THE CALABI–YAU CONJECTURES FOR EMBEDDED SURFACES

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In this talk I will discuss the proof of the Calabi-Yau conjectures for embedded surfaces. This is joint work with Bill Minicozzi, [CM9].

The Calabi-Yau conjectures about surfaces date back to the 1960s. Much work has been done on them over the past four decades. In particular, examples of Jorge-Xavier from 1980 and Nadirashvili from 1996 showed that the immersed versions were false; we will show here that for embedded surfaces, i.e., injective immersions, they are in fact true.

Their original form was given in 1965 in [Ca] where E. Calabi made the following two conjectures about minimal surfaces (see also S.S. Chern, page 212 of [Ch]):

Conjecture 1. "Prove that a complete minimal hypersurface in $\mathbf{R}^n$ must be unbounded."

Calabi continued: "It is known that there are no compact minimal submanifolds of $\mathbf{R}^n$ (or of any simply connected complete Riemannian manifold with sectional curvature ≤ 0). A more ambitious conjecture is":

Conjecture 2. "A complete minimal hypersurface in $\mathbf{R}^n$ has an unbounded projection in every $(n-2)$–dimensional flat subspace."

These conjectures were revisited in S.T. Yau's 1982 problem list (see problem 91 in [Ya1]) by which time the Jorge-Xavier paper had appeared:

Question 3. "Is there any complete minimal surface in $\mathbf{R}^3$ which is a subset of the unit ball?

This was asked by Calabi, [Ca]. There is an example of a complete minimally immersed surface between two parallel planes due to L. Jorge and F. Xavier, [JXa2]. Calabi has also shown that such an example exists in $\mathbf{R}^4$. (One takes an algebraic curve in a compact complex surface covered by the ball and lifts it up.)"

The <u>immersed</u> versions of these conjectures turned out to be false. As mentioned above, Jorge and Xavier, [JXa2], constructed minimal immersions contained between two parallel planes in 1980, giving a counter-example to the immersed version of the more ambitious conjecture 2; see also [RoT]. Another significant development came in 1996, when N. Nadirashvili, [Na1], constructed a complete immersion of a minimal disk into the unit ball in $\mathbf{R}^3$, showing that Conjecture 1 also failed for immersed surfaces; see [MaMo1], [LMaMo1], [LMaMo2], for other topological types than disks.

The conjectures were again revisited in Yau's 2000 millenium lecture (see page 360 in [Ya2]) where Yau stated:

Question 4. "It is known [Na1] that there are complete minimal surfaces properly immersed into the [open] ball. What is the geometry of these surfaces? Can they be embedded?..."

We show in fact considerably more than Calabi's conjectures. This is in part because the conjectures are closely related to properness. Recall that an immersed surface in an open

subset Ω of Euclidean space $\mathbf{R}^3$ (where Ω is all of $\mathbf{R}^3$ unless stated otherwise) is *proper* if the pre-image of any compact subset of Ω is compact in the surface. A well-known question generalizing Calabi's first conjecture asks when is a complete <u>embedded</u> minimal surface proper? (See for instance question 4 in [MeP], or the "Properness Conjecture", conjecture 5, in [Me], or question 5 in [CM7].)

Our main result is an effective version of properness for disks, giving a chord arc bound. Obviously, intrinsic distances are larger than extrinsic distances, so the significance of a chord arc bound is the reverse inequality, i.e., a bound on intrinsic distances from above by extrinsic distances. This is accomplished in the next theorem:

Theorem 5. There exists a constant $C > 0$ so that if $\Sigma \subset \mathbf{R}^3$ is an embedded minimal disk, $\mathcal{B}_{2R} = \mathcal{B}_{2R}(0)$ is an intrinsic ball in $\Sigma \setminus \partial \Sigma$ of radius $2R$, and $\sup_{\mathcal{B}_{r_0}} |A|^2 > r_0^{-2}$ where $R > r_0$, then for $x \in \mathcal{B}_R$

$$C \operatorname{dist}_\Sigma(x, 0) < |x| + r_0. \tag{6}$$

The assumption of a lower curvature bound, $\sup_{\mathcal{B}_{r_0}} |A|^2 > r_0^{-2}$, in the theorem is a necessary normalization for a chord arc bound. This can easily be seen by rescaling and translating the helicoid.

Properness of a complete embedded minimal disk is an immediate consequence of Theorem 5. Namely, by (6), as intrinsic distances go to infinity, so do extrinsic distances. Precisely, if Σ is flat, and hence a plane, then obviously Σ is proper and if it is non-flat, then $\sup_{\mathcal{B}_{r_0}} |A|^2 > r_0^{-2}$ for some $r_0 > 0$ and hence Σ is proper by (6). In sum, we get the following corollary:

Corollary 7. A complete embedded minimal disk in $\mathbf{R}^3$ must be proper.

Corollary 7 in turn implies that the first of Calabi's conjectures is true for <u>embedded</u> minimal disks. In particular, Nadirashvili's examples cannot be embedded. We also get from it an answer to Yau's questions (Question 3 and Question 4).

Another immediate consequence of Theorem 5 together with the one-sided curvature estimate of [CM6] (i.e., theorem 0.2 in [CM6]) is the following version of that estimate for intrinsic balls; see question 3 in [CM7] where this was conjectured:

Corollary 8. There exists $\epsilon > 0$, so that if

$$\Sigma \subset \{x_3 > 0\} \subset \mathbf{R}^3 \tag{9}$$

is an embedded minimal disk with $\mathcal{B}_{2R}(x) \subset \Sigma \setminus \partial \Sigma$ and $|x| < \epsilon R$, then

$$\sup_{\mathcal{B}_R(x)} |A_\Sigma|^2 \leq R^{-2}. \tag{10}$$

As a corollary of this intrinsic one-sided curvature estimate we get that the second, and more ambitious, of Calabi's conjectures is also true for <u>embedded</u> minimal disks. In particular, Jorge-Xavier's examples cannot be embedded. Namely, letting $R \to \infty$ in Corollary 8 gives the following halfspace theorem:

Corollary 11. The plane is the only complete embedded minimal disk in $\mathbf{R}^3$ in a halfspace.

We will also see that our results for disks imply both of Calabi's conjectures and properness also for embedded surfaces with finite topology. Recall that a surface Σ is said to have finite topology if it is homeomorphic to a closed Riemann surface with a finite set of points removed or "punctures". Each puncture corresponds to an end of Σ.

The following generalization of the halfspace theorem gives Calabi's second, more ambitious, conjecture for embedded surfaces with finite topology:

Corollary 12. The plane is the only complete embedded minimal surface with finite topology in $\mathbf{R}^3$ in a halfspace.

Likewise, we get the properness of embedded surfaces with finite topology:

Corollary 13. A complete embedded minimal surface with finite topology in $\mathbf{R}^3$ must be proper.

One may ask if bounds on topology, as in the theorem and corollaries above, are needed to ensure properness. This seems to be the case. Indeed Meeks has sketched an approach to constructing non-proper minimal embeddings with infinite genus using N. Kapouleas' desingularization techniques.

It is clear from the definition of proper that a proper minimal surface in $\mathbf{R}^3$ must be unbounded, so the examples of Nadirashvili are not proper. Much less obvious is that the plane is the only complete proper immersed minimal surface in a halfspace. This is however a consequence of the strong halfspace theorem of D. Hoffman and W. Meeks, [HoMe], and implies that also the examples of Jorge-Xavier are not proper.

There has been extensive work on both properness (as in Corollary 7) and the halfspace property (as in Corollary 11) assuming various curvature bounds. Jorge and Xavier, [JXa1] and [JXa2], showed that there cannot exist a complete immersed minimal surface with bounded curvature in $\cap_i \{x_i > 0\}$; later Xavier proved that the plane is the only such surface in a halfspace, [Xa]. Recently, G.P. Bessa, Jorge and G. Oliveira-Filho, [BJO], and H. Rosenberg, [Ro], have shown that if such a surface is embedded, then it must be proper. This properness was extended to embedded minimal surfaces with locally bounded curvature and finite topology by Meeks and Rosenberg in [MeRo]; finite topology was subsequently replaced by finite genus in [MePRs] by Meeks, J. Perez and A. Ros.

Inspired by Nadirashvili's examples, F. Martin and S. Morales constructed in [MaMo2] a complete bounded minimal immersion which is proper in the (open) unit ball. That is, the preimages of compact subsets of the (open) unit ball are compact in the surface and the image of the surface accumulates on the boundary of the unit ball. They extended this in [MaMo3] to show that any strictly convex, possibly noncompact or nonsmooth, region of $\mathbf{R}^3$ admits a proper complete minimal immersion of the unit disk. In contrast, Nadirashvili had shown earlier in [Na2] that this was impossible in the unit cube.

Finally, we note that Calabi and P. Jones, [Jo], have constructed bounded complete holomorphic (and hence minimal) embeddings in higher codimension. Jones' example is a graph and he used purely analytic methods (including the Fefferman-Stein duality theorem between H^1 and BMO) while, as mentioned in Question 3, Calabi's approach was algebraic: Calabi considered the lift of an algebraic curve in a complex surface covered by the unit ball.

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