psi .$$

Applying this first to ε of the form $2^{-k}, k \ge 0$ and using the Borel-Cantelli Lemma and the monotonicity of $\mathcal{N}_\Gamma(\varepsilon)$, we see that a.s., on the event $\{\varepsilon_3(\delta) \ge \varepsilon_3\}$ for every $\varepsilon > 0$ small enough, $\mathcal{N}_\Gamma(\varepsilon) < \varepsilon^{-(1-\delta)}$. Since $\varepsilon_3(\delta) > 0$ a.s., we obtain the same result without the condition $\varepsilon_3(\delta) \ge \varepsilon_3$. This proves Proposition 8, and it remains to prove Lemmas 18 and 19.

4.2 Back to geodesic stars in discrete maps

Our strategy to prove Lemmas 18 and 19 is to relate them back to asymptotic properties of random quadrangulations. In turn, these properties can be obtained by using a bijection between quadrangulations and certain maps with a simpler structure, for which the scaling limits can be derived (and do not depend on the subsequence (n_k) used to define the space (S, D)).

We start by reformulating slightly the statements of Lemmas 18 and 19, in a way that is more symmetric in the points ρ, x_1, x_2, x_0 that are involved. For this, we use the invariance under re-rooting of the Brownian map [15, Theorem 8.1] stating that ρ has the same role as a uniformly chosen point in S according to the distribution λ. So we let $x_0, x_1, x_2, x_3, x_4, \dots$ be a sequence of independent such points (from now on, x_3 will perform the role of ρ, which will never be mentioned again), and let γ_{ij} be the a.s. unique geodesic from x_i to x_j, for $i < j$.

Both events $\mathcal{A}_1(\varepsilon, \beta), \mathcal{A}_2(\varepsilon)$ that are involved in Lemmas 18 and 19 deal with properties of "geodesic ε-stars" in random maps, in which the different arms of the geodesic stars separate quickly, say after a distance at most ε, rather than being necessarily pairwise disjoint. Indeed, the event $\mathcal{A}_1(\varepsilon, \beta)$ states in particular that the random point x_0 lies at distance ε from a certain point y of γ_{12}, and this point can be seen as a 2-star point from which emanate the segments of γ_{12} from y to x_1, x_2. This does imply that the geodesics γ_{01}, γ_{02} are disjoint outside of the ball $B_D(x_0, \varepsilon)$, as is easily checked. The similar property for the geodesics $\gamma_{03}, \gamma_{01}, \gamma_{02}$ under the event $\mathcal{A}_2(\varepsilon)$ is part of the definition of the latter. Therefore, we need to estimate the probability of events of the following form:

$$\mathcal{G}(\varepsilon, k) = \{\forall\, i, j \in \{1, 2, \dots, k\} \text{ with } i < j,\ \gamma_{0i} \text{ is disjoint from } \gamma_{0j} \text{ outside } B_D(x_0, \varepsilon)\} . \tag{8}$$

More precisely, we define discrete analogs for the events $\mathcal{A}_1(\varepsilon, \beta)$ and $\mathcal{A}_2(\varepsilon)$. Let $\mathcal{A}_1^{(n)}(\varepsilon, \beta)$ be the event that

- any geodesic chain from v_1 to v_2 in Q_n intersects $B_{d_{Q_n}}(v_0, \varepsilon(8n/9)^{1/4})$,
- it either holds that any geodesic chain from v_0 to v_1, visits at least one vertex v with $d_{Q_n}(v, v_0) > \varepsilon^{1-\beta}(8n/9)^{1/4}$, such that (v_1, v, v_2) are not aligned, **or that** any geodesic chain from v_0 to v_2, visits at least one vertex v with $d_{Q_n}(v, v_0) > \varepsilon^{1-\beta}(8n/9)^{1/4}$, such that (v_1, v, v_2) are not aligned

- the vertices v_0, v_1, v_2, taken in any order, are not aligned in Q_n, and $d_{Q_n}(v_0, v_1) \wedge d_{Q_n}(v_0, v_2) \geq 3\varepsilon^{1-\beta}(8n/9)^{1/4}$

and $\mathcal{A}_2^{(n)}(\varepsilon)$ be the event that

- any geodesic chain from v_1 to v_2 in Q_n intersects $B_{d_{Q_n}}(v_0, \varepsilon(8n/9)^{1/4})$,
- no two geodesic chains respectively from v_0 to v_i and from v_0 to v_j share a common vertex outside $B_{d_{Q_n}}(v_0, \varepsilon(8n/9)^{1/4})$, for every $i \neq j$ in $\{1, 2, 3\}$.
- any three vertices among v_0, v_1, v_2, v_3, taken in any order, are not aligned in Q_n, and $d_{Q_n}(v_0, v_1) \wedge d_{Q_n}(v_0, v_2) \wedge d_{Q_n}(v_0, v_3) \geq 3\varepsilon(8n/9)^{1/4}$

Proposition 20 *Let Q_n be a uniform quadrangulation in $\mathbf{Q}_n$, and conditionally given Q_n, let v_0, v_1, v_2, v_3 be uniformly chosen points in $V(Q_n)$. Then*

$$P(\mathcal{A}_1(\varepsilon, \beta)) \leq \limsup_{n\to\infty} P(\mathcal{A}_1^{(n)}(\varepsilon, \beta)),$$

and

$$P(\mathcal{A}_2(\varepsilon)) \leq \limsup_{n\to\infty} P(\mathcal{A}_2^{(n)}(\varepsilon)).$$

Proof. We rely on results of [15], see also [25], stating that the marked quadrangulations $(V(Q_n), (8n/9)^{-1/4} d_{Q_n}, (v_0, v_1, \ldots, v_k))$ converge in distribution along (n_k) for the so-called $(k+1)$-pointed Gromov-Hausdorff topology to $(S, D, (x_0, x_1, \ldots, x_k))$. Assuming, by using the Skorokhod representation theorem that this convergence holds almost-surely, this means that for every $\eta > 0$, and for every n large enough, it is possible to find a correspondence $\mathcal{R}_n$ between $V(Q_n)$ and S, such that $(v_i, x_i) \in \mathcal{R}_n$ for every $i \in \{0, 1, \ldots, k\}$, and such that

$$\sup_{(v,x),(v',x')\in\mathcal{R}_n} \left| \left(\frac{9}{8n}\right)^{1/4} d_{Q_n}(v, v') - D(x, x') \right| \leq \eta. \tag{9}$$

Let us now assume that $\mathcal{A}_1(\varepsilon, \beta)$ holds, and apply the preceding observations for $k = 2$. Assume by contradiction that with positive probability, along some (random) subsequence, there exists a geodesic chain $\gamma_{(n)}$ in Q_n from v_1 to v_2 such that no vertex of this chain lies at distance less than $\varepsilon(8n/9)^{1/4}$ from v_0. Now choose $v^q_{(n)}$ on $\gamma_{(n)}$, in such a way that $(8n/9)^{-1/4} d_{Q_n}(v_0, v^q_{(n)})$ converges to some $q \in (0, D(x_1, x_2)) \cap \mathbb{Q}$. This entails in particular that $(v_1, v^q_{(n)}, v_2)$ are aligned (along the extraction considered above). Then, let $x^q_{(n)}$ be such that $(v^q_{(n)}, x^q_{(n)}) \in \mathcal{R}_n$. By diagonal extraction, we can assume that for every q, $x^q_{(n)}$ converges to some $x^q \in S$, and using (9) entails both that (x_1, x^q, x_2) are aligned, with $D(x_1, x^q) = q$, and $D(x^q, x^{q'}) = |q' - q|$. One concludes that the points x^q are dense in the image of a geodesic from x_1 to x_2, but this geodesic is a.s. unique and has to be γ. By assumption, $d_{Q_n}(v_0, v^q_{(n)}) \geq \varepsilon(8n/9)^{1/4}$ for every rational q and n chosen along the same extraction, so that (9) entails that $D(x_0, x^q) \geq \varepsilon$ for every q, hence that γ does not intersect $B_D(x_0, \varepsilon/2)$, a contradiction.

Similarly, assume by contradiction that with positive probability, for infinitely many values of n, there exists a geodesic chain $\gamma_{(n)}$ from v_0 to v_1 such that every v on this geodesic with $d_{Q_n}(v, v_0) > \varepsilon^{1-\beta}(8n/9)^{1/4}$, satisfies also that (v_1, v, v_2) are aligned (and similarly with the roles of v_1 and v_2 interchanged). Similarly to the above, choose $v^q_{(n)}$ on $\gamma_{(n)}$, in such a way that $(8n/9)^{-1/4} d_{Q_n}(v_0, v^q_{(n)})$ converges to some $q \in (\varepsilon^{1-\beta}, D(x_0, x_1)) \cap \mathbb{Q}$. This entails in particular that $(v_1, v^q_{(n)}, v_2)$ are aligned (along some extraction). By using the correspondence $\mathcal{R}_n$ and diagonal extraction, this allows to construct a portion of geodesic in (S, D) from x_0 to x_1 lying outside of $B_D(x_0, \varepsilon^{1-\beta})$, which visits only points that are aligned with x_1 and x_2. The uniqueness of geodesics allows to conclude that all points x on γ_{01} outside $B_D(x_0, \varepsilon^{1-\beta})$ are in γ. The same holds for γ_{02} instead of γ_{01} by the same argument, so $\mathcal{A}_1(\varepsilon, \beta)$ does not occur.

Next, we know that a.s. x_0, x_1, x_2 have no alignement relations, and by (9) this is also the case of v_0, v_1, v_2 for every large n. A last use of (9) shows that $d_{Q_n}(v_0, v_1) \wedge d_{Q_n}(v_0, v_2) \geq 3\varepsilon^{1-\beta}(8n/9)^{1/4}$ for n large, from the fact that $D(x_0, x_1) \wedge D(x_0, x_2) \geq 7\varepsilon^{1-\beta}$ on the event $\mathcal{A}_1(\varepsilon, \beta)$.

Putting things together, we have obtained the claim on $\mathcal{A}_1(\varepsilon, \beta)$. The statement concerning $\mathcal{A}_2(\varepsilon)$ is similar and left to the reader. □

5 Coding by labeled maps

Our main tool for studying geodesic ε-stars with k arms is a bijection [25] between *multi-pointed delayed quadrangulations* and a class of labeled maps, which extends the celebrated Cori-Vauquelin-Schaeffer bijection. The multi-pointed bijection was used in [25] to prove a uniqueness result for typical geodesics that is related to the result of [15] that we already used in the present work. It was also used in [5] to obtain the explicit form of the joint law of distances between three randomly chosen vertices in the Brownian map (S, D). The way in which we use the multi-pointed bijection is in fact very much inspired from the approach of [5].

5.1 The multi-pointed bijection

5.1.1 Basic properties

Let $\mathbf{q} \in \mathbf{Q}$, and $\mathbf{v} = (v_0, v_1, \ldots, v_k)$ be $k+1$ vertices of $\mathbf{q}$. Let also $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_k)$ be *delays* between the points $v_i, 0 \leq i \leq k$, i.e. relative integers such that for every $i, j \in \{0, 1, \ldots, k\}$ with $i \neq j$,

$$|\tau_i - \tau_j| < d_{\mathbf{q}}(v_i, v_j)\,, \tag{10}$$

$$d_{\mathbf{q}}(v_i, v_j) + \tau_i - \tau_j \in 2\mathbb{N}\,. \tag{11}$$

Such vertices and delays exist as soon as $d_{\mathbf{q}}(v_i, v_j) \geq 2$ for every $i \neq j$ in $\{0, 1, \ldots, k\}$. We let $\mathbf{Q}^{(k+1)}$ be the set of triples $(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau})$ as described, and we let $\mathbf{Q}_n^{(k+1)}$ be the subset of those triples such that $\mathbf{q}$ has n faces.

On the other hand, a *labeled map* with $k+1$ faces is a pair $(\mathbf{m}, \mathbf{l})$ such that $\mathbf{m}$ is a rooted map with $k+1$ faces, named $f_0, f_1, \dots, f_k$, while $\mathbf{l} : V(\mathbf{m}) \to \mathbb{Z}_+$ is a labeling function such that $|\mathbf{l}(u) - \mathbf{l}(v)| \leq 1$ for every u, v linked by an edge of $\mathbf{m}$. If $\mathbf{m}$ has n edges, then $\mathbf{m}$ has $n-k+1$ vertices by Euler's formula. We should also mention that the function $\mathbf{l}$ and the delays $\boldsymbol{\tau}$ are defined up to a common additive constant, but we are always going to consider particular representatives in the sequel.

The bijection of [25] associates with $(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau}) \in \mathbf{Q}_n^{(k+1)}$ a labeled map $(\mathbf{m}, \mathbf{l})$ with n edges, denoted by $\Phi^{(k+1)}(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau})$, in such a way that $V(\mathbf{m}) = V(\mathbf{q}) \setminus \{v_0, v_1, \dots, v_k\}$: This identification will be implicit from now on. Moreover, the function $\mathbf{l}$ satisfies

$$\mathbf{l}(v) = \min_{0 \leq i \leq k} (d_{\mathbf{q}}(v, v_i) + \tau_i) , \qquad v \in V(\mathbf{m}) , \tag{12}$$

where on the right-hand side one should understand that the vertex v is a vertex of $V(\mathbf{q})$. The function $\mathbf{l}$ is extended to $V(\mathbf{q})$ in the obvious way, by letting $\mathbf{l}(v_i) = \tau_i, 0 \leq i \leq k$. This indeed extends (12) by using (10), and we see in passing that $\tau_i + 1$ is the minimal label $\mathbf{l}(v)$ for all vertices v incident to f_i. The interpretation of the labels is the following. Imagine that v_i is a source of liquid, which starts to flow at time τ_i. The liquid then spreads in the quadrangulation, taking one unit of time to traverse an edge. The different liquids are not miscible, so that they end up entering in conflict and becoming jammed. The vertices v such that

$$\mathbf{l}(v) = d_{\mathbf{q}}(v, v_i) + \tau_i < \min_{j \in \{0,1,\dots,k\} \setminus \{i\}} (d_{\mathbf{q}}(v, v_j) + \tau_j)$$

should be understood as the set of vertices that have only been attained by the liquid starting from v_i.

The case where there are ties is a little more elaborate as we have to give *priority rules* to liquids at first encounter. The property that we will need is the following. The label function $\mathbf{l}$ is such that $|\mathbf{l}(u) - \mathbf{l}(v)| = 1$ for every adjacent $u, v \in V(\mathbf{q})$, so that there is a natural orientation of the edges of $\mathbf{q}$, making them point toward the vertex of lesser label. We let $\vec{E}^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$ be the set of such oriented edges. Maximal oriented chains made of edges in $\vec{E}^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$ are then geodesic chains, that end at one of the vertices $v_0, v_1, \dots, v_k$. For every oriented edge $e \in \vec{E}^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$, we consider the oriented chain starting from e, and turning to the left as much as possible at every step. For $i \in \{0, 1, \dots, k\}$, the set $\vec{E}_i^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$ then denotes the set of $e \in \vec{E}^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$ for which this leftmost oriented path ends at v_i. One should see $\vec{E}_i^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$ as the set of edges that are traversed by the liquid emanating from source v_i.

5.1.2 The reverse construction

We will not specify how $(\mathbf{m}, \mathbf{l})$ is constructed from an element $(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau}) \in \mathbf{Q}^{(k+1)}$, but it is important for our purposes to describe how one goes back from the labeled map $(\mathbf{m}, \mathbf{l})$ to the original map with $k+1$ vertices and delays. We first set a couple of extra notions.

For each $i \in \{0, 1, \dots, k\}$, we can arrange the oriented edges of $\vec{E}(f_i)$ cyclically in the so-called *facial order*: Since f_i is located to the left of the incident edges, we can view the faces as polygons bounded by the incident edges, oriented so that they turn around the face counterclockwise, and this order is the facial order. If e, e' are distinct oriented edges incident to the same face, we let $[e, e']$ be the set of oriented edges appearing when going from e to e' in facial order, and we let $[e, e] = \{e\}$. Likewise, the oriented edges incident to a given vertex v are cyclically ordered in counterclockwise order when turning around v. The *corner* incident to the oriented edge e is a small angular sector with apex e_-, that is delimited by e and the edge that follows around v: These sectors should be simply connected and chosen small enough so that they are paiwise disjoint. We will often assimilate e with its incident corner. The label of a corner is going to be the label of the incident vertex. In particular, we will always adopt the notation $\mathbf{l}(e) = \mathbf{l}(e_-)$.

The converse construction from $(\mathbf{m}, \mathbf{l})$ to $(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau})$ goes as follows. Inside the face f_i of $\mathbf{m}$, let us first add an extra vertex v_i, with label

$$\mathbf{l}(v_i) = \tau_i = \min_{v \in V(f_i)} \mathbf{l}(v) - 1\,, \tag{13}$$

consistently with (12) and the discussion below. We view v_i as being incident to a single corner c_i. For every $e \in \vec{E}(\mathbf{m})$, we let f_i be the face of $\mathbf{m}$ incident to e, and define the *successor* of e as the first corner e' following e in the facial order, such that $\mathbf{l}(e') = \mathbf{l}(e) - 1$, we let $e' = s(e)$. If there are no such e', we let $s(e) = c_i$. The corners c_i themselves have no successors.

For every $e \in \vec{E}(\mathbf{m})$, we draw an *arc* between the corner incident to e and the corner incident to $s(e)$. It is possible to do so in such a way that the arcs do not intersect, nor cross an edge of $\mathbf{m}$. The graph with vertex set $V(\mathbf{m}) \cap \{v_0, v_1, \ldots, v_k\}$ and edge-set the set of arcs (so that the edges of $\mathbf{m}$ are excluded), is then a quadrangulation $\mathbf{q}$, with distinguished vertices $v_0, v_1, \ldots, v_k$ and delays $\tau_0, \tau_1, \ldots, \tau_k$ defined by (13). More precisely, if e is incident to f_i, then the arc from e to $s(e)$ is an oriented edge in $E_i^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$, and every such oriented edge can be obtained in this way: In particular, the chain $e, s(e), s(s(e)), \ldots$ from e_- to v_i is the leftmost geodesic chain described when defining the sets $E_i^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$.

To be complete, we should describe how the graph made of the arcs is rooted, but we omit the exact construction as it is not going to play an important role here. What is relevant is that for a given map $(\mathbf{m}, \mathbf{l})$, there are two possible rooting conventions for $\mathbf{q}$. Therefore, the mapping $\Phi^{(k+1)}$ associating a labeled map $(\mathbf{m}, \mathbf{l})$ with a delayed quadrangulation $(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau})$ is two-to-one. Consequently, the mapping $\Phi^{(k+1)}$ pushes forward the counting measure on $\mathbf{Q}_n^{(k+1)}$ to *twice* the counting measure on labeled maps with n edges, as far as we are interested in events that do not depend on the root of $\mathbf{q}$.

5.2 Geodesic r-stars in quadrangulations

We want to apply the previous considerations to the estimation of the probabilities of the events $\mathcal{A}_1^{(n)}(\varepsilon, \beta), \mathcal{A}_2^{(n)}(\varepsilon)$ of Section 4.2. To this end, we will have to specify the appropriate discrete counterpart to the event $\mathcal{G}(\varepsilon, k)$ of (8). Contrary to the continuous case, in quadrangulation there are typically many geodesic chains between two vertices. In uniform quadrangulations with n faces, the geodesic chains between two typical vertices will however form a thin pencil (of width $o(n^{1/4})$), that will degenerate to a single geodesic path in the limit.

Let $k, r > 0$ be integers. We denote by $G(r, k)$ set of pairs $(\mathbf{q}, \mathbf{v})$ with $\mathbf{q} \in \mathbf{Q}$, $\mathbf{v} = (v_0, v_1, \ldots, v_k) \in V(\mathbf{q})^{k+1}$, and such that

- if (v_0, v, v_i) are aligned for some $i \in \{1, 2, \ldots, k\}$ and $d_{\mathbf{q}}(v, v_0) \geq r$, then (v_0, v, v_j) are not aligned for every $j \neq i$,
- no three distinct vertices in $\{v_0, v_1, \ldots, v_k\}$, taken in any order, are aligned in $\mathbf{q}$, and $\min\{d_{\mathbf{q}}(v_0, v_i), 1 \leq i \leq k\} \geq 3r$,

Let $(\mathbf{q}, \mathbf{v}) \in G(r, k)$, and fix $r' \in \{r+1, r+2, \ldots, 2r\}$. We let $\boldsymbol{\tau}^{(r')} = (\tau_0^{(r')}, \tau_1^{(r')}, \ldots, \tau_k^{(r')})$ be defined by

$$\begin{cases} \tau_0^{(r')} = -r' \\ \tau_i^{(r')} = -d_{\mathbf{q}}(v_0, v_i) + r' \,, \qquad 1 \leq i \leq k \,. \end{cases} \tag{14}$$

Lemma 21 *If $(\mathbf{q}, \mathbf{v}) \in G(r, k)$ and $r' \in \{r+1, \ldots, 2r\}$, then $(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau}^{(r')}) \in \mathbf{Q}^{(k+1)}$.*

Proof. Let us verify (10). We write $\boldsymbol{\tau}$ instead of $\boldsymbol{\tau}^{(r')}$ for simplicity. We have $\tau_0 - \tau_i = d_{\mathbf{q}}(v_0, v_i) - 2r'$, and by the assumption that $d_{\mathbf{q}}(v_0, v_i) \geq 3r > r'$, we immediately get $|\tau_0 - \tau_i| < d_{\mathbf{q}}(v_0, v_i)$. Next, if $i, j \in \{1, \ldots, k\}$ are distinct, we have $|\tau_i - \tau_j| = |d_{\mathbf{q}}(v_0, v_j) - d_{\mathbf{q}}(v_0, v_i)| < d_{\mathbf{q}}(v_i, v_j)$, since (v_i, v_j, v_0) are not aligned, and neither are (v_j, v_i, v_0).

We now check (11). First note that for every $i \in \{1, \ldots, k\}$, $d_{\mathbf{q}}(v_0, v_j) + \tau_i - \tau_0 = 2r'$ which is even. Next, for $i, j \in \{1, \ldots, k\}$ distinct, consider the mapping $h : v \in V(\mathbf{q}) \mapsto d_{\mathbf{q}}(v_i, v_j) - d_{\mathbf{q}}(v, v_i) + d_{\mathbf{q}}(v, v_j)$. We have $h(v_j) = 0$, which is even. Moreover, since $\mathbf{q}$ is a bipartite graph, we have $h(u) - h(v) \in \{-2, 0, 2\}$ if u and v are adjacent vertices. Since $\mathbf{q}$ is a connected graph, we conclude that h takes all its values in $2\mathbb{Z}$, so $h(v_0)$ is even, and this is (11). $\square$

Under the hypotheses of Lemma 21, let $(\mathbf{m}, \mathbf{l}) = \Phi^{(k+1)}(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau}^{(r')})$, where $\Phi^{(k+1)}$ is the mapping described in Section 5.1. The general properties of this mapping entail that

$$\min_{v \in V(f_0)} \mathbf{l}(v) = -r' + 1 > -2r \tag{15}$$

and

$$\min_{v \in f_i} \mathbf{l}(v) = -d_{\mathbf{q}}(v_0, v_i) + r' - 1 < 0 \quad \text{ for every } \quad i \in \{1, \ldots, k\} \,.$$

We now state a key combinatorial lemma. Let $\mathbf{LM}^{(k+1)}$ set of labeled maps $(\mathbf{m}, \mathbf{l})$ with $k+1$ faces such that for every $i \in \{1, \ldots, k\}$, the face f_0 and f_i have at least one common incident vertex, and $\min_{v \in V(f_0 \cap f_i)} \mathbf{l}(v) = 0$.

Lemma 22 *Let $(\mathbf{q}, \mathbf{v}) \in G(r, k)$, and $r' \in \{r+1, r+2, \ldots, 2r\}$. Then the labeled map $(\mathbf{m}, \mathbf{l}) = \Phi^{(k+1)}(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau}^{(r')})$ belongs to $\mathbf{LM}^{(k+1)}$.*

Proof. Consider a geodesic chain $\gamma_i = (e_1, \ldots, e_{d_{\mathbf{q}}(v_0,v_i)})$ from v_0 to v_i, and let $e = e_{r'+1}, e' = \overline{e}_{r'}$ and $v = e_- = e'_-$ be the vertex visited by this geodesic at distance r' from v_0. Then it holds that $d_{\mathbf{q}}(v, v_i) = d_{\mathbf{q}}(v_0, v_i) - r'$, so that

$$d_{\mathbf{q}}(v, v_0) + \tau_0 = 0 = d_{\mathbf{q}}(v, v_i) + \tau_i \,.$$

Let us show that $v \in V(f_0 \cap f_i)$. Since $(\mathbf{q}, \mathbf{v}) \in G(r, k)$, we know that v is not on any geodesic from v_0 to v_j, for $j \in \{1, \ldots, k\} \setminus \{i\}$. Therefore,

$$d_{\mathbf{q}}(v, v_j) + \tau_j = d_{\mathbf{q}}(v, v_j) - d_{\mathbf{q}}(v_0, v_j) + r' > -d_{\mathbf{q}}(v_0, v) + r' = 0 \,.$$

From this, we conclude that $\mathbf{l}(v) = 0$, and that v can be incident only to f_0 or f_i. It is then obvious, since γ_i is a geodesic chain, that $e \in \vec{E}_i^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$ and $e' \in \vec{E}_0^{\mathbf{v},\boldsymbol{\tau}}(\mathbf{q})$. From the reverse construction, we see that e and e' are arcs drawn from two corners of the same vertex, that are incident to f_i and f_0 respectively. $\square$

In fact, the proof shows that all the geodesic chains from v_0 to v_i in $\mathbf{q}$ visit one of the vertices of label 0 in $V(f_0 \cap f_i)$. This will be useful in the sequel.

We let $\mathbf{LM}_n^{(k+1)}$ be the subset of elements with n edges, and $\mathsf{LM}_n^{(k+1)}$ be the counting measure on $\mathbf{LM}_n^{(k+1)}$, so its total mass is $\#\mathbf{LM}_n^{(k+1)}$. We want to consider the asymptotic behavior of this measure as $n \to \infty$, and for this, we need to express the elements of $\mathbf{LM}^{(k+1)}$ in a form that is appropriate to take scaling limits.

5.3 Decomposition of labeled maps in $\mathbf{LM}^{(k+1)}$

It is a standard technique both in enumerative combinatorics and in probability theory to decompose maps in simpler objects: Namely, a homotopy type, or *scheme*, which is a map of fixed size, and a labeled forest indexed by the edges of the scheme. See [26, 7, 25, 2] for instance. Due to the presence of a positivity constraint on the labels of vertices incident to f_0 and f_i, this decomposition will be more elaborate than in these references, it is linked in particular to the one described in [5].

5.3.1 Schemes

From this point on, the notation k will always stand for an integer $k \geq 2$. We call *pre-scheme* with $k+1$ faces, an unrooted map $\mathfrak{s}_0$ with $k+1$ faces named $f_0, f_1, \ldots, f_k$, in which every vertex has degree at least 3, and such that for every $i \in \{1, 2, \ldots, k\}$, the set $V(f_0 \cap f_i)$ of vertices incident to both f_0 and f_i is not empty.

It is easy to see, by applying the Euler formula, that there are only a finite number of pre-schemes with $k+1$ faces: Indeed, it has at most $3k-3$ edges and $2k-2$ vertices, with equality if and only if all vertices have degree exactly 3, in which case we say that $\mathfrak{s}_0$ is *dominant*, following [7].

A *scheme* with $k+1$ faces is an unrooted map that can be obtained from a pre-scheme $\mathfrak{s}_0$ in the following way. For every edge of $\mathfrak{s}_0$ that is incident to f_0 and some face f_i with $i \in \{1, 2, \ldots, k\}$, we allow the possibility to split it into two edges, incident to a common,

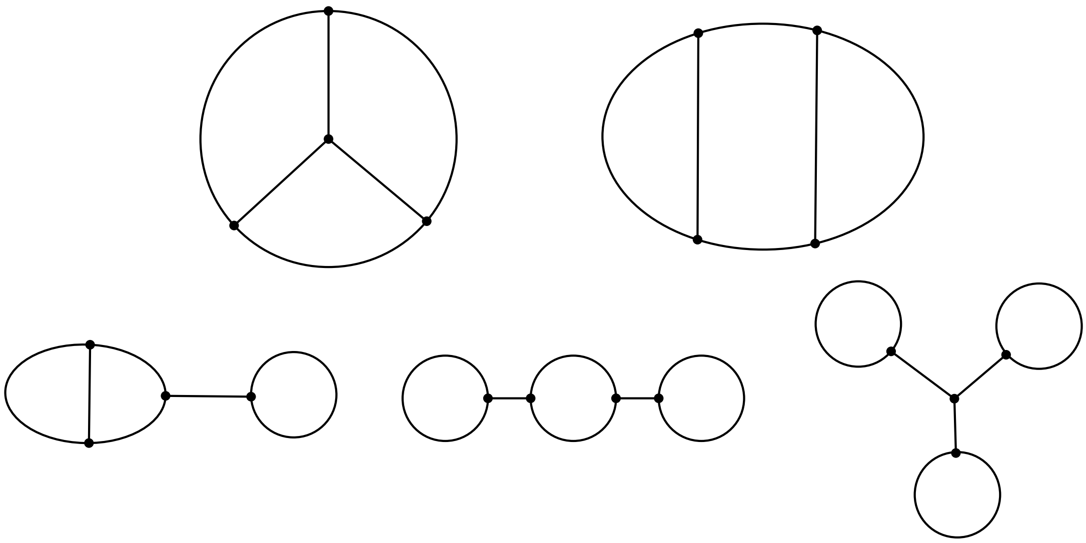

Figure 1: The five dominant pre-schemes with 4 faces, where f_0 is the unbounded face, and after forgetting the names f_1, f_2, f_3 of the other faces (there are sixteen dominant pre-schemes with 4 faces). In the first two cases, the boundary of the exterior face is a Jordan curve, while in the last three it is not simple.

distinguished new vertex of degree 2 called a *null-vertex*. Likewise, some of the vertices of $V(f_0 \cap f_i)$ with $i \in \{1, 2, \ldots, k\}$ are allowed to be distinguished as null-vertices. These operations should be performed in such a way that every face f_i, for $i \in \{1, 2, \ldots, k\}$, is incident to at least one null-vertex. Furthermore, a null vertex of degree 2 is not allowed to be adjacent to any other null vertex (of any degree).

In summary, a scheme is a map $\mathfrak{s}$ with $k+1$ faces labeled $f_0, f_1, \ldots, f_k$, such that

- For every $i \in \{1, 2, \ldots, k\}$, the set $V(f_0 \cap f_i)$ is not empty,
- the vertices of $\mathfrak{s}$ have degrees greater than or equal to 2,
- vertices of degree 2 are all in $\bigcup_{i=1}^{k} V(f_0 \cap f_i)$, and no two vertices of degree 2 are adjacent to each other,
- every vertex of degree 2, plus a subset of the other vertices in $\bigcup_{i=1}^{k} V(f_0 \cap f_i)$, are distinguished as null-vertices, in such a way that every face f_i with $i \in \{1, 2, \ldots, k\}$ is incident to a null-vertex, and no degree-2 vertex is adjacent to another null vertex.

Since the number of pre-schemes with $k+1$ faces is finite, the number of schemes is also finite. Indeed, passing from a pre-scheme to a scheme boils down to specifying a certain subset of edges and vertices of this pre-scheme. We say that the scheme is *dominant* if it can be obtained from a dominant pre-scheme, and if is has exactly k null vertices, which are all of degree 2. Since a dominant pre-scheme has $3k-3$ edges and $2k-2$ vertices, we see that a dominant scheme has $4k-3$ edges and $3k-2$ vertices. We let $\mathbf{S}^{(k+1)}$ be the set of schemes with $k+1$ faces, and $\mathbf{S}_d^{(k+1)}$ the subset of dominant ones.

The vertices of a scheme can be partitioned into three subsets:

- The set $V_N(\mathfrak{s})$ of null vertices,
- The set $V_I(\mathfrak{s})$ of vertices that are incident to f_0 and to some other face among $f_1, \ldots, f_k$, but which are not in $V_N(\mathfrak{s})$, and
- The set $V_O(\mathfrak{s})$ of all other vertices.

Similarly, the edges of $\mathfrak{s}$ can be partitioned into

- the set $E_N(\mathfrak{s})$ of edges incident to f_0 and to some other face among $f_1, \ldots, f_k$, and having at least one extremity in $V_N(\mathfrak{s})$,
- the set $E_I(\mathfrak{s})$ of edges that are incident to f_0 and some other face in $f_1, \ldots, f_k$, but that are not in $E_N(\mathfrak{s})$, and
- the set $E_O(\mathfrak{s})$ of all other edges.

It will be convenient to adopt once and for all an orientation convention valid for every scheme, meaning that every element of $E(\mathfrak{s})$ comes with a privileged orientation. We add the constraint that an edge in $E_N(\mathfrak{s})$ is always oriented towards a vertex of $V_N(\mathfrak{s})$, with priority given to those with degree 2: That is, if an edge of $E_N(\mathfrak{s})$ links two vertices of $V_N(\mathfrak{s})$, the orientation is arbitrary if both vertices have degree at least 3, and points towards to unique incident vertex of degree 2 otherwise. The other orientations are arbitrary, as in Figure 2. We let $\check{E}(\mathfrak{s})$ be the orientation convention of $\mathfrak{s}$.

For every null vertex v of degree 2, there is only one $e \in \check{E}(\mathfrak{s})$ satisfying both $e_+ = v$, and $e \in \vec{E}(f_0)$ (by definition, the face incident to $\overline{e}$ is then some other face among $f_1, \ldots, f_k$). The corresponding (non-oriented) edge is distinguished as a *thin edge*. Similarly, any edge of $E_N(\mathfrak{s})$ incident to at least one null vertex of degree at least 3 is counted as a thin edge. We let $E_T(\mathfrak{s})$ be the set of thin edges. Dominant schemes are the ones having exactly k null-vertices, which are all of degree 2: These also have k thin edges.

5.3.2 Labelings and edge-lengths

An *admissible labeling* for a (planted) scheme $\mathfrak{s}$ is a family $(\ell_v, v \in V(\mathfrak{s})) \in \mathbb{Z}^{V(\mathfrak{s})}$ such that

1. $\ell_v = 0$ for every $v \in V_N(\mathfrak{s})$,
2. $\ell_v > 0$ for every $v \in V_I(\mathfrak{s})$,

A *family of edge-lengths* for a scheme $\mathfrak{s}$ is a family $(r_e, e \in E(\mathfrak{s})) \in \mathbb{N}^{E(\mathfrak{s})}$ of positive integers indexed by the edges of $\mathfrak{s}$. A family of edge-lengths can be naturally seen as being indexed by oriented edges rather than edges, by setting $r_e = r_{\overline{e}}$ to be equal to the edge-length of the edge with orientations $e, \overline{e}$.

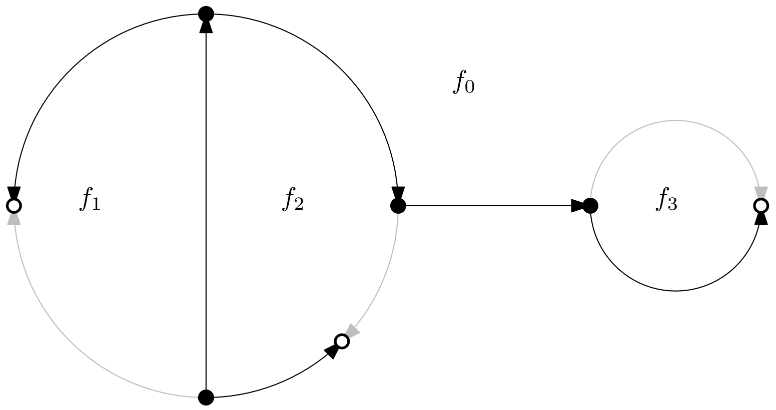

Figure 2: A scheme in $\mathbf{S}_d^{(3)}$, where we indicated thin edges in light gray, and specified the orientation conventions. Here the cardinalities of $V_N(\mathfrak{s}), V_I(\mathfrak{s}), V_O(\mathfrak{s})$ are respectively $3, 4, 0$, and the cardinalities of $E_N(\mathfrak{s}), E_I(\mathfrak{s}), E_O(\mathfrak{s})$ are respectively $6, 1, 2$.

5.3.3 Walk networks

We call *Motzkin walk*[1] a finite sequence $(M(0), M(1), \ldots, M(r))$ with values in $\mathbb{Z}$, where the integer $r \geq 0$ is the duration of the walk, and

$$M(i) - M(i-1) \in \{-1, 0, 1\}, \qquad 1 \leq i \leq r.$$

Given a scheme with admissible labeling $(\ell_v, v \in V(\mathfrak{s}))$ and edge-lengths $(r_e, e \in \vec{E}(\mathfrak{s}))$, a *compatible walk network* is a family $(M_e, e \in \vec{E}(\mathfrak{s}))$ of Motzkin walks indexed by $\vec{E}(\mathfrak{s})$, where for every $e \in \vec{E}(\mathfrak{s})$,

1. one has $M_e(i) = M_{\overline{e}}(r_e - i)$ for every $0 \leq i \leq r_e$,
2. one has $M_e(0) = \ell_{e_-}$,
3. if $e \in E_N(\mathfrak{s}) \cup E_I(\mathfrak{s})$, then M_e takes only non-negative values. If moreover e is not a thin edge, then all the values taken by M_e are positive, except $M_e(r_e)$ (when e is canonically oriented towards the only null vertex is is incident to).

The first condition says that the walks can really be defined as being labeled by edges of $\mathfrak{s}$ rather than oriented edges: The family $(M_e, e \in \vec{E}(\mathfrak{s}))$ is indeed entirely determined by $(M_e, e \in \check{E}(\mathfrak{s}))$, where $\check{E}(\mathfrak{s})$ is the orientation convention on $E(\mathfrak{s})$. Also, note that 1. and 2. together imply that $M_e(r_e) = \ell_{e_+}$, so we see that $M_e(r_e) = 0$ whenever $e \in E_N(\mathfrak{s})$ (with orientation pointing to a vertex of $V_N(\mathfrak{s})$, which is the canonical orientation choice we made). The distinction arising in 3. between thin edges and non-thin edges in $E_N(\mathfrak{s})$ is slightly annoying, but unavoidable as far as exact counting is involved. Such distinctions will disappear in the scaling limits studied in Section 6.

[1]This is not a really standard denomination in combinatorics, where Motzkin paths usually denote paths that are non-negative besides the properties we require

5.3.4 Forests and discrete snakes

Our last ingredient is the notion of *plane forest*. We will not be too formal here, and refer the reader to [25, 2] for more details. A plane tree is a rooted plane map with one face, possibly reduced to a single vertex, and a plane forest is a finite sequence of plane trees $(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_r)$. We view a forest itself as a plane map, by adding an oriented edge from the root vertex of $\mathbf{t}_i$ to the root vertex of $\mathbf{t}_{i+1}$ for $1 \le i \le r-1$, and adding another such edge from the root vertex of $\mathbf{t}_r$ to an extra vertex. These special oriented edges are called *floor edges*, and their incident vertices are the $r+1$ floor vertices. The vertex map, made of a single vertex and no edge, is considered as a forest with no tree.

With a scheme $\mathfrak{s}$, an admissible labeling $(\ell_v, v \in V(\mathfrak{s}))$ and a compatible walk network $(M_e, e \in \vec{E}(\mathfrak{s}))$, a *compatible labeled forest* is the data, for every $e \in \vec{E}(\mathfrak{s})$, of a plane forest F_e with r_e trees, and with an integer-valued labeling function $(L_e(u), u \in V(F_e))$ such that $L_e(u) = M_e(i)$ if u is the $i+1$-th floor vertex of F_e, for $0 \le i \le r_e$ and $L_e(u) - L_e(v) \in \{-1, 0, 1\}$ whenever u and v are adjacent vertices in the same tree of F_e, or adjacent floor vertices.

In order to shorten the notation, we can encode labeled forests in discrete processes called *discrete snakes*. If $\mathbf{t}$ is a rooted plane tree with n edges, we can consider the facial ordering $(e^{(0)}, e^{(1)}, \ldots, e^{(2n-1)})$ of oriented edges starting from its root edge, and let

$$C_{\mathbf{t}}(i) = d_{\mathbf{t}}(e_-^{(0)}, e_-^{(i)})\,, \qquad 0 \le i \le 2n-1\,,$$

and then $C_{\mathbf{t}}(2n) = 0$ and $C_{\mathbf{t}}(2n+1) = -1$. The sequence $(C_{\mathbf{t}}(i), 0 \le i \le 2n+1)$ is called the contour sequence of $\mathbf{t}$, and we turn it into a continuous function defined on the time interval $[0, 2n+1]$ by linearly interpolating between values taken at the integers. Roughly speaking, for $0 \le s \le 2n$, $C_{\mathbf{t}}(s)$ is the distance from the root of the tree at time s of a particle going around the tree at unit speed, starting from the root.

If F is a plane forest with trees $\mathbf{t}_1, \ldots, \mathbf{t}_r$, the contour sequence C_F is just the concatenation of $r + C_{\mathbf{t}_1}, r-1+C_{\mathbf{t}_2}, \ldots, 1 + C_{\mathbf{t}_r}$, starting at r and finishing at 0 at time $r+2n$, where n is the total number of edges in the forest distinct from floor edges. Note the fact that the sequence visits $r-i$ for the first time when it starts exploring the $i+1$-th tree. For simplicity, we still denote the sequence by F. If L is a labeling function on F, the label process is defined by letting $L_F(i)$ be the label of the corner explored at the i-th step of the exploration. Both processes C_F, L_F are extended by linear interpolation between integer times.

The information carried by (F, L) can be summarized into one unique path-valued process

$$(W_{F,L}(i), 0 \le i \le 2n+r)\,, \qquad \text{with} \qquad W_{F,L}(i) = (W_{F,L}(i,j), 0 \le j \le \zeta_F(i))\,,$$

where $\zeta_F(i)$ is the distance to the floor of the vertex $u^{(i)} = e_-^{(i)}$ of F visited at the i-th step in contour order, and $W_{F,L}(i,j)$ is the label $L(u)$ of the ancestor u of $u^{(i)}$ at distance j from the floor. So, for every $i \in \{0, 1, \ldots, 2n+r\}$, $W_{F,L}(i)$ is a finite sequence with length $\zeta_F(i)$, and this sequence is a Motzkin walk. The initial value $W_{F,L}(0)$ is the Motzkin

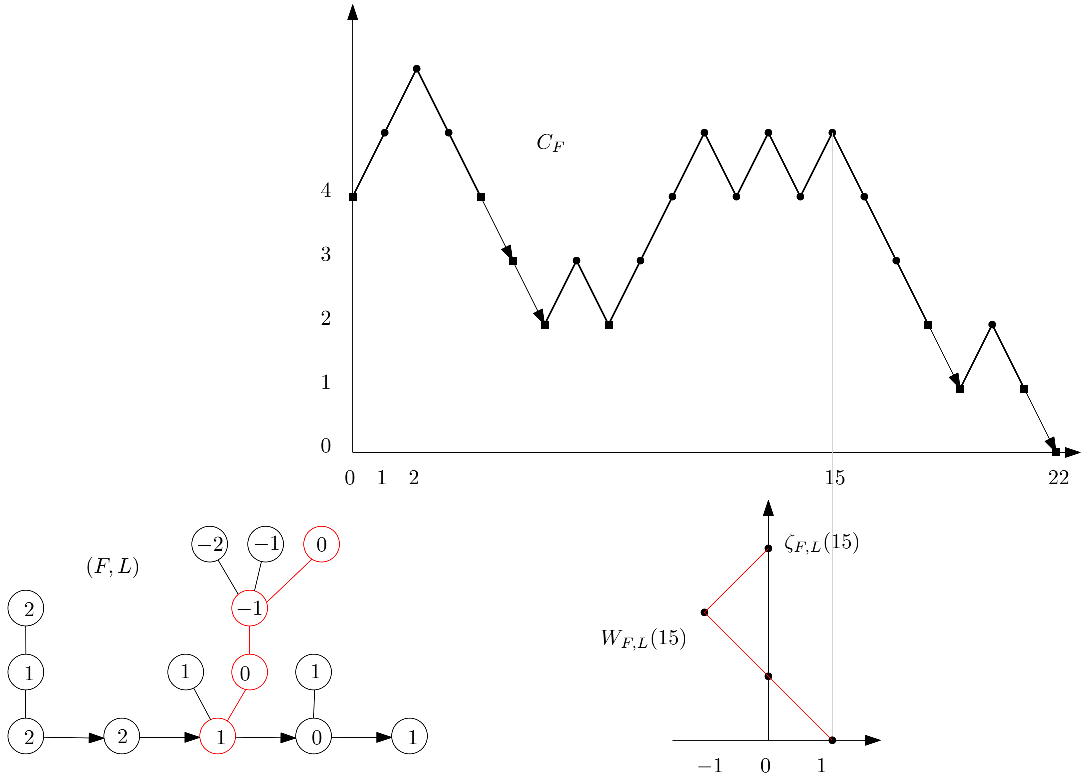

Figure 3: A labeled forest with 4 trees and 22 oriented edges, its contour process, and the associated discrete snake evaluated at time 15

walk $(M(0), M(1), \ldots, M(r))$ given by the labels of the floor vertices of F. Finally, we extend $W_{F,L}$ to a process in $\mathcal{C}(\mathcal{C}(\mathbb{R}))$ by interpolation: $W_{F,L}(i)$ is now really a path in $\mathcal{C}(\mathbb{R})$ obtained by interpolating linearly between integer times $(W_{F,L}(i,j), 0 \leq j \leq \zeta_F(i))$, and for $s \in [i, i+1]$ we simply let $W_{F,L}(s)$ be the path $(W_{F,L}(i+1,t), 0 \leq t \leq \zeta_F(i)+s-i)$ if $\zeta_F(i) < \zeta_F(i+1)$, and $(W_{F,L}(i,t), 0 \leq t \leq \zeta_F(i) - s + i)$ if $\zeta_F(i) > \zeta_F(i+1)$.

We define a *discrete snake* to be a process $W_{F,L}$ obtained in this way from some labeled forest (F, L). From $W_{F,L}$, it is possible to recover (F, L). In particular, if $\underline{\zeta}_F(i) = \min_{0 \leq i' \leq i} \zeta_F(i')$, then for $i \in \{0, 1, \ldots, 2n+r-1\}$, $r + 1 - \underline{\zeta}_F(i)$ is the label of the tree of F visited at time i in the contour process. In particular, we have

$$\begin{aligned} C_F(i) &= r + 1 + \zeta_F(i) - \underline{\zeta}_F(i)\,, & 0 \leq i \leq 2n + r\,, \\ L_F(i) &= W_{F,L}(i, \zeta_F(i))\,, & 0 \leq i \leq 2n + r\,, \\ M(j) &= W_{F,L}(\inf\{i \geq 0 : \zeta_F(i) = r - j\}, r - j)\,, & 0 \leq j \leq r\,. \end{aligned}$$

From this, the data of a scheme $\mathfrak{s}$, an admissible labeling $(\ell_v, v \in V(\mathfrak{s}))$, admissible edge-lengths $(r_e, e \in E(\mathfrak{s}))$, a compatible walk network $(M_e, e \in E(\mathfrak{s}))$, and compatible labeled forests $((F_e, L_e), e \in \vec{E}(\mathfrak{s}))$ can be summed up in the family $(\mathfrak{s}, (W_e, e \in \vec{E}(\mathfrak{s})))$ where W_e is the discrete snake associated with (F_e, L_e). We call a family $(W_e, e \in \vec{E}(\mathfrak{s}))$ obtained in this way an *admissible family of discrete snakes* on the scheme $\mathfrak{s}$.

5.3.5 The reconstruction

Let us reconstruct an element of $\mathbf{LM}^{(k+1)}$, starting from a scheme $\mathfrak{s}$, and an admissible family of discrete snakes $(W_e, e \in \vec{E}(\mathfrak{s}))$. The latter defines labeling, edge-lengths, a walk network and a family of labeled forests, and we keep the same notation as before.

First, we label every vertex of $\mathfrak{s}$ according to $(\ell_v, v \in V(\mathfrak{s}))$. Second, we replace every edge e of $\mathfrak{s}$ with a chain of r_e edges. Since $r_e = r_{\overline{e}}$, this is really an operation on edges rather than oriented edges. Then, the vertices inside each of these chains are labeled according to $M_e(0), M_e(1), \ldots, M_e(r_e)$. Since $M_e(0) = \ell_{e_-}$ and $M_e(r_e) = \ell_{e+}$, these labelings are consistent with the labeling at the vertices of $\mathfrak{s}$. At this stage, we get a map with $k+1$ faces and labeled vertices of degree at least 2.

Then, we graft the labeled forest (F_e, L_e) in such a way that the r_e floor edges of F_e coincide with the oriented edges of the chain with r_e edges corresponding to e, and are all incident to the same face as e (i.e. the face located to the left of e). Note that at this step the construction depends strongly on the orientation of e, and that F_e and $F_{\overline{e}}$ can be very different, despite having the same number of trees, and the fact that the labels of the floor vertices are given by M_e and $M_{\overline{e}}$ respectively.

This yields a labeled map $(\mathbf{m}, \mathbf{l}) \in \mathbf{LM}^{(k+1)}$. To be completely accurate, we need to specify the root of $\mathbf{m}$. To this end, we mark one of the oriented edges of the forest F_{e^*} (comprising the r *oriented* floor edges) for some $e_* \in \vec{E}(\mathfrak{s})$. This edge specifies the root edge of $\mathbf{m}$. Note that the number of oriented edges of $\mathbf{m}$ equals

$$\#\vec{E}(\mathbf{m}) = \sum_{e \in \vec{E}(\mathfrak{s})} \#\vec{E}(F_e)\,. \tag{16}$$

This construction can be inverted: Starting from a labeled map $(\mathbf{m}, \mathbf{l}) \in \mathbf{LM}^{(k+1)}$, we can erase the degree-1 vertices inductively, hence removing families of labeled forests grafted on maximal chains, joining two vertices with degrees ≥ 3 by passing only through degree-2 vertices. Any given such chain is incident to one or two faces only. The resulting map has only vertices of degrees at least 2, we call it $\mathbf{m}'$. In turn, the degree-2 vertices of $\mathbf{m}'$ can be deleted, ending with a pre-scheme $\mathfrak{s}_0$, each edge of which corresponds to a maximal chain of vertices with degree 2 in $\mathbf{m}'$. Its faces are the same as $\mathbf{m}, \mathbf{m}'$ and inherit their names $f_0, \ldots, f_k$. If a maximal chain in $\mathbf{m}'$ is incident to f_0 and f_i for some $i \in \{1, 2, \ldots, k\}$, and if the label of one of the degree-2 vertices of this chain is 0 while the labels of both extremities are strictly positive, then we add an extra degree-2 vertex to the corresponding edge of $\mathfrak{s}_0$, that is distinguished as a null-vertex. Likewise, a vertex of label 0 with degree at least 3 in $\mathbf{m}'$ that is incident to f_0 and some other face is distinguished as a null-vertex. This family of extra null vertices turns $\mathfrak{s}_0$ into a scheme $\mathfrak{s}$, since by definition of $\mathbf{LM}^{(k+1)}$, every face f_i with index $i \in \{1, 2, \ldots, k\}$ has at least an incident vertex with label 0 that is also incident to f_0.

It remains to construct walks indexed by the edges of the scheme. Consider a maximal chain as in the previous paragraph, corresponding to an edge $e \in E(\mathfrak{s})$. If e is not incident to a null-vertex of degree 2, then the labels of the successive vertices of the chain define a walk M_e with positive duration, equal to the number of edges in the chain. By default, the chain can be oriented according to the orientation convention of e in $\check{E}(\mathfrak{s})$, which specifies the order in which we should take the labels to define M_e (changing the orientation would define $M_{\overline{e}}$, which amounts to property 1. in the definition of compatible walk networks). On the other hand, if e is incident to a null vertex of $\mathfrak{s}$ with degree 2, then e is obtained by splitting an edge e'' of $\mathfrak{s}_0$ in two sub-edges, e, e'. For definiteness, we will assume that e is the thin edge, meaning that when e is oriented so that it points towards the null vertex with degree 2, then f_0 lies to its left. The canonical orientation for e' makes it point to the null vertex of degree 2 as well, as is now customary. The chain of $\mathbf{m}'$ that corresponds to the edge e'', when given the same orientation as e, has vertices labeled $l_0, l_1, \dots, l_r$, in such a way that $\{i \in \{1, \dots, r-1\} : k_i = 0\}$ is not empty. Let $T = \max\{i \in \{1, \dots, r-1\} : l_i = 0\}$, and set

$$M_e = (l_0, \dots, l_T)\,, \qquad M_{e'} = (l_r, l_{r-1}, \dots, l_T)\,.$$

This defines two walks with positive durations ending at 0, the second one with only positive values except at the ending point, and the first one takes non-negative values (the starting value being positive). We end up with a scheme, admissible labelings and edge-lengths, and compatible families of walks and forests.

To sum up our study, we have the following result.

Proposition 23 *The data of*

- *a scheme* $\mathfrak{s}$,
- *an admissible family of discrete snakes* $(W_e, e \in \vec{E}(\mathfrak{s}))$,
- *an extra distinguished oriented edge in* F_{e_*}, *for some* $e_* \in \vec{E}(\mathfrak{s})$

determines a unique element in $\mathbf{LM}^{(k+1)}$, *and every such element can be uniquely determined in this way.*

The only part that requires further justification is the word "uniquely" in the last statement: We have to verify that two different elements $(\mathfrak{s}, (W_e, e \in \vec{E}(\mathfrak{s}))), (\mathfrak{s}', (W'_e, e \in \vec{E}(\mathfrak{s})))$ cannot give rise to the same element in $\mathbf{LM}^{(k+1)}$. This is due to the fact that schemes have a trivial automorphism group, i.e. every map automorphism of $\mathfrak{s}$ that preserves the labeled faces is the identity automorphism. To see this, note that there are at least two distinct indices $i, j \in \{0, 1, \dots, k\}$ such that f_i and f_j are incident to a common edge. If there is a unique oriented edge e incident to f_i with $\overline{e}$ incident to f_j, then any map automorphism preserving the labeled faces should preserve this edge, hence all the edges by standard properties of maps. On the other hand, if there are several edges incident to both f_i and f_j, then there are multiple edges between the vertices corresponding to f_i and f_j in the dual graph of $\mathfrak{s}$, and by the Jordan Curve Theorem these edges split the sphere into a collection $G_1, G_2, \dots, G_p$ of 2-gons. Since $k \geq 2$, a scheme has at least three faces, so there exists $r \in \{0, 1, 2, \dots, k\} \setminus \{i, j\}$, and we can assume that the vertex corresponding to f_r in the dual graph of $\mathfrak{s}$ lies in G_1, up to renumbering. But then, a graph automorphism preserving the labeled faces, which boils down to a graph automorphism of the dual preserving the labeled vertices, should preserve G_1, and in particular, it should fix its two boundary edges, oriented from f_i to f_j. Therefore, this automorphism has to be the identity.

Note that this argument is only valid in planar geometry, and does not apply to surfaces of positive genus.

5.4 Using planted schemes to keep track of the root

In this section we present a variant of the preceding description, that allows to keep track of the root of the labeled map by an operation on the scheme called *planting*. A *planted scheme* satisfies the same definition as scheme, except that we allow exactly one exceptional vertex v_{**} with degree 1. In the canonical orientation, the edge incident to the only degree-one vertex always points towards this vertex. We define the sets $V_N(\mathfrak{s}), V_I(\mathfrak{s}), V_O(\mathfrak{s})$ and $E_N(\mathfrak{s}), E_I(\mathfrak{s}), E_O(\mathfrak{s})$ in the same way as we did for schemes. In particular, one will note that v_{**} is always in $V_O(\mathfrak{s})$, since it can only be incident to a single face, while the only edge e_{**} incident to v_{**} is in $E_O(\mathfrak{s})$ for the same reason. We let $\dot{\mathbf{S}}^{(k+1)}$ be the set of planted schemes with $k+1$ faces, and $\dot{\mathbf{S}}_d^{(k+1)}$ the set of dominant planted schemes, i.e. those having exactly k null-vertices, all with degree 2, all other vertices except v_{**} being of degree 3. Due to the distinguished nature of the edge incident to v_{**}, planted schemes always have a trivial automorphism group (not only in planar geometry).

The definition of admissible labelings, edge-lengths, walk networks and discrete snakes associated with a planted scheme is the same as in the non-planted case, with only one exception, stating that the path $M_{e_{**}}$ is allowed to have duration $r_{e_{**}}$ equal to 0, as opposed to all others having positive durations, and the number of trees in the labeled forests encoded by $W_{e_{**}}, W_{\bar{e}_{**}}$ are both equal to $r_{e_{**}}+1$. Moreover, if e' is the edge of $\vec{E(\mathfrak{s})}$ that comes after e_{**} in clockwise order, then we lower the number of trees of the forest $F_{e'}$ to $r_{e'}-1$ instead of $r_{e'}$.

Let us explain how to construct a pair $(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})})$, with $\mathfrak{s}$ a planted scheme and $(W_e)_{e\in E(\mathfrak{s})}$ an admissible family of discrete snakes, starting from a rooted labeled map $(\mathbf{m}, \mathbf{l})$. The construction is very similar as in the preceding section, so we will omit some details, and focus on how we obtain the extra edge e_{**}, and the associated snakes $W_{e_{**}}, W_{\bar{e}_{**}}$.

Let e_* be the root edge of $\mathbf{m}$ and $v_* = (e_*)_-$ be its origin. Recall that $\mathbf{m}'$ is the map obtained from $\mathbf{m}$ after inductively removing inductively all the vertices of degree 1, and that $\mathbf{m}$ can be obtained by grafting a tree component at each corner of $\mathbf{m}'$. The root e_* belongs to one of these trees, let us call it $\mathbf{t}_*$, and we consider the chain $v^{(0)} = v_*, v^{(1)}, \ldots, v^{(r_{e_{**}})}$ down from v_* to the root of this tree (it can occur that $r_{e_{**}}$, meaning that v_* belongs to $V(\mathbf{m}')$). This chain is the one that gives rise to the extra edge e_{**} (i.e. even if it has zero length), and the labels are $\ell_{(e_{**})_-} = \mathbf{l}(v^{(r_{e_{**}})})$ and $\ell_{(e_{**})_+} = \mathbf{l}(v^{(0)})$. Likewise, the labels along the chain define the path $M_{e_{**}}$.

The tree $\mathbf{t}_*$ and the distinguished chain between the root corner of $\mathbf{t}_*$ and e_* can be seen in turn as being made of two forests $F_{e_{**}}, F_{\overline{e}_{**}}$ with the same (positive) number of trees, equal to $r_{e_{**}}$, and in which we decide to forget the last floor edge, which does not play a role here. The last tree explored in $F_{\overline{e}_{**}}$ is then "stolen" from the forest $F_{e'}$, as explained in Figure 4. We leave the last details of the construction as an exercise to the reader.

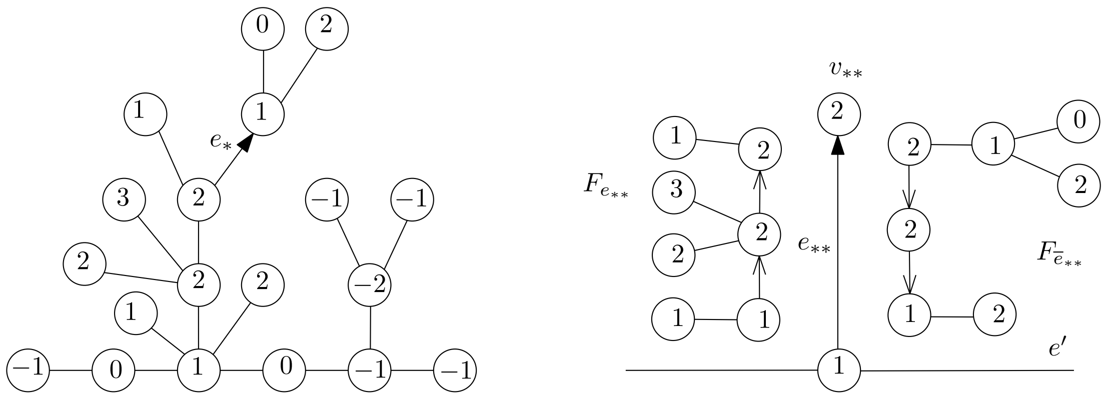

Figure 4: The planting convention: On the left, a small portion of $\mathbf{m}$ around its root e_* is represented, in which the bottom vertices and edges all belong to the map $\mathbf{m}'$. The root e_* in $\mathbf{m}$ determines the location of the special edge e_{**}, and determines two forests with a positive number of trees, but no terminating floor edge. Observe that e_* could well be an oriented edge in $\vec{E}(\mathbf{m}')$, which occurs precisely when $F_{\overline{e}_{**}}$ is made of a single tree ($r_{e_{**}} = 0$) made of a single vertex. The forest associated with e' has one tree less than the forest associated with $\overline{e}'$, this tree being "stolen" as the last tree of the forest $F_{\overline{e}_{**}}$.

6 Scaling limits of labeled maps

The description of labeled maps from Proposition 23 is particularly appropriate when one is interested in taking scaling limits. We first introduce the proper notion of "continuum labeled map".

6.1 Continuum measures on labeled maps

Let (E, d) be a metric space. We let $\mathcal{C}(E)$ be the set of E-valued continuous paths, i.e. of continuous functions $f : [0, \zeta] \to \mathbb{R}$, for some $\zeta = \zeta(f) \geq 0$ called the *duration* of f. This space is endowed with the distance

$$\mathrm{dist}(f, g) = \sup_{t \geq 0} d\big(f(t \wedge \zeta(f)), g(t \wedge \zeta(g))\big) + |\zeta(f) - \zeta(g)|\,,$$

that makes it a Polish space if E is itself Polish. Sometimes, we will also have to consider continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ with infinite duration, so we let $\overline{\mathcal{C}}(E)$ be the set of continuous functions with finite or infinite duration.

We let $\mathbf{CLM}^{(k+1)}$ be the set of pairs of the form $(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})})$, where $\mathfrak{s}$ is a scheme with $k+1$ faces, and for every $e \in \vec{E}(\mathfrak{s})$, W_e is an element of $\mathcal{C}(\mathcal{C}(\mathbb{R}))$, meaning that for every $s \in [0, \zeta(W_e)]$, $W_e(s)$ is a function in $\mathcal{C}(\mathbb{R})$ that can be written $(W_e(s,t), 0 \le t \le \zeta(W_e(s)))$. The space $\mathbf{CLM}^{(k+1)}$ is a Polish space, a complete metric being for instance the one letting $(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})})$ and $(\mathfrak{s}', (W'_e)_{e\in\vec{E}(\mathfrak{s}')})$ be at distance 1 if $\mathfrak{s} \neq \mathfrak{s}'$, and at distance $\max_{e\in\vec{E}(\mathfrak{s})} \operatorname{dist}(W_e, W'_e)$ if $\mathfrak{s} = \mathfrak{s}'$.

In order to make clear distinctions between quantities like $\zeta(W_e), \zeta(W_e(s))$, we will adopt the following notation in the sequel. For $e \in \vec{E}(\mathfrak{s})$, we let

$$M_e = W_e(0) \in \mathcal{C}(\mathbb{R}) , \quad r_e = \zeta(M_e) , \quad \sigma_e = \zeta(W_e) .$$

The measure that will play a central role is a *continuum measure* $\mathsf{CLM}^{(k+1)}$ on labeled maps. To define it, we first need to describe the continuum analogs of the walks and discrete snakes considered in the previous section.

6.1.1 Bridge measures

In the sequel, we let $(X_t, t \in [0, \zeta(X)])$ be the canonical process on the space $\overline{\mathcal{C}}(\mathbb{R})$ of $\mathbb{R}$-valued continuous functions with finite or infinite duration. The infimum process of X is the process $\underline{X}$ defined by

$$\underline{X}_t = \inf_{0\le s\le t} X_s , \qquad 0 \le t \le \zeta(f) .$$

For every $x \in \mathbb{R}$, we also let $T_x = \inf\{t \ge 0 : X_t = x\} \in [0, \infty]$ be the first hitting time of x by X.

The building blocks of $\mathsf{CLM}^{(k+1)}$ are going to be several paths measures. We let $\mathbb{P}_x$ be the law of standard 1-dimensional Brownian motion started from x, and for $y < x$, we let $\mathbb{E}_x^{(y,\infty)}$ be the law of $(X_t, 0 \le t \le T_y)$ under $\mathbb{P}_x$, that is, Browian motion killed at first exit time of (y, ∞).

Next, let $\mathbb{P}^t_{x\to y}$ be the law of the 1-dimensional Brownian bridge from x to y with duration t:

$$\mathbb{P}^t_{x\to y}(\cdot) = \mathbb{P}_x((X_s)_{0\le s\le t} \in \cdot \,|\, X_t = y) .$$

A slick way to properly define this singular conditioned measure is to let $\mathbb{P}^t_{x\to y}$ be the law of $(X_s + (y - X_t)s/t, 0 \le s \le t)$ under $\mathbb{P}_x$, see [31, Chapter I.3]. We also let $\overline{\mathbb{P}}^t_x$ be the law of the *first-passage Brownian bridge* from x to 0 with duration t, defined formally by

$$\overline{\mathbb{P}}^t_x(\cdot) = \mathbb{P}_x((X_s, 0 \le s \le t) \in \cdot \,|\, T_0 = t) .$$

Regular versions for this singular conditioning can be obtained using space-time Doob h-transforms. We refer to [2] for a recent and quite complete treatment of first-passage bridges and limit theorems for their discrete versions, which will be helpful to us later on.

For $t > 0$ let

$$\begin{aligned} p_t(x,y) &= \frac{\mathbb{P}_x(X_t \in \mathrm{d}y)}{\mathrm{d}y} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \qquad x,y \in \mathbb{R}\,, \\ \overline{p}_t(x,0) &= \frac{\mathbb{P}_x(T_0 \in \mathrm{d}t)}{\mathrm{d}t} = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \qquad x > 0\,. \end{aligned}$$

The *bridge measure* from x to y is then the σ-finite measure $\mathbb{B}_{x\to y}$ defined by

$$\mathbb{B}_{x\to y}(\mathrm{d}X) = \int_0^\infty \mathrm{d}t\, p_t(x,y)\mathbb{P}^t_{x\to y}(\mathrm{d}X)\,.$$

We also let $\mathbb{B}^+_{x\to y}$ be the restriction of $\mathbb{B}_{x\to y}$ to the set $\mathcal{C}^+(\mathbb{R}) = \{f \in \mathcal{C}(\mathbb{R}) : f \geq 0\}$. The reflexion principle entails that $\mathbb{P}^t_{x\to y}(\mathcal{C}^+(\mathbb{R})) = 1 - e^{-2xy/t}$, and it is a simple exercise to show that for $x, y > 0$, letting $p^+_t(x,y) = p_t(x,y) - p_t(x,-y) = \mathbb{P}_x(X_t \in \mathrm{d}y, \underline{X}_t \geq 0)/\mathrm{d}y$,

$$\mathbb{B}^+_{x\to y}(\mathrm{d}X) = \int_0^\infty \mathrm{d}t\, p^+_t(x,y)\mathbb{P}^t_{x\to y}(\mathrm{d}X \,|\, \mathcal{C}^+(\mathbb{R}))\,, \tag{17}$$

We now state two path decomposition formulas for the measures $\mathbb{B}_{x\to y}$ that will be useful for our later purposes. If μ is a measure on $\mathcal{C}$, we let $\widehat{\mu}$ be the image measure of μ under the time-reversal operation $f \mapsto \widehat{f}$. If μ and μ' are two probability measures on $\mathcal{C}$ such that $\mu(X(\zeta(X)) = z) = \mu'(X(0) = z) = 1$ for some $z \in \mathbb{R}$, then we let $\mu \bowtie \mu'$ be the image measure of $\mu \otimes \mu'$ under the concatenation operation $(f,g) \mapsto f \bowtie g$ from $\mathcal{C}(\mathbb{R})^2$ to $\mathcal{C}(\mathbb{R})$, where

$$(f \bowtie g)(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq \zeta(f) \\ g(t - \zeta(f)) & \text{if } \zeta(f) \leq t \leq \zeta(f) + \zeta(g)\,. \end{cases}$$

The *agreement formula* of [29, Corollary 3] states that

$$\mathbb{B}_{x\to y}(\mathrm{d}X) = \int_{-\infty}^{x\wedge y} \mathrm{d}z\, \big(\mathbb{E}^{(z,\infty)}_x \bowtie \widehat{\mathbb{E}}^{(z,\infty)}_y\big)(\mathrm{d}X)\,. \tag{18}$$

In particular,

$$\mathbb{B}^+_{x\to y}(\mathrm{d}X) = \int_0^{x\wedge y} \mathrm{d}z\, \big(\mathbb{E}^{(z,\infty)}_x \bowtie \widehat{\mathbb{E}}^{(z,\infty)}_y\big)(\mathrm{d}X)\,.$$

This decomposition should be seen as one of the many versions of Williams' decompositions formulas for Brownian paths: here, the variable z plays the role of the minimum of the generic path X.

A second useful path decomposition states that $x, y \in \mathbb{R}$,

$$\int_{\mathbb{R}} \mathrm{d}z\, (\mathbb{B}_{x\to z} \bowtie \mathbb{B}_{z\to y})(\mathrm{d}X) = \zeta(X)\mathbb{B}_{x\to y}(\mathrm{d}X)\,. \tag{19}$$

To see this, write

$$\zeta(X)\mathbb{B}_{x\to y}(\mathrm{d}X) = \int_0^\infty \mathrm{d}r\, p_r(x,y) \int_0^r \mathrm{d}s\, \mathbb{P}^r_{x\to y}(\mathrm{d}X)\,,$$

then note that for every $r \geq s \geq 0$,

$$\mathbb{P}^r_{x\to y}(X_s \in \mathrm{d}z) = \frac{p_s(x,z)p_{r-s}(z,y)}{p_r(x,y)}\mathrm{d}z\,,$$

and use the fact that given $X_s = z$, the paths $(X_u, 0 \leq u \leq s)$ and $(X_{s+u}, 0 \leq u \leq r-s)$ under $\mathbb{P}^r_{x\to y}$ are independent Brownian bridges, by the Markov property.

6.1.2 Brownian snakes

Let W be the canonical process on $\overline{\mathcal{C}}(\mathcal{C}(\mathbb{R}))$. That is to say, for every $s \in [0, \zeta(W)]$ (or just $s \geq 0$ if $\zeta(W) = \infty$), $W(s)$ is an element of $\mathcal{C}(\mathbb{R})$. For simplicity, we let $\zeta_s = \zeta(W(s))$ be the duration of this path, and let $W(s,t) = W(s)(t)$ for $0 \leq t \leq \zeta_s$. The process $(\zeta_s, 0 \leq s \leq \zeta(W))$ is called the *lifetime process* of W. In order to clearly distinguish the duration of W with that of $W(s)$ for a given s, we will rather denote the duration $\zeta(W)$ of W by the letter $\sigma(W)$.

Let us describe the law of Le Gall's Brownian snake (see [13] for an introduction to the subject). Conditionally given the lifetime process (ζ_s), the process W under the Brownian snake distribution is a non-homogeneous Markov process with the following transition kernel. Given $W(s) = (w(t), 0 \leq t \leq \zeta_t)$, the law of the path $W(s+s')$ is that of the path $(w'(t), 0 \leq t \leq \zeta_{s+s'})$ defined by

$$w'(t) = \begin{cases} W(s,t) & \text{if } 0 \leq t \leq \check{\zeta}_{s,s+s'} \\ W(s, \check{\zeta}(s,s+s')) + B_{t-\check{\zeta}(s,s+s')} & \text{if } \check{\zeta}_{s,s+s'} \leq t \leq \zeta_{s+s'} \end{cases}\,,$$

where $(B_t, t \geq 0)$ is a standard Brownian motion independent of ζ, and

$$\check{\zeta}_{s,s+s'} = \inf_{s \leq u \leq s+s'} \zeta_u\,.$$

We let $\mathbb{Q}_w$ be the law of the process W started from the path $w \in \mathcal{C}$, and driven by a Brownian motion started from $\zeta(w)$ and killed at first hitting of 0 (i.e. a process with law $\mathbb{E}^{(0,\infty)}_{\zeta(w)}$).

For our purposes, the key property of Brownian snake will be the following representation using Poisson random measures, which can be found for instance in [13, 19]. Namely, let $w \in \mathcal{C}(\mathbb{R})$. Recall that $(\zeta_s, 0 \leq s \leq \sigma(W))$ is the lifetime process driving the canonical process W, and that under $\mathbb{Q}_w$ we have $W(0) = w$ and $\zeta_0 = \zeta(w)$. For $0 \leq r \leq \zeta_0$, define $\Sigma_r = \inf\{s \geq 0 : \zeta_s = \zeta_0 - r\}$, so in particular $\sigma = \Sigma_{\zeta_0}$, and the process $(\Sigma_r, 0 \leq r \leq \zeta_0)$ is non-decreasing and right-continuous. For every $r \in [0, \zeta_0]$ such that $\Sigma_r > \Sigma_{r-}$, let

$$W^{(r)}(s) = (W(\Sigma_{r-} + s, t + r), 0 \leq t \leq \zeta_{\Sigma_{r-}+s} - r)\,, \qquad 0 \leq s \leq \Sigma_r - \Sigma_{r-}\,. \tag{20}$$

In this way, every $W^{(r)}$ is an element of $\mathcal{C}(\mathcal{C}(\mathbb{R}))$. Then, under $\mathbb{Q}_w$, we consider the measure

$$\mathcal{M}_w = \sum_{0 \leq r \leq \zeta(w)} \delta_{(r, W^{(r)})} \mathbf{1}_{\{T_r > T_{r-}\}}\,.$$

Lemma 5 in [13, Chapter V] states that under $\mathbb{Q}_w$, the measure $\mathcal{M}_w$ is a Poisson random measure on $[0, \zeta(w)] \times \mathcal{C}$, with intensity measure given by

$$2\,\mathrm{d}r\,\mathbf{1}_{[0,\zeta(w)]}(r)\,\mathbb{N}_{w(r)}(\mathrm{d}W)\,. \tag{21}$$

Here, the measure $\mathbb{N}_x$ is called the *Itô excursion measure* of the Brownian snake started at x. It is the σ-finite "law" of the Brownian snake started from the (trivial) path w with $w(0) = x$ and $\zeta(w) = 0$, and driven by a trajectory which is a Brownian excursion under the Itô measure of the positive excursions of Brownian motion $n(\mathrm{d}\zeta)$. See [31, Chapter XII] for the properties of the excursion measure n, which is denoted by n_+ in this reference. It is important that we fix the normalization of this measure so that the factor 2 in (21) makes sense, and we choose it so that $n(\sup \zeta > x) = (2x)^{-1}$ for every $x > 0$.

6.1.3 The measure $\mathsf{CLM}^{(k+1)}$

For every dominant scheme $\mathfrak{s} \in \mathbf{S}_d^{(k+1)}$, we let $\lambda_\mathfrak{s}$ be the measure on $\mathbb{R}^{V(\mathfrak{s})}$ defined by

$$\lambda_\mathfrak{s}(\mathrm{d}(\ell_v)_{v\in V(\mathfrak{s})}) = \prod_{v\in V_N(\mathfrak{s})} \delta_0(\mathrm{d}\ell_v) \prod_{v\in V_I(\mathfrak{s})} \mathrm{d}\ell_v \mathbf{1}_{\{\ell_v>0\}} \prod_{v\in V_O(\mathfrak{s})} \mathrm{d}\ell_v\,,$$

called the Lebesgue measure on admissible labelings of $\mathfrak{s}$. We define the continuum measure $\mathsf{CLM}^{(k+1)}$ on $\mathbf{CLM}^{(k+1)}$ by

$$\begin{aligned}\mathsf{CLM}^{(k+1)}(\mathrm{d}(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})})) \;=\;& \mathsf{S}_d^{(k+1)}(\mathrm{d}\mathfrak{s}) \int_{\mathbb{R}^{V(\mathfrak{s})}} \lambda_\mathfrak{s}(\mathrm{d}(\ell_v)_{v\in V(\mathfrak{s})}) \\ &\times \prod_{e\in E_N(\mathfrak{s})} \int_{\mathcal{C}^+(\mathbb{R})} \mathbb{E}^{(0,\infty)}_{\ell_{e_-}}(\mathrm{d}M_e)\mathbb{Q}_{\widehat{M_e}}(\mathrm{d}W_e)\mathbb{Q}_{M_e}(\mathrm{d}W_{\overline{e}}) \\ &\times \prod_{e\in E_I(\mathfrak{s})} \int_{\mathcal{C}^+(\mathbb{R})} \mathbb{B}^+_{\ell_{e_-}\to\ell_{e_+}}(\mathrm{d}M_e)\mathbb{Q}_{\widehat{M_e}}(\mathrm{d}W_e)\mathbb{Q}_{M_e}(\mathrm{d}W_{\overline{e}}) \\ &\times \prod_{e\in E_O(\mathfrak{s})} \int_{\mathcal{C}(\mathbb{R})} \mathbb{B}_{\ell_{e_-}\to\ell_{e_+}}(\mathrm{d}M_e)\mathbb{Q}_{\widehat{M_e}}(\mathrm{d}W_e)\mathbb{Q}_{M_e}(\mathrm{d}W_{\overline{e}})\,.\end{aligned} \tag{22}$$

where $\mathsf{S}_d^{(k+1)}$ is the counting measure on $\mathbf{S}_d^{(k+1)}$, and $\widehat{f}(t) = f(\zeta(f) - t), 0 \le t \le \zeta(f)$ is the time-reversed function obtained from $f \in \mathcal{C}(\mathbb{R})$. Note that, since $E_N(\mathfrak{s}), E_I(\mathfrak{s})$ and $E_O(\mathfrak{s})$ partition $E(\mathfrak{s})$, and since we have fixed an orientation convention for the edges in these sets, the oriented edges $e, \overline{e}$ exhaust $\vec{E}(\mathfrak{s})$ when e varies along $E_N(\mathfrak{s}), E_I(\mathfrak{s}), E_O(\mathfrak{s})$.

We also define a measure $\mathsf{CLM}_1^{(k+1)}$, which is rougly speaking a conditioned version of $\mathsf{CLM}^{(k+1)}$ given $\sum_{e\in\vec{E}(\mathfrak{s})} \sigma_e = 1$. Contrary to $\mathsf{CLM}^{(k+1)}$, this is a probability measure on $\mathbf{CLM}^{(k+1)}$. Its description is more elaborate than $\mathsf{CLM}^{(k+1)}$, because it does not have a simple product structure. It is more appropriate to start by defining the trace of

$\mathsf{CLM}_1^{(k+1)}$ on $(\mathfrak{s},(\ell_v),(r_e))$, i.e. its push-forward by the mapping $(\mathfrak{s},(W_e))\mapsto(\mathfrak{s},(\ell_v),(r_e))$, where $r_e=\zeta(W_e(0))$ and $\ell_v=W_e(0,0)$ whenever $v=e_-$. This trace is

$$\frac{1}{\Upsilon^{(k+1)}}\mathsf{S}_d^{(k+1)}(\mathrm{d}\mathfrak{s})\,\lambda_{\mathfrak{s}}((\mathrm{d}\ell_v)_{v\in V(\mathfrak{s})})\Big(\prod_{e\in E(\mathfrak{s})}\mathrm{d}r_e\,p^{(e)}_{r_e}(\ell_{e_-},\ell_{e_+})\Big)\overline{p}_1(2r,0)\,, \tag{23}$$

where $\Upsilon^{(k+1)}\in(0,\infty)$ is the normalizing constant making it a probability distribution, and $p^{(e)}_r(x,y)$ is defined by

- $p^{(e)}_r(x,y)=p_r(x,y)$ if $e\in E_O(\mathfrak{s})$,
- $p^{(e)}_r(x,y)=p^+_r(x,y)$ if $e\in E_I(\mathfrak{s})$, and
- $p^{(e)}_r(x,0)=\overline{p}_r(x,0)$ if $e\in E_N(\mathfrak{s})$ (entailing automatically $\ell_{e_+}=0$).

Finally, in the last displayed expression, we let $r=\sum_{e\in E(\mathfrak{s})}r_e$. Then, conditionally given $(\mathfrak{s},(\ell_v),(r_e))$, the processes $(M_e,e\in\check{E}(\mathfrak{s}))$ are independent, and respectively

- M_e has law $\mathbb{P}^{r_e}_{\ell_{e_-}\to\ell_{e_+}}$ if $e\in E_O(\mathfrak{s})$,
- M_e has law $\mathbb{P}^{r_e}_{\ell_{e_-}\to\ell_{e_+}}(\cdot\,|\,\mathcal{C}^+(\mathcal{R}))$ if $e\in E_I(\mathfrak{s})$
- M_e has law $\overline{\mathbb{P}}^{r_e}_{\ell_{e_-}}$ if $e\in E_N(\mathfrak{s})$.

As usual, the processes $(M_{\overline{e}},e\in\check{E}(\mathfrak{s}))$ are defined by time-reversal: $M_{\overline{e}}=\widehat{M}_e$. Finally, conditionally given $(M_e,e\in E(\mathfrak{s}))$, the processes $(W_e,W_{\overline{e}},e\in\check{E}(\mathfrak{s}))$ are independent Brownian snakes respectively started from $\widehat{M}_e,M_e$, conditioned on the event that the sum σ of their durations is equal to 1.

This singular conditioning is obtained in the following way. First consider a Brownian snake $(W^\circ(s),0\leq s\leq 1)$ with lifetime process given by a first-passage bridge $(\zeta(s),0\leq s\leq 1)$ from $2r=2\sum_{e\in\vec{E}(\mathfrak{s})}r_e$ to 0 with duration 1, and such that $W^\circ(0)$ is the constant path 0 with duration $2r$. Let $e_1,\ldots,e_{4k-3}$ be an arbitrary enumeration of $\vec{E}(\mathfrak{s})$, and let $r_i=r_{e_1}+\ldots+r_{e_i}$ for every $i\in\{0,1,\ldots,4k-3\}$, with the convention $r_0=0$. Then, let $\kappa_i=\inf\{s\geq 0:\zeta(s)=2r-r_i\}$, and for $1\leq i\leq 4k-3$,

$$W^\circ_{e_i}(s,t)=W^\circ(s+\kappa_{i-1},t+2r-r_i)\,,\qquad 0\leq s\leq\kappa_i-\kappa_{i-1}\,,\quad 0\leq t\leq\zeta(s+\kappa_{i-1})-\zeta(\kappa_i)\,. \tag{24}$$

The processes $(W^\circ_e,e\in\vec{E}(\mathfrak{s}))$ are then independent Brownian snakes started from the constant trajectories 0 with respective durations r_e, and conditioned on having total duration $\sigma=\sum_e\sigma_e$ equal to 1. We finally let ζ°_e be the lifetime process of W°_e, and

$$W_e(s,t)=\widehat{M}_e(t)+W^\circ_e(s,t)\,,\qquad 0\leq s\leq\sigma_e\,,\quad 0\leq t\leq\inf_{0\leq u\leq s}\zeta^\circ_e(u)\,, \tag{25}$$

that is, we change the initial value of W°_e to $\widehat{M}_e$. The different processes $(W_e,e\in\vec{E}(\mathfrak{s}))$ are then conditionally independent snakes started respectively from $\widehat{M}_e$, conditioned on

$\sigma = 1$. Hence, the snakes $(W_e, e \in \vec{E}(\mathfrak{s}))$ can be obtained by "cutting into bits of initial lengths r_e" a snake with lifetime process having law $\mathbb{E}^{(0,\infty)}_{2r}[\cdot \mid T_0 = 1]$ and started from the constant zero trajectory, to which we superimpose (independently) the initial trajectories $\widehat{M}_e$.

The relation between $\mathsf{CLM}^{(k+1)}$ and $\mathsf{CLM}^{(k+1)}_1$ goes as follows. For every $c > 0$, define a scaling operation $\Psi^{(k+1)}_c$ on $\mathbf{CLM}^{(k+1)}$, sending $(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})})$ to $(\mathfrak{s}, (W^{[c]}_e)_{e\in\vec{E}(\mathfrak{s})})$, where

$$W^{[c]}_e(s,t) = c^{1/4} W_e(s/c, t/c^{1/2}) , \qquad 0 \le s \le c\sigma_e , \quad 0 \le t \le c^{1/2}\zeta_e(s/c) .$$

Then, we let $\mathsf{CLM}^{(k+1)}_c$ be the image measure of $\mathsf{CLM}^{(k+1)}_1$ by $\Psi^{(k+1)}_c$. Sometimes we will abuse notation and write $\Psi^{(k+1)}_c((W_e)_{e\in\vec{E}(\mathfrak{s})})$ instead of $(W^{[c]}_e)_{e\in\vec{E}(\mathfrak{s})}$.

Proposition 24 *It holds that*

$$\mathsf{CLM}^{(k+1)} = \Upsilon^{(k+1)} \int_0^\infty \mathrm{d}\sigma\, \sigma^{k-9/4}\, \mathsf{CLM}^{(k+1)}_\sigma .$$

Proof. The idea is to disintegrate formula (22) with respect to $\sigma = \sum_{e\in\vec{E}(\mathfrak{s})} \sigma_e$. Note that conditionally given $\mathfrak{s}, (M_e, e \in E(\mathfrak{s}))$, the snakes $(W_e)_{e\in\vec{E}(\mathfrak{s})}$ are independent Brownian snakes started respectively from the trajectories $M_{\overline{e}}$, so that their lifetime processes are independent processes with respective laws $\mathbb{E}^{(0,\infty)}_{r_e}(\mathrm{d}\zeta_e)$. In particular, the lifetime of W_e is an independent variable with same law as T_{-r_e}, the first hitting time of $-r_e$ under $\mathbb{P}_0$. From this, we see that (still conditionally given $\mathfrak{s}, (M_e, e \in E(\mathfrak{s}))$), the total lifetime σ has same distribution as T_{-2r} under $\mathbb{P}_0$, where $r = \sum_{e\in E(\mathfrak{s})} r_e$, and this has a distribution given by $\overline{p}_\sigma(2r, 0)\mathrm{d}\sigma$ (in several places in this proof, we will not differentiate the random variable σ from its generic value, and will do a similar abuse of notation for the elements (ℓ_v) and (r_e), not to introduce new notation). Consequently, we obtain that the trace of $\mathsf{CLM}^{(k+1)}$ on $(\mathfrak{s}, (\ell_v), (r_e), \sigma)$ equals

$$\begin{aligned}
&\mathsf{S}^{(k+1)}_d(\mathrm{d}\mathfrak{s})\lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v)_{v\in V(\mathfrak{s})}) \quad \prod_{e\in E_N(\mathfrak{s})} \int_{\mathcal{C}^+(\mathbb{R})} \mathbb{E}^{(0,\infty)}_{\ell_{e_-}}(\zeta(M_e) \in \mathrm{d}r_e)\\
&\qquad \times \prod_{e\in E_I(\mathfrak{s})} \int_{\mathcal{C}^+(\mathbb{R})} \mathbb{B}^+_{\ell_{e_-}\to\ell_{e_+}}(\zeta(M_e) \in \mathrm{d}r_e)\\
&\qquad \times \prod_{e\in E_O(\mathfrak{s})} \int_{\mathcal{C}(\mathbb{R})} \mathbb{B}_{\ell_{e_-}\to\ell_{e_+}}(\zeta(M_e) \in \mathrm{d}r_e)\\
&\qquad \times \overline{p}_\sigma(2r, 0)\mathrm{d}\sigma\\
&= \mathsf{S}^{(k+1)}_d(\mathrm{d}\mathfrak{s})\lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v)_{v\in V(\mathfrak{s})}) \quad \Big(\prod_{e\in E(\mathfrak{s})} \mathrm{d}r_e\, p^{(e)}_{r_e}(\ell_{e_-}, \ell_{e_+}) \Big) \mathrm{d}\sigma\, \overline{p}_\sigma(2r, 0) .
\end{aligned}$$

Conditionally given $(\mathfrak{s}, (\ell_v), (r_e), \sigma)$, the path M_e has law $\mathbb{P}^{r_e}_{\ell_{e_-}\to\ell_{e_+}}, \mathbb{P}^{r_e}_{\ell_{e_-}\to\ell_{e_+}}(\cdot \mid \mathcal{C}^+(\mathbb{R}))$ or $\overline{\mathbb{P}}^{r_e}_{\ell_{e_-}}$ according to whether e belongs to $E_O(\mathfrak{s}), E_I(\mathfrak{s})$ or $E_N(\mathfrak{s})$, and these paths are independent. Finally, the paths W_e are independent Brownian snakes respectively starting from $M_{\overline{e}}$, conditioned on the sum of their durations being σ. These paths can be defined by applying the scaling operator $\Psi_\sigma^{(k+1)}$ to a family of independent Brownian snakes started from the paths $(\sigma^{-1/4}M_e(\sigma^{1/2}t), 0 \le t \le \sigma^{-1/2}r_e)$, and conditioned on the sum of their durations being 1.

Let us change variables $r'_e = \sigma^{-1/2}r_e$ and $\ell'_v = \sigma^{-1/4}\ell_v$ and let $r' = \sum_{e\in E(\mathfrak{s})} r'_e$. We have obtained that $\mathsf{CLM}^{(k+1)}$ is alternatively described by a trace on $(\mathfrak{s}, (\ell_v), (r_e), \sigma)$ equal to

$$\mathsf{S}_d^{(k+1)}(\mathrm{d}\mathfrak{s})\lambda_{\mathfrak{s}}(\mathrm{d}(\sigma^{1/4}\ell'_v)_{v\in V(\mathfrak{s})})\Big(\prod_{e\in E(\mathfrak{s})} \sigma^{1/2}\mathrm{d}r'_e\, p^{(e)}_{\sigma^{1/2}r'_e}(\sigma^{1/4}\ell'_{e_-}, \sigma^{1/4}\ell'_{e_+})\Big)\mathrm{d}\sigma\, \overline{p}_\sigma(2r'\sigma^{1/2}, 0)\,,$$

and conditionally given these quantities, the processes (M_e) are chosen as in the previous paragraph, while the snakes (W_e) are the image under $\Psi_\sigma^{(k+1)}$ of independent snakes respectively started from $M_{\overline{e}}$, conditioned on the sum of their durations being 1. We then use the fact that for every $r, c > 0$ and every $x, y \in \mathbb{R}$ such that the following expressions make sense,

$$p_{cr}(c^{1/2}x, c^{1/2}y) = c^{-1/2}p_r(x,y)\,, \qquad p^+_{cr}(c^{1/2}x, c^{1/2}y) = c^{-1/2}p^+_r(x,y)\,,$$

and

$$\overline{p}_{cr}(c^{1/2}x, 0) = c^{-1}p^+_r(x, 0)\,,$$

which is a simple consequence of Brownian scaling. Finally, for dominant schemes we have $3k-2$ vertices, among which k are in $V_N(\mathfrak{s})$ do not contribute in $\lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v))$, and $4k-3$ edges among which $2k$ are in $E_N(\mathfrak{s})$, yielding that the trace of $\mathsf{CLM}^{(k+1)}$ on $(\mathfrak{s}, (\ell_v), (r_e), \sigma)$ is

$$\sigma^{\frac{2(k-1)}{4}+\frac{2k-3}{4}-1}\,\mathrm{d}\sigma\, \mathsf{S}_d^{(k+1)}(\mathrm{d}\mathfrak{s})\lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v)_{v\in V(\mathfrak{s})})\Big(\prod_{e\in E(\mathfrak{s})} \mathrm{d}r_e\, p^{(e)}_{r_e}(\ell_{e_-}, \ell_{e_+})\Big)\,\overline{p}_1(2r, 0)\,,$$

and we recognize $\Upsilon^{(k+1)}\sigma^{k-9/4}\mathrm{d}\sigma$ times the trace of $\mathsf{CLM}_1^{(k+1)}$ on $(\mathfrak{s}, (\ell_v), (r_e))$. Since $\mathsf{CLM}_\sigma^{(k+1)}$ is the image measure of $\mathsf{CLM}_1^{(k+1)}$ by $\Psi_\sigma^{(k+1)}$, we have finally obtained that the trace of $\mathsf{CLM}^{(k+1)}$ on the variable σ is $\Upsilon^{(k+1)}\sigma^{k-9/4}\mathrm{d}\sigma$, and that $\mathsf{CLM}_\sigma^{(k+1)}, \sigma > 0$ are conditional measures of $\mathsf{CLM}^{(k+1)}$ given σ, as wanted. □

6.2 Limit theorems

For every $n \ge 1$, we define a new scaling operator on $\mathbf{CLM}^{(k+1)}$ by $\psi_n^{(k+1)}(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})}) = (\mathfrak{s}, (W_e^{\{n\}})_{e\in\vec{E}(\mathfrak{s})})$, where

$$W_e^{\{n\}}(s,t) = \Big(\frac{9}{8n}\Big)^{1/4} W_e(2ns, \sqrt{2n}t)\,, \qquad 0 \le s \le \frac{\sigma_e}{2n}\,, \quad 0 \le t \le \frac{1}{\sqrt{2n}}\zeta_e(2ns)\,.$$

So this is almost the same as $\Psi_{1/2n}$, except that we further multiply the labels in W_e by $\sqrt{3/2} = (9/4)^{1/4}$. Recall that $\mathsf{LM}_n^{(k+1)}$ is the counting measure over $\mathbf{LM}_n^{(k+1)}$, we view it also as a measure on $\mathbf{CLM}^{(k+1)}$ by performing the decomposition of Proposition 23. Similar abuse of notation will be used in the sequel.

Proposition 25 *We have the following weak convergence of finite measures on* $\mathbf{CLM}^{(k+1)}$*:*

$$\left(\frac{9}{2}\right)^{1/4} \frac{\psi_n^{(k+1)}{}_* \mathsf{LM}_n^{(k+1)}}{6^k \cdot 12^n n^{k-5/4}} \underset{n\to\infty}{\longrightarrow} \Upsilon_{k+1} \mathsf{CLM}_1^{(k+1)} .$$

This is proved in a very similar way to [25, 2], but one has to pay extra care in manipulating elements of $\mathbf{LM}^{(k+1)}$, because of the required positivity of ℓ_v and M_e when $v \in V_I(\mathfrak{s})$ and $e \in E_N(\mathfrak{s}) \cup E_I(\mathfrak{s})$. We start with a preliminary observation that justifies our definition of *dominant* schemes.

Lemma 26 *It holds that* $\mathsf{LM}_n^{(k+1)}(\{(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})}) : \mathfrak{s} \in \mathbf{S}_d^{(k+1)}\}) = O(12^n n^{k-5/4})$*, while* $\mathsf{LM}_n^{(k+1)}(\{(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})}) : \mathfrak{s} \notin \mathbf{S}_d^{(k+1)}\}) = O(12^n n^{k-3/2})$

Proof. Let $\mathfrak{s}$ be an element of $\mathbf{S}^{(k+1)}$, that is a scheme with $k+1$ faces. Then

$$\mathsf{LM}_n^{(k+1)}(\{(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})}) : (W_e)_{e\in\vec{E}(\mathfrak{s})} \text{ a compatible family of discrete snakes}\}) \tag{26}$$

is just the number of elements $(\mathbf{m}, \mathbf{l})$ in $\mathbf{LM}^{(k+1)}$ that induce the scheme $\mathfrak{s}$ via the construction of Proposition 23, and which have n edges in total. By the discussion of Section 5.3, an element $(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})})$ can be viewed as a walk network $(M_e, e \in E(\mathfrak{s}))$ and a family of labeled forests $(F_e, L_e)_{e\in\vec{E}(\mathfrak{s})}$ compatible with this walk network. Now, once the walks $(M_e, e \in E(\mathfrak{s}))$ are determined, we know that for $e \in \vec{E}(\mathfrak{s})$, the forest F_e has r_e trees, where r_e is the duration of M_e. The labels of the floor vertices of F_e are $M_e(0), M_e(1), \dots, M_e(r_e)$. For a given F_e, there are exactly 3^{n_e} possible labelings compatible with the labeling of the floor vertices, where n_e is the number of edges in the forest that are distinct from the floor edges, coming from the fact that the label difference along an edge of each tree belongs to $\{-1, 0, 1\}$. Therefore, if we let $r = \sum_{e\in E(\mathfrak{s})} r_e$, then by concatenating the forests $F_e, e \in \vec{E}(\mathfrak{s})$ in some given order that is fixed by convention, we obtain a forest with $2r$ trees and a total of $n - r$ edges (distinct from the floor edges) by (16). Once the labeling of the roots is fixed, there are 3^{n-r} different labelings for these forests. This construction can be easily inverted: Starting from a forest with $2r$ trees and $n - r$ edges (distinct from the floor edges), we can reconstruct $(F_e, L_e)_{e\in\vec{E}(\mathfrak{s})}$.

From this, and a classical counting result for plane forests [28], we deduce that for a given walk network $(M_e, e \in \check{E}(\mathfrak{s}))$, there are exactly

$$3^{n-r} \frac{2r}{2n} \binom{2n}{n-r}$$

labeled forests $(F_e, L_e)_{e\in\vec{E}(\mathfrak{s})}$ compatible with this walk network. Since, in the decomposition of Proposition 23, we still have to select one of the oriented edges in the forests (including floor edges), we will obtain an extra factor of $2n$ in the end. At this point, we have obtained that (26) equals

$$2 \sum_{(M_e, e\in E(\mathfrak{s}))} 3^{n-r} r \binom{2n}{n-r} ,$$

where the sum is over all walk networks compatible with the scheme $\mathfrak{s}$.

Let $\mathcal{W}(a,b;r)$ be the number of Motzkin walks with duration r, starting at a and ending at b. Likewise, we let $\mathcal{W}^+(a,b;r)$ be the number of such paths that are strictly positive, except perhaps at their endpoints (so that $\#\mathcal{W}^+(0,0;1)=1$ for instance). Then the formula for (26) becomes

$$2\sum_{(\ell_v)}\sum_{(r_e)}3^{n-r}r\binom{2n}{n-r}\prod_{e\in\check{E}(\mathfrak{s})}\mathcal{W}^{(e)}(\ell_{e_-},\ell_{e_+};r_e)\,, \tag{27}$$

where the first sum is over all admissible labelings for $\mathfrak{s}$, the second sum is over all edge-lengths on $\mathfrak{s}$, and the superscript (e) accounts for the constraint on the path M_e, namely,

- $\mathcal{W}^{(e)}(a,b;r)=\mathcal{W}(a,b;r)$ if $e\in E_O(\mathfrak{s})$,
- $\mathcal{W}^{(e)}(a,b;r)=\mathcal{W}^+(a,b;r)$ if $e\in E_I(\mathfrak{s})$,
- $\mathcal{W}^{(e)}(a,0;r)=\mathcal{W}^+(a,0;r)$ if $e\in E_N(\mathfrak{s})\setminus E_T(\mathfrak{s})$
- $\mathcal{W}^{(e)}(a,0;r)=\mathcal{W}^+(a+1,0;r+1)$ if $e\in E_T(\mathfrak{s})$.

To explain the last point, recall that the walks M_e indexed by the thin edges are non-negative and finish at 0, so we can turn it into a walk taking positive values except at the last point, by translating labels by 1 and adding an extra "virtual" -1 step in the end.

Note that $q_r(a,b):=3^{-r}\mathcal{W}(a,b;r)$ (resp. $q_r^+(a,b):=3^{-r}\mathcal{W}^+(a,b;r)$) is the probability that a uniform Motzkin walk with r steps started at a finishes at b (resp. without taking non-negative values, except possibly at the start and end point).

At this point, we will need the following consequence of the local limit theorem of [27], stating that for every $a,b\in\mathbb{Z}$ and $p\geq 1$, there exists a finite constant $C>0$ depending only on p such that for every $r\geq 1$,

$$\sqrt{r}q_r(a,b)\leq\frac{C}{1+\left|\frac{b-a}{\sqrt{r}}\right|^p}\,. \tag{28}$$

Likewise, by viewing $4^{-n}\binom{2n}{n-r}$ as the probability that a simple random walk attains $2r$ in $2n$ steps, a similar use of the local limit theorem allows to show

$$\frac{\sqrt{2n}}{4^n}\binom{2n}{n-r}\leq\frac{C}{1+\left(\frac{r}{\sqrt{n}}\right)^p}\,. \tag{29}$$

On the other hand, the reflection principle entails that for $a,b,r>0$,

$$q_r^+(a,b)=q_r(a,b)-q_r(a,-b)\,, \tag{30}$$

and the cyclic lemma [28] entails that for $a,r>0$,

$$q_r^+(a,0)=\frac{a}{r}q_r(a,0)\,,\qquad\text{and}\qquad q_r^+(0,0)=\frac{1}{3}\mathbf{1}_{\{r=1\}}+\frac{1}{r-1}q_{r-1}(1,0)\mathbf{1}_{\{r>1\}}\,. \tag{31}$$

These considerations imply that for a fixed p, (26) is bounded from above by

$$C\,12^n \sum_{(\ell_v)} \sum_{(r_e)} \frac{\frac{r}{\sqrt{n}}}{1+\left(\frac{r}{\sqrt{n}}\right)^{p+1}} \tag{32}$$

$$\times \prod_{e\in E_F(\mathfrak{s})} \frac{1}{\sqrt{r_e}} \cdot \frac{1}{1+\left(\frac{\ell_{e_+}-\ell_{e_-}}{\sqrt{r_e}}\right)^2} \prod_{e\in E_N^{(1)}(\mathfrak{s})} \frac{1}{r_e} \cdot \frac{\frac{\ell_{e_-}+\mathbf{1}_{\{e\in E_T(\mathfrak{s})\}}}{\sqrt{r_e+\mathbf{1}_{\{e\in E_T(\mathfrak{s})\}}}}}{1+\left|\frac{\ell_{e_-}+\mathbf{1}_{\{e\in E_T(\mathfrak{s})\}}}{\sqrt{r_e+\mathbf{1}_{\{e\in E_T(\mathfrak{s})\}}}}\right|^3} \prod_{e\in E_N^{(0)}(\mathfrak{s})} \frac{1}{r_e^{3/2}}\,,$$

where we let $E_N^{(0)}(\mathfrak{s})$ be the set of edges in $E_N(\mathfrak{s})$ with both extremities in $V_N(\mathfrak{s})$, $E_N^{(1)}(\mathfrak{s}) = E_N(\mathfrak{s}) \setminus E_N^{(0)}(\mathfrak{s})$, and finally

$$E_F(\mathfrak{s}) = E(\mathfrak{s}) \setminus E_N(\mathfrak{s}) = E_I(\mathfrak{s}) \cup E_O(\mathfrak{s})$$

(here the subscript F stands for *free*). Of course, the constant C above depends only on p, but not on n. As before, the sums ae over all admissible labelings and edge-lengths. At this point, since we are only interested in taking upper-bounds, we may (and will) in fact sum for (ℓ_v) belonging to the set $\mathbb{Z}^{V(\mathfrak{s})\setminus V_N(\mathfrak{s})} \times \{0\}^{V_N(\mathfrak{s})}$, i.e. we lift the positivity constraint on vertices in $V_I(\mathfrak{s})$.

Now, let $r' = \sum_{e\in E(\mathfrak{s})\setminus E_N^{(0)}(\mathfrak{s})} r_e$, so that $r' \le r$, and note that

$$\frac{\frac{r}{\sqrt{n}}}{1+\left(\frac{r}{\sqrt{n}}\right)^{p+1}} \le \frac{C}{1+\left(\frac{r}{\sqrt{n}}\right)^{p}} \le \frac{C}{1+\left(\frac{r'}{\sqrt{n}}\right)^{p}}\,.$$

Therefore, we can sum out the edge-lengths in the last product of (32), and use elementary inequalities in the second product (as simple as $l+1 \le 2l$ for every integer $l \ge 1$) to get the upper-bound

$$C\,12^n \sum_{(\ell_v)} \sum_{(r_e)}{}' \frac{1}{1+\left(\frac{r'}{\sqrt{n}}\right)^{p}} \prod_{e\in E_F(\mathfrak{s})} \frac{1}{\sqrt{r_e}} \cdot \frac{1}{1+\left(\frac{\ell_{e_+}-\ell_{e_-}}{\sqrt{r_e}}\right)^2} \prod_{e\in E_N^{(1)}(\mathfrak{s})} \frac{1}{r_e} \cdot \frac{1}{1+\left(\frac{\ell_{e_-}}{\sqrt{r_e}}\right)^2}\,,$$

the symbol $\sum'_{(r_e)}$ meaning that we sum only over all positive integers r_e indexed by edges $e \in E(\mathfrak{s}) \setminus E_N^{(0)}(\mathfrak{s})$. At this point, we write this as an integral

$$C\,12^n \int \mathrm{d}(\ell_v) \int{}' \mathrm{d}(r_e) \frac{1}{1+\left(\frac{\sum'_e \lfloor r_e\rfloor}{\sqrt{n}}\right)^{p}}$$

$$\times \prod_{e\in E_F(\mathfrak{s})} \frac{\mathbf{1}_{\{r_e\ge 1\}}}{\sqrt{\lfloor r_e\rfloor}} \cdot \frac{1}{1+\left(\frac{\lfloor \ell_{e_+}\rfloor-\lfloor \ell_{e_-}\rfloor}{\sqrt{\lfloor r_e\rfloor}}\right)^2} \prod_{e\in E_N^{(1)}(\mathfrak{s})} \frac{\mathbf{1}_{\{r_e\ge 1\}}}{\lfloor r_e\rfloor} \cdot \frac{1}{1+\left(\frac{\lfloor \ell_{e_-}\rfloor}{\sqrt{\lfloor r_e\rfloor}}\right)^2}\,,$$

the first integral being with respect to the measure $\prod_{v\in V(\mathfrak{s})\setminus V_N(\mathfrak{s})} \mathrm{d}\ell_v \prod_{v\in V_N(\mathfrak{s})} \delta_0(\mathrm{d}\ell_v)$, and the second over $\prod_{e\in E(\mathfrak{s})\setminus E_N^{(0)}(\mathfrak{s})} \mathrm{d}r_e \mathbf{1}_{\{r_e\ge 0\}}$. Using the fact that all the quantities r_e in the integrand are greater than or equal to 1, the integral is bounded by

$$C\,12^n \int \mathrm{d}(\ell_v) \int' \mathrm{d}(r_e) \frac{1}{1+\left(\frac{r'}{\sqrt{n}}\right)^p} \prod_{e\in E_F(\mathfrak{s})} \frac{1}{\sqrt{r_e}} \cdot \frac{1}{1+\left(\frac{\ell_{e_+}-\ell_{e_-}}{\sqrt{r_e}}\right)^2} \prod_{e\in E_N^{(1)}(\mathfrak{s})} \frac{1}{r_e} \cdot \frac{1}{1+\left(\frac{\ell_{e_-}}{\sqrt{r_e}}\right)^2}\,.$$

We perform a linear change of variables, dividing ℓ_v by $n^{1/4}$ and r_e by $n^{1/2}$, which gives after simplification

$$C12^n n^{(\#V_F(\mathfrak{s})+\#E_F(\mathfrak{s}))/4} \mathscr{I}(\mathfrak{s})\,,$$

where

$$V_F(\mathfrak{s}) = V(\mathfrak{s}) \setminus V_N(\mathfrak{s}) = V_I(\mathfrak{s}) \cup V_O(\mathfrak{s})\,,$$

is the set of *free vertices*, and

$$\mathscr{I}(\mathfrak{s}) = \int \mathrm{d}(\ell_v) \int' \mathrm{d}(r_e) \frac{1}{1+(r')^p} \prod_{e\in E_F(\mathfrak{s})} \frac{1}{\sqrt{r_e}} \cdot \frac{1}{1+\left(\frac{\ell_{e_+}-\ell_{e_-}}{\sqrt{r_e}}\right)^2} \prod_{e\in E_N^{(1)}(\mathfrak{s})} \frac{1}{r_e} \cdot \frac{1}{1+\left(\frac{\ell_{e_-}}{\sqrt{r_e}}\right)^2}\,.$$

Provided $\mathscr{I}(\mathfrak{s})$ is finite for every scheme $\mathfrak{s}$, which we will check in a moment, we can conclude that (26) has a dominant behavior as $n \to \infty$ whenever $\mathfrak{s}$ is such that $\#V_F(\mathfrak{s}) + \#E_F(\mathfrak{s})$ is the largest possible. Assume that at least one of the vertices of $V_N(\mathfrak{s})$ has degree at least 3. If we "free" this vertex by declaring it in V_I instead, then we have increased $\#V_F(\mathfrak{s})$ by 1, but to make sure that the resulting map is also a scheme, we should also add a degree-2 null vertex at the center of each edge of $E_N^{(1)}(\mathfrak{s})$ incident to v. Each of these operations adds an edge in $E_N^{(1)}$, but does not decrease the cardinality of $E_F(\mathfrak{s})$. Therefore, the maximal value of $\#V_F(\mathfrak{s}) + \#E_F(\mathfrak{s})$ is attained for schemes in which all null-vertices have degree 2 (and in particular, $E_N^{(0)}$ is empty). Furthermore, if there are more than $k+1$ null-vertices with degree 2, then at least one of them can be removed without breaking the condition that $\mathfrak{s}$ is a scheme, because at least two of them are incident to the same face f_i for some $i \in \{1, 2, \dots, k\}$. Removing a degree-2 null vertex increases $\#E_F$ by 1 and leaves $\#V_F$ unchanged. Therefore, the optimal schemes are those having k null vertices, which are all of degree 2. But now, the optimal schemes will be obviously those with the largest number of vertices (or edges), and by definition, these are the dominant ones. Since a dominant scheme has $2k-2$ free vertices and $2k-3$ free edges, we obtain that $\#V_F(\mathfrak{s}) + \#E_F(\mathfrak{s}) \le 4k-5$, with equality if and only if $\mathfrak{s}$ is dominant. Dominant schemes thus have a contribution $O(12^n n^{k-5/4})$ to (26), while non-dominant ones have a contribution $O(12^n n^{k-5/4-1/4})$, and by summing over all such schemes we get the result.

It remains to justify that $\mathscr{I}(\mathfrak{s})$ is finite for every scheme $\mathfrak{s}$. To see this, we first integrate with respect to the variables (ℓ_v). We view $E_N^{(1)}$ and the corresponding incident vertices as a subgraph of $\mathfrak{s}$. Let $\mathfrak{a}$ be a spanning tree of $\mathfrak{s}/E_N^{(1)}$, that is, every vertex in $V_F(\mathfrak{s})$ is linked to a vertex of the subgraph $E_N^{(1)}$ by a unique injective chain, canonically oriented toward $E_N^{(1)}$.

We then bound

$$\prod_{e\in E_F(\mathfrak{s})}\frac{1}{\sqrt{r_e}}\cdot\frac{1}{1+\left(\frac{\ell_{e_+}-\ell_{e_-}}{\sqrt{r_e}}\right)^2}\leq\prod_{e\notin E(\mathfrak{a})\cup E_N(\mathfrak{s})}\frac{1}{\sqrt{r_e}}\prod_{e\in E(\mathfrak{a})}\frac{1}{\sqrt{r_e}}\cdot\frac{1}{1+\left(\frac{\ell_{e_+}-\ell_{e_-}}{\sqrt{r_e}}\right)^2}.$$

In this form, we can then perform the change of variables $x_{e_-}=\ell_{e_+}-\ell_{e_-}$ for every $e\in E(\mathfrak{a})$, and $x_v=\ell_v$ whenever $v\in V_F(\mathfrak{s})$ is incident to an edge in $E_N^{(1)}$. This change of variable is triangular with Jacobian 1. This allows to bound $\mathscr{I}(\mathfrak{s})$ by

$$\begin{aligned}\int'\mathrm{d}(r_e)\frac{1}{1+(r')^p}&\prod_{e\notin E(\mathfrak{a})\cup E_N(\mathfrak{s})}\frac{1}{\sqrt{r_e}}\prod_{e\in E(\mathfrak{a})}\int\frac{\mathrm{d}x_{e_-}/\sqrt{r_e}}{1+\left(\frac{x_{e_-}}{\sqrt{r_e}}\right)^2}\prod_{v\in V_F(\mathfrak{s})}\int\prod_{\substack{e\in E_N^{(1)}(\mathfrak{s})\\ e_-=v}}\frac{1}{r_e}\cdot\frac{\mathrm{d}x_v}{1+(x_v/\sqrt{r_e})^2}\\ &=C\int'\mathrm{d}(r_e)\frac{1}{1+(r')^p}\prod_{e\notin E(\mathfrak{a})\cup E_N(\mathfrak{s})}\frac{1}{\sqrt{r_e}}\prod_{v\in V_F(\mathfrak{s})}\int\prod_{\substack{e\in E_N^{(1)}(\mathfrak{s})\\ e_-=v}}\frac{1}{r_e}\cdot\frac{\mathrm{d}x_v}{1+(x_v/\sqrt{r_e})^2}\end{aligned}$$

The last integral is of the form (for positive $r_1,\dots,r_l$)

$$\begin{aligned}\int\mathrm{d}x\prod_{i=1}^{l}\frac{1}{r_i}\cdot\frac{1}{1+(x/\sqrt{r_i})^2}&\leq C\int\mathrm{d}x\prod_{i=1}^{l}\frac{1}{r_i}\wedge\frac{1}{x^2}\\ &\leq C\frac{\sqrt{\min(r_1,\dots,r_l)}}{\prod_{i=1}^{l}r_i},\end{aligned}$$

for some constant C depending on l, the second step being easy to prove by first assuming that $r_1<r_2<\dots<r_l$ and decomposing the integral along the intervals $[0,\sqrt{r_1}],[\sqrt{r_1},\sqrt{r_2}],\dots,[\sqrt{r_l},\infty[$. Note that the last upper-bound is integrable on $[0,1]^l$ with respect to $\mathrm{d}r_1\dots\mathrm{d}r_l$, since

$$\int_{[0,1]^l}\prod\frac{\mathrm{d}r_i}{r_i}\sqrt{\min(r_1,\dots,r_i)}=l\int_0^1\frac{\mathrm{d}r_1}{\sqrt{r_1}}\int_{[r_1,1]^{l-1}}\prod_{i=2}^{l}\frac{\mathrm{d}r_i}{r_i}=l\int_0^1\frac{\mathrm{d}r_1}{\sqrt{r_1}}\log(1/r_1)^{l-1},$$

which is finite. By choosing $p>\#E(\mathfrak{s})$ (since there are only a finite number of elements of $\mathbf{S}^{(k+1)}$, we can choose a single p valid for every scheme with $k+1$ faces), we finally obtain that $\mathscr{I}(\mathfrak{s})$ is bounded from above by an integral in (r_e), whose integrand is both integrable in a neighborhood of 0 and of infinity. Therefore $\mathscr{I}(\mathfrak{s})$ is finite, as wanted. □

We now prove Proposition 25. By Lemma 26, it suffices to consider the restriction of $\mathsf{LM}_n^{(k+1)}$ to the labeled maps whose associated scheme is in $\mathbf{S}_d^{(k+1)}$. We start by considering the image of the measure $\mathsf{LM}_n^{(k+1)}$ under the mapping $(\mathfrak{s},(W_e)_{e\in\vec{E}_\mathfrak{s}})\mapsto(\mathfrak{s},(\ell_v)_{v\in V(\mathfrak{s})},(r_e)_{e\in E(\mathfrak{s})})$, which we still write $\mathsf{LM}_n^{(k+1)}$ by abuse of notation. Let f be a continuous function on $\mathbf{S}_d^{(k+1)}\times\mathbb{R}^{V(\mathfrak{s})}\times\mathbb{R}^{E(\mathfrak{s})}$, we can assume that $f(\mathfrak{s}',(\ell_v),(r_e))$ is non-zero only when $\mathfrak{s}'$ equals some particular dominant scheme $\mathfrak{s}\in\mathbf{S}_d^{(k+1)}$, and we drop the mention of the first component of f in the sequel.

Then we have, by similar arguments as in the derivation of (27),

$$\psi_n^{(k+1)}{}_*\mathsf{LM}_n^{(k+1)}(f) = 2{\cdot}3^{n+k}\sum_{(\ell_v)}\sum_{(r_e)} r\binom{2n}{n-r}\prod_{e\in\check{E}(\mathfrak{s})} q_{r_e}^{(e)}(\ell_{e_-},\ell_{e_+})f\Big(\Big(\Big(\frac{9}{8n}\Big)^{1/4}\ell_v\Big),\Big(\frac{r_e}{\sqrt{2n}}\Big)\Big),$$

where $q_r^{(e)}(a,b) = 3^{-r}\mathcal{W}^{(e)}(a,b;r)$ and $r=\sum_e r_e$. Here, the factor 3^k comes from the fact that there are k edges in $E_T(\mathfrak{s})$, because $\mathfrak{s}$ is dominant, and each such edge corresponds to a Motzkin walk with a final "virtual" step in the end, participating an extra factor 3, as explained in the proof of Proposition 26. We write this as an integral

$$2{\cdot}12^n 3^k\int\lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v))\int\mathrm{d}(r_e)\frac{[r]}{4^n}\binom{2n}{n-[r]}\prod_{e\in\check{E}(\mathfrak{s})} q_{\lfloor r_e\rfloor}^{(e)}(\lfloor\ell_{e_-}\rfloor,\lfloor\ell_{e_+}\rfloor)f\Big(\Big(\Big(\frac{9}{8n}\Big)^{1/4}\lfloor\ell_v\rfloor\Big),\Big(\frac{\lfloor r_e\rfloor}{\sqrt{2n}}\Big)\Big),$$

where $[r]=\sum_{e\in E(\mathfrak{s})}\lfloor r_e\rfloor$, and where we omitted to write indicators $\mathbf{1}_{\{r_e\geq 1\}}$ and $\mathbf{1}_{\{\ell_v\geq 1\}}$ whenever $v\in V_I(\mathfrak{s})$ to lighten the expression. We perform a linear change of variables, dividing r_e by $\sqrt{2n}$ and ℓ_v by $(8n/9)^{1/4}$, yielding

$$2\cdot 12^n 3^k (2n)^{\#E(\mathfrak{s})/2}(8n/9)^{\#V_F(\mathfrak{s})/4}\int\lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v))\int\mathrm{d}(r_e)\frac{[r\sqrt{2n}]}{4^n}\binom{2n}{n-[r\sqrt{2n}]}$$
$$\prod_{e\in\check{E}(\mathfrak{s})} q_{\lfloor r_e\sqrt{2n}\rfloor}^{(e)}\Big(\Big\lfloor\ell_{e_-}\Big(\frac{8n}{9}\Big)^{1/4}\Big\rfloor,\Big\lfloor\ell_{e_+}\Big(\frac{8n}{9}\Big)^{1/4}\Big\rfloor\Big)f\Big(\Big(\Big(\frac{9}{8n}\Big)^{1/4}\Big\lfloor\ell_v\Big(\frac{8n}{9}\Big)^{1/4}\Big\rfloor\Big),\Big(\frac{\lfloor r_e\sqrt{2n}\rfloor}{\sqrt{2n}}\Big)\Big).$$

Now, by the local limit theorem, it holds that for every fixed $r>0$ and $\ell,\ell'\in\mathbb{R}$, we have

$$\Big(\frac{8n}{9}\Big)^{1/4} q_{\lfloor r_e\sqrt{2n}\rfloor}\Big(\Big\lfloor\Big(\frac{8n}{9}\Big)^{1/4}\ell\Big\rfloor,\Big\lfloor\Big(\frac{8n}{9}\Big)^{1/4}\ell'\Big\rfloor\Big)\underset{n\to\infty}{\longrightarrow} p_{r_e}(\ell,\ell'),$$

and similarly with q^+ and p^+ instead of q and p (recall (30)), whenever $\ell,\ell'>0$, while for $\ell>0$, using (31) and the local limit theorem,

$$\sqrt{2n}q^+_{\lfloor r_e\sqrt{2n}\rfloor}\Big(\Big\lfloor\Big(\frac{8n}{9}\Big)^{1/4}\ell\Big\rfloor,0\Big)\underset{n\to\infty}{\longrightarrow}\overline{p}_{r_e}(\ell,0)\,,.$$

A final use of the local limit theorem (or Stirling's formula), as in (29) shows that for positive $r_e, e\in E(\mathfrak{s})$, with same notation as above,

$$\frac{[r\sqrt{2n}]}{4^n}\binom{2n}{n-[r\sqrt{2n}]}\underset{n\to\infty}{\longrightarrow}\overline{p}_1(2r,0)\,.$$

Finally, since dominant schemes have $4k-3$ edges (of which $2k-3$ are free) and $3k-2$ vertices ($2k-2$ free) we see after simplifications that the integral is equivalent to

$$3^k 2^k (2/9)^{1/4} 12^n n^{k-5/4}\int\lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v))\int\mathrm{d}(r_e)\overline{p}_1(2r,0)\Big(\prod_{e\in\check{E}(\mathfrak{s})} p_{r_e}^{(e)}(\ell_{e_-},\ell_{e_+})\Big)f((\ell_v),(r_e))\,,$$

at least if we can justify to take the limit inside the integral. This is done by dominated convergence: We bound f by its supremum norm, and apply the very same argument as in the proof of Lemma 26, using (28) and (29) to bound the integrand by the integrand of the quantity $\mathscr{I}(\mathfrak{s})$. Details are left to the reader.

In summary, by definition of $\mathsf{CLM}_1^{(k+1)}$, we have proven that

$$\left(\frac{9}{2}\right)^{1/4} \frac{\psi_n^{(k+1)}{}_*\mathsf{LM}_n^{(k+1)}}{6^k \cdot 12^n n^{k-5/4}}(F) \underset{n\to\infty}{\longrightarrow} \Upsilon_{k+1}\mathsf{CLM}_1^{(k+1)}(F)\,,$$

when F is a continuous and bounded function of the form $F(\mathfrak{s},(W_e)) = f(\mathfrak{s},(\ell_v),(r_e))$. In order to prove the full result, we note that Proposition 23 entails that under the measure $\mathsf{LM}_n^{(k+1)}$, conditionally given $(\mathfrak{s},(\ell_v),(r_e))$, the paths $(M_e, e \in \check{E}(\mathfrak{s}))$ are independent discrete walks, respectively from ℓ_{e_-} to ℓ_{e_+} and with duration r_e, which are conditioned to be positive (except possibly at their final point) when $e \in E_N(\mathfrak{s}) \cup E_I(\mathfrak{s})$. This is in fact not perfectly right since paths M_e with $e \in E_T(\mathfrak{s})$ come with one extra negative step, but this is of no incidence to what follows. Since we have shown that r_e scales like $\sqrt{2n}$ and ℓ_v like $(8n/9)^{1/4}$, it is quite standard (see [2, Lemmas 10 and 14] for a recent and thorough exposition) that the paths M_e, after applying the operation $\psi_n^{(k+1)}$, converge in distribution to independent Brownian bridges (conditioned to be positive if $e \in E_I(\mathfrak{s})$, or first-passage bridges when $e \in E_N(\mathfrak{s})$), in accordance with the definition of $\mathsf{CLM}_1^{(k+1)}$.

Lastly, conditionally on the paths $(M_e, e \in E(\mathfrak{s}))$ under the discrete measure $\mathsf{LM}_n^{(k+1)}$, the snakes $(W_e, e \in \vec{E}(\mathfrak{s}))$ are associated with independent labeled forests (F_e, L_e) with respectively r_e trees, conditioned on having a total number of oriented edges equal to $2n$. Moreover, the labels are uniform among all admissible labelings, with the constraint that the labels of the root vertices in the forest associated with the edge $e \in \vec{E}(\mathfrak{s})$ are given by the path M_e. By subtracting these labels we obtain forests with root labels 0, and uniform labelings among the admissible ones, so that we are only concerned in the convergence of the discrete snakes associated with these forests to independent Brownian snakes started respectively from the constant trajectory equal to 0 and duration r_e, and conditioned on $\sigma = \sum_{e\in\vec{E}(\mathfrak{s})} \sigma_e = 1$, under the law $\mathsf{CLM}_1^{(k+1)}$ conditionally on $(\mathfrak{s},(\ell_v),(r_e))$.

But the forests F_e can be obtained from one single forest with $2r$ trees and total number of oriented edges (comprising floor edges) equal to $2n$, by cutting the floor into segments with respectively r_e trees, for $e \in \vec{E}(\mathfrak{s})$. The random snake associated with this forest converges, once rescaled according to the operation $\psi_n^{(k+1)}$, to a Brownian snake with total duration 1, starting from the constant trajectory equal to 0 and with duration $2r$, by [2, Proposition 15]. It is then easy to see that the snakes obtained by "cutting this snake into bits of lengths r_e", as explained in the definition of $\mathsf{CLM}_1^{(k+1)}$ in Section 6.1.3, is indeed the limiting analog of the discrete cutting just mentioned. A completely formal proof requires to write the discrete analogs of (24) and (25) and verify that they pass to the limit, which is a little cumbersome and omitted. □

6.3 The case of planted schemes

Recall that in a planted scheme $\mathfrak{s} \in \dot{\mathbf{S}}^{(k+1)}$ with $k+1$ faces, there is a unique vertex of degree 1, the others being of degree 3. This vertex is an element of $V_O(\mathfrak{s})$, and the edge incident to this vertex is an element of $E_O(\mathfrak{s})$. The elements of $\dot{\mathbf{S}}_d^{(k+1)}$ (the dominant planted schemes) have $3k$ vertices, of which k are null vertices (all of degree 2), and $4k-1$ edges, of which $2k$ are elements of $E_N(\mathfrak{s})$.

On the space

$$\mathbf{C\dot{L}M}^{(k+1)} = \left\{ (\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})}) : \mathfrak{s} \in \dot{\mathbf{S}}^{(k+1)}, (W_e)_{e\in\vec{E}(\mathfrak{s})} \in \mathcal{C}(\mathcal{C}(\mathbb{R}))^{\vec{E}(\mathfrak{s})} \right\},$$

the "planted continuum measure" $\mathsf{C\dot{L}M}^{(k+1)}$ and its conditioned counterpart $\mathsf{C\dot{L}M}_1^{(k+1)}$ are defined as in formulas (22) and (23) for $\mathsf{CLM}^{(k+1)}$ and $\mathsf{CLM}_1^{(k+1)}$, only replacing the counting measure $\mathsf{S}_d^{(k+1)}$ by the counting measure $\dot{\mathsf{S}}_d^{(k+1)}$ on planted scheme, and changing $\Upsilon^{(k+1)}$ by the proper normalization constant $\dot{\Upsilon}^{(k+1)}$. These are measures on the space $\mathbf{CLM}^{(k+1)}$ that constitutes in the pairs $(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})})$, where $\mathfrak{s}$ is a planted scheme, and $(W_e)_{e\in\vec{E}(\mathfrak{s})}$ is an admissible family of discrete snakes.

The analogous statements to Propositions 24 and 26 goes as follows. We define the scaling operations $\dot{\Psi}_c$ and $\dot{\psi}_n^{(k+1)}$ on $\mathbf{C\dot{L}M}^{(k+1)}$ by the same formulas as $\Psi_c^{(k+1)}$ and $\psi_n^{(k+1)}$ in sections 6.1.3 and 6.2. In the following statement, we use the decomposition of Section 5.4 rather than Proposition 23, and we view LM_n as a measure on $\mathbf{C\dot{L}M}^{(k+1)}$.

Proposition 27 *It holds that*

$$\mathsf{C\dot{L}M}^{(k+1)} = \int_0^\infty \mathrm{d}\sigma\, \sigma^{k-5/4} \mathsf{C\dot{L}M}_\sigma^{(k+1)}.$$

Moreover, we have the following weak convergence of finite measures on $\mathbf{C\dot{L}M}^{(k+1)}$*:*

$$\left(\frac{9}{2}\right)^{1/4} \frac{\dot{\psi}_n^{(k+1)}{}_* \mathsf{LM}_n^{(k+1)}}{6^k \cdot 12^n n^{k-5/4}} \underset{n\to\infty}{\longrightarrow} \dot{\Upsilon}^{(k+1)} \mathsf{C\dot{L}M}_1^{(k+1)}.$$

The proof follows exactly the same lines as Proposition 25, so we leave it to the reader. One has to be a little careful about the small variations in the construction of Section 5.4, compared to Proposition 23, when we are dealing with the distinguished edge and its adjacent edges, but these variations disappear in the scaling limit. Also, note that the reason why the measure $\mathrm{d}\sigma\, \sigma^{k-5/4}$ appears in the disintegration of $\mathsf{C\dot{L}M}^{(k+1)}$, instead of $\mathrm{d}\sigma\, \sigma^{k-9/4}$, is that $\mathsf{C\dot{L}M}^{(k+1)}$ carries intrinsically the location of a distinguished point (corresponding to the only vertex of the scheme that has degree 1). This marked point should be seen as the continuum counterpart of the root edge in discrete maps, so it is natural that if the continuum object has a total "mass" σ, then there marking introduces a further factor σ.

In fact, we could recover Propositions 24 and 25 from Proposition 27, by considering the natural operation from $\dot{\mathbf{S}}^{(k+1)}$ to $\mathbf{S}^{(k+1)}$ that erases the distinguished edge incident to the degree-1 vertex. We leave to the reader to properly formulate and prove such a statement, which we are not going to need in the sequel. The reason why we are dealing with planted schemes (which are "richer" objects) only now is that we find them a little harder to understand and manipulate than schemes. Using non-planted schemes will also simplify the proofs in Section 7.4.

7 Proof of the key lemmas

We finally use the results of Sections 5 and 6 to prove Lemmas 18 and 19. Recall from Proposition 20 how the probabilities of events $\mathcal{A}_1(\varepsilon,\beta)$ and $\mathcal{A}_2(\varepsilon)$ are dominated by the limsup of probabilities of related events $\mathcal{A}_1^{(n)}(\varepsilon,\beta)$ and $\mathcal{A}_2^{(n)}(\varepsilon)$ for quadrangulations. The latter events were defined on a probability space supporting the random variables Q_n and marked vertices $v_0, v_1, \ldots, v_k$, in the sequel we will see them as sets of marked quadrangulations $(\mathbf{q},\mathbf{v})$ with $\mathbf{q} \in \mathbf{Q}_n$ and $\mathbf{v} \in V(\mathbf{q})^{k+1}$, by abuse of notation.

It turns out that the latter events have a tractable translation in terms of the labeled map $(\mathbf{m},\mathbf{l})$ associated with the random quadrangulation Q_n by the multi-pointed bijection $\Phi^{(k+1)}$ of Section 5.1.

7.1 Relation to labeled maps

Let us first consider $\mathcal{A}_1^{(n)}(\varepsilon,\beta)$. In the sequel, if $X \in \mathcal{C}(\mathbb{R})$ we let $\underline{X} = \inf_{0\le t\le\zeta(X)} X(t)$, and if $W \in \mathcal{C}(\mathcal{C}(\mathbb{R}))$ we let $\underline{W} = \inf_{0\le s\le\sigma(W)} \underline{W(s)}$. We let $\mathcal{B}_1(\varepsilon,\beta)$ be the event on $\dot{\mathbf{CLM}}^{(3)}$ that

- for every $e \in \vec{E}(\mathfrak{s})$ incident to f_0, it holds that $\underline{W}_e \ge -2\varepsilon$,
- if $e \in E(\mathfrak{s})$ is incident to f_1 and f_2, then $\underline{M}_e \ge 0$,
- there exists $e \in E_I(\mathfrak{s})$ such that, if we orient e in such a way that it is incident to f_1 or f_2, then $\underline{W}_e \le -\varepsilon^{1-\beta}$.

Let $(\mathbf{q},\mathbf{v}) \in \mathcal{A}_1^{(n)}(\varepsilon,\beta)$, $r = \lfloor \varepsilon(8n/9)^{1/4}\rfloor - 1$, and $r' \in \{r+1, r+2, \ldots, 2r\}$. We note that $\mathcal{A}_1^{(n)}(\varepsilon,\beta)$ contains the event $G(r,2)$, so by Lemma 21, the labeled map $(\mathbf{m},\mathbf{l}) = \Phi^{(3)}(\mathbf{q},\mathbf{v},\boldsymbol{\tau}^{(r')})$ is an element of $\mathbf{LM}^{(3)}$. We let $(\mathfrak{s},(W_e, e\in\vec{E}(\mathfrak{s})))$ be the element of $\dot{\mathbf{CLM}}^{(k+1)}$ associated with $(\mathbf{m},\mathbf{l})$ as in Section 5.4 (note that we choose to take planted schemes here). Recall the definition of the scaling functions $\psi_n^{(k+1)}$ and $\dot{\psi}_n^{(k+1)}$ from Sections 6.2 and 6.3.

Lemma 28 *With the above notation, for every $\varepsilon > 0$ small enough, $(\mathfrak{s},(W_e, e\in\vec{E}(\mathfrak{s})))$ belongs to the event $\mathcal{B}_1^{(n)}(\varepsilon,\beta) = (\dot{\psi}_n^{(3)})^{-1}(\mathcal{B}_1(\varepsilon,\beta))$*

Proof. The first point is a consequence of the fact (15) that the minimal label of a corner incident to f_0 is equal to $\tau_0^{(r')}+1$, which is $-r'+1 \ge -2\varepsilon(8n/9)^{1/4}$ by our choice.

For the second point, since $\tau_i^{(r')} = -d_{\mathbf{q}}(v_i, v_0) - \tau_0^{(r')}$, we deduce that for every $v \in B_{d_{\mathbf{q}}}(v_0, \varepsilon(8n/9)^{1/4})$,

$$d_{\mathbf{q}}(v, v_0) + \tau_0^{(r')} < 0 < d_{\mathbf{q}}(v, v_i) + \tau_i^{(r')} ,$$

since $d_{\mathbf{q}}(v_0, v_i) - d_{\mathbf{q}}(v, v_i) \le d_{\mathbf{q}}(v, v_0) < -\tau_0^{(r')}$ by the triangle inequality. Consequently, we obtain that $B_{d_{\mathbf{q}}}(v_0, \varepsilon(8n/9)^{1/4}) \setminus \{v_0\}$ is included in the set of vertices of $\mathbf{m}$ that are incident to the face f_0, and to no other face. Under the event $\mathcal{A}_1^{(n)}(\varepsilon, \beta)$, we thus obtain that any geodesic from v_1 to v_2 has to visit vertices incident to f_0 exclusively.

Therefore, such a geodesic has to visit a vertex incident to f_0 and to either f_1 or f_2. By definition of $\mathbf{LM}^{(3)}$, such a vertex v has non-negative label $l \ge 0$, and the length of the geodesic chain has to be at least (we let $\min_{f_1} \mathbf{l} = \min_{w \in V(f_1)} \mathbf{l}(w)$)

$$d_{\mathbf{q}}(v_1, v_2) \ge \Big(\mathbf{l}(v) - \min_{f_1} \mathbf{l} + 1\Big) + \Big(\mathbf{l}(v) - \min_{f_2} \mathbf{l} + 1\Big) \ge 2 - \min_{f_1} \mathbf{l} - \min_{f_2} \mathbf{l} . \tag{33}$$

Now assume that v a vertex incident to both f_1 and f_2, and let l be its label. Choose two corners e, e' incident to v that belong respectively to f_1 and f_2. By drawing the leftmost geodesic chains from e, e' to v_1, v_2 visiting the consecutive successors of e, e' as in Section 5.1.2, and concatenating these chains, we obtain a chain visiting only vertices that are incident to f_1 or f_2. This chain cannot be geodesic between v_1 and v_2 because it does not visit vertices incident to f_0 exclusively. But its length is given by

$$\Big(l - \min_{f_1} \mathbf{l} + 1\Big) + \Big(l - \min_{f_2} \mathbf{l} + 1\Big) \ge d_{\mathbf{q}}(v_1, v_2) .$$

Comparing with (33), we see that necessarily, $l \ge 0$. Hence all labels of vertices incident to f_1 and f_2 are non-negative, implying the second point in the definition of $\mathcal{B}_1^{(n)}(\varepsilon, \beta)$.

For the third point, we argue by contradiction, and assume that for every edge $e \in \vec{E}(f_1)$ of $\mathfrak{s}$ with $\overline{e} \in \vec{E}(f_0)$, (resp. $e \in \vec{E}(f_2)$ with $\overline{e} \in \vec{E}(f_0)$), the labels in the forest F_e that is branched on e are all greater than or equal to $-\varepsilon^{1-\beta}(8n/9)^{1/4}$. Note that every dominant pre-scheme with 3 faces is such that there is a unique edge incident both to f_0 and f_1 (resp. f_0 and f_2). Consider a geodesic chain γ from v_2 to v_1. Since this chain has to visit vertices exclusively contained in f_0, we can consider the last such vertex in the chain. Then the following vertex v is necessarily incident to f_1: Otherwise, it would be incident to f_2, and the concatenation of a leftmost geodesic from v to v_2 with the remaining part of γ between v and v_1 would be a shorter chain from v_2 to v_1 not visiting strictly f_0.

Necessarily, there is a vertex v' with label 0 incident to f_1 and f_0. Let e, e' be corners of $\mathbf{m}$ incident to f_1 and respectively incident to v and v'. By hypothesis, when visiting one of the intervals $[e, e']$ or $[e', e]$, we encounter only corners with labels greater than or equal to $-\varepsilon^{1-\beta}(8n/9)^{1/4}$. Consider the left-most geodesics $\tilde{\gamma}, \tilde{\gamma}'$ from e and e' to v_1, visiting their consecutive successors. By replacing the portion of γ from v to v_1 by $\tilde{\gamma}$, we still get a geodesic from v_2 to v_1. On the other hand, $\tilde{\gamma}'$ is a portion of a geodesic from v_0 to v_1, in which an initial segment of length at most $2r \le 2\varepsilon(8n/9)^{1/4}$ has been removed. So we have found a geodesic from v_0 to v_1 such that every vertex v'' on this geodesic that lies outside of $B_{d_{\mathbf{q}}}(v_0, (2\varepsilon + \varepsilon^{1-\beta})(8n/9)^{1/4})$ is such that (v_2, v'', v_1) are aligned. The same holds with the roles of v_1 and v_2 interchanged, and for ε small enough we have $2\varepsilon \leq \varepsilon^{1-\beta}$ therefore, $\mathcal{A}_1^{(n)}(\varepsilon, \beta)$ does not hold. $\square$

As a corollary, we obtain that for every $\beta \in (0,1)$ and $\varepsilon > 0$ small enough,

$$\limsup_{n\to\infty} P(\mathcal{A}_1^{(n)}(\varepsilon,\beta)) \leq \frac{C}{\varepsilon}\dot{\mathsf{CLM}}_1^{(3)}(\mathcal{B}_1(\varepsilon,\beta)) . \tag{34}$$

To justify this, we write, since Q_n is uniform in $\mathbf{Q}_n$ and $(v_0, v_1, \ldots, v_k)$ are independent uniformly chosen points in $V(Q_n)$, if we let $r = \lfloor \varepsilon(8n/9)^{1/4} \rfloor - 1$,

$$\begin{aligned} P(\mathcal{A}_1^{(n)}(\varepsilon,\beta)) &= \frac{1}{n^{k+1}\#\mathbf{Q}_n} \sum_{\mathbf{q}\in\mathbf{Q}_n, \mathbf{v}\in V(\mathbf{q})^{k+1}} \mathbf{1}_{\{(\mathbf{q},\mathbf{v})\in\mathcal{A}_1^{(n)}(\varepsilon,\beta)\}} \\ &= \frac{1}{rn^{k+1}\#\mathbf{Q}_n} \sum_{\mathbf{q}\in\mathbf{Q}_n, \mathbf{v}\in V(\mathbf{q})^{k+1}} \sum_{r'=r+1}^{2r} \mathbf{1}_{\{(\mathbf{q},\mathbf{v})\in\mathcal{A}_1^{(n)}(\varepsilon,\beta)\}} . \end{aligned}$$

Since $\Phi^{(k+1)}$ is two-to-one, and by Lemma 22, this is bounded from above by

$$\frac{2}{rn^{k+1}\#\mathbf{Q}_n} \sum_{(\mathbf{m},\mathbf{l})\in\mathbf{LM}_n^{(3)}} \mathbf{1}_{\{(\Phi^{(k+1)})^{-1}(\mathbf{m},\mathbf{l})\in\mathcal{A}_1^{(n)}(\varepsilon,\beta)\}} \leq \frac{C}{\varepsilon 12^n n^{k-5/4}} \dot{\psi}_n^{(3)}{}_*\mathsf{LM}_n(\mathcal{B}_1(\varepsilon,\beta)) ,$$

where we finally used Lemma 28 and the well-known fact that

$$\#\mathbf{Q}_n = \frac{2}{n+2} \cdot \frac{3^n}{n+1} \binom{2n}{n} .$$

Finally, Proposition 27 entails (34), since $\mathcal{B}_1(\varepsilon,\beta)$ is a closed set. We will estimate the upper-bound in (34) in Section 7.3.

Let us now bound the probability of $\mathcal{A}_2^{(n)}(\varepsilon)$. We say that a dominant scheme $\mathfrak{s} \in \mathbf{S}_d^{(k+1)}$ is *predominant* if f_0 has minimal degree in $\mathfrak{s}$. For $k = 3$, we see that the top-left pre-scheme of Figure 1 is the only one that gives rise to predominant schemes with 4 faces by adding a null-vertex in the middle of the three edges incident to f_0: All the dominant schemes constructed from the four others pre-schemes will have at least 7 oriented edges incident to f_0. We let $\mathfrak{P}$ be the set of predominant schemes.

We consider the event $\mathcal{A}_3 \subset \mathbf{LM}^{(4)}$ that

- the scheme $\mathfrak{s}$ associated with the map belongs to $\mathfrak{P}$,
- for every $v \in V(f_1 \cap f_2)$, it holds that $\mathbf{l}(v) \geq 0$,
- for every $v \in V(f_3 \cap f_1)$ and $v' \in V(f_3 \cap f_2)$, and e, e' two corners of f_3 incident to v, v', if we take the convention that $[e, e']$ is the set of corners between e and e' in facial order around f_3, that passes through the corners of the unique edge of $\mathfrak{s}$ incident to f_3 and f_0, then it holds that

$$\mathbf{l}(v) + \mathbf{l}(v') + 2 \geq \min_{e''\in[e,e']} \mathbf{l}(e'') . \tag{35}$$

For instance, in Figure 5 below, the interval $[e, e']$ is represented as the dotted contour inside the face f_3.

Let $(\mathbf{q}, \mathbf{v}) \in \mathcal{A}_2^{(n)}(\varepsilon)$, $r = \lfloor \varepsilon(8n/9)^{1/4} \rfloor - 1$ and $r' \in \{r+1, r+2, \ldots, 2r\}$. By Lemma 21, the labeled map $(\mathbf{m}, \mathbf{l}) = \Phi^{(4)}(\mathbf{q}, \mathbf{v}, \boldsymbol{\tau}^{(r')})$ is an element of $\mathbf{LM}^{(4)}$, and we let $(\mathfrak{s}, (W_e, e \in \vec{E}(\mathfrak{s}))$ be the element of $\mathbf{CLM}^{(4)}$ associated with it via Proposition 23.

Lemma 29 *Under these hypotheses, it always hold that* $\min_{e \in \vec{E}(f_0)} \underline{W}_e \geq -2\varepsilon(8n/9)^{1/4}$, *and moreover, if* $\mathfrak{s} \in \mathfrak{P}$ *then* $(\mathbf{m}, \mathbf{l}) \in \mathcal{A}_3$.

Proof. The lower-bound on $\min_{e \in \vec{E}(f_0)} \underline{W}_e$ is just (15). Next, let us assume that $\mathfrak{s} \in \mathfrak{P}$, and let us check that $\mathcal{A}_3$ is satisfied. The second point is derived in exactly the same way as we checked the second point of $\mathcal{B}_1(\varepsilon, \beta)$ in the derivation of Lemma 28. Indeed, on $\mathcal{A}_2(\varepsilon)$, for n large enough, all geodesic paths from v_1 to v_2 have to pass through f_0, and so they have length at least $-\min_{v \in f_1} \mathbf{l}(v) - \min_{v \in V(f_2)} \mathbf{l}(v) + 2$. So if $\mathbf{l}(v) < 0$ for some $v \in V(f_1 \cap f_2)$, then by drawing the successive arcs starting from two corners of f_1 and f_2 incident to v, until v_1 and v_2 are reached, we would construct a path with length at most $-\min_{v \in V(f_1)} \mathbf{l}(v) - \min_{v \in V(f_2)} \mathbf{l}(v)$ between v_1 and v_2, a contradiction.

For the third point, note that if we draw the two leftmost geodesic chains from e, e' to v_3 inside the face f_3, then these two chains coalesce at a distance from v_3 which is precisely $\min_{e'' \in [e,e']} \mathbf{l}(e'') + 2$. Therefore, it is possible to build a geodesic chain from v_1 to v, with length $\mathbf{l}(v) - \min_{u \in V(f_1)} \mathbf{l}(u) + 1$, and to concatenate it with a chain of length $\mathbf{l}(v) + \mathbf{l}(v') - 2\min_{e'' \in [e,e']} \mathbf{l}(e'') + 2$ from v to v', and then with a geodesic chain from v' to v_2, with length $\mathbf{l}(v') - \min_{u \in V(f_2)} \mathbf{l}(u) + 1$. Since the resulting path cannot be shorter than a geodesic from v_1 to v_2, we obtain the third required condition. $\square$

We now translate the event $\mathcal{A}_3$ in terms of the encoding processes $(\mathfrak{s}, (W_e)_{e \in \vec{E}(\mathfrak{s})})$. If $\mathfrak{s}$ is a predominant scheme with 4 faces, then the associated pre-scheme is the first one of Figure 1. Therefore, there is a single edge e_{ij} incident to f_i and f_j for every $i < j$ in $\{1, 2, 3\}$. We let $\mathcal{B}_2(\varepsilon) \subset \mathbf{CLM}^{(4)}$ be the event that

1. the scheme $\mathfrak{s}$ belongs to $\mathfrak{P}$,
2. $\min_{e \in \vec{E}(f_0)} \underline{W}_e \geq -2\varepsilon$,
3. $\underline{M}_{e_{12}} \geq 0$,
4. for every $t \in [0, r_{e_{13}}]$ and $t' \in [0, r_{e_{23}}]$, it holds that

$$\min_{e \in \vec{E}_3(\mathfrak{s}) : \bar{e} \in \vec{E}_0(\mathfrak{s})} \underline{W}_e \wedge \inf\{\underline{W}_{e_{13}}^{(s)}, 0 \leq s \leq t\} \wedge \inf\{\underline{W}_{e_{23}}^{(s)}, 0 \leq s \leq t'\} \leq M_{e_{13}}(t) + M_{e_{23}}(t') \tag{36}$$

Here and in the remainder of the paper, we use a slightly unusual convention, that $W_{e_{13}}^{(s)}, W_{e_{23}}^{(s)}$ are the snake excursions (the tree components in the continuum forest) branching on e_{13} and e_{23} that lie inside f_3, where the orientation of e_{13}, e_{23} points away from f_0. For instance, in Figure 5, these are the tree components that lie to the *right* of e_{13}

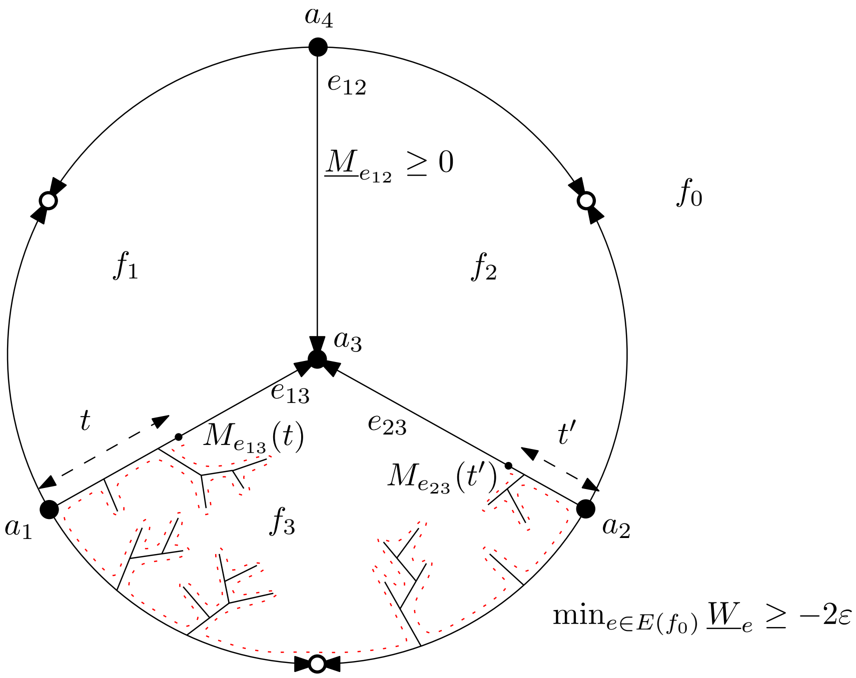

Figure 5: Illustration for the event $\mathcal{B}_2(\varepsilon)$. Only a small portion of the snakes branching on the scheme have been represented here. The scheme underlying this element of $\mathbf{CLM}^{(4)}$ is a predominant scheme with 4 faces, and all the others are obtained by obvious symmetries. All labels along the edge e_{12} are non-negative, all labels in f_0 are greater than or equal to -2ε, and for every $t \in [0, r_{e_{13}}], t' \in [0, r_{e_{23}}]$, the minimal label along the dotted contour is at most $M_{e_{13}}(t) + M_{e_{23}}(t')$. The blank vertices indicate elements of $V_N(\mathfrak{s})$, and the labels a_1, a_2, a_3, a_4 are the integration variables appearing in the proof of Lemma 37 below.

instead of the left, as our usual conventions would require: In this case, we should really read $W_{\bar{e}_{13}}^{(r_{e_{13}}-s)}$ rather than $W_{e_{13}}^{(s)}$.

Although it is a little tedious, it is really a matter of definitions to check that if $(\mathbf{m}, \mathbf{l}) \in \mathcal{A}_3$ and $\min_{e\in\vec{E}(f_0)} \underline{W}_e \geq -2\varepsilon(8n/9)^{1/4}$ then $(\mathfrak{s}, (W_e)_{e\in\vec{E}(\mathfrak{s})})$ belongs to $(\psi_n^{(4)})^{-1}(\mathcal{B}_2(\varepsilon))$. To be perfectly accurate, there is a small difference coming from the $+2$ term in (35), which does not appear anymore in (36). A way to circumvent this would be to re-define the discrete snake processes associated with a labeled map $(\mathbf{m}, \mathbf{l})$ by shifting them by 1. Such a modification obviously does not change the limit theorems of Section 6.

By similar reasoning as in the derivation of (34), using Proposition 25 rather than Proposition 27, we deduce from the above discussion that

$$\limsup_{n\to\infty} P(\mathcal{A}_2^{(n)}(\varepsilon)) \leq \frac{C}{\varepsilon}\Big(\mathsf{CLM}_1^{(4)}(\mathfrak{s} \notin \mathfrak{P}, \min_{e\in\vec{E}(f_0)} \underline{W}_e \geq -2\varepsilon) + \mathsf{CLM}_1^{(4)}(\mathcal{B}_2(\varepsilon))\Big)\,. \qquad (37)$$

7.2 Some estimates for bridges and snakes

Here we gather the technical estimates that will be needed to estimate the upper-bounds in (34) and (37). Recall the notation from Section 6.1.1 and 6.1.2 for bridge and snake measures.

Lemma 30 *Fix $\lambda > 0$. Then*

$$\sup_{x,y>0} \mathbb{B}_{x\to y}[e^{-\lambda\zeta(X)}] < \infty \qquad \text{and} \qquad \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \mathrm{d}y\, \mathbb{B}_{x\to y}[e^{-\lambda\zeta(X)}] < \infty\,. \tag{38}$$

Moreover, for every $x, y \geq 0$, it holds that

$$\mathbb{B}_{x\to y}[e^{-\lambda\zeta(X)}\mathbf{1}_{\{\underline{X}\geq 0\}}] \leq 2(x\wedge y)\,, \tag{39}$$

and

$$\mathbb{B}_{x\to y}[\zeta(X)e^{-\lambda\zeta(X)}\mathbf{1}_{\{\underline{X}\geq 0\}}] \leq \sqrt{\frac{2}{\lambda}}\,xy\,. \tag{40}$$

Proof. On the one hand, we have

$$\mathbb{B}_{x\to y}[e^{-\lambda\zeta(X)}] = \int_0^\infty \mathrm{d}r\, p_r(x,y)e^{-\lambda r} \leq \int_0^\infty \frac{\mathrm{d}r\, e^{-\lambda r}}{\sqrt{2\pi r}}\,,$$

which is finite and independent of x, y. Moreover, by Fubini's theorem and the fact that $p_r(x, y)$ is a probability density, we have

$$\int_{\mathbb{R}} \mathrm{d}y\, \mathbb{B}_{x\to y}[e^{-\lambda\zeta(X)}] = \int_0^\infty \mathrm{d}r\, e^{-\lambda r}\,,$$

which is again independent of x. This gives (38).

Next, as in Section 6.1.1, we use the following consequence of the reflection principle:

$$\mathbb{P}^r_{x\to y}(\underline{X} \geq 0) = 1 - e^{-2xy/r}\,, \qquad \text{for every} \quad x, y \geq 0 \text{ and} r > 0\,.$$

Assuming first that $x, y > 0$ and $x \neq y$, this gives

$$\begin{aligned}\mathbb{B}_{x\to y}(e^{-\lambda\zeta(X)}\mathbf{1}_{\{\underline{X}\geq 0\}}) &= \int_0^\infty \mathrm{d}r\, p_r(x,y)e^{-\lambda r} - \int_0^\infty \mathrm{d}r\, p_r(x,y)e^{-\lambda r}e^{-2xy/r} \\ &= \mathbb{E}_0\left[\frac{T_{|x-y|}}{|x-y|}e^{-\lambda T_{|x-y|}}\right] - \mathbb{E}_0\left[\frac{T_{x+y}}{x+y}e^{-\lambda T_{x+y}}\right],\end{aligned}$$

where we used the well-known fact that $(a/r)p_r(0, a)\mathrm{d}r = \mathbb{P}_0(T_a \in \mathrm{d}r)$. From the Laplace transform of T_a, given by $\mathbb{E}_0[e^{-uT_a}] = e^{-a\sqrt{2u}}$, we immediately get that for $a > 0$,

$$\mathbb{E}_0\left[\frac{T_a}{a}e^{-uT_a}\right] = \frac{1}{\sqrt{2u}}e^{-a\sqrt{2u}}\,,$$

from which we obtain

$$\mathbb{B}_{x\to y}(e^{-\lambda\zeta(X)}\mathbf{1}_{\{\underline{X}\geq 0\}}) = \frac{e^{-|x-y|\sqrt{2\lambda}}}{\sqrt{2\lambda}}(1-e^{-2\sqrt{2\lambda}(x\wedge y)}) \leq 2(x\wedge y)\,,$$

as wanted. This remains true for $x=y$ or for $xy=0$ by a continuity argument, yielding (39). The proof of (40) is similar, writing

$$\begin{aligned}\mathbb{B}_{x\to y}(\zeta(X)e^{-\lambda\zeta(X)}\mathbf{1}_{\{\underline{X}\geq 0\}}) &= \int_0^\infty \mathrm{d}r\, r\, p_r(x,y)e^{-r} - \int_0^\infty \mathrm{d}r\, r\, p_r(x,y)e^{-r}e^{-2xy/r} \\ &= \mathbb{E}_0\Big[\frac{T_{|x-y|}^2}{|x-y|}e^{-\lambda T_{|x-y|}}\Big] - \mathbb{E}_0\Big[\frac{T_{x+y}^2}{x+y}e^{-\lambda T_{x+y}}\Big]\,,\end{aligned}$$

and using again Laplace transforms to get, for $a,u>0$,

$$\mathbb{E}_0\Big[\frac{T_a^2}{a}e^{-uT_a}\Big] = \frac{1}{2\sqrt{2u^3}}(a\sqrt{2u}+1)e^{-a\sqrt{2ua}}\,.$$

Now assume without loss of generality that $0<y\leq x$ and write

$$\begin{aligned}&\mathbb{B}_{x\to y}(\zeta(X)e^{-\lambda\zeta(X)}\mathbf{1}_{\{\underline{X}\geq 0\}}) \\ &= \frac{1}{2\sqrt{2\lambda^3}}\Big(((x-y)\sqrt{2\lambda}+1)e^{(-x+y)\sqrt{2\lambda}} - ((x+y)\sqrt{2\lambda}+1)e^{-(x+y)\sqrt{2\lambda}}\Big) \\ &= \frac{e^{-x\sqrt{2\lambda}}\cosh(y\sqrt{2\lambda})}{\sqrt{2\lambda^3}}\Big((x\sqrt{2\lambda}+1)\tanh(y\sqrt{2\lambda}) - y\sqrt{2\lambda}\Big) \\ &\leq \sqrt{\frac{2}{\lambda}}xy\,,\end{aligned}$$

as wanted. □

Lemma 31 *Let x,y,z be positive real numbers. Then*

$$\mathbb{E}_x^{(y,\infty)}[\mathbb{Q}_X(\underline{W}\geq 0)] \leq \Big(\frac{y}{x}\Big)^2 \qquad \textit{if} \quad 0<y<x\,, \tag{41}$$

and there exists a finite $C>0$ such that

$$\mathbb{E}_x^{(y,\infty)}[\mathbb{Q}_X(\underline{W}\geq 0)\mathbb{Q}_X(\underline{W}<-z)] \leq Cy^2\Big(\frac{1}{z}\wedge\frac{1}{x}\Big)^2 \qquad \textit{if} \quad 0<y<x\,. \tag{42}$$

Finally, for every $\beta\in[-2,3]$, it holds that

$$\mathbb{B}_{x\to y}[\mathbb{Q}_X(\underline{W}\geq 0)] \leq \frac{1}{5}x^\beta y^{1-\beta}\,. \tag{43}$$

Proof. We use the Poisson point process description of the Brownian snake. Namely, recall the notation from Section 6.1.2 and the fact that the snake W under $\mathbb{Q}_X$ can be decomposed in excursions $W^{(r)}, 0 \le r \le \zeta(X)$, in such a way that

$$\sum_{0\le r\le \zeta(X)} \delta_{(r,W^{(r)})}\mathbf{1}_{\{T_r > T_{r-}\}}$$

is a Poisson random measure on $[0,\zeta(X)]\times\mathcal{C}(\mathbb{R})$, with intensity measure given by

$$2\,\mathrm{d}r\,\mathbf{1}_{[0,\zeta(X)]}(r)\,\mathbb{N}_{X(r)}(\mathrm{d}W)\,.$$

From this and the known formula [19]

$$\mathbb{N}_0(\underline{W} < -y) = \frac{3}{2y^2}\,, \qquad y > 0\,,$$

we obtain, using standard properties of Poisson measures,

$$\begin{aligned}
\mathbb{Q}_X(\underline{W}\ge 0) &= \exp\Big(-2\int_0^{\zeta(X)}\mathrm{d}r\,\mathbb{N}_{X(r)}(\underline{W}<0)\Big)\\
&= \exp\Big(-2\int_0^{\zeta(X)}\mathrm{d}r\,\mathbb{N}_0(\underline{W}<-X(r))\Big)\\
&= \exp\Big(-\int_0^{\zeta(X)}\frac{3\,\mathrm{d}r}{X(r)^2}\Big)\,. \qquad (44)
\end{aligned}$$

We deduce that for every $x > y > 0$,

$$\mathbb{E}_x^{(y,\infty)}[\mathbb{Q}_X(\underline{W}\ge 0)] = \mathbb{E}_x^{(y,\infty)}\Big[\exp\Big(-\int_0^{T_y}\frac{3\,\mathrm{d}r}{X(r)^2}\Big)\Big]\,.$$

Recalling that reflected Brownian motion is a 1-dimensional Bessel process, we now use the absolute continuity relations between Bessel processes with different indices, due to Yor [31, Exercise XI.1.22] (see also [19] for a similar use of these absolute continuity relations). The last expectation then equals

$$\Big(\frac{x}{y}\Big)^3\mathbb{P}_x^{\langle 7\rangle}(T_y<\infty)\,,$$

where, for $\delta \ge 0$, $\mathbb{P}_x^{\langle\delta\rangle}$ is the law of the δ-dimensional Bessel process started from $x > 0$. Recall that for $\delta \ge 2$, the δ-dimensional Bessel process started from $x > 0$ is the strong solution (starting from x) of the stochastic differential equation driven by the standard Brownian motion $(B_t, t \ge 0)$:

$$\mathrm{d}Y_t = \mathrm{d}B_t + \frac{\delta-1}{2Y_t}\mathrm{d}t\,.$$

One can show that $Y_t > 0$ for all t, hence that the drift term is well-defined, whenever $\delta \geq 2$. Showing that $\mathbb{P}_x^{\langle\delta\rangle}(T_y < \infty) = (y/x)^{\delta-2}$ for every $x > y > 0$ is now a simple exercise, using the fact that $(Y_t^{2-\delta}, t \geq 0)$ is a local martingale by Itô's formula. Putting things together, we get (41).

Let us now turn to (42). By (44) and an easy translation invariance argument, we have

$$\begin{aligned}
&\mathbb{E}_x^{(y,\infty)}[\mathbb{Q}_X(\underline{W} \geq 0)\mathbb{Q}_X(\underline{W} < -z)] \\
&= \mathbb{E}_x^{(y,\infty)}\Big[\exp\Big(-\int_0^{T_y}\frac{3\,\mathrm{d}r}{X(r)^2}\Big)\Big(1-\exp\Big(-\int_0^{T_y}\frac{3\,\mathrm{d}r}{(X(r)+z)^2}\Big)\Big)\Big] \\
&\leq 3\,\mathbb{E}_x^{(y,\infty)}\Big[\exp\Big(-\int_0^{T_y}\frac{3\,\mathrm{d}r}{X(r)^2}\Big)\Big(1\wedge\frac{T_y}{z^2}\Big)\Big] \\
&= 3\Big(\frac{x}{y}\Big)^3\mathbb{E}_x^{\langle 7\rangle}\Big[\mathbf{1}_{\{T_y<\infty\}}\Big(1\wedge\frac{T_y}{z^2}\Big)\Big] \\
&\leq 3\Big(\frac{x}{y}\Big)^3\Big(\mathbb{P}_x^{\langle 7\rangle}(T_y<\infty)\wedge\frac{1}{z^2}\mathbb{E}_x^{\langle 7\rangle}[T_y\mathbf{1}_{\{T_y<\infty\}}]\Big),
\end{aligned}$$

where we have used again the absolute continuity relations for Bessel processes at the third step. We already showed that $\mathbb{P}_x^{\langle 7\rangle}(T_y < \infty) = (y/x)^5$, so to conclude it suffices to show that $\mathbb{E}_x^{\langle 7\rangle}[T_y\mathbf{1}_{\{T_y<\infty\}}] \leq Cy^5/x^3$ for some finite constant C. By the Markov property, and using again the formula for the probability that $T_y < \infty$ for the 7-dimensional Bessel process, we have

$$\begin{aligned}
\mathbb{E}_x^{\langle 7\rangle}[T_y\mathbf{1}_{\{T_y<\infty\}}] &= \int_0^\infty \mathrm{d}s\,\mathbb{P}_x^{\langle 7\rangle}(s<T_y<\infty) \\
&= \int_0^\infty \mathrm{d}s\,\mathbb{E}_x^{\langle 7\rangle}\big[\mathbf{1}_{\{s<T_y\}}\mathbb{P}_{X(s)}^{\langle 7\rangle}(T_y<\infty)\big] \\
&\leq \int_0^\infty \mathrm{d}s\,\mathbb{E}_x^{\langle 7\rangle}\Big[\Big(\frac{y}{X(s)}\Big)^5\Big].
\end{aligned}$$

Using the fact that the 7-dimensional Bessel process has same distribution as the Euclidean norm of the 7-dimensional Brownian motion, and the known form of the latter's Green function, we obtain that if $u = (1,0,0,0,0,0,0) \in \mathbb{R}^7$, it holds that

$$\mathbb{E}_x^{\langle 7\rangle}[T_y\mathbf{1}_{\{T_y<\infty\}}] \leq \int_{\mathbb{R}^7}\frac{\mathrm{d}z}{|z-xu|^5}\cdot\Big(\frac{y}{|z|}\Big)^5 = \frac{y^5}{x^3}\int_{\mathbb{R}^7}\frac{\mathrm{d}z}{|z-u|^5|z|^5},$$

and the integral is finite, as wanted.

We now prove (43), by using the agreement formula (18), entailing that

$$\begin{aligned}
\mathbb{B}_{x\to y}[\mathbb{Q}_X(\underline{W}\geq 0)] &= \int_{-\infty}^{x\wedge y}\mathrm{d}z\,(\mathbb{E}_x^{(z,\infty)}\bowtie\widehat{\mathbb{E}}_y^{(z,\infty)})[\mathbb{Q}_X(\underline{W}\geq 0)] \\
&= \int_0^{x\wedge y}\mathrm{d}z\,\mathbb{E}_x^{(z,\infty)}[\mathbb{Q}_X(\underline{W}\geq 0)]\mathbb{E}_y^{(z,\infty)}[\mathbb{Q}_X(\underline{W}\geq 0)] \\
&\leq \int_0^{x\wedge y}\mathrm{d}z\,\Big(\frac{z}{x}\Big)^2\cdot\Big(\frac{z}{y}\Big)^2 = \frac{(x\wedge y)^5}{5x^2y^2},
\end{aligned}$$

where we used (41) in the penultimate step. The conclusion follows easily. □

Let us consider once again the Poisson point measure representation $(W^{(t)}, 0 \le t \le \zeta(X))$ of the Brownian snake under the law $\mathbb{Q}_X$, as explained around (20). Fix $y > 0$. We are interested in the distribution of the random variable $\inf_{0\le t\le T_{-z}} \underline{W}^{(t)}$ for $0 \le z \le y$, as well as in bounding expectations of the form

$$\mathbb{E}_0^{(-y,\infty)}\Big[\mathbb{Q}_X\Big(-x \wedge \inf_{0\le r\le T_{-z}} \underline{W}^{(r)} \le -2z \text{ for every } z \in [0,y]\Big)\Big].$$

To this end, we perform yet another Poisson measure representation for these quantities.

Lemma 32 *Let $y > 0$ be fixed. Let X be the canonical process on $\mathcal{C}(\mathbb{R})$, and W be the canonical process on $\mathcal{C}(\mathcal{C}(\mathbb{R}))$ started from $W(0) = X$. Recall that $T_x = \inf\{t \ge 0 : X(t) = x\}$ for every $x \in \mathbb{R}$. For every $z \ge 0$, let*

$$I_z = -z - \inf\{\underline{W}^{(t)} : T_{(-z)-} \le t \le T_{-z}\},$$

which is taken to be 0 by convention if $T_{-z} = \infty$. Then under $\mathbb{P}_0^{(-y,\infty)}(\mathrm{d}X)\mathbb{Q}_X(\mathrm{d}W)$, the point measure

$$\sum_{0\le z\le y} \delta_{(z,I_z)} \mathbf{1}_{\{I_z>0\}}$$

is a Poisson random measure on $[0, y] \times \mathbb{R}_+$ with intensity $\mathrm{d}z\mathbf{1}_{[0,y]}(z) \otimes 2\mathrm{d}a/a^2$.

Proof. By Itô's excursion theory, under the distribution $\mathbb{E}_0^{(-y,\infty)}(\mathrm{d}X)$, if we let

$$X^{(z)} = z + X(T_{(-z)-} + t), \qquad 0 \le t \le T_{-z} - T_{(-z)-}$$

for every $z \in (0, y)$ such that $T_{-z} > T_{(-z)-}$, then the measure

$$\sum_{0\le z\le y} \delta_{(z,X^{(z)})} \mathbf{1}_{\{T_{(-z)} > T_{(-z)-}\}}$$

is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{C}(\mathbb{R})$ with intensity $\mathrm{d}z\mathbf{1}_{(0,y)}(z) \otimes 2n(\mathrm{d}X)$. For every $z \in [0, y]$ such that $T_{(-z)} > T_{(-z)-}$, we can interpret $(z + W^{(t+T_{(-z)-})}, 0 \le t \le T_{-z} - T_{(-z)-})$ as an independent mark on the excursion $X^{(z)}$, given by a snake with distribution $\mathbb{Q}_{X^{(z)}}(\mathrm{d}W)$. By the marking properties for Poisson measures and symmetry, we obtain that $\sum_{0\le z\le y} \delta_{(z,I_z)}$ is itself a Poisson measure with intensity

$$\mathrm{d}z\mathbf{1}_{\{0\le z\le y\}} \otimes \int_{\mathcal{C}(\mathbb{R})} 2n(\mathrm{d}X)\mathbb{Q}_X(-\underline{W} \in \mathrm{d}a).$$

Now, we use (44) and the fact ([31, Exercise XII.2.13]) that the image measure of $n(\mathrm{d}X)$ under the scaling operation $X \mapsto a^{-1}X(a^2\cdot)$ is $a^{-1}n(\mathrm{d}X)$, to get

$$\begin{aligned}\int_{\mathcal{C}(\mathbb{R})} 2n(\mathrm{d}X)\mathbb{Q}_X(-\underline{W} > a) &= \int_{\mathcal{C}(\mathbb{R})} 2n(\mathrm{d}X)\Big(1 - \exp\Big(-\int_0^{\zeta(X)} \frac{3\mathrm{d}r}{(X(r)+a)^2}\Big)\Big)\\ &= \int_{\mathcal{C}(\mathbb{R})} 2n(\mathrm{d}X)\Big(1 - \exp\Big(-\int_0^{\zeta(X)/a^2} \frac{3\mathrm{d}r}{(a^{-1}X(a^2r)+1)^2}\Big)\Big)\\ &= \frac{K}{a},\end{aligned}$$

where

$$K=\int_{\mathcal{C}(\mathbb{R})}2n(\mathrm{d}X)\Big(1-\exp\Big(-\int_0^{\zeta(X)}\frac{3\mathrm{d}r}{(X(r)+1)^2}\Big)\Big).$$

To compute it explicitly, write

$$K=\int_{\mathcal{C}(\mathbb{R})}2n(\mathrm{d}X)\int_0^{\zeta(X)}\frac{3\mathrm{d}t}{(X(t)+1)^2}\exp\Big(-\int_t^{\zeta(X)}\frac{3\mathrm{d}r}{(X(r)+1)^2}\Big)$$

and use the Bismut decomposition [31, Theorem XII.4.7] to obtain

$$K=6\int_0^\infty\frac{\mathrm{d}a}{(a+1)^2}\mathbb{E}_a^{(0,\infty)}\Big[\exp\Big(-\int_0^{T_0}\frac{3\mathrm{d}r}{(X(r)+1)^2}\Big)\Big].$$

By translating the process X by 1 and arguing as in the proof of Lemma 31, we obtain that the expectation inside the integral equals $(a+1)^{-2}$. This ends the proof. $\square$

By definition of $I_z, z\geq 0$, we have that the process $(\inf_{0\leq r\leq T_{-z}}\underline{W}^{(r)}, 0\leq z\leq y)$ under $\mathbb{P}_z^{(-y,\infty)}$ is equal to $(-\sup_{0\leq r\leq z}I_r, 0\leq z\leq y)$. Dealing with such random variables and processes is classical in extreme values theory [30]. In the sequel, we will let $\sum_z \delta_{(z,\Delta_z)}\mathbf{1}_{\{\Delta_z>0\}}$ be a Poisson random measure on $\mathbb{R}_+\times(0,\infty)$ with intensity $\mathrm{d}z\otimes K\mathrm{d}a/a^2$, for some $K>0$ (not necessarily equal to 2), and defined on some probability space $(\Omega,\mathcal{F},P)$. We also let $\overline{\Delta}_z=\sup_{0\leq r\leq z}\Delta_r$ (the process $\overline{\Delta}$ is called a *record process*). Note that for every $t>0$, the process $\overline{\Delta}$ remains constant equal to $\overline{\Delta}_t$ on a small neighborhood to the right of t, and that infinitely many jumps times accumulate near $t=0$. Hence, the process $(\overline{\Delta}_t, t>0)$ is a jump-hold process.

By standard properties of Poisson random measures, the one-dimensional marginal laws of this process are so-called *Fréchet laws*, given by

$$P(\overline{\Delta}_t\leq x)=\exp\Big(-\frac{Kt}{x}\Big),\qquad t,x\geq 0.$$

Moreover, the process $\overline{\Delta}$ satisfies the homogeneous scaling relation

$$\Big(\frac{\overline{\Delta}_{at}}{a},t\geq 0\Big)\overset{(d)}{=}(\overline{\Delta}_t,t\geq 0).$$

For $x,t\geq 0$, consider the event

$$H(x,t)=\{x\vee\overline{\Delta}_s\geq s\,,\,0\leq s\leq t\}=\{\overline{\Delta}_s\geq s\,,\,x\leq s\leq t\}.$$

By scaling, we have $P(H(x,t))=P(H(x/t,1))$.

Lemma 33 *For every $0\leq x\leq 1$, it holds that*

$$P(H(x,1))\leq x^{e^{-K}/K}.$$

Proof. Let $(J_n, D_n, n \in \mathbb{Z})$ be a (measurable) enumeration of the jump times of the process $\overline{\Delta}$, and values of $\overline{\Delta}$ at these jump times, in such a way that

$$\ldots < J_{-2} < J_{-1} < J_0 < J_1 < J_2 < \ldots, \qquad D_n = \overline{\Delta}_{J_n}, \qquad n \in \mathbb{Z}.$$

In particular, note that J_n is always the coordinate of the first component of the Poisson measure used to construct $\overline{\Delta}$, and D_n is the corresponding second component, since at time J_n, by definition, the process $\overline{\Delta}$ achieves a new record.

We use the fact [30] that the measure $\mathcal{M} = \sum_{n\in\mathbb{Z}} \delta_{(D_n, J_{n+1}-J_n)}$ is a Poisson random measure on $(0,\infty)^2$, with intensity measure given by

$$\mu(\mathrm{d}y\mathrm{d}u) = \frac{e^{-Ku/y}}{y^2}\mathrm{d}y\mathrm{d}u.$$

Note that on the event $H(x,1)$, it must hold that for every n such that $D_n \in [x,1]$, we have $J_{n+1}-J_n \le D_n$ (since otherwise, we have $\overline{\Delta}_{J_{n+1}-} = D_n < J_{n+1}$ with $D_n \in [x,1]$, so $H(x,1)$ cannot hold). This means that the measure $\mathcal{M}$ has no atom in $\{(y,u) : y \in [x,1], u > y\}$. Therefore, we have

$$\begin{aligned} P(H(x,1)) &\le \exp(-\mu(\{(y,u) : y \in [x,1], u > y\})) \\ &= \exp(K^{-1}e^{-K}\log(x)), \end{aligned}$$

as wanted. □

7.3 Fast confluence of geodesics

We now turn to the proof of Lemma 18.

If $\mathfrak{s}$ is a planted dominant scheme, we call it predominant if the degree of f_0 is minimal, as for non-planted schemes. Figure 6 displays in first position the possible predominant schemes (without face labels: There are two possible such labelings depending on the location of f_1 and f_2 as inside faces).

Lemma 34 *There exists a constant $C \in (0,\infty)$ such that for every $\varepsilon > 0$,*

$$\dot{\mathsf{CLM}}_1^{(3)}\Big(\mathfrak{s} \notin \mathfrak{P}, \min_{e\in\vec{E}(f_0)} \underline{W}_e \ge -\varepsilon, \min_{e\in E(f_1\cap f_2)} \underline{M}_e \ge 0\Big) \le C\varepsilon^5.$$

Proof. By an elementary scaling argument using Proposition 24, we have,

$$\begin{aligned} &\dot{\mathsf{CLM}}_1^{(3)}\Big(\mathfrak{s} \notin \mathfrak{P}, \min_{e\in\vec{E}(f_0)} \underline{W}_e \ge -\varepsilon, \min_{e\in E(f_1\cap f_2)} \underline{M}_e \ge 0\Big) \qquad (45) \\ &\le C \cdot \dot{\mathsf{CLM}}^{(3)}\Big(\mathbf{1}_{[1/2,1]}(\sigma)\,;\, \mathfrak{s} \notin \mathfrak{P}, \min_{e\in\vec{E}(f_0)} \underline{W}_e \ge -\varepsilon, \min_{e\in E(f_1\cap f_2)} \underline{M}_e \ge 0\Big) \\ &\le C \cdot \dot{\mathsf{CLM}}^{(3)}\Big(e^{-\sigma(W)}\,;\, \mathfrak{s} \notin \mathfrak{P}, \min_{e\in\vec{E}(f_0)} \underline{W}_e \ge -\varepsilon, \min_{e\in E(f_1\cap f_2)} \underline{M}_e \ge 0\Big). \end{aligned}$$

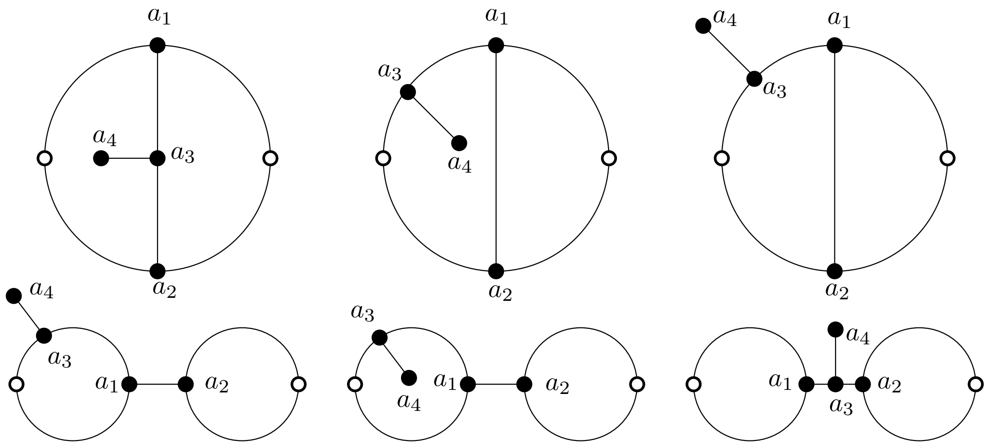

Figure 6: The dominant planted schemes with 3 faces, where f_0 is the outside face, considered up re-labeling of faces, and obvious symmetries. Elements of $V_N(\mathfrak{s})$ are indicated by blank vertices. The first one is a predominant scheme, the other ones are not. The notation a_1, a_2, a_3, a_4 refers to the integration variables appearing in the proof of Lemmas 34 and 35.

In this form, we can take advantage of the fact that $\dot{\mathsf{CLM}}^{(3)}$ is a sum over $\dot{\mathbf{S}}_d^{(3)}$ of product measures, as expressed in (22). The contributions of the second and third schemes in Figure 6 to the above upper-bound are then at most

$$C \int_{\mathbb{R}_+^4} \mathrm{d}a_1 \, \mathrm{d}a_2 \, \mathrm{d}a_3 \, \mathrm{d}a_4 \mathbb{E}_{a_1}^{(0,\infty)}[\mathbb{Q}_X(\underline{W} \geq -\varepsilon)]\mathbb{E}_{a_3}^{(0,\infty)}[\mathbb{Q}_X(\underline{W} \geq -\varepsilon)] \\ \times \mathbb{E}_{a_2}^{(0,\infty)}[\mathbb{Q}_X(\underline{W} \geq -\varepsilon)]^2 \mathbb{B}_{a_1 \to a_3}(\underline{W} \geq -2\varepsilon)\mathbb{B}_{a_1 \to a_2}(e^{-\zeta(X)}\mathbf{1}_{\{\underline{X} \geq 0\}})\mathbb{B}_{a_3 \to a_4}(e^{-\zeta(X)})$$

In this expression, we can integrate out the variable a_4 using the second expression of (38). We bound the terms of the form $\mathbb{E}_a^{(0,\infty)}[\mathbb{Q}_X(\underline{W} \geq -\varepsilon)]$ by using (41), and the terms involving measures $\mathbb{B}_{a \to b}$ by using (43) with $\beta = 1/3$, and (39) together with the fact that $a_1 \wedge a_2 \leq \sqrt{a_1 a_2}$. This gives an upper-bound

$$C \int_{\mathbb{R}_+^3} \mathrm{d}a_1 \, \mathrm{d}a_2 \, \mathrm{d}a_3 \Big(\frac{\varepsilon}{\varepsilon + a_1}\Big)^2 \Big(\frac{\varepsilon}{\varepsilon + a_2}\Big)^4 \Big(\frac{\varepsilon}{\varepsilon + a_3}\Big)^2 (a_1 + \varepsilon)^{1/3}(a_3 + \varepsilon)^{2/3}\sqrt{a_1 a_2} \\ \leq C\varepsilon^5 \int_{\mathbb{R}_+^3} \frac{\mathrm{d}a_1 \, \mathrm{d}a_2 \, \mathrm{d}a_3}{(1 + a_1)^{7/6}(1 + a_2)^{7/2}(1 + a_3)^{4/3}},$$

and the integral is finite, as wanted.

It remains to evaluate the contribution of the bottom three schemes of Figure 6. These do not have edges that are incident both to f_1 and f_2, so that the condition $\min_{e \in E(f_1 \cap f_2)} \underline{M}_e \geq 0$ can be removed. The contribution of the fourth and fifth schemes is then bounded by (we skip some intermediate steps analogous to the above)

$$\begin{aligned} C \int_{\mathbb{R}_+^4} & \mathrm{d}a_1\,\mathrm{d}a_2\,\mathrm{d}a_3\,\mathrm{d}a_4 \Big(\frac{\varepsilon}{\varepsilon+a_1}\Big)^2\Big(\frac{\varepsilon}{\varepsilon+a_2}\Big)^4\Big(\frac{\varepsilon}{\varepsilon+a_3}\Big)^2 \\ & \times(a_1+\varepsilon)^{\beta}(a_2+\varepsilon)^{1-\beta}(a_1+\varepsilon)^{\beta'}(a_3+\varepsilon)^{1-\beta'}\mathbb{B}_{a_3\to a_4}(e^{-\zeta(X)}) \\ \leq\ & C\int_{\mathbb{R}_+^3}\mathrm{d}a_1\,\mathrm{d}a_2\,\mathrm{d}a_3\Big(\frac{\varepsilon}{\varepsilon+a_1}\Big)^2\Big(\frac{\varepsilon}{\varepsilon+a_2}\Big)^4\Big(\frac{\varepsilon}{\varepsilon+a_3}\Big)^2 \\ & \times(a_1+\varepsilon)^{-1}(a_2+\varepsilon)^{2}(a_1+\varepsilon)^{1/2}(a_3+\varepsilon)^{1/2} \end{aligned}$$

taking $\beta=-1$ and $\beta'=1/2$, and we conclude similarly. The contribution of the sixth scheme of Figure 6 is bounded above by

$$\begin{aligned} C \int_{\mathbb{R}_+^4} \mathrm{d}a_1\,\mathrm{d}a_2\,\mathrm{d}a_3\,\mathrm{d}a_4 \Big(\frac{\varepsilon}{\varepsilon+a_1}\Big)^4\Big(\frac{\varepsilon}{\varepsilon+a_2}\Big)^4 (a_1+\varepsilon)^{\beta}(a_2+\varepsilon)^{\beta'}(a_3+\varepsilon)^{2-\beta-\beta'}\mathbb{B}_{a_3\to a_4}(e^{-\zeta(X)}) \\ \leq C\int_{\mathbb{R}_+^3}\mathrm{d}a_1\,\mathrm{d}a_2\,\mathrm{d}a_3\Big(\frac{\varepsilon}{\varepsilon+a_1}\Big)^4\Big(\frac{\varepsilon}{\varepsilon+a_2}\Big)^4\Big(\frac{\varepsilon}{\varepsilon+a_3}\Big)^2(a_1+\varepsilon)^{2}(a_2+\varepsilon)^{2}(a_3+\varepsilon)^{-2}, \end{aligned}$$

taking $\beta=\beta'=2$. Once again, we conclude in the same way as above. □

Lemma 35 *For every $c,\beta>0$, there exists a finite $C>0$ such that for every $\varepsilon>0$,*

$$\dot{\mathsf{CLM}}_1^{(3)}\Big(\mathfrak{s}\in\mathfrak{P},\ \min_{e\in\vec{E}(f_0)}\underline{W}_e\geq-\varepsilon,\ \min_{e\in E(f_1\cap f_2)}\underline{M}_e\geq 0, \exists e\in E_I(\mathfrak{s}),\underline{W}_e\leq -c\varepsilon^{1-\beta}\Big)\leq C\varepsilon^{4+\beta}, \tag{46}$$

where in the last part of the event, it is implicit that e is oriented so that it is incident to f_1 or f_2, but not f_0.

Proof. By using the same scaling argument as in (45), up to changing c by $2^{1/4}c$ and C by a larger constant, it suffices to prove a similar bound for the measure $\dot{\mathsf{CLM}}^{(3)}(e^{-\sigma}\cdot)$ instead of $\dot{\mathsf{CLM}}_1^{(3)}$, and use formula (22) to estimate this quantity. Furthermore, for obvious symmetry reasons, up to increasing the constant C by a factor 8 in the end, it suffices to estimate the contribution of the scheme $\mathfrak{s}$ corresponding to the first picture of Figure 6, where f_1 is the left internal face and f_2 is the right internal face, and specifying that among the four edges $e\in E_I(\mathfrak{s})$, the top-left one is such that $\underline{W}_e\leq-\varepsilon^{1-\beta}$, the others being unconstrained. The contribution of this event to (46) is then bounded by

$$\begin{aligned} C\int_{\mathbb{R}_+^2}\mathrm{d}a_1\,\mathrm{d}a_2\Big(\frac{\varepsilon}{a_1+\varepsilon}\Big)^2\Big(\frac{\varepsilon}{a_2+\varepsilon}\Big)^4\mathbb{E}_{a_1}^{(0,\infty)}[\mathbb{Q}_X(\underline{W}\geq-\varepsilon)\mathbb{Q}_X(\underline{W}\leq-\varepsilon^{1-\beta})] \\ \times\int_{\mathbb{R}}\mathrm{d}a_3\,\mathbb{B}_{a_2\to a_3}[e^{-\zeta(X)}\mathbf{1}_{\{\underline{X}\geq 0\}}]\mathbb{B}_{a_1\to a_3}[e^{-\zeta(X)}\mathbf{1}_{\{\underline{X}\geq 0\}}]\int_{\mathbb{R}}da_4\,\mathbb{B}_{a_3\to a_4}[e^{-\zeta(X)}]\,. \end{aligned}$$

The last integral is bounded by (38), independently on a_1,a_2,a_3. We then apply (19) to the integral in the variable a_3, which gives rise to the factor $\mathbb{B}_{a_1\to a_2}[\zeta(X)e^{-\zeta(X)\sqrt{2}}\mathbf{1}_{\{\underline{X}\geq 0\}}]$, and this is bounded by Ca_1a_2 by (40). After an elementary change of variables and a translation and scaling in last remaining expectation, we obtain the bound

$$C\varepsilon^4\int_{\mathbb{R}_+^2}\mathrm{d}a_1\,\mathrm{d}a_2\frac{a_1a_2}{(a_1+1)^2(a_2+1)^4}\mathbb{E}^{(1,\infty)}_{a_1+1}[\mathbb{Q}_X(\underline{W}\geq 0)\mathbb{Q}_X(\underline{W}\leq -\varepsilon^{-\beta}+1)]\,,$$

in which the contribution of a_2 can be integrated out. Therefore, taking ε small enough so that $\varepsilon^{-\beta}-1>\varepsilon^{-\beta}/2$ and using (42), this bound is less than or equal to

$$\begin{aligned}
&C\varepsilon^4\int_0^\infty \mathrm{d}a_1\frac{a_1}{(a_1+1)^2}\Big(\frac{1}{a_1+1}\wedge\frac{1}{\varepsilon^{-\beta}-1}\Big)^2\\
\leq\ & C\varepsilon^4\Big(\varepsilon^{2\beta}\int_0^{\varepsilon^{-\beta}}\frac{\mathrm{d}a_1}{a_1+1}+\int_{\varepsilon^{-\beta}}^\infty\frac{\mathrm{d}a_1}{(a_1+1)^3}\Big)\\
\leq\ & C\varepsilon^4(\varepsilon^{2\beta}\log(1+\varepsilon^{-\beta})+(\varepsilon^{-\beta}+1)^{-2})\\
\leq\ & C\varepsilon^{4+\beta}
\end{aligned}$$

This bound remains true for every $\varepsilon>0$, possibly up to changing the constant C. Note that we could have in fact obtained a bound of the form $C\varepsilon^{4+2\beta'}$ for any $\beta'\in(0,\beta)$ with this method. □

Combining Lemmas 34 and 35 with (34), this completes the proof of Lemma 18 (note that we changed the second condition in the definition of the event $\mathcal{B}_1(\varepsilon,\beta)$ by $\min_{e\in\vec{E}(f_0)}\underline{W}_e\geq-\varepsilon$, this was for lightening the notation but is of no impact, as is easily checked).

7.4 ε-geodesic stars

We finally prove Lemma 19. In these proofs, up to considering $\mathcal{B}_2(\varepsilon/2)$ instead of $\mathcal{B}_2(\varepsilon)$, we will replace the condition that $\min_{e\in\vec{E}(\mathfrak{s})}\underline{W}_e\geq-2\varepsilon$ by the similar bound with $-\varepsilon$ without impacting the result.

Lemma 36 *There exists some constant $C>0$ such that*

$$\mathsf{CLM}_1^{(4)}\Big(\mathfrak{s}\notin\mathfrak{P},\ \min_{e\in\vec{E}(f_0)}\underline{W}_e\geq-\varepsilon\Big)\leq C\varepsilon^5\,.$$

Proof. As in Lemma 34, a scaling argument shows that it suffices to prove the same bound with the measure $\mathsf{CLM}^{(4)}(e^{-\sigma}\cdot)$ replacing $\mathsf{CLM}_1^{(4)}$. Note that for any $k\geq2$, we have

$$\begin{aligned}
&\mathsf{CLM}^{(k+1)}\Big(e^{-\sigma}\,;\,\mathfrak{s}\notin\mathfrak{P}\,,\ \min_{e\in\vec{E}(f_0)}\underline{W}_e\geq-\varepsilon\Big)\\
=\ &\sum_{\mathfrak{s}\notin\mathfrak{P}}\int\lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v)_{v\in V(\mathfrak{s})})\prod_{e\in E_N(\mathfrak{s})}\mathbb{E}^{(0,\infty)}_{\ell_{e_-}}[\mathbb{Q}_X[e^{-\sigma(W)}\mathbf{1}_{\{\underline{W}\geq-\varepsilon\}}]\mathbb{Q}_X[e^{-\sigma(W)}]]\\
&\times\prod_{e\in E_J(\mathfrak{s})}\mathbb{B}^+_{\ell_{e_-}\to\ell_{e_+}}[\mathbb{Q}_X[e^{-\sigma(W)}\mathbf{1}_{\{\underline{W}\geq-\varepsilon\}}]\mathbb{Q}_X[e^{-\sigma(W)}]]\\
&\times\prod_{e\in E_I(\mathfrak{s})}\mathbb{B}_{\ell_{e_-}\to\ell_{e_+}}[\mathbb{Q}_X[e^{-\sigma(W)}\mathbf{1}_{\{\underline{W}\geq-\varepsilon\}}]^2]\prod_{e\in E_O(\mathfrak{s})}\mathbb{B}_{\ell_{e_-}\to\ell_{e_+}}[\mathbb{Q}_X[e^{-\sigma(W)}]^2]\,.
\end{aligned}$$

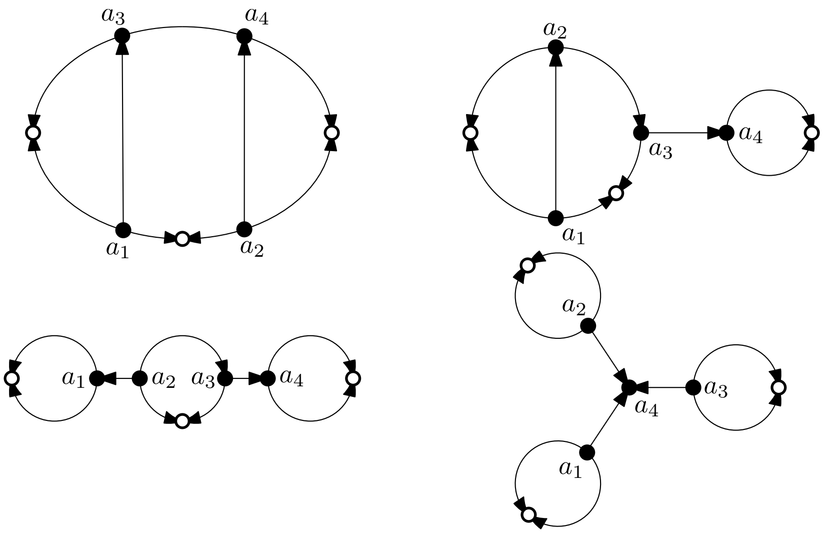

Figure 7: The four dominant, non-predominant schemes with 4 faces, f_0 being the external face, and considered up to obvious symmetries. Blank vertices indicate elements of $V_N(\mathfrak{s})$. The labels a_1, a_2, a_3, a_4 are the integration variables used in the proof of Lemma 36.

By Lemmas 30 and 31, we obtain

$$\begin{aligned} &\mathsf{CLM}^{(k+1)}\Big(e^{-\sigma}\,;\, \mathfrak{s}\notin\mathfrak{P}\,,\ \min_{e\in\vec{E}(f_0)} \underline{W}_e \geq -\varepsilon\Big) \\ &\leq\ C\sum_{\mathfrak{s}\notin\mathfrak{P}}\int \lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v)_{v\in V(\mathfrak{s})}) \prod_{e\in E_N(\mathfrak{s})}\Big(\frac{\varepsilon}{\ell_{e_-}+\varepsilon}\Big)^2 \prod_{e\in E_I(\mathfrak{s})\cup E_J(\mathfrak{s})}(\ell_{e_-}+\varepsilon)^{\beta_e}(\ell_{e_+}+\varepsilon)^{1-\beta_e}\end{aligned}$$

for any choice of $\beta_e\in[-2,3]$, that can depend on e. We then divide every variable ℓ_v, with v incident to f_0, by ε. We obtain

$$\begin{aligned}\mathsf{CLM}^{(k+1)}\Big(e^{-\sigma}\,;\,\mathfrak{s}\notin\mathfrak{P}\,,\ \min_{e\in\vec{E}(f_0)} W_e\geq-\varepsilon\Big) &\leq C\sum_{\mathfrak{s}\notin\mathfrak{P}}\varepsilon^{\#V_I(\mathfrak{s})+\#E_I(\mathfrak{s})+\#E_J(\mathfrak{s})} \qquad (47)\\ \times\int\lambda_{\mathfrak{s}}(\mathrm{d}(\ell_v)_{v\in V(\mathfrak{s})})\prod_{e\in E_N(\mathfrak{s})}\frac{1}{(\ell_{e_-}+1)^2}&\prod_{e\in E_I(\mathfrak{s})\cup E_J(\mathfrak{s})}(\ell_{e_-}+1)^{\beta_e}(\ell_{e_+}+1)^{1-\beta_e}\,.\end{aligned}$$

Now let us focus again on the case $k=3$. The predominant schemes are the ones that are obtained from the first pre-scheme of Figure 1, by adding three null vertices inside each edge incident to f_0, and then labeling the three "interior" faces by f_1, f_2, f_3, and choosing a root.

All other (dominant) schemes are not predominant, and are indicated in Figure 7. Let us consider the first scheme in this figure. In this case, we have $\#V_I(\mathfrak{s})=4, \#E_J(\mathfrak{s})=1, \#E_I(\mathfrak{s})=0$, and taking $\beta_e=1/2$, where e is the unique edge of $\#E_J(\mathfrak{s})$, the contribution to the upper-bound (47), is bounded by

$$\varepsilon^5 \int_{\mathbb{R}_+^4} \frac{\mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4}{(a_1+1)^4 (a_2+1)^4 (a_3+1)^{3/2} (a_4+1)^{3/2}} \le C\varepsilon^5 .$$

Let us turn to the dominant schemes corresponding to the third pre-scheme of Figure 1. In this case, one has $\#V_I(\mathfrak{s}) = 4, \#E_J(\mathfrak{s}) = 1, \#E_I(\mathfrak{s}) = 1$, and, taking $\beta_e = 1/3$, the contribution to (47) is bounded by

$$\varepsilon^6 \int_{\mathbb{R}_+^4} \frac{\mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4}{(a_1+1)^4 (a_2+1)^{4/3} (a_3+1)^{4/3} (a_4+1)^{10/3}} \le C\varepsilon^6 .$$

The dominant schemes corresponding to the fourth pre-scheme of Figure 1 have $\#V_I(\mathfrak{s}) = 4, \#E_J(\mathfrak{s}) = 1, \#E_I(\mathfrak{s}) = 2$, and taking alternatively $\beta_e \in \{1/3, 1/2\}$ for the three edges in $E_I(\mathfrak{s}) \cup E_J(\mathfrak{s})$, the contribution to (47) is at most

$$\varepsilon^7 \int_{\mathbb{R}_+^4} \frac{\mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4}{(a_1+1)^{10/3} (a_2+1)^{7/6} (a_3+1)^{7/6} (a_4+1)^{10/3}} \le C\varepsilon^7 .$$

The dominant schemes corresponding to the fifth pre-scheme of Figure 1 have $\#V_I(\mathfrak{s}) = 4, \#E_J(\mathfrak{s}) = 0, \#E_I(\mathfrak{s}) = 3$, and taking $\beta_e = -1$ for every edge in $E_I(\mathfrak{s})$, the contribution to (47) is bounded by

$$\varepsilon^7 \int_{\mathbb{R}_+^4} \frac{\mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3 \mathrm{d}a_4}{(a_1+1)^2 (a_2+1)^2 (a_3+1)^2 (a_4+1)^3} \le C\varepsilon^7 .$$

This entails the result. □

Lemma 37 *There exist finite $C, \chi > 0$ such that for every $\varepsilon > 0$,*

$$\mathsf{CLM}_1^{(4)}(\mathcal{B}_2(\varepsilon)) \le C\varepsilon^{4+\chi} .$$

Proof. In all this proof, the mention of 1., 2., 3., 4., will refer to the four points in the definition of $\mathcal{B}_2(\varepsilon)$, for some $\varepsilon > 0$.

Using scaling again as in (22), it suffices to prove a similar bound for $\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2(\varepsilon)})$. Let us introduce some notation. We let $T_z^{e_{13}} = \inf\{s \ge 0 : M_{e_{13}}(s) = z\} \in [0, \infty]$ for every $z \in \mathbb{R}$. We define $T_z^{e_{23}}$ in an analogous way. Let

$$\Xi = -\Big(\min_{e \in \vec{E}(f_3) : \overline{e} \in \vec{E}(f_0)} \underline{W}_e \wedge \inf\{\underline{W}_{e_{13}}^{(s)}, 0 \le s \le T_0^{e_{13}}\} \wedge \inf\{\underline{W}_{e_{23}}^{(s)}, 0 \le s \le T_0^{e_{23}}\} \Big)$$

and $\overline{\Delta}_y = \sup_{0 \le z \le y} \Delta_z$, where

$$\Delta_z = -z - \left(\inf\{\underline{W}_{e_{13}}^{(t)}, T_{(-z)-}^{e_{13}} \le t \le T_{-z}^{e_{13}}\} \wedge \inf\{\underline{W}_{e_{23}}^{(t)}, T_{(-z)-}^{e_{23}} \le s \le T_{-z}^{e_{23}}\} \right) .$$

Then, by taking t in (36) to be of the form $T^{e_{13}}_{-y}$, as long as $y>0$ is such that $T^{e_{13}}_{-y}<\infty$, and by choosing $t'=T^{e_{23}}_{-y}$ in a similar way, we obtain that

$$(-\Xi)\wedge(-\overline{\Delta}_y-y)\le(-\Xi)\wedge\inf_{0\le z\le y}(-\Delta_z-z)\le -2y\,.$$

Let α,η,η',η'' be positive numbers, all strictly larger than ε, and such that $\eta>\eta'$. Their values will be fixed later to be appropriate powers of ε. We observe that $\mathcal{B}_2(\varepsilon)$ is contained in the union of the following three events $\mathcal{B}'_2(\varepsilon),\mathcal{B}''_2(\varepsilon),\mathcal{B}'''_2(\varepsilon)$, which are defined by points 1., 2., 3. in the definition of $\mathcal{B}_2(\varepsilon)$, together with $4'$., $4''$. and $4'''$. respectively, where

$4'$. it holds that either $r_{e_{13}}\le\eta$ or $r_{e_{23}}\le\eta$ or $T^{e_{13}}_{-\alpha}>\eta'$ or $T^{e_{23}}_{-\alpha}>\eta'$,

$4''$. it holds that $r_{e_{13}}\wedge r_{e_{23}}>\eta$, $T^{e_{13}}_{-\alpha}\vee T^{e_{23}}_{-\alpha}\le\eta'$, and $\Xi>\eta''$,

$4'''$. it holds that $r_{e_{13}}\wedge r_{e_{23}}>\eta$, $T^{e_{13}}_{-\alpha}\vee T^{e_{23}}_{-\alpha}\le\eta'$, and $\eta''\vee\overline{\Delta}_y\ge y$ for every $y\in[0,\eta']$

It remains to estimate separately the quantities $\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}'_2(\varepsilon)})$, $\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}''_2(\varepsilon)})$ and $\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}'''_2(\varepsilon)})$. For this, it suffices to restrict our attention to the event that $\mathfrak{s}$ is the predominant scheme of Figure 5, since the other predominant schemes are the same up to symmetries. Moreover, let us observe that the points $4'$., $4''$. and $4'''$. only involve the snakes W_e where e or its reversal is incident to f_3, and not the others. Therefore, when writing the above three quantities according to the defintion (22) of $\mathsf{CLM}^{(4)}$, there are going to be a certain number of common factors, namely, those which correspond to the contribution of points 2., 3. to the 5 edges of $\mathfrak{s}$ that are not incident to f_3. Renaming the labels $\ell_v, v\in V(\mathfrak{s})\setminus V_N(\mathfrak{s})$ as a_1,a_2,a_3,a_4 as indicated in Figure 5, we obtain that these common factors are

- a factor
$$\mathbb{E}^{(0,\infty)}_{a_4}\left[\mathbb{Q}_X[e^{-\sigma(W)}]\mathbb{Q}_X[e^{-\sigma(W)}\mathbf{1}_{\{\underline{W}\ge-\varepsilon\}}]\right]^2\le\left(\frac{\varepsilon}{a_4+\varepsilon}\right)^4$$
corresponding to the contribution of 2. to the two edges incident to f_0 and ending at the vertex with label a_4, where we used (41),

- a factor
$$\mathbb{E}^{(0,\infty)}_{a_1}\left[\mathbb{Q}_X[e^{-\sigma(W)}]\mathbb{Q}_X[e^{-\sigma(W)}\mathbf{1}_{\{\underline{W}\ge-\varepsilon\}}]\right]\mathbb{E}^{(0,\infty)}_{a_2}\left[\mathbb{Q}_X[e^{-\sigma(W)}]\mathbb{Q}_X[e^{-\sigma(W)}\mathbf{1}_{\{\underline{W}\ge-\varepsilon\}}]\right]$$
$$\le\left(\frac{\varepsilon}{a_1+\varepsilon}\right)^2\left(\frac{\varepsilon}{a_2+\varepsilon}\right)^2$$
corresponding to the contribution to 2. of the two edges incident to f_1 and f_2 that end at the vertices with labels a_1 and a_2, where we used (41) again,

- a factor
$$\mathbb{B}_{a_4\to a_3}\left[\mathbb{Q}_X[e^{-\sigma(W)}]^2\mathbf{1}_{\{\underline{X}\ge0\}}\right]\le 2a_4$$
corresponding to the contribution of 3. to the edge e_{12}, and where we used the fact that $\mathbb{Q}_X[e^{-\sigma(W)}]=e^{-\zeta(X)\sqrt{2}}$ and (39).

Let us now bound $\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2'(\varepsilon)})$. The condition 4′. involves only the edges e_{13} and e_{23}, so that the contribution of 2. to the two edges incident to f_0 and f_3 will bring further factors bounded by $(\varepsilon/(\varepsilon+a_1))^2$ and $(\varepsilon/(\varepsilon+a_2))^2$. Also, by symmetry, up to a factor 2, we can estimate the contribution only of the event $\{r_{13}\leq\eta \text{ or } T_{-\alpha}^{e_{13}}>\eta'\}$, and ignore the rest, so that the edge e_{23} participates by a factor $\mathbb{B}_{a_1\to a_3}[\mathbb{Q}_X[e^{-\sigma(W)}]^2]$, which is bounded by (38), while e_{13} contributes by a factor

$$\begin{aligned}&\mathbb{B}_{a_1\to a_3}\big[\mathbb{Q}_X[e^{-\sigma(W)}]^2\mathbf{1}_{\{\zeta(X)\leq\eta \text{ or } T_{-\alpha}>\eta'\}}\big]\\ &\quad\leq\ \mathbb{B}_{a_1\to a_3}\big[\mathbb{Q}_X[e^{-\sigma(W)}]^2(\mathbf{1}_{\{\zeta(X)\leq\eta\}}+\mathbf{1}_{\{\zeta(X)>\eta,T_{-\alpha}>\eta'\}})\big]\\ &\quad\leq\ \mathbb{B}_{a_1\to a_3}[\zeta(X)\leq\eta]+\mathbb{B}_{a_1\to a_3}[e^{-\zeta(X)\sqrt{2}}\mathbf{1}_{\{\zeta(X)>\eta,T_{-\alpha}>\eta'\}}]\end{aligned}\tag{48}$$

Now on the one hand, since $p_r(x,y)$ is a probability density, we have

$$\int_{\mathbb{R}_+}\mathrm{d}a_3\,\mathbb{B}_{a_1\to a_3}[\zeta(X)\leq\eta]\leq\int_0^\eta\mathrm{d}r\int_{\mathbb{R}}\mathrm{d}a_3\,p_r(a_1,a_3)=\eta\,,$$

and on the other hand

$$\mathbb{B}_{a_1\to a_3}[e^{-\zeta(X)\sqrt{2}}\mathbf{1}_{\{\zeta(X)>\eta,T_{-\alpha}>\eta'\}}]\leq\int_\eta^\infty\mathrm{d}r\,e^{-r\sqrt{2}}p_r(a_1,a_3)\mathbb{P}^r_{a_1}(T_{-\alpha}>\eta')\,.$$

From the Markovian bridge construction of [12], we have, for every $r>\eta$,

$$\mathbb{P}^r_{a_1\to a_3}(T_{-\alpha}>\eta')=\mathbb{E}_{a_1}\left[\mathbf{1}_{\{T_{-\alpha}>\eta'\}}\frac{p_{r-\eta'}(X(\eta'),a_3)}{p_r(a_1,a_3)}\right],$$

so that

$$\int_{\mathbb{R}_+}\mathrm{d}a_3\int_\eta^\infty\mathrm{d}r\,e^{-r\sqrt{2}}p_r(a_1,a_3)\mathbb{P}^r_{a_1\to a_3}(T_{-\alpha}>\eta')\leq\mathbb{P}_{a_1}(T_{-\alpha}>\eta')\int_0^\infty\mathrm{d}r\,e^{-r\sqrt{2}}\,.$$

Now, we have, by symmetry and scaling of Brownian motion, and since T_1 has same distribution as X_1^{-2} under $\mathbb{P}_0$,

$$\begin{aligned}\mathbb{P}_{a_1}(T_{-\alpha}>\eta')&=\mathbb{P}_0(T_{a_1+\alpha}>\eta')\\ &=\mathbb{P}_0\Big(T_1>\frac{\eta'}{(a_1+\alpha)^2}\Big)\\ &=\mathbb{P}_0\Big(|X_1|<\frac{a_1+\alpha}{\sqrt{\eta'}}\Big)\\ &\leq C\,\frac{a_1+\alpha}{\sqrt{\eta'}}\,,\end{aligned}$$

for every $\varepsilon\in(0,1)$ and $a_1>0$. Hence the integral of (48) with respect to $a_3\in\mathbb{R}$ is bounded by $C(\eta+(a_1+\alpha)/\sqrt{\eta'})$. By putting together all the factors, recalling that

$\alpha > \varepsilon$, we have obtained,

$$\begin{aligned}
&\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2'(\varepsilon)}) \\
&\le C\int_{\mathbb{R}_+^3} \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_4 \Big(\frac{\varepsilon}{a_1+\varepsilon}\Big)^4\Big(\frac{\varepsilon}{a_2+\varepsilon}\Big)^4\Big(\frac{\varepsilon}{a_4+\varepsilon}\Big)^4 a_4\Big(\eta+\frac{a_1+\alpha}{\sqrt{\eta'}}\Big) \\
&\le C\varepsilon^4\int_{\mathbb{R}_+}\frac{\mathrm{d}a_1}{(a_1+1)^4}\Big(\eta+\frac{\varepsilon a_1+\alpha}{\sqrt{\eta'}}\Big). \\
&\le C\varepsilon^4\Big(\eta\vee\frac{\alpha}{\sqrt{\eta'}}\Big)\int_{\mathbb{R}_+}\frac{\mathrm{d}a_1}{(a_1+1)^4}\Big(2+\frac{\varepsilon}{\alpha}a_1\Big) \\
&\le C\varepsilon^4\Big(\eta\vee\frac{\alpha}{\sqrt{\eta'}}\Big) \qquad (49)
\end{aligned}$$

Let us now turn to the estimation of $\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2''(\varepsilon)})$. We first observe the following absolute continuity-type bound: For any non-negative measurable function F and $\lambda > 0$

$$\int_{\mathbb{R}}\mathrm{d}y\,\mathbb{B}_{x\to y}[e^{-\lambda\zeta(X)}F(X(s),0\le s\le T_{-\alpha})\mathbf{1}_{\{\zeta(X)>\eta,T_{-\alpha}\le\eta'\}}] \le \frac{1}{\lambda}\mathbb{E}_x^{(-\alpha,\infty)}[F(X)]\,. \qquad (50)$$

Indeed, note that by using again the Markovian bridge description of [12],

$$\begin{aligned}
&\int_{\mathbb{R}}\mathrm{d}y\,\mathbb{B}_{x\to y}[e^{-\lambda\zeta(X)}F(X(s),0\le s\le T_{-\alpha})\mathbf{1}_{\{\zeta(X)>\eta,T_{-\alpha}\le\eta'\}}] \\
&= \int_{\mathbb{R}}\mathrm{d}y\int_\eta^\infty e^{-\lambda r}\mathrm{d}r\,\mathbb{E}_x[F(X(s),0\le s\le T_{-\alpha})\mathbf{1}_{\{T_{-\alpha}\le\eta'\}}p_{r-\eta'}(X(\eta'),y)] \\
&\le \int_0^\infty e^{-\lambda r}\mathrm{d}r\,\mathbb{E}_x[F(X(s),0\le s\le T_{-\alpha})]\,,
\end{aligned}$$

as wanted. Back to $\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2''(\varepsilon)})$, note that there are exactly two edges incident to f_3 with reversal incident to f_0, and we let e_1 be the one that is linked to the vertex with label a_1, and e_2 the one linked to the vertex with label a_2. Note that the event $\{\Xi > \eta''\}$ is then included in the union

$$\{\underline{W}_{e_1} < -\eta''\}\cup\{\underline{W}_{e_2} < -\eta''\}\cup\Big\{\inf_{0\le s\le T_0^{e_{13}}}\underline{W}_{e_{13}}^{(s)} < -\eta''\Big\}\cup\Big\{\inf_{0\le s\le T_0^{e_{23}}}\underline{W}_{e_{23}}^{(s)} < -\eta''\Big\}.$$

For symmetry reasons, the first two have same contribution (after integrating with respect to a_1, a_2, a_3, a_4), as well as the last two. The contribution of the edges e_1 and e_2 to $\{\underline{W}_{e_1} < -\eta''\}$ is then bounded by (we use also 2., and recall that $\eta > \eta' > \varepsilon$)

$$\mathbb{E}_{a_1}^{(0,\infty)}[\mathbb{Q}_X(\underline{W}\ge-\varepsilon)\mathbb{Q}_X(\underline{W}\le-\eta'')]\Big(\frac{\varepsilon}{a_2+\varepsilon}\Big)^2 \le C\Big(\frac{\varepsilon}{\eta''-\varepsilon}\Big)^2\Big(\frac{\varepsilon}{a_2+\varepsilon}\Big)^2$$

by (42), while the edges e_{13} and e_{23} contribute factors of the form $\mathbb{B}_{a_1\to a_3}[\mathbb{Q}_X[e^{-\sigma(W)}]^2]$ and $\mathbb{B}_{a_2\to a_3}[\mathbb{Q}_X[e^{-\sigma(W)}]^2]$, which are bounded after integration of the variable a_3, by (38).

Hence,

$$\begin{aligned}
&\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2''(\varepsilon)}\mathbf{1}_{\{\underline{W}_{e_1}<-\eta''\}}) \\
&\quad\le\; C\Big(\frac{\varepsilon}{(\eta''-\varepsilon)}\Big)^2\int_{\mathbb{R}_+^3}\mathrm{d}a_1\,\mathrm{d}a_2\,\mathrm{d}a_4\,\Big(\frac{\varepsilon}{a_1+\varepsilon}\Big)^2\Big(\frac{\varepsilon}{a_2+\varepsilon}\Big)^4\Big(\frac{\varepsilon}{a_4+\varepsilon}\Big)^4 a_4 \\
&\quad\le\; C\varepsilon^4\Big(\frac{\varepsilon}{\eta''-\varepsilon}\Big)^2,
\end{aligned}$$

On the other hand, the edges e_1 and e_2 do not contribute to $\{\inf\{\underline{W}^{(s)}_{e_{13}}, 0\le s\le T_0^{e_{13}}\}<-\eta''\}$, and involve only, via 2., a factor $(\varepsilon/(a_1+\varepsilon))^2(\varepsilon/(a_2+\varepsilon))^2$. The contribution of e_{13} and e_{23}, integrated in a_3, is bounded by

$$\begin{aligned}
&\int_{\mathbb{R}}\mathrm{d}a_3\,\mathbb{B}_{a_1\to a_3}\Big[\mathbb{Q}_X[e^{-\sigma}]\mathbb{Q}_X\big[\inf_{0\le s\le T_0}\underline{W}^{(s)}<-\eta''\big]\mathbf{1}_{\{\zeta(X)>\eta,T_{-\alpha}\le\eta'\}}\Big]\mathbb{B}_{a_2\to a_3}[e^{-\sigma}] \\
&\quad\le C\mathbb{E}_{a_1}^{(-\alpha,\infty)}\Big[\mathbb{Q}_X\big[\inf_{0\le s\le T_0}\underline{W}^{(s)}<-\eta''\big]\Big] \\
&\quad\le CP(\overline{\Delta}_{a_1}>\eta''+a_1) \\
&\quad\le C\Big(1-\exp\Big(-\frac{2a_1}{a_1+\eta''}\Big)\Big) \\
&\quad\le C\frac{a_1}{\eta''},
\end{aligned}$$

where we used (48) in the second step, and Lemma 32 in the third step: Here, under P, $(\Delta_t, t\ge 0)$ is a Poisson process with intensity $2\mathrm{d}a/a^2$, and $\overline{\Delta}$ is its supremum process. We have obtained

$$\begin{aligned}
&\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2''(\varepsilon)}\mathbf{1}_{\{\inf\{\underline{W}^{(s)},0\le s\le T_0^{e_{13}}\}<-\eta''\}}) \\
&\quad\le\; C\int_{\mathbb{R}_+^3}\mathrm{d}a_1\,\mathrm{d}a_2\,\mathrm{d}a_4\Big(\frac{\varepsilon}{a_1+\varepsilon}\Big)^4\frac{a_1}{\eta''}\Big(\frac{\varepsilon}{a_2+\varepsilon}\Big)^4\Big(\frac{\varepsilon}{a_4+\varepsilon}\Big)^4 a_4 \\
&\quad\le\; C\frac{\varepsilon^5}{\eta''}\int_{\mathbb{R}_+}\mathrm{d}a_1\frac{a_1}{(a_1+1)^4} \\
&\quad\le\; C\frac{\varepsilon^5}{\eta''}.
\end{aligned}$$

These estimations, together with our preliminary remarks, entail that

$$\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2''(\varepsilon)})\le C\varepsilon^4\Big(\Big(\frac{\varepsilon}{\eta''-\varepsilon}\Big)^2+\frac{\varepsilon}{\eta''}\Big). \tag{51}$$

Finally, let us consider $\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2'''(\varepsilon)})$. Points 2. and 3. induce contributions of the edges incident to f_0, as well as the edge e_{12}, that are bounded by

$$2\Big(\frac{\varepsilon}{a_1+\varepsilon}\Big)^4\Big(\frac{\varepsilon}{a_2+\varepsilon}\Big)^4\Big(\frac{\varepsilon}{a_4+\varepsilon}\Big)^4 a_4\,.$$

Now, point 4‴. involves only the edges e_{13} and e_{23}, and contributes by a factor bounded above by

$$\mathbb{B}_{a_1\to a_3}\Big[e^{-\zeta(X)\sqrt{2}}\mathbf{1}_{\{\zeta(X)>\eta, T_{-\alpha}\le\eta'\}}\int_{\mathcal{C}(\mathcal{C}(\mathbb{R}))}\mathbb{Q}_X(\mathrm{d}W)\mathbb{Q}_X(\mathrm{d}W')\mathbf{1}_{\{\eta''\vee\overline{I}_y\vee\overline{I}'_y\ge y, 0\le y\le\alpha\}}\Big],$$

where I was defined in Lemma 32, $\overline{I}_y=\sup_{0\le z\le y}I_y$, and $I',\overline{I}'$ are defined in a similar way from the trajectory W'. Now we use again a bound with same spirit as (48). Namely, for every $\lambda>0$, every $x,y\in\mathbb{R}$ and for non-negative measurable F, we have

$$\begin{aligned}&\int_{\mathbb{R}}\mathrm{d}z\int_{\mathcal{C}(\mathbb{R})^2}\mathbb{B}_{x\to z}(\mathrm{d}X)\mathbb{B}_{y\to z}(\mathrm{d}X')\,e^{-\lambda\zeta(X)-\lambda\zeta(X')}\\ &\qquad\times\mathbf{1}_{\{\zeta(X)\wedge\zeta(X')>\eta, T_{-\alpha}\vee T'_{-\alpha}\le\eta'\}}F\big((X(s))_{0\le s\le T_{-\alpha}},(X'(s))_{0\le s\le T'_{-\alpha}}\big)\\ &\le\frac{1}{\lambda^2\sqrt{2\pi(\eta-\eta')}}\int_{\mathcal{C}(\mathbb{R})^2}\mathbb{P}_x^{(-\alpha,\infty)}(\mathrm{d}X)\mathbb{P}_y^{(-\alpha,\infty)}(\mathrm{d}X')F(X,X'),\end{aligned}$$

where T'_z is the first hitting time of z by X'. This is obtained in a similar way to (48), writing the left-hand side as

$$\begin{aligned}&\int_{\mathbb{R}}\mathrm{d}z\int_{(\eta,\infty)^2}\mathrm{d}r\mathrm{d}r'e^{-\lambda(r+r')}\int_{\mathcal{C}(\mathbb{R})^2}\mathbb{P}_x(\mathrm{d}X)\mathbb{P}_y(\mathrm{d}X')\\ &\qquad\times\mathbf{1}_{\{T_{-\alpha}\vee T'_{-\alpha}<\eta'\}}F\big((X(s))_{0\le s\le T_{-\alpha}},(X'(s))_{0\le s\le T'_{-\alpha}}\big)p_{r-\eta'}(X(\eta'),z)p_{r-\eta'}(X'(\eta'),z)\end{aligned}$$

This is obtained by first checking this for F of a product form and using the Markovian description of bridges, and then applying a monotone class argument. We then use the bound $p_{r-\eta'}(X'(\eta'),z)\le(2\pi(\eta-\eta'))^{-1/2}$, valid for $r\ge\eta>\eta'$, and use Fubini's theorem to integrate $p_{r-\eta'}(X(\eta'),z)$ with respect to z, as in the derivation of (50). Therefore, after integrating with respect to the variables a_3, we obtain that the edges e_{13} and e_{23} together contribute by

$$C\int_{\mathcal{C}(\mathbb{R})^2}\mathbb{P}_{a_1}^{(-\alpha,\infty)}(\mathrm{d}X)\mathbb{P}_{a_2}^{(-\alpha,\infty)}(\mathrm{d}X')\int_{\mathcal{C}(\mathcal{C}(\mathbb{R}))^2}\mathbb{Q}_X(\mathrm{d}W)\mathbb{Q}_{X'}(\mathrm{d}W')\mathbf{1}_{\{\eta''\vee\overline{I}_y\vee\overline{I}'_y\ge y\,,\,0\le y\le\alpha\}},$$

where I' is defined from X' as I was defined from X. This equals

$$C\int_{\mathcal{C}(\mathbb{R})^2}\mathbb{P}_0^{(-\alpha,\infty)}(\mathrm{d}X)\mathbb{P}_0^{(-\alpha,\infty)}(\mathrm{d}X')\int_{\mathcal{C}(\mathcal{C}(\mathbb{R}))^2}\mathbb{Q}_X(\mathrm{d}W)\mathbb{Q}_{X'}(\mathrm{d}W')\mathbf{1}_{\{\eta''\vee\overline{I}_y\vee\overline{I}'_y\ge y\,,\,0\le y\le\alpha\}},$$

by an application of the Markov property, noticing that $\overline{I}_y$ and $\overline{I}'_y$ only involve the processes $W^{(s)}$ for $T_0\le s\le T_{-y}$, and similarly for $\overline{I}'_y$ (we skip the details). By Lemma 32 and standard properties of Poisson measures, the process $(I_y\vee I'_y, 0\le y\le\alpha)$ under the law $\mathbb{E}_0^{(-\alpha,\infty)}\otimes\mathbb{E}_0^{(-\alpha,\infty)}[\mathbb{Q}_X(\mathrm{d}W)\mathbb{Q}_X(\mathrm{d}W')]$ is a Poisson process on the time-interval $[0,\alpha]$ with intensity $4\mathrm{d}a/a^2$. By Lemma 33 we obtain that the last displayed quantity is less than $(\eta''/\alpha)^{e^{-4}/4}$, and we conclude that

$$\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2'''(\varepsilon)})\le C\varepsilon^4\frac{1}{\sqrt{\eta-\eta'}}\Big(\frac{\eta''}{\alpha}\Big)^{e^{-4}/4}$$

This, together with (49) and (51), finally entails that

$$\mathsf{CLM}^{(4)}(e^{-\sigma}\mathbf{1}_{\mathcal{B}_2(\varepsilon)}) \leq C\varepsilon^4\left(\eta \vee \frac{\alpha}{\sqrt{\eta'}} + \left(\frac{\varepsilon}{\eta'' - \varepsilon}\right)^2 + \frac{\varepsilon}{\eta''} + \frac{1}{\sqrt{\eta - \eta'}}\left(\frac{\eta''}{\alpha}\right)^{e^{-4}/4}\right).$$

Let us now choose $\alpha, \eta, \eta', \eta''$ of the form

$$\alpha = \varepsilon^{\beta}, \quad \eta = \varepsilon^{\nu}, \quad \eta' = \varepsilon^{\nu'}, \quad \eta'' = \varepsilon^{\nu''},$$

with $\beta, \nu, \nu', \nu'' \in (0,1)$, and let us assume for the time being that $\varepsilon < 1$. Then the condition $\eta > \eta'$ amounts to $\nu < \nu'$, and by picking ε even smaller if necessary (depending on the choice of ν'') we can assume that $\eta - \eta' > \eta/2$ and $\eta'' - \varepsilon > \eta''/2$. The above bound then becomes

$$C\varepsilon^4(\varepsilon^{\nu} + \varepsilon^{\beta - \nu'/2} + \varepsilon^{1-\nu''} + \varepsilon^{(\nu''-\beta)e^{-4}/4 - \nu/2}).$$

Therefore, it suffices to choose ν', β, ν'' so that $0 < \nu' < 2\beta < 2\nu'' < 2$, and then ν so that $0 < \nu < \nu' \wedge (\nu'' - \beta)e^{-4}/2$, to obtain a bound of the form $C\varepsilon^{4+\chi}$ with positive χ, as wanted. One the choice is made, this bound remains obviously valid without restriction on ε, by taking C larger if necessary. □

The combination of (37) and Lemmas 36 and 37 finally entail Lemma 19.

8 Concluding remarks

Beyond quadrangulations. An important problem in random map theory is the question of *universality*. It is expected that Theorem 1 remains valid for much more general families of random maps than quadrangulations, namely, the *regular critical* Boltzmann maps introduced in [20, 22] should admit the Brownian map as a scaling limit. In particular, in his work mentioned at the end of the Introduction, Le Gall proves Theorem 1 for $2p$-angulations for any given $p \geq 2$, that is maps with faces of degree $2p$. Up to a deterministic, p-dependent multiplicative factor, the limit is still the Brownian map.

We believe that our method is robust enough to tackle this more general problem. In particular, the results of Sections 5 could be adapted to more general maps using variants of the Bouttier-Di Francesco-Guitter bijection [4] that incorporate multiple points. The labeled maps that would intervene would be more complicated than the ones of the present paper, but we expect that the scaling limits of Section 6 remain valid. Indeed, such generalizations hold for the basic case of trees, by the invariance principles developed in [20, 23]. This program seems reasonable to carry out at least in the case where quadrangulations are replaced by random *bipartite* maps. Since the currently available invariance principles for trees coding non-bipartite maps are considerably weaker than those for bipartite maps, the case of non-bipartite maps (including random triangulations) would probably be notably harder to analyze.

Once results analogous to those of Sections 5 and 6 are obtained, our strategy of proof would be valid without much change as soon as we have the prior knowledge that (loosely speaking)

1. any subsequential scaling limit (S, D) of the family of maps under consideration is homeomorphic to (S, D^*)
2. the estimates for the volumes of balls in (S, D) of Section 3 hold
3. typical geodesics in (S, D) are a.s. unique.

All these points are also quite robust, at least in the bipartite case. They are known for $2p$-angulations, and likely extend to more general cases using the approaches of [14, 15, 25].

Last but not least, we believe that similar methods extend to higher genera, probably to the cost of technical complication. In particular, the analogs of points 1., 2., 3. above have been derived for maps on orientable compact surfaces in [25, 2, 3].

Stable maps. A one-parameter family of scaling limits of maps, different from the Brownian map, can be obtained by considering Boltzmann distributions on maps for which the degrees of faces have heavy tails [16]. The problem of the uniqueness of the scaling limit is still open for these maps. We do not know if the methods of the present paper can be adapted to tackle this problem.

Geodesic stars in the Brownian map. The methods of Sections 5, 6 and 7 allow to give estimates on the probability of the event $\mathcal{G}(\varepsilon, k)$ that, if $x_0, x_1, \dots, x_k$ are uniformly chosen points in S, the geodesics from x_0 to $x_1, \dots, x_k$ are pairwise disjoint outside of $B_{D_*}(x_0, \varepsilon)$. Using similar arguments as in Section 7.1, one finds that this probability is of the same order as

$$\frac{1}{\varepsilon}\mathsf{CLM}_1^{(k+1)}\Big(\min_{e \in \vec{E}(f_0)} \underline{W}_e \geq -\varepsilon\Big) .$$

By using bounds of the type (47), this is bounded by $C\varepsilon^{k-1}$, and we think that the exponent is sharp. Since one needs about ε^{-4} balls of radius ε to cover (S, D^*), possibly up to slowly varying terms, this estimate seems to indicate that there is an order of ε^{k-5} such ε-geodesic stars with k arms in the Brownian map. Letting $\varepsilon \to 0$, and making a leap of faith, this suggests that the probability that there exist points $x_1, \dots, x_k$ satisfying the geodesic star event $\mathcal{G}(S; x_1, \dots, x_k)$ of Definition 13 is 1 as long as $k \in \{1, 2, 3, 4\}$, and 0 when $k \geq 6$, the case $k = 5$ being critical and harder to settle. We hope to study more detailed aspects of geodesic stars in the Brownian map in future work.

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