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The Cai Model with Time Delay: Existence of Periodic Solutions and Asymptotic Analysis

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Abstract: The economic growth model with endogenous labor shift under a dual economy proposed by Cai [Applied Mathematics Letters **21**, 774-779 (2008)] is generalized in this paper by introducing a time delay in the physical capital. By choosing the delay as a bifurcation parameter, it is proved that the delayed model has unique nonzero equilibrium and a Hopf bifurcation is proven to exist as the delay crosses a critical value. Moreover the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are investigated in this paper by applying the center manifold theorem and the normal form theory.

Keywords: Delay, Hopf bifurcation, dual economy, oscillations, center manifold theorem

1. Introduction

Recently an economic growth model with endogenous labor shift under a dual economy has been proposed by Cai [11]. Specifically the role of the industry (sector I) and of the agriculture (sector II) are taken into account and the economy is based on the assumption that the labor shift from the agricultural sector to the industrial sector is a necessary process for developing countries to realize industrialization. Accordingly, the mathematical model of Cai [11] reads:

$$\begin{cases} \dot{K} = sF(K, L_0 + M) - \delta K, \\ \dot{M} = G\left(w\left(K, L_0 + M\right) - w_0\right) - H(M), \end{cases} \tag{1}$$

where:

– K denotes the physical capital,
– F is a neoclassical production function,
– δ is the depreciation rate of capital stock,
– s is the saving rate,
– M represents the total labor shifted from the sector I to the sector II,
– L_0 is the initial labor in the agricultural sector,
– $w\left(K, L_0 + M\right)$ is the wage rate of sector I,
– w_0 is the survival wage rate of sector II.

From the mathematical viewpoint G and H are C^1 functions of their arguments. Moreover

– $G(0) = H(0) = 0$,
– $G'(\cdot) > 0$, $H'(\cdot) > 0$,
– $\lim_{M\to\infty} H(M) = \infty$.

The main result of paper [11] is the proof of the existence and uniqueness of an asymptotically stable equilibrium point of the mathematical model (1). Moreover Cai shows that the initial capital and labor in the industrial sector have a substantial effect on the capital growth of the industrial sector and the labor shift.

This paper is concerned with a generalization of the mathematical model (1). Specifically the production occurs with a delay while new capital is installed, namely at time t the productive capital stock happens to be $K(t-\tau) = K_d$. It is worth stressing that the idea of production taking time was firstly studied by Hayek [16], with an analysis based on Böhm-Bawerk's work [10], and later rigorously formulated by Kalecki [18]. Other works-related were done by Frisch and Holme [13], Kydland and Prescott [20], and Rustichini [23].

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Nowadays, the introduction of a lag has become a common approach in biology and economics, see, among others, the review papers [1,2], papers [3,9,12,14,15,19,21,24,25] and the references therein.

Assuming, for simplicity, the following Cobb-Douglas production function:

$$F(K_d, L_0 + M) = K_d^\alpha (L_0 + M)^{1-\alpha}, \quad \alpha \in (0,1),$$

we have that the economy is now described by the following system of two non-linear delayed differential equations:

$$\begin{cases} \dot{K} = sK_d^\alpha (L_0 + M)^{1-\alpha} - \delta K, \\ \dot{M} = G\left((1-\alpha)K_d^\alpha (L_0 + M)^{-\alpha} - w_0\right) - H(M). \end{cases} \tag{2}$$

It is known that the critical points of the system (2) correspond to those with vanishing delay. In particular it is easy to show (see Cai [11]) that there exists a unique non-trivial equilibrium (K_*, M_*) such that:

$$sK_*^{\alpha-1}(L_0 + M_*)^{1-\alpha} = \delta,$$
$$G\left((1-\alpha)K_*^\alpha (L_0 + M_*)^{-\alpha} - w_0\right) = H(M_*).$$

In this paper the asymptotic dynamics of the system (2) is studied in terms of local stability. Taking the delay as a bifurcation parameter, a Hopf bifurcation is proven to exist as the delay crosses a critical value. Additionally an explicit algorithm is established for determining the direction of Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions.

The present paper is organized as follows. Section 2 deals with the local stability result of the positive equilibrium and the occurrence of the Hopf bifurcation is also investigated. Section 3 is devoted to investigations on the direction of Hopf bifurcation and the stability of bifurcating periodic solutions by using normal form and center manifold method introduced by Hassard et al. [17]. Finally, conclusions are drawn in Section 4.

2. Asymptotic stability and existence of Hopf bifurcation

The first step for studying the local stability of the equilibrium (K_*, M_*) is to linearize the system (2) near (K_*, M_*). Therefore the linear system reads:

$$\begin{cases} \dot{K} = a(K - K_*) + b(M - M_*) + c(K_d - K_*), \\ \dot{M} = d(M - M_*) + n(K_d - K_*), \end{cases} \tag{3}$$

where a, b, c, d, n have the following expressions:

$$a = -\delta < 0, \quad b = (1-\alpha)\delta K_*(L_0 + M_*)^{-1} > 0,$$

$$d = -\frac{(1-\alpha)\alpha\delta K_*(L_0 + M_*)^{-2} G_M^*}{s} - H_M^* < 0,$$

$$n = \frac{(1-\alpha)\alpha\delta(L_0 + M_*)^{-1} G_{K_d}^*}{s} > 0, \quad c = \alpha\delta > 0,$$

where

$$G_v^* = G_v\left((1-\alpha)K_*^\alpha (L_0 + M_*)^{-\alpha} - w_0\right), \quad v \in \{K_d, M\},$$

and

$$H_M^* = H_M(M_*).$$

The characteristic equation of the system (3) is

$$\lambda^2 - (a+d)\lambda + ad + [c(d-\lambda) - bn]\, e^{-\lambda\tau} = 0. \tag{4}$$

It is well known that the steady state of the system (3) is asymptotically stable if all roots of Eq. (4) have negative real parts, and unstable if Eq. (4) has a root with positive real part. We shall now investigate the distribution of the roots of Eq. (4). First, we consider the case $\tau = 0$. In this case, the characteristic equation (4) reduces to

$$\lambda^2 - (a+c+d)\lambda + (a+c)d - bn = 0. \tag{5}$$

By applying the Routh-Hurwitz criterion, we obtain the following lemma.

Lemma 2.1. All roots of Eq. (5) have negative real parts if and only if $a+c+d < 0$ and $(a+c)d - bn > 0$.

Hereafter we assume that the conditions stated in the above Lemma hold. By Corollary 2.4 in Ruan and Wei paper [22], it follows that if instability occurs for a particular value of the delay $\tau > 0$, a characteristic root of (4) must intersect the imaginary axis. Suppose that (4) has a purely imaginary root $i\omega$, with $\omega > 0$. Then, by separating real and imaginary parts in (5), we have

$$\begin{cases} \omega^2 - ad = (cd - bn)\cos\omega\tau - c\omega\sin\omega\tau, \\ -(a+d)\omega = (cd - bn)\sin\omega\tau + c\omega\cos\omega\tau. \end{cases} \tag{6}$$

Therefore

$$\omega^4 + (a^2 - c^2 + d^2)\omega^2 + (ad)^2 - (cd - bn)^2 = 0. \tag{7}$$

Setting $p = a^2 - c^2 + d^2$, $q = (ad)^2 - (cd - bn)^2$ and $r = \omega^2$, Eq. (7) rewrites as

$$r^2 + pr + q = 0. \tag{8}$$

Lemma 2.2.

i) If $q < 0$, or $q = 0$ and $p < 0$, Eq. (8) has exactly one positive root.
ii) If $q > 0$, $p < 0$ and $p^2 > 4q$, Eq. (8) has two positive roots.
iii) If $p < 0$ and $p^2 = 4q$, Eq. (8) has only one positive root.
iv) If $q \geq 0$ and $p \geq 0$ or $p < 0$ and $p^2 < 4q$, Eq. (8) has no real roots or non-positive roots.

Without loss of generality, we can assume that Eq. (8) has two positive roots r_1 and r_2, with the possibility of

$r_1 = r_2$. Then Eq. (7) has only two positive roots $\omega_1 = \sqrt{r_1}$ and $\omega_2 = \sqrt{r_2}$. From (6), we can define

$$\tau_l = \frac{1}{\omega_l}\arcsin\left\{\frac{-c\omega_l^3 + [acd + (a+d)(n-c)b]\,\omega_l}{c^2\omega_l^2 + (cd-bn)^2}\right\},$$

where $l \in \{1, 2\}$. Setting

$$\tau_0 = \tau_{l_0} \equiv \min_{l\in\{1,2\}}\{\tau_l\} \text{ and } \omega_0 = \omega_{l_0},$$

then (τ_0, ω_0) solves Eqs. (6).

Lemma 2.3. Let $\tau = \tau_0$. Then Eq. (5) has a unique pair of simple purely imaginary roots $\pm i\omega_0$. Furthermore,

$$\left.\frac{d\,(Re\lambda)}{d\tau}\right|_{\tau=\tau_0} > 0 \quad \text{if } g(\omega_0^2) > 0,$$

where

$$g(\omega_0^2) = (a+d)^2\left[(cd-bn)^2 + 2(\omega_0^2 - ad)\right] + c^2\left[\omega_0^4 - (ad)^2\right].$$

Proof. Let $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ denote the root of (5) such that $\mu(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. Substituting $\lambda(\tau)$ into Eq. (5) and differentiating both sides of it with respect to τ yields:

$$\left\{2\lambda - [c(d-\lambda) - bn]\,\tau e^{-\lambda\tau} - (a+d) - ce^{-\lambda\tau}\right\}\frac{d\lambda}{d\tau} = [c(d-\lambda) - bn]\,\lambda e^{-\lambda\tau}. \quad (9)$$

From Eq. (9) and using Eq. (5), we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0} = \frac{a+d-2\lambda}{\lambda\left[\lambda^2 - (a+d)\lambda + ad\right]} - \frac{c}{\lambda\left[c(d-\lambda) - bn\right]} - \frac{\tau}{\lambda}.$$

Therefore

$$sign\left\{\left.\frac{d\,(Re\lambda)}{d\tau}\right|_{\tau=\tau_0}\right\} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0}\right\} = sign\left\{(a+d)^2\left[(cd-bn)^2 + 2(\omega_0^2 - ad)\right] + c^2\left[\omega_0^4 - (ad)^2\right]\right\},$$

and then the proof. □

Applying Lemmas 2.1 − 2.3, we can draw the following conclusions on the stability of the equilibrium point (K_*, M_*) of the system (2) and the existence of Hopf bifurcation at (K_*, M_*).

Theorem 2.4. If condition iv) in Lemma 2.2 is satisfied, then the equilibrium (K_*, M_*) is locally asymptotically stable for all values of $\tau \geq 0$. If either conditions i) or ii) or iii) of Lemma 2.2 hold and $g(\omega_0^2) \neq 0$, then there exists a critical value $\tau_0 > 0$ such that the equilibrium (K_*, M_*) is locally asymptotically stable when $\tau \in [0, \tau_0)$, and it is unstable when $\tau > \tau_0$. Moreover, a Hopf bifurcation occurs at (K_*, M_*) when $\tau = \tau_0$.

Remark 2.5. If $g(\omega_0^2) \neq 0$, then $[d\,(Re\lambda)\,/d\tau]_{\tau=\tau_0} > 0$. The proof is by contradiction. Assume $d\,(Re\lambda)\,/d\tau < 0$ for $\tau = \tau_0$ and τ close to τ_0. Then Eq. (5) has a characteristic root $\lambda(\tau) = \mu(\tau) + i\omega(\tau)$ with $\mu(\tau) > 0$, which contradicts the fact that all roots of Eq. (5) have negative real parts when $\tau < \tau_0$. Thus, when the delay τ near τ_0 is increased, the root of Eq. (5) crosses the imaginary axis from left to right.

Remark 2.6. If $g(\omega_0^2) = 0$, then the equilibrium (K_*, M_*) is locally asymptotically stable while $\tau \in [0, \tau_0)$. However, when $\tau = \tau_0$, we cannot conclude to stability or instability of the equilibrium. In this case, from Lemma 2.3, we have $[d\,(Re\lambda)\,/d\tau]_{\tau=\tau_0} = 0$.

3. Direction and stability of Hopf bifurcation

In this section, we investigate the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions based on the normal form approach theory and center manifold theory introduced by Hassard et al [17]. For notational convenience, let $\tau = \tau_0 + \mu$, $\mu \in \mathbb{R}$. Then $\mu = 0$ is the Hopf bifurcation value for the system (2). By the transformation $x = K - K_*$, $y = M - M_*$, $t = t/\tau$, our system is equivalent to the following functional differential equation system in $C = C([-1, 0], \mathbb{R}^2)$:

$$\dot{u} = L_\mu(u_t) + f(\mu, u_t), \quad (10)$$

where $u = (u_1, u_2)^T = (x, y)^T \in \mathbb{R}^2$, $u_t(\theta) = u(t+\theta) \in C$, and $L_\mu : C \to \mathbb{R}^2$, $f : \mathbb{R} \times C \to \mathbb{R}^2$ are defined, respectively, as follows:

$$L_\mu(\varphi) = (\tau_0 + \mu)\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\varphi(0) + (\tau_0 + \mu)\begin{bmatrix} c & 0 \\ n & 0 \end{bmatrix}\varphi(-1)$$

and

$$f(\mu, \varphi) = (\tau_0 + \mu)\begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix}.$$

Here, $\varphi = (\varphi_1, \varphi_2) \in C$, and the expansion of f near the equilibrium point is given by

$$\begin{aligned} f^{(1)} = {} & \frac{1}{2}\left[(\alpha-1)cK_*^{-1}\varphi_1(-1)^2 \right.\\ & + 2(1-\alpha)c(L_0 + M_*)^{-1}\varphi_1(-1)\varphi_2(0) \\ & \left. - \alpha b(L_0 + M_*)^{-1}\varphi_2(0)^2\right] \\ & + \frac{1}{3!}\left[(1-\alpha)(2-\alpha)cK_*^{-2}\varphi_1(-1)^3 \right.\\ & + 3(\alpha^2 - 1)(L_0 + M_*)^{-1}cK_*^{-1}\varphi_1(-1)^2\varphi_2(0) \\ & + 3\alpha(\alpha-1)c(L_0 + M_*)^{-2}\varphi_1(-1)\varphi_2(0)^2 \\ & \left. + \alpha(1+\alpha)b(L_0 + M_*)^{-2}\varphi_2(0)^3\right] \end{aligned}$$

and

$$f^{(2)} = \frac{1}{2}\left[P^*_{x_dx_d}\varphi_1(-1)^2 + 2P^*_{x_dy}\varphi_1(-1)\varphi_2(0) + P^*_{yy}\varphi_2(0)^2\right] + \frac{1}{3!}\left[P^*_{x_dx_dx_d}\varphi_1(-1)^3 + 3P^*_{x_dx_dy}\varphi_1(-1)^2\varphi_2(0) + 3P^*_{x_dyy}\varphi_1(-1)\varphi_2(0)^2 + P^*_{yyy}\varphi_2(0)^3\right].$$

We use the notation

$$P(x_d, y) \equiv G\left(w\left(x_d + K_*, L_0 + y + M_*\right) - w_0\right) - H(y + M_*),$$

with $P^*_{x_dx_d} \equiv P_{x_dx_d}(K_*, L_0 + M_*)$ and so on. By the Riesz representation theorem, there exists a matrix function whose components are functions $\eta(\theta)$ of the bounded variation in $\theta \in [-1, 0]$ such that:

$$L_\mu\varphi = \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta), \qquad \varphi \in C.$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_0 + \mu)\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} D(\theta) + (\tau_0 + \mu)\begin{bmatrix} c & 0 \\ n & 0 \end{bmatrix} D(\theta + 1),$$

where $D(\theta) = 0$ if $\theta \neq 0$, $D(\theta) = 1$ if $\theta = 0$. For $\varphi \in C([-1, 0], \mathbb{R}^2)$, we define

$$A(\mu)\varphi = \begin{cases} \dfrac{d\varphi}{d\theta}, & \theta \in [-1, 0), \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \varphi), & \theta = 0. \end{cases}$$

Then system (10) can be written as follows:

$$\dot{u} = A(\mu)u_t + R(\mu)u_t, \tag{11}$$

where $u_t = u(t + \theta)$, for $\theta \in [-1, 0]$.

For $\psi \in C([0, 1,], (\mathbb{R}^2)^*)$, we define the adjoint operator A^* of A as

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0, 1], \\ \displaystyle\int_{-1}^{0} d\eta(r, \mu)\psi(-r), & s = 0, \end{cases}$$

and the following bilinear inner product

$$< \psi(s), \varphi(\theta) > = \bar{\psi}(0)\varphi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$ and the bar means complex conjugate transpose of a vector. From the discussions in the previous section, we know that $\pm i\omega_0\tau_0$ are eigenvalues of A. Thus, they are also eigenvalues of A^*. Next, we calculate the eigenvector $q(\theta)$ of A belonging to $i\omega_0\tau_0$ and eigenvector $q^*(s)$ of A^* belonging to the eigenvalue $-i\omega_0\tau_0$. Then, it is not difficult to show that

$$q(\theta) = (1, \rho)^T e^{i\omega_0\tau_0\theta}, \qquad q^*(s) = B(1, \sigma)e^{i\omega_0\tau_0 s},$$

where

$$\rho = \frac{ne^{-i\omega_0\tau_0}}{i\omega_0 - d}, \qquad \sigma = -\frac{b}{i\omega_0 - d},$$

$$B = \frac{1}{1 + \bar{\rho}\sigma + \tau_0(n\sigma + c)e^{i\omega_0\tau_0}}.$$

Moreover, $< q^*(s), q(\theta) >= 1$ and $< q^*(s), \bar{q}(\theta) >= 0$. Next, we study the stability of bifurcated periodic solutions. By using the approach of Hassard et al. [17], we first compute the coordinates to describe the center manifold at $\mu = 0$. We define

$$z =< q^*, u_t > \text{ and } W(t, \theta) = u_t(\theta) - 2Re\left[zq(\theta)\right]. \tag{12}$$

On the center manifold, we have

$$W(t, \theta) = W(z, \bar{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{13}$$

In fact, z and $\bar{z}$ are local coordinates for the center manifold in the direction of q^* and $\bar{q}^*$. Note that W is real if u_t is real. We consider only real solutions. For the solution u_t, since $\mu = 0$, from (11) we have

$$\dot{z} = i\omega_0\tau_0 z + < \bar{q}^*(\theta), f\left(0, W(z, \bar{z}, \theta)2Re\left[zq(\theta)\right]\right) > = i\omega_0\tau_0 z + \bar{q}^*(0)f\left(0, W(z, \bar{z}, 0)2Re\left[zq(0)\right]\right). \tag{14}$$

We rewrite the above equation as

$$\dot{z} = i\omega_0\tau_0 z + g(z, \bar{z}), \tag{15}$$

where

$$g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) \tag{16}$$

$$= g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\bar{z}^2}{2} + g_{21}(\theta)\frac{z^2\bar{z}}{2} + \cdots. \tag{17}$$

From Eq. (11) and Eq. (14), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2Re\left\{\bar{q}^*(0)fq(\theta)\right\}, & \theta \in [-1, 0), \\ AW - 2Re\left\{\bar{q}^*(0)fq(0)\right\} + f, & \theta = 0, \end{cases} \tag{18}$$

Rewrite Eq. (18) as follows

$$\dot{W} = AW + \Omega(z, \bar{z}, \theta), \tag{19}$$

where

$$\Omega(z,\bar{z},\theta) = \Omega_{20}(\theta)\frac{z^2}{2} + \Omega_{11}(\theta)z\bar{z} + \Omega_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots . \quad (20)$$

Substituting the corresponding series into Eq. (19) and comparing the coefficients, we obtain

$$(A - 2i\omega_0\tau_0)\, W_{20}(\theta) = -\Omega_{20}(\theta),$$
$$AW_{11}(\theta) = -\Omega_{11}(\theta).$$

Noticing that $q(\theta) = (1,\rho)^T e^{i\omega_0\tau_0\theta}$, $q^*(0) = B(1,\sigma)$, from (12) we get

$$u_{1t}(0) = z + \bar{z} + W^{(1)}(t,0),$$
$$u_{2t}(0) = \rho z + \overline{\rho z} + W^{(2)}(t,0),$$
$$u_{1t}(-1) = e^{-i\omega_0\tau_0}z + e^{i\omega_0\tau_0}\bar{z} + W^{(1)}(t,0),$$
$$u_{2t}(-1) = e^{-i\omega_0\tau_0}\rho z + e^{i\omega_0\tau_0}\overline{\rho z} + W^{(2)}(t,0).$$

According to Eq. (14) and Eq. (15), we have

$$g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}).$$

By Eq. (13), expanding and comparing the coefficients with those in Eq. (16), it follows:

$$g_{20} = \tau_0\bar{B}\left\{\left[(\alpha-1)cK_*^{-1}e^{-2i\omega_0\tau_0} + 2\alpha(1-\alpha)\delta^2K_*(L_0+M_*)^{-1}\rho e^{-i\omega_0\tau_0} - \alpha b(L_0+M_*)^{-1}\rho^2\right] + \bar{\sigma}(P^*_{x_d}e^{-i\omega_0\tau_0} + P^*_y\rho)^{(2)}\right\}$$

$$g_{02} = \tau_0\bar{B}\left\{\left[(\alpha-1)cK_*^{-1}e^{2i\omega_0\tau_0} + 2\alpha(1-\alpha)\delta^2K_*(L_0+M_*)^{-1}\bar{\rho} e^{i\omega_0\tau_0} - \alpha b(L_0+M_*)^{-1}\bar{\rho}^2\right] + \bar{\sigma}(P^*_{x_d}e^{i\omega_0\tau_0} + P^*_y\bar{\rho})^{(2)}\right\}$$

$$g_{11} = \tau_0\bar{B}\left\{(\alpha^2-1)(L_0+M_*)^{-1}cK_*^{-1} + \bar{\sigma}P^*_{x_dx_d} + (e^{-i\omega_0\tau_0}\bar{\rho} + e^{i\omega_0\tau_0}\rho)\left[(1-\alpha)(L_0+M_*)^{-1}c + \bar{\sigma}P^*_{x_dy}\right] + |\rho|^2\left[-\alpha b(L_0+M_*)^{-1} + \bar{\sigma}P^*_{yy}\right]\right\}$$

$$\begin{aligned} g_{21} = {} & W^{(1)}_{20}(-1)\left\{e^{i\omega_0\tau_0}\left[(\alpha-1)cK_*^{-1} + \bar{\sigma}P^*_{x_dx_d}\right] + \frac{\bar{\rho}(1+\bar{\sigma})}{2}\right\} + W^{(1)}_{11}(-1) \\ & \times\left\{2e^{-i\omega_0\tau_0}\left[(\alpha-1)cK_*^{-1} + \bar{\sigma}P^*_{x_dx_d}\right] + 2\rho(1+\bar{\sigma})\right\} \\ & + W^{(2)}_{11}(0)\left\{2e^{-i\omega_0\tau_0}(1+\bar{\sigma}) + 2\rho\left[-\alpha b(L_0+M_*)^{-1} + \sigma P^*_{yy}\right]\right\} + W^{(2)}_{20}(0) \\ & \left\{e^{i\omega_0\tau_0}(1+\bar{\sigma}) + \bar{\rho}\left[-\alpha b(L_0+M_*)^{-1} + \bar{\sigma}P^*_{yy}\right]\right\} \\ & + \tau_0\bar{B}\left\{(e^{-2i\omega_0\tau_0}\bar{\rho} + 2\rho)\left[(\alpha^2-1)(L_0+M_*)^{-1}cK_*^{-1} + \bar{\sigma}P^*_{x_dx_dy}\right]\right. \\ & \left. + \left(2|\rho|^2e^{-i\omega_0\tau_0} + e^{i\omega_0\tau_0}\rho^2\right)\left[\alpha(\alpha-1)c(L_0+M_*)^{-2} + \bar{\sigma}P^*_{x_dyy}\right]\right\}, \end{aligned} \quad (21)$$

where

$$(P^*_{x_d}e^{-i\omega_0\tau_0} + P^*_y\rho)^{(2)} \equiv P^*_{x_dx_d}e^{-2i\omega_0\tau_0} + 2P^*_{x_dy}e^{-i\omega_0\tau_0}\rho + P^*_{yy}\rho^2.$$

It remains to compute $W_{20}(\theta)$ and $W_{11}(\theta)$ that appear in g_{21}. From Eq. (18) and Eq. (19), we have

$$\Omega(z,\bar{z},\theta) = -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -gq(\theta) - \overline{gq}(\theta).$$

Comparing the coefficients with Eq. (20) gives

$$\begin{cases} \Omega_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ \Omega_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{cases} \quad (22)$$

Hence, from Eqs. (22) it follows

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta),$$
$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta).$$

Solving with respect $W_{20}(\theta)$ and $W_{11}(\theta)$, we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_1e^{2i\omega_0\tau_0\theta},$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_2,$$

where $E_1, E_2 \in \mathbb{R}^2$ are constant vectors.

Following Hassard et al. [17], by using the found expressions for g_{20} and g_{11}, we can build the following two equations:

$$\begin{aligned} & \left[2i\omega_0\tau_0I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta}d\eta(\theta,0)\right]E_1 = M(2)E_1 \\ & = \left[2\alpha(1-\alpha)\delta^2K_*(L_0+M_*)^{-1}\rho e^{-i\omega_0\tau_0} + (\alpha-1)cK_*^{-1}e^{-2i\omega_0\tau_0} - \alpha b(L_0+M_*)^{-1}\rho^2\right] \\ & + \bar{\sigma}(P^*_{x_d}e^{-i\omega_0\tau_0} + P^*_y\rho)^{(2)} \end{aligned}$$

and

$$\begin{aligned} & \left[-\int_{-1}^{0} d\eta(\theta,0)\right]E_2 = M(0)E_2 \\ & = \left[2\alpha(1-\alpha)\delta^2K_*(L_0+M_*)^{-1}\bar{\rho} e^{i\omega_0\tau_0} + (\alpha-1)cK_*^{-1}e^{2i\omega_0\tau_0} - \alpha b(L_0+M_*)^{-1}\bar{\rho}^2\right] \\ & + \bar{\sigma}(P^*_{x_d}e^{i\omega_0\tau_0} + P^*_y\bar{\rho})^{(2)}, \end{aligned}$$

where

$$M(\zeta) = \begin{bmatrix} i\omega_0\zeta - a - ce^{-i\omega_0\tau_0\zeta} & -b \\ -ne^{-i\omega_0\tau_0\zeta} & i\omega_0\zeta - d \end{bmatrix}, \quad \zeta \in \{0, 2\}.$$

Solving this set of equations, we can derive E_1 and E_2, respectively. According to the above analysis, $W_{20}(0)$ and $W_{11}(0)$ are determined. Thus, all g_{ij} are known and we can compute the following quantities:

$$c_1(0) = \frac{i}{2\omega_0\tau_0}\left[g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right] + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{Re\,[c_1(0)]}{Re\,[\lambda'(\tau_0)]},$$

$$\beta_2 = 2Re\,[c_1(0)]\,,$$

$$\tau_2 = -\frac{Im\,[c_1(0)] + \mu_2 Im\,[\lambda'(\tau_0)]}{\omega_0\tau_0}.$$

Using the method of Hassard et al. [17], we obtain the following result.

Theorem 3.1.

1. The sign of μ_2 determines the direction of Hopf bifurcation: if $\mu_2 > 0$ $(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0$ $(\tau < \tau_0)$.
2. β_2 determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ $(\beta_2 > 0)$.
3. τ_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $\tau_2 > 0$ $(\tau_2 < 0)$.

4. Conclusion

In this paper we have investigated the local stability and Hopf bifurcation of the positive equilibrium of an endogenous labor shift model under a dual economy with time delay. Specifically, we have generalized the model analyzed in Cai [11] by assuming the growth of capital at time t to be a function of the amount of capital held at time $t-\tau$. Regarding the delay τ as a parameter and applying the local Hopf bifurcation theory, we have studied the existence of periodic oscillations for the delayed model.

In detail, we have showed that if some conditions are satisfied, then different scenarios arise. The positive equilibrium may be asymptotically stable for all $\tau \geq 0$, or as the delay τ increases, it can lose its stability, and a Hopf bifurcation occurs at the positive equilibrium, i.e. a family of periodic orbits bifurcates from it. Furthermore, by using the normal form theory and center manifold theorem, we have derived an explicit algorithm and sufficient conditions for the stability of the bifurcating periodic solutions.

It is worth stressing that further generalizations of the model proposed in the present paper can be performed by introducing a thermostat which can guarantee the conservation of some quantities in the system, see papers [5,8], and an internal structure that takes care of the ability of the interacting entities in the system, see papers [4,6]. Finally the analysis should be addressed to the comparison with empirical data following the suggestions proposed in the paper [7].

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