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THE BURGERS EQUATION WITH A NOISY FORCE AND THE STOCHASTIC HEAT EQUATION

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Abstract

We consider the multidimensional Burgers equation with a viscosity term and a random force modelled by a functional of time–space white noise, $\{w_k(t,x)\}$:

$$\text{(B)} \qquad \frac{\partial u_k}{\partial t} + \lambda \sum_{j=1}^{n} u_j \frac{\partial u_k}{\partial x_j} = \nu \Delta u_k + w_k(t,x); \;\; 1 \le k \le n, \;\; (t,x) \in \mathbf{R}^{n+1}$$

We discuss the equation in the framework of a class of distribution valued stochastic processes called *functional processes*, and interpret the products $u_j \frac{\partial u_k}{\partial x_j}$ as Wick products. Then we show that the nonlinear equation (B) can be transformed into a linear, stochastic heat equation with a noisy potential. This heat equation is solved explicitly in the following two cases

a) For a white noise potential

b) For a positive noise potential.

1 Introduction

Starting in 1940 Burgers [Bu] initiated an extensive analysis of the nonlinear initial value problem

$$u_t + uu_x = \nu u_{xx} \qquad (\text{where } u_t = \tfrac{\partial u}{\partial t} \text{ etc.}) \tag{1.1}$$

as a model for turbulence. The equation is a simplified version of the Navier-Stokes equations with $R = 1/\nu$ corresponding to the Reynolds number.

Despite its apparent simple form, the Burgers equation (1.1) encompasses many of the important features of fluid flow, and has furthermore found many applications in other areas.

A crucial property of (1.1) is that it admits a linearization by a nonlinear transformation, the Forsyth-Florin-Hopf-Cole transformation [Fo, p. 101], [Fl]

$$u(t,x) = -2\nu \frac{\varphi_x(t,x)}{\varphi(t,x)} .$$

This transformation reduces (1.1) to the linear diffusion equation

$$\begin{aligned} &\varphi_t = \nu \varphi_{xx} \\ &\varphi(x,0) = \exp\Big(-\frac{1}{2\nu} \int_0^x u(0,y)dy \Big) . \end{aligned} \tag{1.2}$$

Using well-known formulas for the solution of (1.2) one can actually derive explicit solutions of (1.1) [Bu].

Frequently lack of information about all properties of the system makes it natural to introduce a stochastic model, for example by representing some of the coefficients of the equation by some type of noise. See e.g. [LØU 1], [LØU 3] where positive noise is introduced as a model for the permeability of a porous medium in connection with fluid flow. For an extensive discussion from a physical point of view of stochastic properties of the Burgers equation we refer to [GMS].

We will in this paper study a multidimensional Burgers equation with a stochastic force term. It is outside the scope of this paper to give a comprehensive discussion of the various physical applications of Burgers equations with a noisy force term. We will, however, mention the Kardar-Parisi-Zhang (KPZ) model [KPZ], [MHVZ] as it will be relevant for our presentation. This model has been introduced to study growth of interfaces, e.g. the way solids form through growth processes on the surface.

Let $h(x,t)$ denote the profile of an interface as measured from a point $x \in \mathbf{R}^d$ on a given reference plane. Assume that the local growth velocity $\frac{\partial h}{\partial t}$ is a sum of a relaxation of the interface by a surface tension ν and a nonlinear function v of ∇h. This can be justified from the Eden model. Taking the simplest function $v(x) = \frac{\lambda}{2}|x|^2$ and adding a stochastic noise term $N = N(x,t)$ we obtain the KPZ model

$$h_t = \nu \Delta h + \frac{\lambda}{2}\left|\nabla h\right|^2 + N \tag{1.3}$$

See [KPZ] and [KS] for further discussion.

By slightly extending the Forsyth-Florin-Hopf-Cole transformation ([KR], [HHF]) using

$$\varphi = \exp\left[\frac{\lambda}{2\nu} h\right] \tag{1.4}$$

we obtain

$$\varphi_t = \nu \Delta \varphi + \frac{\lambda}{2\nu} N \varphi\,. \tag{1.5}$$

Observe that the additive noise in the KPZ model has been turned into a multiplicative noise as a potential in the heat equation. The relation to Burgers equation can be seen by writing

$$u = -\nabla h\,, \qquad w = -\nabla N \tag{1.6}$$

thus obtaining

$$u_t + \lambda(u, \nabla)u = \nu \Delta u + w \tag{1.7}$$

(where $u(t,x) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ and $(u, \nabla) = \sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}$) which is a multidimensional system of nonlinear partial differential equations with a random force w, generalizing Burgers' equation (1.1).

To give a rigorous analysis of (1.7) with a noise term of white noise type one has to provide a careful interpretation of (1.7) and in particular in what sense one can find solutions to (1.7). As explained below, we interpret the product $(u, \nabla)u$ as a *Wick product*, writing $(u \diamond \nabla)u$, which corresponds to a renormalization.

Recently Sinai [Si] has studied asymptotic properties as $t \to \infty$ of the scalar one-dimensional Burgers equation with noise. In a future paper we would like to apply our results to study the behaviour of the solutions u of (1.7) when the viscosity ν approaches 0.

In this paper we will study the following stochastic variant of the (multi–dimensional) Burgers equation:

$$\frac{\partial u_k}{\partial t} + \lambda \sum_{j=1}^{n} u_j \frac{\partial u_k}{\partial x_j} = \nu \Delta u_k + w_k(t,x)\,; \qquad 1 \le k \le n \tag{1.8}$$

where λ and ν are constants, $\nu > 0$, $t \in \mathbf{R}$, $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ and $\Delta u_k = \sum_{j=1}^{n} \frac{\partial^2 u_k}{\partial x_j^2}$. We assume that $w_1, \dots, w_n$ are given $(n+1)$–parameter *functional processes* (to be defined below) and we seek a solution $\overline{u}(t,x) = (u_1(t,x), \dots, u_n(t,x))$. We may regard $\overline{u}$ as the velocity field of a (vorticity free) fluid with viscosity ν, being exposed to the random force $\overline{w}(t,x) = (w_1(t,x), \dots, w_n(t,x))$.

Here we would like to point out that it is not clear what one should mean with a solution of (1.8). Already for the stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = w(t,x)\,; \qquad (t,x) \in \mathbf{R} \times \mathbf{R}^n \tag{1.9}$$

where w is $(n+1)$–parameter white noise and $n \ge 2$, it was shown by Walsh [W] that a solution u only exists as a *distribution* valued stochastic process. One would expect that a similar statement holds for (1.8). But if $\{u_k\}_{k=1}^{n}$ is a distribution valued process, how does one define the products $u_j \frac{\partial u_k}{\partial x_j}$?

Our answer is that the distributions are represented in the Colombeau sense and the products are interpreted as *Wick products* $u_j \diamond \frac{\partial u_k}{\partial x_j}$ (see definition below). There are several reasons for this: The Wick product corresponds to a form of renormalization commonly used in quantum physics. Moreover, the use of Wick product is already (indirectly) present in ordinary Ito calculus. More precisely, if Y_t is a bounded, adapted (real) stochastic process, and B_t is 1-dimensional Brownian motion, then it was proved in [LØU 2] that

$$\int_0^T Y_t dB_t = \int_0^T Y_t \diamond W_t dt \qquad \text{for all } T < \infty \tag{1.10}$$

where W_t denotes (1-parameter) white noise. The left hand side of (1.10) denotes the Ito integral. In fact, (1.10) remains true for nonadapted processes Y_t if the left hand side is interpreted as the *Skorohod* integral. We refer to [GHLØUZ] for a survey of some properties and applications of the Wick product.

In §2 we give some background material and we formulate a class of distribution valued stochastic processes (the L^p *functional processes*), from which we seek a solution of (1.8).

In §3 we show that the (nonlinear) equation (1.8) can be transformed into a *linear* stochastic heat equation with a noisy potential. This is achieved by introducing a *Wick exponential* substitution.

In §4 we solve the linear stochastic heat equation in two cases:

a) When the potential is white noise
b) When the potential is positive noise

The basic ingredient in our method is the use of the *Hermite transform* and its inverse. The Hermite transform changes the stochastic equation into a deterministic equation with complex–valued parameters.

2 White noise, Hermite transform and functional processes

Here we recall some basic concepts, definitions and results which are used later in the paper.

The white noise probability space.

A general reference here is [HKPS]. Let $\mathcal{S} = \mathcal{S}(\mathbf{R}^d)$ denote the Schwartz space of rapidly decrasing smooth functions on $\mathbf{R}^d$ and let $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^d)$ denote the dual of $\mathcal{S}$. Then $\mathcal{S}'$ is the space of *tempered distributions* on $\mathbf{R}^d$. By the Bochner–Minlos theorem [GV] there exists a probability measure μ on $\mathcal{S}'$ such that

$$\int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \qquad \text{for all } \phi \in \mathcal{S} \tag{2.1}$$

where $\langle \omega, \phi \rangle = \omega(\phi)$ denotes the action of $\omega \in \mathcal{S}'$ on $\phi \in \mathcal{S}$ and $\|\phi\| = \|\phi\|_{L^2(\mathbf{R}^d)} = \int_{\mathbf{R}^d} |\phi(x)|^2 dx$ (dx = Lebesgue measure). μ is called *the white noise probability measure* and the triple $(\mathcal{S}', \mathcal{B}, \mu)$ where $\mathcal{B}$ denotes the family of Borel sets in $\mathcal{S}'$, is called the *white noise probability space*.

The white noise process is the map

$$W : \mathcal{S} \times \mathcal{S}' \to \mathbf{R}$$

given by

$$W(\phi, \omega) = W_\phi(\omega) = \langle \omega, \phi \rangle \qquad \text{for } \phi \in \mathcal{S},\ \omega \in \mathcal{S}' \tag{2.2}$$

It is a consequence of (2.1) that the *Ito isometry* holds, i.e.

$$\|W_\phi\|_{L^2(\mu)} = \|\phi\|\,; \qquad \phi \in \mathcal{S} \tag{2.3}$$

From this we see that if $\psi \in L^2(\mathbf{R}^d) = L^2(\mathbf{R}^d, dx)$ and we choose $\phi_n \in \mathcal{S}$ such that $\phi_n \to \psi$ in $L^2(\mathbf{R}^d)$ then

$$W_\psi := \lim_{n\to\infty} W_{\phi_n} \quad \text{exists in } L^2(\mu), \tag{2.4}$$

and the limit is independent of the choice of $\{\phi_n\}$. In particular, if we define

$$\tilde{B}_x(\omega) = \tilde{B}_{x_1,\dots,x_d}(\omega) = \langle \omega, \chi_{[0,x_1]\times\cdots\times[0,x_d]} \rangle \tag{2.5}$$

(where χ denotes the indicator function), then $\tilde{B}_x$ has a continuous version B_x which then becomes a *d–parameter Brownian motion.*

The *d-parameter Wiener-Ito integral* of $\phi \in L^2(\mathbf{R}^d)$ is defined by

$$\int_{\mathbf{R}^d} \phi(y) dB_y(\omega) = W_\phi(\omega). \tag{2.6}$$

From the integration by parts formula for Wiener-Ito integrals we see that white noise may be regarded as the distributional derivative of Brownian motion

$$W = \frac{\partial^d B_{x_1,\dots,x_d}}{\partial x_1 \partial x_2 \dots \partial x_d} \tag{2.7}$$

For more details see [LØU 1] or [HLØUZ].

The Wiener-Ito chaos expansion.

Let h_n be the *n'th order Hermite polynomial* defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}); \qquad n = 0, 1, 2, 3 \dots \tag{2.8}$$

and for $n = 1, 2, \dots$ let ξ_n be *the Hermite function of order* n defined by

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x)\,; \qquad x \in \mathbf{R} \tag{2.9}$$

Then $\{\xi_n\}_{n=1}^\infty$ forms an orthonormal basis for $L^2(\mathbf{R})$. Therefore the family of tensor products

$$e_\alpha := e_{\alpha_1,\dots,\alpha_d} := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_d} \tag{2.10}$$

(where α denotes the multi-index $(\alpha_1, \ldots, \alpha_d)$) forms an orthonormal basis for $L^2(\mathbf{R}^d)$. This is the basis we will use throughout this paper. With a slight abuse of notation let $e_1, e_2, \ldots$ denote a fixed ordering of the family $\{e_\alpha\}_\alpha$ from now on. Put

$$\theta_j = \theta_j(\omega) = \int_{\mathbf{R}^d} e_j(x) dB_x(\omega) \tag{2.11}$$

and define, for each multi–index $\alpha = (\alpha_1, \ldots, \alpha_m)$,

$$H_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j) \tag{2.12}$$

A version of *the Wiener-Ito chaos theorem* states that the family $\{H_\alpha\}_\alpha$ forms an orthogonal basis for $L^2(\mu)$. Therefore any $X \in L^2(\mu)$ has the (unique) representation

$$X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \tag{2.13}$$

(the sum being taken over all multi-indices of non-negative integers). Moreover, we have the isometry

$$\|X\|^2_{L^2(\mu)} = \sum_\alpha \alpha! c_\alpha^2, \tag{2.14}$$

where $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_m!$ if $\alpha = (\alpha_1, \ldots, \alpha_m)$. See [HKPS] for more details.

The Wick product

If $X = \sum_\alpha a_\alpha H_\alpha$ and $Y = \sum_\beta b_\beta H_\beta$ are two functions in $L^2(\mu)$ we define *the Wick product* $X \diamond Y$ as follows:

$$X \diamond Y = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta} = \sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \Big) H_\gamma \quad \text{(when convergent)} \tag{2.15}$$

For general $X, Y \in L^2(\mu)$ this sum may or may not converge in L^p for some $p \geq 1$.

EXAMPLE 2.1.

If $X(\omega) = W_\phi(\omega) = \int_{\mathbf{R}^d} \phi(x) dB_x$ and $Y(\omega) = W_\psi(\omega) = \int_{\mathbf{R}^d} \psi(x) dB_x$ with $\phi, \psi \in L^2(\mathbf{R}^d)$, then

$$X \diamond Y(\omega) = \int_{\mathbf{R}^d \times \mathbf{R}^d} \phi \hat{\otimes} \psi(x, y) dB^{\otimes 2}_{x,y} \tag{2.16}$$

where $\hat{\otimes}$ denotes the symmetric tensor product (i.e. $\phi \hat{\otimes} \psi(x,y) = \frac{1}{2}[\phi(x)\psi(y) + \psi(y)\psi(x)]$) and the term on the right hand side is *the double Ito integral* (see e.g. [GHLØUZ] for more details)

The Wick product can be extended to $L^1(\mu)$ in the following sense:
Suppose $X, Y \in L^1(\mu)$ and suppose there exists $X_n, Y_n \in L^2(\mu)$ such that

$$X_n \to X\,, \quad Y_n \to Y \quad \text{in } L^1(\mu)$$

and

$$X_n \diamond Y_n \ \text{ converges in } \ L^1(\mu) \text{ to } Z, \text{ say.}$$

Then we define

$$X \diamond Y = Z \tag{2.17}$$

This definition does not depend on the sequences $\{X_n\}$, $\{Y_n\}$. Moreover, we have

$$E[X \diamond Y] = E[X]E[Y] \tag{2.18}$$

(See [HLØUZ].)

If $X \in L^1(\mu)$ and for all $k = 2, 3, \ldots$ *the Wick powers*

$$X^{\diamond k} = X \diamond X \diamond \cdots \diamond X \quad (k \text{ factors})$$

exist and belong to $L^1(\mu)$. We can define *the Wick exponential* of X, $\operatorname{Exp} X$, by

$$\operatorname{Exp} X = \sum_{k=0}^{\infty} \frac{1}{k!} X^{\diamond k} \tag{2.19}$$

provided the sum converges in $L^1(\mu)$.

EXAMPLE 2.2.

The Wick exponential of white noise, $\operatorname{Exp} W_\phi$, turns out to have the simple form

$$K = \operatorname{Exp} W_\phi = \exp(W_\phi - \tfrac{1}{2}\|\phi\|^2)\,; \qquad \phi \in \mathcal{S} \tag{2.20}$$

In particular, $\operatorname{Exp} W_\phi \geq 0$ a.s. μ. This and other properties of $\operatorname{Exp} W_\phi$ make it a suitable model for *positive noise* in many applications. See [LØU1], [LØU 3]. For general X, $\operatorname{Exp} X$ need not be positive (see [GHLØUZ]).

Note that, when defined, the Wick exponential function shares many of the properties of the ordinary exponential. In particular, we have

$$\operatorname{Exp}(X) \diamond \operatorname{Exp}(Y) = \operatorname{Exp}(X+Y),$$
$$\frac{\partial}{\partial x_j}\operatorname{Exp}(X(x)) = \operatorname{Exp}(X(x)) \diamond \frac{\partial X}{\partial x_j} \quad \text{etc. (see §3).}$$

The Hermite transform

The Hermite transform $\mathcal{H}$ transforms functions $X \in L^2(\mu)$ into analytic functions $\mathcal{H}X(z_1, z_2, \ldots)$ of infinitely many complex variables $z_1, z_2, \ldots$ Using the representation (2.12) the transform can easily be described as follows:

$$\text{If } X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \text{ then } \mathcal{H}X(z) = \tilde{X}(z) = \sum_\alpha c_\alpha z^\alpha, \tag{2.21}$$

where $z = (z_1, z_2, \ldots) \in \mathbf{C}_0^{\mathbf{N}}$ (the set of all finite sequences of complex numbers) and we have used the multi–index notation $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ if $\alpha = (\alpha_1, \ldots, \alpha_m)$.

EXAMPLE 2.3.

To find the Hermite transform of white noise $X(\omega) = W_\phi(\omega)$ we proceed as follows:

$$\begin{aligned} W_\phi(\omega) &= \int_{\mathbf{R}^d} \phi(x) dB_x(\omega) = \sum_j (\phi, e_j) \int_{\mathbf{R}^d} e_j(x) dB_x(\omega) = \sum_j (\phi, e_j)\theta_j \\ &= \sum_j (\phi, e_j) h_1(\theta_j) = \sum_j (\phi, e_j) H_{\varepsilon_j}(\omega), \end{aligned}$$

with $(\phi, e_j) = \int_{\mathbf{R}^d} \phi(y) e_j(y) dy$ and $\varepsilon_j = (0, 0, \ldots, 1)$ with 1 on the j'th place; $j = 1, 2, \ldots$

Hence by (2.21)

$$\widetilde{W}_\phi(z) = \sum_j (\phi, e_j) z_j \ ; \ z = (z_1, z_2, \ldots) \tag{2.22}$$

REMARK.

The Hermite transform $\mathcal{H}$ is related to *the $\mathcal{S}$–transform* ([HKPS]) by the identity

$$\mathcal{H}X(z_1, z_2, \ldots) = \mathcal{S}X(z_1 e_1 + z_2 e_2 + \ldots) \ ; \ (z_1, z_2, \ldots) \in \mathbf{C}_0^{\mathbf{N}} \tag{2.23}$$

See e.g. [GHLØUZ].

A fundamental property of the Hermite transform is that it transforms the Wick product into an ordinary, complex product:

$$\mathcal{H}(X \diamond Y)(z) = \mathcal{H}X(z) \cdot \mathcal{H}Y(z)\,; \qquad z \in \mathbf{C}_0^{\mathbf{N}} \tag{2.24}$$

(See [LØU 1], [HLØUZ])

Moreover, we have an explicit *inverse Hermite transform*:

If $\tilde{X}(z)$ is the Hermite transform of $X \in L^2(\mu)$ then we can recover X from $\tilde{X}$ by

$$X(\omega) = \int_{\mathbf{R}^{\mathbf{N}}} \tilde{X}(\theta_1 + iy_1, \theta_2 + iy_2, \ldots) d\lambda(y) \tag{2.25}$$

where $d\lambda(y) = d\lambda(y_1, y_2, \ldots)$ is the probability measure on $\mathbf{R}^{\mathbf{N}}$ defined by

$$\int_{\mathbf{R}^{\mathbf{N}}} f(y_1, \ldots, y_n) d\lambda(y) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} f(y) e^{-\frac{1}{2}|y|^2} dy \tag{2.26}$$

if f is a bounded function depending only on the first n coordinates $(y_1, \ldots, y_n)$ of y.

Again we refer to [LØU 1], [HLØUZ] for more information. As explained in the latter reference, the inversion (2.25) should be interpreted as a limit of the inversions of truncations of $\tilde{X}$.

Functional processes.

Finally we describe the type of distribution valued/generalized stochastic processes we are about to consider:

Let $p \geq 1$. An *L^p functional process* is a map

$$X:\ \mathcal{S} \times \mathbf{R}^d \times \mathcal{S}' \to \mathbf{R}$$

such that

(i) $x \to X(\phi, x, \omega)$ is (Borel) measurable for all $\phi \in \mathcal{S},\ \omega \in \mathcal{S}'$

and

(ii) $\omega \to X(\phi, x, \omega)$ belongs to $L^p(\mu)$ for all $\phi \in \mathcal{S},\ x \in \mathbf{R}^d$

Intuitively $X(\phi, x, \omega)$ is the result we get if we measure the quantity X using the test function (or "window") ϕ shifted to the point x and in the "experiment" ω. By "ϕ shifted to x" we mean the function $\phi_x(\cdot)$ defined by

$$\phi_x(y) = \phi(y - x)$$

EXAMPLE 2.4.

White noise W may be regarded as an L^p functional process (for any $p < \infty$) by putting

$$W(\phi, x, \omega) = W_{\phi_x}(\omega) = \int_{\mathbf{R}^d} \phi(y-x) dB_y(\omega)$$

Note that the distributional derivatives of $W(\cdot, x, \omega)$ at ϕ coincide with the derivatives w.r.t. x of $W(\phi, x, \omega)$. In view of this and the general interpretation of $X(\phi, x, \omega)$ given above, we will regard partial derivatives of a functional process $X(\phi, x, \omega)$ as taken w.r.t. x for a fixed ϕ.

For brevity we sometimes suppress ϕ and/or ω and write $X(x, \omega)$ or simply $X(x)$ for $X(\phi, x, \omega)$. Similarly, in a time-space situation we write $X(t, x)$ for $X(\phi, t, x, \omega)$.

3 From the stochastic Burgers equation to the stochastic heat equation

We are now ready to consider (the Wick interpretation of) the stochastic Burgers equation, i.e.

$$\frac{\partial u_k}{\partial t} + \lambda \sum_{j=1}^{n} u_j \diamond \frac{\partial u_k}{\partial x_j} = \nu \Delta u_k + w_k(t, x) ; \qquad u_k(0, x) = g_k(x) \tag{3.1}$$

More precisely, we seek n L^p functional processes $u_k = u_k(\phi, \overline{x}, \omega)$, with $\overline{x} = (t, x)$, such that for all $\phi \in \mathcal{S}(\mathbf{R}^{n+1})$

$$\frac{\partial}{\partial t} u_k(\phi, t, x, \cdot) + \lambda \sum_{j=1}^{n} u_j(\phi, t, x, \cdot) \diamond \frac{\partial}{\partial x_j} u_k(\phi, t, x, \cdot) =$$

$$\nu \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u_k(\phi, t, x, \cdot) + w_k(\phi, t, x, \cdot) ; \qquad 1 \leq k \leq n \tag{3.2}$$

where w_k are given $(n+1)$-parameter functional processes.

Suppose such a solution $u = (u_1, \ldots, u_n)$ exists, for some $p \geq 1$. We now make two assumptions:

Assumption 1 There exists an L^p functional process X such that

$$u_k = -\frac{\partial X}{\partial x_k} ; \qquad 1 \leq k \leq n \tag{3.3}$$

Assumption 2 There exists an $(n+1)$-parameter functional process N such that

$$w_k = -\frac{\partial N}{\partial x_k}; \qquad 1 \le k \le n \tag{3.4}$$

REMARK.

Let $N = W_\varphi(\omega)$ be 1-dimensional white noise in the $n+1$ parameters $t, x_1, \ldots, x_n$ as described in §2. Choose

$$\varphi(t, x_1, \ldots, x_n) = \varphi_0(t)\varphi_1(x_1)\cdots\varphi_n(x_n) \qquad \text{with } \varphi_k \in \mathcal{S}(\mathbf{R}).$$

Put

$$v_k := \frac{\partial N}{\partial x_k} = W_{\frac{\partial \varphi}{\partial x_k}} = W_{\psi_k},$$

where $\psi_k = \varphi_0\varphi_1\varphi_2\cdots\varphi_k'\cdots\varphi_n$, $1 \le k \le n$

Then each v_k is a white noise. Moreover, if $k \ne m$ then v_k and v_m are independent, since ψ_k and ψ_m are orthogonal in $L^2(\mathbf{R}^{n+1})$. Therefore, if the given noise $w = (w_1, \ldots, w_n)$ in (3.1) is white, then we can always find a *representation* of w of the form (3.4) in Assumption 2. Adopting the terminology from ordinary stochastic differential equations, we might say that this corresponds to considering the stochastic partial differential equation (3.1) in the *weak* sense.

If we substitute (3.3), (3.4) in (3.1) we get

$$-\frac{\partial}{\partial x_k}\Big(\frac{\partial X}{\partial t}\Big) + \lambda \sum_j \frac{\partial X}{\partial x_j} \diamond \frac{\partial}{\partial x_j}\Big(\frac{\partial X}{\partial x_k}\Big) = -\nu \sum_j \frac{\partial^2}{\partial x_j^2}\Big(\frac{\partial X}{\partial x_k}\Big) - \frac{\partial N}{\partial x_k} \tag{3.5}$$

or

$$\frac{\partial X}{\partial t} = \frac{\lambda}{2}\sum_j \Big(\frac{\partial X}{\partial x_j}\Big)^{\diamond 2} + \nu\Delta X + N + C, \tag{3.6}$$

where $C = C(t, \omega)$ does not depend on x.

Next assume that the following functional processes

$$Y := \operatorname{Exp}\Big(\frac{\lambda}{2\nu}X\Big), \quad Y \diamond X \quad \text{and} \quad Y \diamond X^{\diamond 2} \tag{3.7}$$

exist and belong to $L^p(\mu)$ for some $p \ge 1$ (for all $\phi \in \mathcal{S}$).

Then from the basic properties of the Wick product we get

$$\frac{\partial Y}{\partial t} = \frac{\lambda}{2\nu} Y \diamond \frac{\partial X}{\partial t}, \qquad \frac{\partial Y}{\partial x_j} = \frac{\lambda}{2\nu} Y \diamond \frac{\partial X}{\partial x_j} \tag{3.8}$$

and hence

$$\begin{aligned}\Delta Y &= \sum_j \frac{\partial}{\partial x_j}\Big(\frac{\partial Y}{\partial x_j}\Big) = \sum_j \frac{\partial}{\partial x_j}\Big(\frac{\lambda}{2\nu} Y \diamond \frac{\partial X}{\partial x_j}\Big) \\ &= \sum_j \Big(\frac{\lambda}{2\nu}\Big)^2 Y \diamond \Big(\frac{\partial X}{\partial x_j}\Big)^{\diamond 2} + \sum_j \frac{\lambda}{2\nu} Y \diamond \frac{\partial^2 X}{\partial x_j^2} \\ &= \frac{\lambda}{2\nu} Y \diamond \Big[\frac{\lambda}{2\nu} \sum_j \Big(\frac{\partial X}{\partial x_j}\Big)^{\diamond 2} + \Delta X\Big] \end{aligned} \tag{3.9}$$

If we Wick-multiply (3.6) by $\frac{\lambda}{2\nu} Y$ and use (3.8), (3.9) we get

$$\frac{\partial Y}{\partial t} = \nu \Delta Y + \frac{\lambda}{2\nu} Y \diamond [N + C] \tag{3.10}$$

We summarize this as follows:

THEOREM 3.1

Suppose $u = (u_1, \ldots, u_n)$ is a functional process solution of the Burgers equation (3.1). Suppose (3.3), (3.4) and (3.7) hold. Then with

$$u_k = -\frac{\partial X}{\partial x_k}; \quad w_k = -\frac{\partial N}{\partial x_k}; \qquad 1 \le k \le n \tag{3.11}$$

the process

$$Y = \operatorname{Exp}\Big(\frac{\lambda}{2\nu} X\Big)$$

solves the equation (3.10).

4 Solution of the stochastic heat equation

In this section we give the solution of some stochastic heat equations of the type that appeared in §3.

First we consider the equation

$$\frac{\partial Y}{\partial t} = \nu \Delta Y + H \diamond Y; \qquad Y(0, x) = f(x) \tag{4.1}$$

where $H = \frac{\lambda}{2\nu}(N + C)$ as in §3, N being 1-dimensional white noise in the $n+1$ parameters $(t, x_1, x_2, \ldots, x_n)$. For simplicity we assume that both $f(x)$ and $C(t)$ are bounded, deterministic functions.

THEOREM 4.1

Equation (4.1) has a unique L^2 functional process solution given by

$$Y(t,x) = \hat{E}^x\Big[f(b_{\alpha t})\mathrm{Exp}\Big(\int_0^t H(s,b_{\alpha s})ds\Big)\Big] \tag{4.2}$$

where $\alpha = \sqrt{2\nu}$ and $(b_t, \hat{P}^x)$ is standard Brownian motion in $\mathbf{R}^n$ ($\hat{E}^x$ denotes expectation w.r.t. $\hat{P}^x$).

Proof. Taking $\mathcal{H}$–transforms of (4.1) we get, with $\tilde{Y} = \mathcal{H}Y$,

$$\frac{\partial \tilde{Y}}{\partial t} = \nu\Delta\tilde{Y} + \tilde{H}\cdot\tilde{Y}\,; \qquad \tilde{Y}(0,x) = f(x). \tag{4.3}$$

Here $\tilde{H} = \tilde{H}(t,x,z)$ is complex and so is $\tilde{Y}(t,x,z)$. But the Feynman-Kac formula is easily seen to extend to the complex case. So the solution of (4.3) is given by

$$\tilde{Y}(t,x,z) = \hat{E}^x\Big[f(\beta_t)\exp\Big(\int_0^t \tilde{H}(s,\beta_s,z)ds\Big)\Big] \tag{4.4}$$

where β_t is the diffusion on $\mathbf{R}^n$ with generator $\nu\Delta = 2\nu\cdot\frac{1}{2}\Delta$. In other words,

$$\beta_t = b_{\alpha t} \qquad \text{with } \alpha = \sqrt{2\nu}$$

where b_t is standard Brownian motion

Since $\frac{2\nu}{\lambda}\tilde{H}(t,x,z) = \sum_j(\phi,e_j)z_j + C(t)$; $z = (z_1,z_2,\ldots)$ (See (2.21), Example 2.3), it is easily seen that $\tilde{Y}(t,x,z) \in L^2(d\lambda(\xi)\times d\lambda(\eta))$ ($z = \xi + i\eta$), where λ is defined in (2.26). Therefore $Y \in L^2(\mu)$ (see Corollary 4.3 in [HLØUZ]). Moreover, we can apply the inverse Hermite transform to (4.4) and (4.2) follows.

Next we consider a different case of independent interest. We represent the potential by the *positive* noise $K = \mathrm{Exp}\,W$ constructed in Example 2.2:

THEOREM 4.2

Assume that f is deterministic. Then the equation

$$\frac{\partial Y}{\partial t} = \frac{1}{2}\Delta Y + \mathrm{Exp}\,W \diamond Y\,; \qquad Y(0,x) = f(x) \tag{4.5}$$

has a unique L^1 functional process solution given by

$$Y(t,x) = \hat{E}^x\Big[f(b_t)\mathrm{Exp}\Big(\int_0^t \mathrm{Exp}(W(\phi_{b_s}))ds\Big)\Big], \tag{4.6}$$

where b_t is as in Theorem 4.1.

REMARK.

The functional process Y defined by (4.6) is not in $L^p(\mu)$ for any $p > 1$. In fact, Y does not even belong to the space $(\mathcal{S})^*$ of *Hida distributions*. (See [HKPS] and [GHLØUZ] for definitions and properties of $(\mathcal{S})^*$; in general we have $L^p(\mu) \subset (\mathcal{S})^*$ for $p > 1$, but not for $p = 1$). To see this we argue as follows: If $Y \in (\mathcal{S})^*$ we can apply the $\mathcal{S}$-transform to it. Then if we evaluate the $\mathcal{S}$-transform at $\phi = ze_1$ for $z \in \mathbf{C}$ we get

$$\mathcal{S}(Y)(ze_1) = \mathcal{S}Y = \hat{E}^x\Big[f(b_t)\exp\Big(\int_0^t \exp \mathcal{S}W(\phi_{b_s})ds\Big)\Big]$$

where $\mathcal{S}W(\phi_{b_s}) = (\phi_{b_s}, e_1)z$. But according to the characterization theorem of Potthoff and Streit [PS] this analytic function of z grows too fast at ∞ to be the $\mathcal{S}$-transform of an element of $(\mathcal{S})^*$.

The proof of Theorem 4.2 will be split into several lemmas. We first consider the process obtained by truncating the series for the (outer) Wick exponential in (4.6), i.e. we put

$$Y_n(t,x) = \sum_{k=0}^{n} \frac{1}{k!}\hat{E}^x\Big[f(b_t)\Big(\int_0^t \operatorname{Exp}(W(\phi_{b_s}))ds\Big)^{\diamond k}\Big]; \quad n=0,1,2,\ldots \tag{4.7}$$

LEMMA 4.3

a) $Y_n(t,x) \in L^2(\mu)$ for all n, t, x

b) For any $T > 0$ we have

$$\|Y_n(t,x) - Y(t,x)\|_{L^1(\mu)} \to 0 \text{ uniformly for } t \le T,\ x \in \mathbf{R}^n.$$

Proof.

a) Note that

$$\Big(\int_0^t \operatorname{Exp}(W(\phi_{b_s}))ds\Big)^{\diamond k} \tag{4.8}$$

$$= \int_0^t \cdots \int_0^t \operatorname{Exp}\Big(W(\phi_{b_{s_1}}) + \cdots + W(\phi_{b_{s_k}})\Big)ds_1 \cdots ds_k$$

$$= \int_0^t \cdots \int_0^t \operatorname{Exp}\Big(W(\phi_{b_{s_1}} + \cdots + \phi_{b_{s_k}})\Big)ds_1 \cdots ds_k \tag{4.9}$$

$$= \int_0^t \cdots \int_0^t \exp\Big(W(\phi_{b_{s_1}} + \cdots + \phi_{b_{s_k}})$$

$$-\frac{1}{2}\|\phi_{b_{s_1}}+\cdots+\phi_{b_{s_k}}\|^2\Big)ds_1\dots ds_k$$

$$\le \quad t^{k/2}\Big[\int_0^t\cdots\int_0^t \exp(2W(\phi_{b_{s_1}}+\cdot+\phi_{b_{s_k}})$$

$$- \quad \|\phi_{b_{s_1}}+\cdots+\phi_{b_{s_k}}\|^2)ds_1\dots ds_k\Big]^{1/2}.$$

Therefore, if $|f(x)|\le C$ for all x we have

$$E\Big[\Big|\hat{E}^x\Big[f(b_t)\Big(\int_0^t \operatorname{Exp}(W(\phi_{b_s}))ds\Big)^{\diamond k}\Big|^2\Big]$$

$$\le C^2t^k\int_0^t\cdots\int_0^t \exp\Big(\|\phi_{b_{s_1}}+\cdots+\phi_{b_{s_k}}\|^2\Big)ds_1\dots ds_k$$

$$\le C^2t^{2k}\exp(k^2\|\phi\|^2).$$

We conclude that $Y_n\in L^2(\mu)$.

To prove b), note that for $l>m$, $t\le T$ and all x we have, using (4.8) and C as above,

$$E\Big[|Y_l(t,x)-Y_m(t,x)|\Big]\le\sum_{k=m+1}^{l}\frac{1}{k!}E\Big[\Big|\hat{E}^x\Big[f(b_t)\Big(\int_0^t \operatorname{Exp}(W(\phi_{b_s}))ds\Big)^{\diamond k}\Big]\Big|\Big]$$

$$\le C\sum_{k=m+1}^{l}\frac{1}{k!}\hat{E}^x\Big[E\Big[\int_0^t\cdots\int_0^t \operatorname{Exp}(W(\phi_{b_{s_1}}+\cdots+\phi_{b_{s_k}}))ds_1\dots ds_k\Big]\Big]$$

$$\le C\sum_{k=m+1}^{l}\frac{1}{k!}T^k\to 0\quad\text{as }\ l,m\to\infty$$

This shows that $Y_m(t,x)\to Y(t,x)$ in $L^1(\mu)$, uniformly for $x\in\mathbf{R}^n$ and $t\le T$.

LEMMA 4.4

a) $\frac{\partial Y_m}{\partial t}\in L^1(\mu)$ for all m

b) As $m\to\infty$ $\frac{\partial Y_m}{\partial t}$ converges in $L^1(\mu)$ uniformly for $t\le T$ and $x\in\mathbf{R}^n$ to

$$\begin{aligned}&\frac{\partial}{\partial t}P_tf(x)+\hat{E}^x\Big[\operatorname{Exp}\Big[\int_0^t \operatorname{Exp}W(\phi_{b_s})ds\diamond \operatorname{Exp}W(\phi_{b_t})f(b_t)\Big]\Big]\\&\quad+\frac{1}{2}\int_0^t\Big\{\hat{E}^x\Big[\operatorname{Exp}\Big[\int_0^s \operatorname{Exp}W(\phi_{b_u})du\\&\quad\diamond \operatorname{Exp}W(\phi_{b_s})\hat{E}^{b_s}\Big[\frac{|b_{t-s}-b_0|^2-n(t-s)}{(t-s)^2}f(b_{t-s})\Big]\Big]\Big]\Big\}ds\end{aligned}\tag{4.10}$$

where

$$P_tf(x)=\hat{E}^x[f(b_t)].\tag{4.11}$$

Proof. Set

$$V_k(t,x) = \hat{E}^x\Big[f(b_t)\Big(\int_0^t \mathrm{Exp}W(\phi_{b_s})ds\Big)^{\diamond k}\Big] \tag{4.12}$$

Then $V_k(t,x) \in L^2(\mu)$ and its Hermite transform is

$$\widetilde{V}_k(t,x) = \hat{E}^x\Big[f(b_t)\Big(\int_0^t \exp\widetilde{W}(\phi_{b_s})ds\Big)^{k}\Big]$$

To compute $\frac{\partial}{\partial t}\widetilde{V}_k$ we put $g(u_1,\ldots,u_k) = \exp(\widetilde{W}(\phi_{b_{u_1}}+\cdots+\phi_{b_{u_k}}))$ and rewrite $\widetilde{V}_k$ as in (4.8):

$$\begin{aligned}\widetilde{V}_k(t,x) &= \hat{E}^x\Big[f(b_t)\int_0^t\cdots\int_0^t g(u_1,\ldots,u_k)du_1\ldots du_k\Big]\\ &= k!\hat{E}^x\Big[f(b_t)\int_{0\le u_1\le\cdots\le u_k\le t}\cdots\int g(u_1,\ldots,u_k)du_1\ldots du_k\Big]\\ &= k!\int_0^t\int_0^{u_k}\cdots\int_0^{u_2}\hat{E}^x[f(b_t)g(u_1,\ldots,u_k)]du_1\ldots du_k\end{aligned}$$

Put

$$p_t(x) = (2\pi t)^{-\frac{n}{2}}\exp\Big(-\frac{|x|^2}{2t}\Big).$$

Then

$$\begin{aligned}&\frac{\partial}{\partial t}\hat{E}^x\Big[f(b_t)g(u_1,\ldots,u_k)\Big]\\ &= \frac{\partial}{\partial t}\int_{\mathbf{R}^{k+1}} g(x_1,\ldots,x_k)f(x_{k+1})p_{u_1}(x_1-x)p_{u_2-u_1}(x_2-x_1)\cdots \qquad (4.13)\\ &\cdots p_{u_k-u_{k-1}}(x_k-x_{k-1})p_{t-u_k}(x_{k+1}-x_k)dx_1\cdots dx_{k+1}\\ &= \frac{1}{2}\hat{E}^x\left[\Big\{\frac{|b_t-b_{u_k}|^2}{(t-u_k)^2}-\frac{n}{t-u_k}\Big\}f(b_t)g(u_1,\ldots,u_k)\right] \qquad (4.14)\end{aligned}$$

In order to differentiate (4.12) under the integral sign, we need to know that

$$J := \int_{0\le u_1\le\ldots\le u_k\le t}\cdots\int\Big|\frac{\partial}{\partial t}\hat{E}^x[f(b_t)g(u_1,\ldots,u_k)]\Big|du_1\ldots du_k < \infty \tag{4.15}$$

Let $\mathcal{F}_u$ denote the σ-algebra generated by $\{b_s(\cdot)\}_{s\le u}$. By (4.13) and the Markov property

$$\begin{aligned}&\frac{\partial}{\partial t}\hat{E}^x[f(b_t)g(u_1,\ldots,u_k)]\\ &= \frac{1}{2}\hat{E}^x\left[\hat{E}^x\Big[\Big\{\frac{|b_t-b_{u_k}|^2}{(t-u_k)^2}-\frac{n}{t-u_k}\Big\}f(b_t)g(u_1,\ldots,u_k)|\mathcal{F}_{u_k}\Big]\right]\\ &= \frac{1}{2}\hat{E}^x\left[g(u_1,\ldots,u_k)\hat{E}^{b_{u_k}}\Big[\Big\{\frac{|b_{t-u_k}-b_0|^2}{(t-u_k)^2}-\frac{n}{t-u_k}\Big\}f(b_{t-u_k})\Big]\right] \qquad (4.16)\end{aligned}$$

But for any $x \in \mathbf{R}^n$ we have

$$\Big|\hat{E}^x\Big[\Big\{\frac{|b_{t-u_k}-b_0|^2}{(t-u_k)^2}-\frac{n}{t-u_k}\Big\}f(b_{t-u_k})\Big]\Big|$$

$$= \Big|\hat{E}^x\Big[\Big\{\frac{|b_{t-u_k}-b_0|^2}{(t-u_k)^2}-\frac{n}{t-u_k}\Big\}(f(b_{t-u_k})-f(b_0))\Big]\Big|$$

$$\leq \frac{1}{(t-u_k)^2}\hat{E}^x\Big[(|b_{t-u_k}-b_0|^2-n(t-u_k))^2\Big]^{1/2}\hat{E}^x\Big[f(b_{t-u_k})-f(b_0))^2\Big]^{1/2} \tag{4.17}$$

$$\leq \frac{C_1}{t-u_k}\hat{E}^\circ\Big[(b_1-n)^2\Big]^{1/2}\hat{E}^x\Big[|b_{t-u_k}-b_0|^2\Big]^{1/2} \leq \frac{C_2}{(t-u_k)^{1/2}} \tag{4.18}$$

Here, and in the following, C_1, C_2 denote constants.

Substituting this in (4.15) and then in (4.14) we get

$$\begin{aligned} J &\leq \frac{1}{2}C_2\int_{0\leq u_1\leq\cdots\leq u_k\leq t}\cdots\int \hat{E}^x\Big[g(u_1,\ldots,u_k)\cdot\frac{1}{(t-u_k)^{1/2}}\Big]du_1\ldots du_k \\ &\leq \frac{1}{2}C_2\int_{0\leq u_1\leq\ldots\leq u_k\leq t}\int \exp\Big(\frac{1}{2}\|\phi_{b_{u_1}}+\cdots+\phi_{b_{u_k}}\|^2\Big)\cdot\frac{1}{(t-u_k)^{1/2}}du_1\ldots du_k \\ &\leq \frac{1}{2}C_2\exp\Big(\frac{k^2}{2}\|\phi\|^2\Big)t^{k-1}\int_0^t\frac{du_k}{(t-u_k)^{1/2}} < \infty \end{aligned}$$

and (4.15) is proved.

This justifies differentiation under the integral sign in (4.12) and we get

$$\begin{aligned} \frac{\partial\widetilde{V}_k(t,x)}{\partial t} &= k!\int_0^t\int_0^{u_{k-1}}\cdot\int_0^{u_2}\hat{E}^x\Big[g(u_1,\ldots,u_{k-1})\cdot\exp\widetilde{W}(\phi_{b_t})f(b_t)\Big]du_1\ldots du_{k-1} \\ &+\frac{1}{2}k!\int_0^t\int_0^{u_k}\cdots\int_0^{u_2}\hat{E}^x\Big[\Big\{\frac{|b_t-b_{u_k}|^2-n(t-u_k)}{(t-u_k)^2}\Big\}g(u_1,\ldots,u_k)f(b_t)\Big]du_1\ldots du_k \\ &= k\hat{E}^x\Big[\Big(\int_0^t\exp\widetilde{W}(\phi_{b_u})du\Big)^{k-1}\cdot\exp\widetilde{W}(\phi_{b_t})f(b_t)\Big] \qquad (4.19) \\ &+\frac{1}{2}k\int_0^t\hat{E}^x\Big(\int_0^{u_k}\exp\widetilde{W}(\phi_{b_u})du\Big)^{k-1}\exp\widetilde{W}(\phi_{b_{u_k}})\cdot \\ &\quad \hat{E}^{b_{u_k}}\Big[\Big\{\frac{|b_{t-u_k}-b_0|^2-n(t-u_k)}{(t-u_k)^2}\Big\}f(b_{t-u_k})\Big]du_k \end{aligned}$$

Taking inverse Hermite transform we get

$$\begin{aligned} &\frac{\partial Y_m(t,x)}{\partial t} = \\ &\frac{\partial}{\partial t}P_t(f(x)+\sum_{k=1}^m\frac{k}{k!}\Big\{\hat{E}^x\Big[\Big(\int_0^t\operatorname{Exp}W(\phi_{b_u})du\Big)^{\diamond(k-1)}\diamond\operatorname{Exp}W(\phi_{b_t})f(b_t)\Big]\Big\} \\ &+\frac{1}{2}\sum_{k=1}^m\frac{k}{k!}\int_0^t\hat{E}^x\Big[\Big(\int_0^{u_k}\operatorname{Exp}W(\phi_{b_u})du\Big)^{\diamond k-1} \\ &\diamond\operatorname{Exp}W(\phi_{b_{u_k}})\hat{E}^{b_{u_k}}\Big[\frac{|b_{t-u_k}-b_0|^2-n(t-u_k)}{(t-u_k)^2}\cdot f(b_{t-u_k})\Big]\Big]du_k \end{aligned}$$

From this it is clear that $\frac{\partial Y_m(t,x)}{\partial t} \in L^1(\mu)$.

Moreover, from (4.16) we get, for $l > m$.

$$E\left[\left|\frac{\partial Y_l}{\partial t} - \frac{\partial Y_m}{\partial t}\right|\right] \leq \sum_{k=m+1}^{l} \frac{1}{(k-1)!} t^{k-1} \|f\|_\infty$$

$$+\frac{1}{2}\sum_{k=m+1}^{l} \frac{1}{(k-1)!} \int_0^t \hat{E}^x\left[u_k^{k-1} \cdot \left|\hat{E}^{b_{u_k}}\left[\frac{(b_{t-u_k} - b_0|^2 - n(t-u_k)}{(t-u_k)^2} \cdot f(b_{t-u_k})\right]\right|\right] du_k$$

$$\leq \sum_{k=m+1}^{l} \frac{1}{(k-1)!} t^{k-1} \|f\|_\infty + C_3 \sum_{k=m+1}^{l} \frac{1}{(k-1)!} t^{k-1} \int_0^t \frac{du_k}{\sqrt{t-u_k}}$$

$\rightarrow 0$ uniformly for $t \leq T$ and $x \in \mathbf{R}^n$,

That completes the proof of Lemma 4.4.

LEMMA 4.5

$$\frac{\partial Y(t,x)}{\partial t} \in L^1(\mu) \quad \text{and is given by (4.10)}$$

Proof. This is a direct consequence of Lemma 4.4.

LEMMA 4.6

a) $\Delta Y_m(t,x) \in L^1(\mu)$ for all m

and

b) $\Delta Y_m(t,x) \rightarrow \frac{\partial Y(t,x)}{\partial t} - Y(t,x) \diamond \operatorname{Exp} W(\phi_x)$ in $L^1(\mu)$

as $m \rightarrow \infty$, uniformly for $t \leq T$ and $x \in \mathbf{R}^n$.

Proof. Let V_k be as in (4.11). Then $\widetilde{V}_k$ is given by (4.12). To compute $\Delta \widetilde{V}_k$ we would like to differentiate under the integral sign in (4.12). This operation will be justified if we can show that

$$\int_{0 \leq u_1 \leq \cdots \leq u_k \leq t} \cdots \int \left|\Delta \hat{E}^x[f(b_t) g(u_1, \ldots, u_k)]\right| du_1 \ldots du_k < \infty \tag{4.20}$$

Since

$$\begin{aligned} D: &= \Delta \hat{E}^x[f(b_t) g(u_1, \ldots, u_k)] \\ &= \hat{E}^x\left[f(b_t) g(u_1, \ldots, u_k)\left\{\frac{|b_{u_1} - x|^2}{u_1^2} - \frac{n}{u_1}\right\}\right], \end{aligned} \tag{4.21}$$

we get by the Markov property that, if $\psi(u) = \sum_{i=2}^{k} \phi_{b_{u_i - u_1}}$,

$$D = \hat{E}^x \Big[\exp \widetilde{W}(\phi_{b_{u_1}}) \Big\{ \frac{|b_{u_1} - x|^2}{u_1^2} - \frac{n}{u_1} \Big\} \hat{E}^{b_{u_1}} [\exp \widetilde{W}(\psi) \cdot f(b_{t-u_1})] \Big]$$

Set

$$F(y) = \exp \widetilde{W}(\phi_y) \hat{E}^y [\exp \widetilde{W}(\psi) \cdot f(b_{t-u_1})].$$

Then $F \in C_b^2(\mathbf{R}^n)$ and

$$\begin{aligned} \Big| \frac{\partial F}{\partial y_i} \Big| &\leq \Big| \hat{E}^y \Big[\frac{b_{u_2-u_1}^{(i)} - y_i}{u_2 - u_1} \exp \widetilde{W}(\psi) \cdot f(b_{t-u_1}) \Big] \Big| + C_1 \\ &\leq C_2 \cdot \Big(\frac{1}{(u_2 - u_1)^{1/2}} + C_3 \Big) \end{aligned}$$

By a calculation similar to the one in (4.16) we find that

$$\begin{aligned} |\Delta \hat{E}^x [f(b_t) \cdot g(u_1, \dots, u_k)]| &= \Big| \hat{E}^x \Big[\frac{|b_{u_1} - x|^2 - n u_1}{u_1^2} \cdot F(b_{u_1}) \Big] \Big| \\ &\leq C_4 \cdot \frac{1}{u_1^{1/2}} \cdot \Big(\frac{1}{(u_2 - u_1)^{1/2}} + C_3 \Big). \end{aligned}$$

Thus (4.18) holds and we get

$$\Delta \widetilde{V}_k(x, t) = \tag{4.22}$$

$$k! \int_{0 \leq u_1 \leq \dots \leq u_k \leq t} \cdots \int \hat{E}^x \Big[g(u_1, \dots, u_k) \Big\{ \frac{|b_{u_1} - x|^2 - n u_1}{u_1^2} \Big\} f(b_t) \Big] du_1 \dots du_k.$$

Finally we work out the relation between $\frac{\partial V_k}{\partial t}(t, x)$ and $\Delta V_k(t, x)$:

Since $\Delta V_k(t, x)$ exists we can write, with P_t as in (4.10),

$$\begin{aligned} \Delta V_k(t, x) &= \lim_{s \to 0} \frac{P_s V_k(t, x) - V_k(t, x)}{s} \\ &= \lim_{s \to 0} \Bigg\{ \frac{V_k(t + s, x) - V_k(t, x)}{s} \\ &\quad + \hat{E}^x \Big[f(b_{t+s}) \frac{1}{s} \Big\{ \Big(\int_s^{t+s} \operatorname{Exp}(W_{\phi_{b_u}}) du \Big)^{\diamond k} - \Big(\int_0^{t+s} \operatorname{Exp}(W_{\phi_{b_u}}) du \Big)^{\diamond k} \Big\} \Big] \Bigg\}. \end{aligned}$$

Therefore

$$\Delta V_k(t, x) = \frac{\partial V_k(t, x)}{\partial t} - k \hat{E}^x \Big[f(b_t) \Big(\int_0^t \operatorname{Exp}(W_{\phi_{b_u}}) du \Big)^{\diamond (k-1)} \Big] \diamond \operatorname{Exp} W_{\phi_x}$$

Multiplying by $\frac{1}{k!}$ and summing from $k=0$ to m we get

$$\begin{aligned}
&\Delta Y_m(t,x) = \\
&\frac{\partial Y_m(t,x)}{\partial t} - \sum_{k=1}^{m} \frac{1}{(k-1)!} \hat{E}^x \Big[f(b_t) \Big(\int_0^t \operatorname{Exp}(W_{\phi_{b_u}}) du \Big)^{\diamond(k-1)} \Big] \diamond \operatorname{Exp} W_{\phi_x} \\
&\to \frac{\partial Y(t,x)}{\partial t} - \hat{E}^x \Big[f(b_t) \operatorname{Exp}\Big(\int_0^t \operatorname{Exp}(W_{\phi_{b_u}}) du \Big) \Big] \diamond \operatorname{Exp} W_{\phi_x} \\
&= \frac{\partial Y(t,x)}{\partial t} - Y(t,x) \diamond \operatorname{Exp} W_{\phi_x},
\end{aligned}$$

convergence being in $L^1(\mu)$, uniformly for $x \in \mathbf{R}^n$, $t \le T$.

This proves Lemma 4.6.

Combining Lemmas 4.3–4.6 we get Theorem 4.2.

CONCLUDING REMARKS

1) The stochastic heat equation

$$\frac{\partial u}{\partial t} = \Delta u + W \diamond u \tag{4.23}$$

(where W denotes white noise in $n+1$ parameters (t,x)), has been studied by Nualart and Zakai [NZ], who proved the existence of a solution of a type they call *generalized Wiener functionals.* By Theorem 4.1 we have seen that there exists a (unique) L^2 functional process solution to this equation.

2) Using the stochastic version (3.7) (based on the Wick exponential $\operatorname{Exp}(\cdot)$) of the Forsyth-Florin-Hopf-Cole transformation, we have transformed the stochastic Burgers equation into a stochastic heat equation. To go the other way, from the solution of the stochastic heat equation to the solution of the stochastic Burgers equation, one would need the concept of a *Wick-logarithm*, $\operatorname{Log}(\cdot)$. We will return to this in a future paper.

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