

THE CALABI-YAU EQUATION ON ALMOST-KÄHLER FOUR-MANIFOLDS

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Abstract

Let (M, ω) be a compact symplectic 4-manifold with a compatible almost complex structure J . The problem of finding a J -compatible symplectic form with prescribed volume form is an almost-Kähler analogue of Yau's theorem and is connected to a programme in symplectic topology proposed by Donaldson. We call the corresponding equation for the symplectic form the *Calabi-Yau equation*. Solutions are unique in their cohomology class. It is shown in this paper that a solution to this equation exists if the Nijenhuis tensor is small in a certain sense. Without this assumption, it is shown that the problem of existence can be reduced to obtaining a C^0 bound on a scalar potential function.

1. Introduction

In 1954 Calabi [Ca] conjectured that any representative of the first Chern class of a compact Kähler manifold (M, ω) can be written as the Ricci curvature of a Kähler metric ω' cohomologous to ω . He showed that any such metrics are unique. Yau [Ya] famously solved Calabi's conjecture around twenty years later. This result, and the immediate corollary that any Kähler manifold with $c_1(M) = 0$ admits a Ricci-flat metric, have had many applications in both mathematics and theoretical physics.

Yau's theorem is equivalent to finding a Kähler metric in a given Kähler class with prescribed volume form. By the $\partial\bar{\partial}$ -Lemma this amounts to solving the complex Monge-Ampère equation

$$(1.1) \quad (\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^F \omega^n,$$

for smooth real ϕ with $\omega + \sqrt{-1}\partial\bar{\partial}\phi > 0$, where $n = \dim_{\mathbb{C}} M$ and F is any smooth function with $\int_M e^F \omega^n = \int_M \omega^n$. Yau solved this equation by considering the family of equations obtained by replacing F by $tF + c_t$ for some constant c_t , for $t \in [0, 1]$ and using the continuity method. This

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requires an openness argument using the implicit function theorem, and, more importantly, a closedness argument which requires his celebrated *a priori* estimates. Yau also generalized Calabi's conjecture: first in the case when the right hand side of (1.1) may have poles or zeros [Ya]; and second, with Tian, in the context of complete non-compact Kähler Ricci-flat metrics [TiYa1, TiYa2]. For other results along these lines, see [Ko], [BaKo1], [BaKo2], [Jo], for example.

The aim of this paper is to attempt to generalize Yau's theorem in a very different direction. We consider the case when the almost complex structure is not integrable. This problem was suggested to the author by Donaldson and is motivated by a wider programme of his on the symplectic topology of 4-manifolds [Do]. Let (M, ω) be a symplectic four-manifold. Then there exists an almost complex structure J which is compatible with ω . This defines a metric g by

$$g(\cdot, \cdot) = \omega(\cdot, J\cdot) > 0.$$

If J is integrable then it is Kähler. In general, the data (M, ω, J) is called an *almost-Kähler manifold* and we will call ω an *almost-Kähler form*. The volume form version of Yau's theorem still makes sense. Given an almost-Kähler 4-manifold (M, ω, J) we ask whether there exists an almost-Kähler form ω' solving the equation

$$(1.2) \quad \omega'^2 = e^F \omega^2,$$

for any function F satisfying

$$(1.3) \quad \int_M e^F \omega^2 = \int_M \omega^2,$$

and we also ask whether ω' can be taken to be cohomologous to ω . We call (1.2) the *Calabi-Yau equation*. Any solution to (1.2) is unique in its cohomology class - this fact was pointed out to the author by Donaldson. A proof is given in section 2.

Following Yau, we use the continuity method to try to obtain the existence of a solution. First, we consider the question of *a priori* estimates for solutions to (1.2). For simplicity, assume that ω' is cohomologous to ω . We show that all the estimates can be reduced to a uniform bound of a scalar potential function ϕ_1 defined, up to a constant, by

$$(1.4) \quad \frac{\omega \wedge \omega'}{\omega'^2} = 1 - \frac{1}{4} \Delta' \phi_1,$$

where Δ' is the Laplacian associated to ω' . The function ϕ_1 belongs to a 1-parameter family of 'almost-Kähler potentials' $\{\phi_s\}_{s \in [0,1]}$, defined in section 2, which all coincide in the Kähler case with the usual Kähler potential.

Theorem 1. *Let (M, ω, J) be a compact almost-Kähler 4-manifold. Suppose that ω' is another almost-Kähler form, cohomologous to ω , and*

satisfying (1.2). Then there exist positive constants K_i depending only on (M, ω, J) , F and $\text{osc}_M \phi_1$ such that $\omega' \geq K_0^{-1} \omega$ and

$$\|\omega'\|_{C^i(g)} \leq K_i \quad \text{for } i = 0, 1, 2, \dots,$$

where $\text{osc}_M \phi_1 = \sup_M \phi_1 - \inf_M \phi_1$.

An analogous result holds even if ω' and ω are not necessarily cohomologous (see Section 7). We turn now to the question of openness in the continuity method. Denote by \mathcal{H}_ω^+ the space of self-dual harmonic 2-forms with respect to ω and by H_ω^+ the corresponding subspace in $H^2(M; \mathbb{R})$. H_ω^+ is a maximal positive subspace for the intersection form on $H^2(M; \mathbb{R})$ and its dimension is $b^+(M)$. Notice that ω is harmonic and self-dual and so $b^+(M) \geq 1$. If $b^+(M) = 1$ then we can show the openness part of the continuity method, remaining in the same cohomology class. In the case where $b^+(M) > 1$ the openness argument still works if we allow the class to vary within H_ω^+ .

Under the assumption that the Nijenhuis tensor $N(J)$ is small in the L^1 norm, the required uniform bound on ϕ_1 can be obtained. So in this case, we can solve equation (1.2).

Theorem 2. *Let (M, ω, J) be a compact almost Kähler 4-manifold.*

- (i) *Suppose $b^+(M) = 1$. Then for $F \in C^\infty(M)$ satisfying (1.3) there exists an almost-Kähler form ω' cohomologous to ω solving (1.2) if*

$$(1.5) \quad \|N(J)\|_{L^1(g)} < \epsilon,$$

for $\epsilon > 0$ depending only on g and $\|F\|_{C^2(g)}$.

- (ii) *If $b^+(M) > 1$ then the same holds except that the solution ω' may lie in a different cohomology class in H_ω^+ .*

With a little work, an explicit ϵ could be written down. However, it is hoped that the condition (1.5) could be removed entirely (cf. [Do]). In addition, it would be interesting to improve on Theorem 2 even further in light of a possible application in symplectic topology described to the author by Donaldson. Given an almost complex structure J_1 on a symplectic 4-manifold, a natural question is: does there exist a symplectic form compatible with J_1 ? In general the answer is negative, as can be seen from the well-known Kodaira-Thurston example [Th], [FeGoGr]. However, it is sensible to ask this question under the (obviously necessary) assumption that there exists some symplectic form Ω taming J_1 . In this case, there exists an almost complex structure J_0 compatible with Ω and, by a well-known result of Gromov [Gr], a smooth path of almost complex structures $\{J_t\}_{t \in [0,1]}$ all taming Ω . Set $\omega_0 = \Omega$ and consider the equation

$$\omega_t^2 = \Omega^2,$$

for ω_t compatible with J_t . Finding a solution for $t = 1$ would solve the problem. To prove this using a continuity method one would require estimates for ω_t depending only on Ω and J_t .

These methods appear to make sense only in four dimensions, since the system of equations is overdetermined in higher dimensions. Nevertheless, it should be noted that many of the estimates here carry over easily to any dimension.

The outline of the paper is as follows: in Section 2, some preliminaries are given, almost-Kähler potentials are defined and uniqueness for the Calabi-Yau equation is proved; in Section 3 an estimate on the metric g' in terms of the potential is given; a Hölder estimate on the metric is proved in Section 4; the higher order estimates and the proof of Theorem 1 are given in Section 5; finally, in Sections 6 and 7, Theorem 2 is proved in the cases $b^+(M) = 1$ and $b^+(M) > 1$ respectively.

Remark 1.1. Delanoë [De] considered, following a suggestion of Gromov, a different problem concerning the equation (1.2). He looked for solutions of $\omega'^n = e^F \omega^n$, on an almost-Kähler manifold (M, ω) of dimension $2n$, of the form $\omega' = \omega + d(Jd\phi)$ for a smooth real function ϕ so that ω' tames J but is not necessarily compatible with J (here, J acts on 1-forms in the usual way). He showed that when $n = 2$, if there exists such a solution for every F , then J is in fact integrable. We do not expect the solutions we obtain in Theorem 2 to be, in general, of the form $\omega' = \omega + d(Jd\phi)$ for any ϕ .

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2. Almost-Kähler geometry and the Calabi-Yau equation

Notation and preliminaries.

We will often work in local coordinates, making use of the Einstein summation convention. The almost complex structure $J = J_i^j dx^i \otimes \frac{\partial}{\partial x^j}$ satisfies, by definition, the condition

$$J_i^k J_k^j = -\delta_i^j.$$

We will lower indices in the usual way using the metric g so that

$$J_{ij} = J_i^k g_{kj} = \omega_{ij}.$$

The condition $d\omega = 0$ can be written as

$$(2.1) \quad \partial_i J_{jk} + \partial_j J_{ki} + \partial_k J_{ij} = 0.$$

It follows that the equation

$$(2.2) \quad \nabla_i J_j^i = 0$$

holds on an almost-Kähler manifold, where ∇ is the Levi-Civita connection associated to the metric g . This implies that ω is harmonic with respect to the metric g .

Define two tensors \mathcal{P} and \mathcal{Q} by

$$\begin{aligned} \mathcal{P}_{kl}^{ij} &= \frac{1}{2}(\delta_k^i \delta_l^j - J_k^i J_l^j) \\ \mathcal{Q}_{kl}^{ij} &= \frac{1}{2}(\delta_k^i \delta_l^j + J_k^i J_l^j). \end{aligned}$$

Then the compatibility of g and ω with J implies that $\mathcal{P}_{kl}^{ij} g_{ij} = 0$ and $\mathcal{P}_{kl}^{ij} J_{ij} = 0$. Considering \mathcal{P} and \mathcal{Q} as operators on two-tensors, we have

$$\mathcal{P} + \mathcal{Q} = \text{Id}.$$

Moreover, at each point, \mathcal{P} and \mathcal{Q} are self-adjoint with respect to g and define projections onto the spaces $\ker \mathcal{Q}$ and $\ker \mathcal{P}$ respectively.

The obstruction to the almost complex structure J being integrable is the Nijenhuis tensor $N : TM \times TM \rightarrow TM$, which is given by

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

In local coordinates, this can be written as

$$N_{jk}^i = J_k^l \partial_l J_j^i + J_l^i \partial_j J_k^l - J_j^l \partial_l J_k^i - J_l^i \partial_k J_j^l.$$

On an almost-Kähler manifold, the Nijenhuis tensor can be written in the simpler form

$$N_{jk}^i = 2(\nabla^i J_j^l) J_{kl}.$$

By the Newlander-Nirenberg theorem, the almost complex structure J is integrable if and only if N vanishes identically and if and only if ∇J vanishes identically.

For later use, we also make the following simple observation. Let Rm denote the Riemmanian curvature tensor of an almost-Kähler metric g . Then

$$\sup_M |\nabla J|^2 \leq C \|\text{Rm}\|_{C^0(g)},$$

for C a constant depending only on dimension. Indeed, using the usual commutation formulae for covariant derivatives along with (2.1) and (2.2),

$$0 = \Delta |J|^2 = 2|\nabla J|^2 + \text{Rm} * J * J,$$

where $*$ denotes some bilinear operation involving tensor products and the metric g . Similarly, by calculating $\Delta |\nabla J|^2$ we see that

$$|\nabla \nabla J|^2 = \nabla \nabla \text{Rm} * J * J + \text{Rm} * \nabla \nabla J * J + \nabla \text{Rm} * J * \nabla J + \text{Rm} * \nabla J * \nabla J,$$

and it follows that $\sup_M |\nabla \nabla J|^2$ can be bounded by a constant depending only on dimension and $\|\text{Rm}\|_{C^2(g)}$.

Almost-Kähler potentials.

Now restrict to four dimensions. Let ω and ω' be two almost-Kähler forms with $[\omega] = [\omega']$. For $s \in [0, 1]$, let $\Omega_s = (1 - s)\omega + s\omega'$ and define the *almost-Kähler potentials* ϕ_s by

$$(1 - 2s)\omega \wedge \omega' + s\omega'^2 - (1 - s)\omega^2 = -\frac{1}{2}\Omega_s \wedge d(Jd\phi_s).$$

Since $-\Omega \wedge d(Jd\phi) = \frac{1}{2}\Delta_\Omega \phi \Omega^2$ for any almost-Kähler form Ω and function ϕ , the existence of the ϕ_s follows from elementary Hodge theory. The ϕ_s are uniquely determined up to the addition of a constant. In the Kähler case, they all coincide with the usual Kähler potential ϕ given by $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\phi$. We are interested in three particular almost-Kähler potentials, corresponding to $s = 0, \frac{1}{2}, 1$. They satisfy:

$$(2.3) \quad \frac{1}{4}\Delta\phi_0 = \frac{\omega \wedge \omega'}{\omega^2} - 1$$

$$(2.4) \quad \frac{1}{4}\Delta'\phi_1 = 1 - \frac{\omega \wedge \omega'}{\omega'^2}$$

$$(2.5) \quad \frac{1}{2}(\Delta_{\frac{1}{2}}\phi_{\frac{1}{2}}) = \frac{\omega'^2 - \omega^2}{\Omega_{\frac{1}{2}}^2},$$

where Δ' and $\Delta_{\frac{1}{2}}$ are the Laplacians associated to ω' and $\Omega_{\frac{1}{2}}$.

In addition, for each s , define a one form a_s by the equations

$$\omega' = \omega - \frac{1}{2}d(Jd\phi_s) + da_s,$$

and $d_s^*a_s = 0$, where d_s^* is the formal adjoint of d associated to the metric Ω_s . Note that a_s is defined only up to the addition of a harmonic 1-form. A short calculation shows that a_s satisfies the elliptic system

$$(2.6) \quad \begin{cases} da_s \wedge \Omega_s &= 0 \\ \mathcal{P}da_s &= \frac{1}{4}(\partial_i J_j^k - \partial_j J_i^k)(\partial_k \phi_s) dx^i \wedge dx^j \\ d_s^*a_s &= 0. \end{cases}$$

It will be convenient to give a different formulation of (2.6). Let $*_s$ be the Hodge-star operator associated to Ω_s . Then the projection $\frac{1}{2}(1 + *_s) : \Lambda^2 \rightarrow \Lambda_s^+$ onto the self-dual two forms can be written

$$\frac{1}{2}(1 + *_s)(\chi) = \left(\frac{\Omega_s \wedge \chi}{\Omega_s^2} \right) \Omega_s + \mathcal{P}\chi, \quad \text{for } \chi \in \Lambda^2.$$

Hence, the system (2.6) can be rewritten as

$$(2.7) \quad \begin{cases} d_s^+ a_s &= \frac{1}{4}(\partial_i J_j^k - \partial_j J_i^k)(\partial_k \phi_s) dx^i \wedge dx^j \\ d_s^* a_s &= 0, \end{cases}$$

for $d_s^+ = \frac{1}{2}(1 + *_s)d : \Lambda^1 \rightarrow \Lambda_s^+$. Observe that, since $d_s^+b = 0$ implies $db = 0$ for any 1-form b (see for example [DoKr], Prop. 1.1.19), the kernel of the operator (d_s^+, d_s^*) consists of the harmonic 1-forms.

The Calabi-Yau equation.

For a manifold of dimension $n = 2m$, linearizing the Calabi-Yau equation gives $a \mapsto \omega^{m-1} \wedge da$ for a one-form a . Combining with the linear operator $a \mapsto \mathcal{P}da$ and imposing $d^*a = 0$ gives a system which is elliptic if $n = 4$ and overdetermined if $n > 4$.

Restricting to four-manifolds, solutions to (1.2) are unique in their cohomology class. To see this, let ω_1 and $\omega_2 = \omega_1 + db$ be cohomologous almost-Kähler forms with $\omega_1^2 = \omega_2^2$. Let $\Omega = \omega_1 + \omega_2$. Then $\Omega \wedge db = 0$ and $\mathcal{P}db = 0$ from which it follows that $d_\Omega^+b = 0$. Then $db = 0$ and so $\omega_1 = \omega_2$.

Finally we mention that the Calabi-Yau equation (1.2) can also be written as

$$(2.8) \quad \text{tr}_g g' = e^F \text{tr}_{g'} g,$$

where we are writing g' for the metric associated to ω' and where

$$\text{tr}_g g' = g^{ij} g'_{ij} = J^{ij} J'_{ij}, \quad \text{and} \quad \text{tr}_{g'} g = g'^{ij} g_{ij} = J'^{ij} J_{ij}.$$

We are using here the obvious notation for lowering and raising indices using the metric g' , so that $J'_{ij} = J_i^k g'_{kj}$ and $J'^{ij} = J_k^j g'^{ki}$.

3. Uniform estimate on the metric

In this section we will prove an estimate on the metric in terms of ϕ_1 , similar to the one proved by Yau [Ya] (see also [Au]) in the Kähler case. The computations here are somewhat more involved because of extra terms arising from the non-integrability of the almost complex structure.

Theorem 3.1. *There exist constants C and A depending only on $\|\text{Rm}(g)\|_{C^2(g)}$, $\sup_M |F|$ and the lower bound of ΔF such that*

$$\text{tr}_g g' \leq C e^{A(\phi_1 - \inf_M \phi_1)}.$$

Proof. We will calculate $\Delta'(\log(\text{tr}_g g') - A\phi_1)$, where A is a constant to be determined, and then apply the maximum principle. For this calculation we will work at a point using normal coordinates for g .

First, the equation (1.2) can be written

$$(3.1) \quad \log \det g' = 2F + \log \det g.$$

Applying the Laplace operator Δ of the metric g , we obtain

$$(3.2) \quad 2\Delta F = g^{ij} g'^{kl} \partial_i \partial_j g'_{kl} - g^{ij} g'^{kq} g'^{pl} (\partial_i g'_{pq})(\partial_j g'_{kl}) - g^{ij} g'^{kl} \partial_i \partial_j g_{kl}.$$

Now calculate

$$\Delta'(\text{tr}_g g') = g'^{kl} g'^{ij} \partial_k \partial_l g'_{ij} + g'^{kl} g'_{ij} \partial_k \partial_l g'^{ij} - g'^{kl} \Gamma_{kl}^p g'^{ij} \partial_p g'_{ij},$$

where we use Γ_{kl}^p to denote the Christoffel symbols corresponding to ∇' . We can rewrite this last term using the equation (valid at a point)

$$g'^{kl} \Gamma_{kl}^p = -J_q^p g'^{kl} \nabla_k J_l^q,$$

which follows from the equation $g'^{kl} \nabla'_k J_l^q = 0$. Thus we have

$$(3.3) \quad \Delta'(\text{tr}_g g') = g'^{kl} g'^{ij} \partial_k \partial_l g'_{ij} + g'^{kl} g'_{ij} \partial_k \partial_l g'^{ij} + J_q^p g'^{kl} (\nabla_k J_l^q) g'^{ij} \partial_p g'_{ij}.$$

The first terms on the right hand side of (3.2) and (3.3) are related as follows:

$$(3.4) \quad \begin{aligned} g'^{kl} g'^{ij} \partial_i \partial_j g'_{kl} &= g'^{kl} g'^{ij} \partial_k \partial_l g'_{ij} - J_r^k g'^{ij} \partial_i \partial_j J_k^r - 2J'_{jr} g'^{kl} g'^{ij} \partial_i \partial_l J_k^r \\ &\quad + J'_{rj} g'^{kl} g'^{ij} \partial_k \partial_l J_i^r - 2J_r^k g'^{ij} \partial_k \partial_j J_i^r \\ &\quad - 2g'^{kl} g'^{ij} (\partial_i J_k^r + \partial_k J_i^r) J_j^q \partial_r g'_{ql} \\ &\quad - 2g'^{ij} (\partial_i J_q^r + \partial_q J_i^r) \partial_r J_j^q \\ &\quad - 2g'^{kl} g'^{ij} (\partial_l J_k^r) J_j^s \partial_i g'_{rs}. \end{aligned}$$

To see this, calculate

$$(3.5) \quad \begin{aligned} g'^{kl} g'^{ij} \partial_i \partial_j g'_{kl} &= -g'^{kl} g'^{ij} \partial_i \partial_j (J_k^r J_{rl}') \\ &= -J_k^r g'^{kl} g'^{ij} \partial_i \partial_j J_{rl}' - J_{rl}' g'^{kl} g'^{ij} \partial_i \partial_j J_k^r \\ &\quad - 2g'^{kl} g'^{ij} (\partial_i J_k^r) (\partial_j J_{rl}'). \end{aligned}$$

We will now apply (2.1) to the first term on the right hand side of the above equation to obtain

$$\begin{aligned} -J_k^r g'^{kl} g'^{ij} \partial_i \partial_j J_{rl}' &= -J_k^r g'^{kl} g'^{ij} \partial_i (\partial_r J_{jl}' + \partial_l J_{rj}') \\ &= J'^{lr} g'^{ij} (\partial_i \partial_l J_{jr}' + \partial_i \partial_l J_{jr}') \\ &= 2J_k^r g'^{kl} g'^{ij} \partial_i \partial_l J_{jr}' \\ &= 2g'^{kl} g'^{ij} \partial_i \partial_l (J_k^r J_{jr}') - 2g'^{kl} g'^{ij} (\partial_i J_k^r) (\partial_l J_{jr}') \\ &\quad - 2g'^{kl} g'^{ij} (\partial_l J_k^r) (\partial_i J_{jr}') - 2J'_{jr} g'^{kl} g'^{ij} \partial_i \partial_l J_k^r. \end{aligned}$$

Then in (3.5) we have

$$(3.6) \quad \begin{aligned} g'^{kl} g'^{ij} \partial_i \partial_j g'_{kl} &= 2g'^{kl} g'^{ij} \partial_i \partial_l g'_{kj} - 2g'^{kl} g'^{ij} (\partial_i J_k^r) (\partial_l J_{jr}') \\ &\quad - 2g'^{kl} g'^{ij} (\partial_l J_k^r) (\partial_i J_{jr}') - 2J'_{jr} g'^{kl} g'^{ij} \partial_i \partial_l J_k^r \\ &\quad - J'_{rl} g'^{kl} g'^{ij} \partial_i \partial_j J_k^r - 2g'^{kl} g'^{ij} (\partial_i J_k^r) (\partial_j J_{rl}'). \end{aligned}$$

Similarly,

$$(3.7) \quad \begin{aligned} g'^{kl} g'^{ij} \partial_k \partial_l g'_{ij} &= 2g'^{ij} g'^{kl} \partial_k \partial_j g'_{il} - 2g'^{ij} g'^{kl} (\partial_k J_i^r) (\partial_j J_l^r) \\ &\quad - 2g'^{ij} g'^{kl} (\partial_j J_i^r) (\partial_k J_l^r) - 2J_{lr}' g'^{ij} g'^{kl} \partial_k \partial_j J_i^r \\ &\quad - J_{rj}' g'^{kl} g'^{ij} \partial_k \partial_l J_i^r - 2g'^{kl} g'^{ij} (\partial_k J_i^r) (\partial_l J_r^j). \end{aligned}$$

Combining (3.6) and (3.7) and again making use of (2.1) gives (3.4). From (3.2), (3.3) and (3.4) we obtain

$$(3.8) \quad \begin{aligned} \Delta'(\log \operatorname{tr}_g g') &= \frac{1}{\operatorname{tr}_g g'} \left(\Delta'(\operatorname{tr}_g g') - \frac{|\nabla' \operatorname{tr}_g g'|^2}{\operatorname{tr}_g g'} \right) \\ &= \frac{1}{\operatorname{tr}_g g'} \left(2\Delta F + g'^{kl} g'^{ij} g^{pq} (\nabla_p g'_{ik}) (\nabla_q g'_{jl}) \right. \\ &\quad + 2g'^{kl} g'^{ij} (\nabla_i J_k^r + \nabla_k J_i^r) J_j^q (\nabla_r g'_{ql}) \\ &\quad + 2g'^{ij} (\nabla_i J_q^r + \nabla_q J_i^r) (\nabla_r J_j^q) \\ &\quad + g'^{kl} (\nabla_k J_l^r) g'^{ij} (2J_j^s \nabla_i g'_{rs} + J_r^s \nabla_s g'_{ij}) \\ &\quad + J_r^k g'^{ij} \nabla_i \nabla_j J_k^r - 2g'^{ij} g'^{kl} g'_{qj} J_r^q \nabla_i \nabla_l J_k^r \\ &\quad - J_r^q g'_{qj} g'^{kl} g'^{ij} \nabla_k \nabla_l J_i^r - 2R \\ &\quad \left. + 2g'^{kl} g'_{ij} g^{pj} R_{lpk}^i - \frac{|\nabla' \operatorname{tr}_g g'|^2}{\operatorname{tr}_g g'} \right), \end{aligned}$$

where, by definition,

$$|\nabla' \operatorname{tr}_g g'|^2 = g'^{kl} (\partial_k \operatorname{tr}_g g') (\partial_l \operatorname{tr}_g g'),$$

and where we are making use of the equations

$$g'^{ij} g'^{kl} \partial_i \partial_j g_{kl} + 2J_r^k g'^{ij} \partial_k \partial_j J_i^r = -2R$$

and

$$\begin{aligned} &2J_{jr}' g'^{kl} g'^{ij} \partial_i \partial_l J_k^r + g'^{kl} g'_{ij} \partial_k \partial_l g'^{ij} \\ &= -2g'^{ij} g'^{kl} g'_{qj} J_r^q \nabla_i \nabla_l J_k^r + 2g'^{kl} g'_{ij} g^{pj} R_{lpk}^i, \end{aligned}$$

which both hold only at a point.

Notice now that (3.8) is an equation of tensors. Since we are going to apply the maximum principle we need to obtain a good lower bound for the right hand side of (3.8). We have to deal with the bad terms that involve derivatives of g' and are not nonnegative: namely, the third, fifth and last terms. First, we need a lemma.

Lemma 3.1. *Define tensors α_{ijp} and β_{ijp} by*

$$\begin{aligned}\alpha_{ijp} &= J'_{pl}(\nabla_i J_j^l - \nabla_j J_i^l) + (\nabla_p J_l^k)(g'_{kj} J_i^l - g'_{ki} J_j^l) \\ &\quad + g'_{kp}((\nabla_l J_j^k) J_i^l - (\nabla_l J_i^k) J_j^l), \\ \beta_{ijp} &= g'_{kl}((\nabla_j J_i^l) J_p^k - (\nabla_j J_p^l) J_i^k) - g'_{kj}((\nabla_p J_l^k) J_i^l - (\nabla_i J_l^k) J_p^l) \\ &\quad - (\nabla_l J_j^k)(g'_{kp} J_i^l - g'_{ki} J_p^l).\end{aligned}$$

Then

$$\begin{aligned}\text{(i)} \quad & 2\mathcal{P}_{ij}^{rs} \nabla_r g'_{sp} - 2\mathcal{P}_{ji}^{rs} \nabla_r g'_{sp} = \alpha_{ijp}, \\ \text{(ii)} \quad & 2\mathcal{Q}_{ij}^{rs} \nabla_r g'_{sp} - 2\mathcal{Q}_{pj}^{rs} \nabla_r g'_{si} = \beta_{ijp}.\end{aligned}$$

Notice that in the Kähler case the tensors α and β vanish identically and the lemma states that $\mathcal{P}_{ij}^{rs} \nabla_r g'_{sp}$ is symmetric in i and j while $\mathcal{Q}_{ij}^{rs} \nabla_r g'_{sp}$ is symmetric in i and p . It is important here that α and β do not contain derivatives of g' .

Proof. We prove (i). The proof of (ii) is similar. From

$$\nabla_i J'_{jk} + \nabla_j J'_{ki} + \nabla_k J'_{ij} = 0,$$

we have

$$J_j^q \nabla_i g'_{qk} + J_k^q \nabla_j g'_{qi} + J_i^q \nabla_k g'_{qj} + g'_{qk} \nabla_i J_j^q + g'_{qi} \nabla_j J_k^q + g'_{qj} \nabla_k J_i^q = 0.$$

Multiplying by J_p^j we obtain

$$\begin{aligned}\nabla_i g'_{pk} &= J_p^j J_k^q \nabla_j g'_{qi} + J_p^j J_i^q \nabla_k g'_{jq} + g'_{qk} J_p^j \nabla_i J_j^q \\ &\quad + g'_{qi} J_p^j \nabla_j J_k^q + g'_{qj} J_p^j \nabla_k J_i^q \\ &= -2\mathcal{P}_{pk}^{rs} \nabla_r g'_{si} + \nabla_p g'_{ki} + \nabla_k g'_{pi} - g'_{qj} J_i^q \nabla_k J_p^j \\ &\quad + g'_{qk} J_p^j \nabla_i J_j^q + g'_{qi} J_p^j \nabla_j J_k^q,\end{aligned}$$

where we have made use of the identity $\mathcal{P}_{pi}^{jq} g'_{jq} = 0$. Then (i) follows easily. q.e.d.

We return to equation (3.8). The third term on the right hand side (ignoring the factor of $1/\text{tr} g'$) can be written

$$\begin{aligned}\text{(3.9)} \quad & 2g'^{kl} g'^{ij} (\nabla_i J_k^r + \nabla_k J_i^r) J_j^q \nabla_r g'_{ql} \\ &= 2g'^{kl} g'^{ij} g'^{rs} (\nabla_i J_{kr} + \nabla_k J_{ir}) \nabla_s J'_j{}^l \\ &\quad - 2g'^{kl} g'^{ij} g'^{rs} (\nabla_i J_{kr} + \nabla_k J_{ir}) g'_{ql} \nabla_s J_j^q.\end{aligned}$$

Making use of the identity $\mathcal{Q}_{ir}^{ab}\nabla_k J_{ab} = 0$ we rewrite the first term on the right hand side of (3.9) as

$$\begin{aligned}
(3.10) \quad & 2g'^{kl}g'^{ij}g'^{rs}(\nabla_i J_{kr} + \nabla_k J_{ir})\nabla_s J'_{jl} \\
& = 4g'^{kl}g'^{ij}g'^{rs}(\nabla_k J_{ir})(\nabla_s J'_{jl}) + 2g'^{kl}g'^{ij}g'^{rs}(\nabla_r J_{ki})(\nabla_s J'_{jl}) \\
& = 4g'^{kl}g'^{ij}g'^{rs}\mathcal{P}_{ir}^{ab}(\nabla_k J_{ab})(\nabla_s J'_{jl}) \\
& \quad + 2g'^{kl}g'^{ij}g'^{rs}\mathcal{P}_{ki}^{ab}(\nabla_r J_{ab})(\nabla_s J'_{jl}) \\
& = (I) + (II),
\end{aligned}$$

where

$$(I) = 4g'^{kl}g'^{ij}g'^{rs}(\nabla_k J_{ir})\mathcal{P}_{js}^{ab}(\nabla_b J'_{al})$$

and

$$(II) = 2g'^{kl}g'^{ij}g'^{rs}(\nabla_r J_{ik})\mathcal{P}_{jl}^{ab}(\nabla_s J'_{ab}).$$

To deal with (I), first note that

$$\begin{aligned}
\mathcal{P}_{js}^{ab}\nabla_b J'_{al} & = \mathcal{P}_{sj}^{ab}\nabla_a J'_{bl} \\
& = -\mathcal{P}_{sj}^{ab}(\nabla_a g'_{bq})J_l^q - \mathcal{P}_{sj}^{ab}g'_{bq}\nabla_a J_l^q \\
& = -\mathcal{P}_{js}^{ab}(\nabla_a g'_{bq})J_l^q - \frac{1}{2}\alpha_{sjq}J_l^q - \mathcal{P}_{sj}^{ab}g'_{bq}\nabla_a J_l^q \\
& = \mathcal{P}_{sj}^{ab}\nabla_b J'_{al} + \mathcal{P}_{js}^{ab}g'_{bq}\nabla_a J_l^q - \frac{1}{2}\alpha_{sjq}J_l^q - \mathcal{P}_{sj}^{ab}g'_{bq}\nabla_a J_l^q.
\end{aligned}$$

That is, the term $\mathcal{P}_{js}^{ab}\nabla_b J'_{al}$ is symmetric in j and s modulo terms that don't involve derivatives of g' . Then

$$\begin{aligned}
(I) & = -4g'^{kl}g'^{ij}g'^{rs}(\nabla_k J_{ri})\mathcal{P}_{sj}^{ab}\nabla_b J'_{al} \\
& \quad + 4g'^{kl}g'^{ij}g'^{rs}(\nabla_k J_{ir})(\mathcal{P}_{js}^{ab}g'_{bq}\nabla_a J_l^q - \frac{1}{2}\alpha_{sjq}J_l^q - \mathcal{P}_{sj}^{ab}g'_{bq}\nabla_a J_l^q),
\end{aligned}$$

and hence, interchanging the indices i and j with r and s in the first term, we have

$$\begin{aligned}
(3.11) \quad (I) & = 2g'^{kl}g'^{ij}g'^{rs}(\nabla_k J_{ir})(\mathcal{P}_{js}^{ab}g'_{bq}\nabla_a J_l^q \\
& \quad - \frac{1}{2}\alpha_{sjq}J_l^q - \mathcal{P}_{sj}^{ab}g'_{bq}\nabla_a J_l^q).
\end{aligned}$$

For (II), calculate

$$\begin{aligned}
(3.12) \quad \mathcal{P}_{jl}^{ab}\nabla_s J'_{ab} & = \frac{1}{2}\left(\nabla_s J'_{jl} - J_j^a J_l^b \nabla_s J'_{ab}\right) \\
& = \frac{1}{2}\left(\nabla_s J'_{jl} - \nabla_s J'_{jl} + J'_{ab} J_l^b \nabla_s J_j^a + J'_{ab} J_j^a \nabla_s J_l^b\right) \\
& = \frac{1}{2}J'_{ab}\left(J_l^b \nabla_s J_j^a + J_j^a \nabla_s J_l^b\right).
\end{aligned}$$

Then from equations (3.9), (3.10), (3.11) and (3.12) we have the following expression for the third term of (3.8):

$$\begin{aligned}
(3.13) \quad & 2g'^{kl}g'^{ij}(\nabla_i J_k^r + \nabla_k J_i^r)J_j^q \nabla_r g'_{ql} \\
& = 2g'^{kl}g'^{ij}g'^{rs}(\nabla_k J_{ir})(\mathcal{P}_{js}^{ab}g'_{bq} \nabla_a J_l^q - \frac{1}{2}\alpha_{sjq}J_l^q - \mathcal{P}_{sj}^{ab}g'_{bq} \nabla_a J_l^q) \\
& \quad + g'^{kl}g'^{ij}g'^{rs}(\nabla_r J_{ik})J'_{ab}(J_l^b \nabla_s J_j^a + J_j^a \nabla_s J_l^b) \\
& \quad - 2g'^{kl}g'^{ij}g'^{rs}(\nabla_i J_{kr} + \nabla_k J_{ir})g'_{ql} \nabla_s J_j^q.
\end{aligned}$$

We deal now with the fifth term on the right hand side of (3.8). Calculate:

$$\begin{aligned}
(3.14) \quad & g'^{kl}(\nabla_k J_l^r)g'^{ij}(2J_j^s \nabla_i g'_{rs} + J_r^s \nabla_s g'_{ij}) \\
& = g'^{kl}(\nabla_k J_l^r)g'^{ij}(2\nabla_i J'_{jr} - J_r^s \nabla_s (J'_{pj} J_i^p)) \\
& = g'^{kl}(\nabla_k J_l^r)g'^{ij}(2\nabla_i J'_{jr} - J_r^s J_i^p \nabla_p J'_{sj} \\
& \quad - J_r^s J_i^p \nabla_j J'_{ps} - J_r^s J'_{pj} \nabla_s J_i^p) \\
& = g'^{kl}(\nabla_k J_l^r)g'^{ij}(2\nabla_i J'_{jr} + J_i^a J_r^b \nabla_a J'_{jb} - \nabla_j J'_{ir} \\
& \quad + J_i^p J'_{ps} \nabla_j J_r^s - J_r^s J'_{pj} \nabla_s J_i^p) \\
& = g'^{kl}(\nabla_k J_l^r)g'^{ij}(2\mathcal{Q}_{ir}^{ab} \nabla_a J'_{jb} + J_i^p J'_{ps} \nabla_j J_r^s - J_r^s J'_{pj} \nabla_s J_i^p) \\
& = g'^{kl}(\nabla_k J_l^r)g'^{ij}(2\mathcal{Q}_{ir}^{ab}(\nabla_a g'_{bq})J_j^q + 2\mathcal{Q}_{ir}^{ab}g'_{bq} \nabla_a J_j^q - g'_{is} \nabla_j J_r^s \\
& \quad - J_r^s J'_{pj} \nabla_s J_i^p) \\
& = g'^{kl}(\nabla_k J_l^r)(\frac{1}{2}\beta_{irq}J^{iq} + 2g'^{ij}\mathcal{Q}_{ir}^{ab}g'_{bq} \nabla_a J_j^q - g'^{ij}g'_{is} \nabla_j J_r^s \\
& \quad - g'^{ij}J_r^s J'_{pj} \nabla_s J_i^p),
\end{aligned}$$

where, for the last line, we have used the identity

$$2\mathcal{Q}_{ir}^{ab}(\nabla_a g'_{bq})J^{iq} = \frac{1}{2}\beta_{irq}J^{iq},$$

which follows immediately from Lemma 3.1.

Finally, we must deal with the last term on the right hand side of (3.8). We do this by making use of the good second term.

Lemma 3.2. *There exists a constant C' depending only on $\sup_M |F|$ and $\|\text{Rm}(g)\|_{C^0(g)}$ such that*

$$\frac{|\nabla'(\text{tr}_g g')|^2}{\text{tr}_g g'} \leq g'^{kl}g'^{ij}g'^{pq}(\nabla_p g'_{ik})(\nabla_q g'_{jl}) + C'(\text{tr}_g g')(\text{tr}_g g').$$

To prove this lemma we will work in a coordinate system (x^1, \dots, x^4) centred at a point p such that the first derivatives of the metric g vanish at p and

$$J_1^3 = J_2^4 = 1 = -J_3^1 = -J_4^2,$$

and all other entries of the matrix (J_i^j) are zero at p . Define local vector fields

$$Z_\alpha = \frac{1}{2} (\partial_\alpha - \sqrt{-1} J_\alpha^i \partial_i),$$

for $\alpha = 1, 2$. Set

$$G_{\alpha\bar{\beta}} = g(Z_\alpha, \bar{Z}_\beta)$$

and

$$G'_{\alpha\bar{\beta}} = g'(Z_\alpha, \bar{Z}_\beta),$$

for $\alpha, \beta = 1, 2$. Then we make a linear change in the coordinates (x^1, \dots, x^4) so that, in addition to the above conditions at p , we also impose that $G_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$ and that $G'_{\alpha\bar{\beta}}$ be diagonal. Notice that in these coordinates, the first derivatives of the g_{ij} vanish at p , but in general, the first derivatives of the $G_{\alpha\bar{\beta}}$ will not.

It will also be useful to consider the local vector fields

$$W_\alpha = \frac{1}{2} (-J_A^i \partial_i - \sqrt{-1} \partial_A),$$

for $\alpha = 1, 2$, where we are setting $A = \alpha + 2$. Notice that, at p , $W_\alpha = Z_\alpha$. In the sequel, we will use the indices A, B, C, D, M, N to denote $\alpha + 2, \beta + 2, \gamma + 2, \delta + 2, \mu + 2, \nu + 2$. Set

$$\tilde{G}_{\alpha\bar{\beta}} = g(W_\alpha, \bar{W}_\beta)$$

and

$$\tilde{G}'_{\alpha\bar{\beta}} = g'(W_\alpha, \bar{W}_\beta).$$

Observe that

$$G_{\alpha\bar{\beta}} = \frac{1}{2} (g_{\alpha\beta} - \sqrt{-1} J_{\alpha\beta}), \quad G'_{\alpha\bar{\beta}} = \frac{1}{2} (g'_{\alpha\beta} - \sqrt{-1} J'_{\alpha\beta}),$$

and

$$\tilde{G}_{\alpha\bar{\beta}} = \frac{1}{2} (g_{AB} - \sqrt{-1} J_{AB}), \quad \tilde{G}'_{\alpha\bar{\beta}} = \frac{1}{2} (g'_{AB} - \sqrt{-1} J'_{AB}).$$

At the point p , $G = \tilde{G}$ and $G' = \tilde{G}'$; this fact will be used later in the proof of Lemma 3.2. Notice also that $g_{ij} = 2\delta_{ij}$ and that g'_{ij} is diagonal. A final word about notation: when we are using the local vector fields Z_α and \bar{Z}_α as differential operators, we will instead write D_α and $D_{\bar{\alpha}}$ respectively. We require some preliminary results before we prove Lemma 3.2.

Lemma 3.3.

$$(3.15) \quad D_\gamma G'_{\alpha\bar{\beta}} = D_\alpha G'_{\gamma\bar{\beta}} + \frac{1}{4} (a_{\gamma\alpha\beta} + \sqrt{-1}b_{\gamma\alpha\beta})$$

$$(3.16) \quad \text{and } D_\gamma \tilde{G}'_{\alpha\bar{\beta}} = D_\alpha \tilde{G}'_{\gamma\bar{\beta}} + \frac{1}{4} (a_{\gamma AB} + \sqrt{-1}b_{\gamma AB}),$$

where

$$a_{\gamma ij} = (\partial_i J_\gamma^k - \partial_\gamma J_i^k) J'_{kj} + (\partial_j J_\gamma^k) J'_{ik} - (\partial_j J_i^k) J'_{\gamma k}$$

$$\text{and } b_{\gamma ij} = g'_{ki} \partial_j J_\gamma^k - g'_{k\gamma} \partial_j J_i^k + J'_{jl} (J_i^k \partial_k J_\gamma^l - J_\gamma^k \partial_k J_i^l).$$

Proof. We will just prove (3.15), since the proof of (3.16) is similar. Calculate

$$(3.17) \quad \begin{aligned} D_\gamma G'_{\alpha\bar{\beta}} &= \frac{1}{4} (\partial_\gamma - \sqrt{-1} J_\gamma^i \partial_i) (g'_{\alpha\beta} - \sqrt{-1} J'_\alpha) \\ &= \frac{1}{4} (\partial_\gamma g'_{\alpha\beta} - J_\gamma^i \partial_i J'_\alpha - \sqrt{-1} J_\gamma^i \partial_i g'_{\alpha\beta} - \sqrt{-1} \partial_\gamma J'_\alpha) \\ &= \frac{1}{4} (\partial_\gamma g'_{\alpha\beta} - J_\gamma^i \partial_\alpha J'_{i\beta} - J_\gamma^i \partial_\beta J'_{\alpha i} - \sqrt{-1} J_\gamma^i \partial_i g'_{\alpha\beta} \\ &\quad - \sqrt{-1} \partial_\alpha J'_{\gamma\beta} - \sqrt{-1} \partial_\beta J'_{\alpha\gamma}) \\ &= \frac{1}{4} (\partial_\gamma g'_{\alpha\beta} + \partial_\alpha g'_{\gamma\beta} + (\partial_\alpha J_\gamma^i) J'_{i\beta} - J_\alpha^i \partial_\beta J'_{\gamma i} - (\partial_\beta J_\alpha^i) J'_{\gamma i} \\ &\quad + (\partial_\beta J_\gamma^i) J'_{\alpha i} - \sqrt{-1} J_\gamma^i \partial_i (J'_{\beta p} J_\alpha^p) - \sqrt{-1} \partial_\alpha J'_{\gamma\beta} \\ &\quad - \sqrt{-1} \partial_\beta J'_{\alpha\gamma}) \\ &= \frac{1}{4} (\partial_\gamma g'_{\alpha\beta} + \partial_\alpha g'_{\gamma\beta} + (\partial_\alpha J_\gamma^i) J'_{i\beta} - J_\alpha^i \partial_i J'_{\gamma\beta} - J_\alpha^i \partial_\gamma J'_{\beta i} \\ &\quad - (\partial_\beta J_\alpha^i) J'_{\gamma i} + (\partial_\beta J_\gamma^i) J'_{\alpha i} - \sqrt{-1} J_\gamma^i J_\alpha^p \partial_i J'_{\beta p} \\ &\quad - \sqrt{-1} J_\gamma^i J'_{\beta p} \partial_i J_\alpha^p - \sqrt{-1} \partial_\alpha J'_{\gamma\beta} - \sqrt{-1} \partial_\beta J'_{\alpha\gamma}), \end{aligned}$$

where to go from the third to the fourth lines we have used the simple identity

$$J_\gamma^i \partial_\beta J'_{\alpha i} + (\partial_\beta J_\gamma^i) J'_{\alpha i} = (\partial_\beta J_\alpha^i) J'_{\gamma i} + J_\alpha^i \partial_\beta J'_{\gamma i}.$$

Notice that

$$\begin{aligned} J_\gamma^i J_\alpha^p \partial_i J'_{\beta p} &= J_\gamma^i J_\alpha^p \partial_\beta J'_{ip} + J_\gamma^i J_\alpha^p \partial_p J'_{\beta i} \\ &= \partial_\beta J'_{\gamma\alpha} - J'_{ip} J_\alpha^p \partial_\beta J_\gamma^i - J'_{ip} J_\gamma^i \partial_\beta J_\alpha^p + J_\alpha^i J_\gamma^p \partial_i J'_{\beta p}. \end{aligned}$$

In (3.17) this gives us

$$\begin{aligned}
& D_\gamma G'_{\alpha\bar{\beta}} \\
&= \frac{1}{4}(\partial_\gamma g'_{\alpha\beta} + \partial_\alpha g'_{\gamma\beta} + (\partial_\alpha J_\gamma^i)J'_{i\beta} - J_\alpha^i \partial_i J'_{\gamma\beta} - J_\alpha^i \partial_\gamma J'_{\beta i} \\
&\quad - (\partial_\beta J_\alpha^i)J'_{\gamma i} + (\partial_\beta J_\gamma^i)J'_{\alpha i} + \sqrt{-1}J'_{i\bar{p}}J_\alpha^p \partial_\beta J_\gamma^i + \sqrt{-1}J'_{i\bar{p}}J_\gamma^i \partial_\beta J_\alpha^p \\
&\quad - \sqrt{-1}J_\alpha^i J_\gamma^p \partial_i J'_{\beta p} - \sqrt{-1}J_\gamma^i J'_{\beta p} \partial_i J_\alpha^p - \sqrt{-1}\partial_\alpha J'_{\gamma\beta}) \\
&= \frac{1}{4}(\partial_\alpha g'_{\gamma\beta} + (\partial_\alpha J_\gamma^i)J'_{i\beta} - J_\alpha^i \partial_i J'_{\gamma\beta} + (\partial_\gamma J_\alpha^i)J'_{\beta i} - (\partial_\beta J_\alpha^i)J'_{\gamma i} \\
&\quad + (\partial_\beta J_\gamma^i)J'_{\alpha i} + \sqrt{-1}g'_{i\alpha} \partial_\beta J_\gamma^i - \sqrt{-1}g'_{p\gamma} \partial_\beta J_\alpha^p - \sqrt{-1}J_\alpha^i \partial_i g'_{\beta\gamma} \\
&\quad + \sqrt{-1}J_\alpha^i J'_{\beta p} (\partial_i J_\gamma^p) - \sqrt{-1}J_\gamma^i J'_{\beta p} (\partial_i J_\alpha^p) - \sqrt{-1}\partial_\alpha J'_{\gamma\beta}) \\
&= D_\alpha G'_{\gamma\bar{\beta}} + \frac{1}{4}((\partial_\alpha J_\gamma^i - \partial_\gamma J_\alpha^i)J'_{i\beta} - (\partial_\beta J_\alpha^i)J'_{\gamma i} + (\partial_\beta J_\gamma^i)J'_{\alpha i} \\
&\quad + \sqrt{-1}(g'_{i\alpha} \partial_\beta J_\gamma^i - g'_{i\gamma} \partial_\beta J_\alpha^i) + \sqrt{-1}J'_{\beta p}(J_\alpha^i \partial_i J_\gamma^p - J_\gamma^i \partial_i J_\alpha^p)),
\end{aligned}$$

as required.

q.e.d.

We will also need the following result.

Lemma 3.4. *At the point p we have*

$$(3.18) \quad 2D_\gamma(G^{\alpha\bar{\beta}}G'_{\alpha\bar{\beta}}) = 2G^{\alpha\bar{\beta}}D_\gamma G'_{\alpha\bar{\beta}} = D_\gamma(g^{ij}g'_{ij}) + E_\gamma \quad \text{and}$$

$$(3.19) \quad 2D_\gamma(\tilde{G}^{\alpha\bar{\beta}}\tilde{G}'_{\alpha\bar{\beta}}) = 2\tilde{G}^{\alpha\bar{\beta}}D_\gamma\tilde{G}'_{\alpha\bar{\beta}} = D_\gamma(g^{ij}g'_{ij}) - E_\gamma,$$

$$\text{where } E_\gamma = \sum_{i=1}^2 g'_{rs}(D_\gamma J_i^r)J_i^s.$$

We are using here the usual notation for the inverse metrics G^{-1} and \tilde{G}^{-1} . Note also that repeated greek indices α, β, \dots are used to denote a sum from 1 to 2, whereas repeated lower case roman letters i, j, \dots denote the usual sum from 1 to 4, unless otherwise indicated.

Proof. We will prove just (3.18). Calculate

$$\begin{aligned}
(3.20) \quad 2D_\gamma(G^{\alpha\bar{\beta}}G'_{\alpha\bar{\beta}}) &= -\frac{1}{2}G^{\alpha\bar{\mu}}G^{\nu\bar{\beta}}(-J_\gamma^i \partial_i J_{\nu\mu} - \sqrt{-1}\partial_\gamma J_{\nu\mu})G'_{\alpha\bar{\beta}} \\
&\quad + 2G^{\alpha\bar{\beta}}D_\gamma G'_{\alpha\bar{\beta}} \\
&= 2G^{\alpha\bar{\beta}}D_\gamma G'_{\alpha\bar{\beta}} \\
&= \frac{1}{2}G^{\alpha\bar{\beta}}(\partial_\gamma g'_{\alpha\beta} - \sqrt{-1}J_\gamma^k \partial_k g'_{\alpha\beta}).
\end{aligned}$$

But

$$\begin{aligned}
(3.21) \quad D_\gamma(g^{ij}g'_{ij}) &= \frac{1}{2}g^{ij}(\partial_\gamma - \sqrt{-1}J_\gamma^k\partial_k)g'_{ij} \\
&= \frac{1}{4}\sum_{i=1}^4\left(\partial_\gamma g'_{ii} - \sqrt{-1}J_\gamma^k\partial_k g'_{ii}\right) \\
&= \frac{1}{2}\sum_{i=1}^2\left(\partial_\gamma g'_{ii} - g'_{rs}(\partial_\gamma J_i^r)J_i^s - \sqrt{-1}J_\gamma^k\partial_k g'_{ii}\right. \\
&\quad \left. + \sqrt{-1}J_\gamma^k g'_{rs}(\partial_k J_i^r)J_i^s\right),
\end{aligned}$$

where we have used the fact that, at p ,

$$\partial_k g'_{AB} - \partial_k g'_{\alpha\beta} = -g'_{rs}(\partial_k J_\alpha^r)J_\beta^s - g'_{rs}(\partial_k J_\beta^s)J_\alpha^r.$$

Comparing (3.20) and (3.21) gives (3.18).

q.e.d.

We need one more lemma before we can prove Lemma 3.2.

Lemma 3.5. *At p ,*

$$g^{ij}a_{\gamma ij} = 0 \quad \text{and} \quad g^{ij}b_{\gamma ij} = 0.$$

Proof. For the first equation, calculate at p :

$$\begin{aligned}
g^{ij}a_{\gamma ij} &= g^{ij}(\nabla_i J_\gamma^k)J'_{kj} - g^{ij}(\nabla_\gamma J_i^k)J'_{kj} \\
&\quad + g^{ij}(\nabla_j J_\gamma^k)J'_{ik} - g^{ij}(\nabla_j J_i^k)J'_{\gamma k} \\
&= -g^{ij}(\nabla_\gamma J_i^k)J'_{kj} \\
&= -g^{ij}(\nabla_\gamma J_i^k)J_k^s g'_{sj} \\
&= g^{ij}J_i^k(\nabla_\gamma J_k^s)g'_{sj} \\
&= -g^{ik}J_i^j(\nabla_\gamma J_k^s)g'_{sj} \\
&= 0,
\end{aligned}$$

by symmetry. For the second, calculate:

$$\begin{aligned}
g^{ij}b_{\gamma ij} &= g^{ij}g'_{ki}(\nabla_j J_\gamma^k) - g^{ij}g'_{k\gamma}(\nabla_j J_i^k) \\
&\quad + g^{ij}J'_{jl}J_i^k(\nabla_k J_\gamma^l) - g^{ij}J'_{jl}J_\gamma^k(\nabla_k J_i^l) \\
&= g^{ij}J'_{kl}J_i^l(\nabla_j J_\gamma^k) - g^{ik}J'_{jl}J_i^j(\nabla_k J_\gamma^l) \\
&\quad - g^{ij}J'_{jl}J_\gamma^k(\nabla_i J_k^l) - g^{ij}J'_{jl}J_\gamma^k(\nabla^l J_{ik}) \\
&= 2g^{ij}J'_{kl}J_i^l(\nabla_j J_\gamma^k) - 2J_\gamma^k J'_{jl}(\nabla^j J_k^l).
\end{aligned}$$

Notice that the second term vanishes since

$$J'_{jl}\nabla^j J_k^l = \frac{1}{2}g'_{lp}J_j^p\nabla_k J^{jl} = 0.$$

But also the first term can be written

$$g^{ij} J'_{kl} J_i{}^l (\nabla_j J_\gamma{}^k) = g^{ij} J_k{}^p g'_{pl} J_i{}^l (\nabla_j J_\gamma{}^k) = -J'_{ip} J_\gamma{}^k (\nabla^i J_k{}^p) = 0,$$

finishing the proof of the lemma.

q.e.d.

Proof of Lemma 3.2.

Using Lemma 3.4 we have

$$\begin{aligned} & g'^{kl} \partial_k (g^{ij} g'_{ij}) \partial_l (g^{pq} g'_{pq}) \\ &= 2G'^{\gamma\bar{\delta}} D_\gamma (g^{ij} g'_{ij}) D_{\bar{\delta}} (g^{pq} g'_{pq}) \\ &= G'^{\gamma\bar{\delta}} (2D_\gamma (G^{\alpha\bar{\beta}} G'_{\alpha\bar{\beta}}) - E_\gamma) (2D_{\bar{\delta}} (G^{\mu\bar{\nu}} G'_{\mu\bar{\nu}}) - \bar{E}_\delta) \\ &\quad + G'^{\gamma\bar{\delta}} (2D_\gamma (\tilde{G}^{\alpha\bar{\beta}} \tilde{G}'_{\alpha\bar{\beta}}) + E_\gamma) (2D_{\bar{\delta}} (\tilde{G}^{\mu\bar{\nu}} \tilde{G}'_{\mu\bar{\nu}}) + \bar{E}_\delta) \\ &= 4G'^{\gamma\bar{\delta}} (G^{\alpha\bar{\beta}} D_\gamma G'_{\alpha\bar{\beta}}) (G^{\mu\bar{\nu}} D_{\bar{\delta}} G'_{\mu\bar{\nu}}) \\ &\quad + 4G'^{\gamma\bar{\delta}} (\tilde{G}^{\alpha\bar{\beta}} D_\gamma \tilde{G}'_{\alpha\bar{\beta}}) (\tilde{G}^{\mu\bar{\nu}} D_{\bar{\delta}} \tilde{G}'_{\mu\bar{\nu}}) - 2G'^{\gamma\bar{\delta}} E_\gamma \bar{E}_\delta. \end{aligned}$$

From Lemma 3.3,

$$\begin{aligned} & g'^{kl} \partial_k (g^{ij} g'_{ij}) \partial_l (g^{pq} g'_{pq}) \\ &= 4G'^{\gamma\bar{\delta}} (G^{\alpha\bar{\beta}} D_\alpha G'_{\gamma\bar{\beta}}) (G^{\mu\bar{\nu}} D_{\bar{\nu}} G'_{\mu\bar{\delta}}) \\ &\quad + 4G'^{\gamma\bar{\delta}} (G^{\alpha\bar{\beta}} D_\alpha \tilde{G}'_{\gamma\bar{\beta}}) (G^{\mu\bar{\nu}} D_{\bar{\nu}} \tilde{G}'_{\mu\bar{\delta}}) \\ &\quad + 2\text{Re} \left\{ G'^{\gamma\bar{\delta}} G^{\alpha\bar{\beta}} (a_{\gamma\alpha\beta} + \sqrt{-1} b_{\gamma\alpha\beta}) G^{\mu\bar{\nu}} D_{\bar{\nu}} G'_{\mu\bar{\delta}} \right. \\ &\quad \left. + G'^{\gamma\bar{\delta}} G^{\alpha\bar{\beta}} (a_{\gamma AB} + \sqrt{-1} b_{\gamma AB}) G^{\mu\bar{\nu}} D_{\bar{\nu}} \tilde{G}'_{\mu\bar{\delta}} \right\} \\ &\quad + \frac{1}{4} G'^{\gamma\bar{\delta}} G^{\alpha\bar{\beta}} (a_{\gamma\alpha\beta} + \sqrt{-1} b_{\gamma\alpha\beta}) G^{\mu\bar{\nu}} (a_{\delta\mu\nu} - \sqrt{-1} b_{\delta\mu\nu}) \\ &\quad + \frac{1}{4} G'^{\gamma\bar{\delta}} G^{\alpha\bar{\beta}} (a_{\gamma AB} + \sqrt{-1} b_{\gamma AB}) G^{\mu\bar{\nu}} (a_{\delta MN} - \sqrt{-1} b_{\delta MN}) \\ &\quad - 2G'^{\gamma\bar{\delta}} E_\gamma \bar{E}_\delta. \end{aligned}$$

But

$$\begin{aligned} & G^{\mu\bar{\nu}} D_{\bar{\nu}} \tilde{G}'_{\mu\bar{\delta}} \\ &= G^{\mu\bar{\nu}} D_{\bar{\nu}} G'_{\mu\bar{\delta}} + \frac{1}{4} G^{\mu\bar{\nu}} (a_{\delta\mu\nu} + a_{\nu MD} - \sqrt{-1} b_{\delta\mu\nu} - \sqrt{-1} b_{\nu MD}) - \bar{E}_\delta, \end{aligned}$$

and so, making use of Lemma 3.5,

$$\begin{aligned}
(3.22) \quad & g'^{kl} \partial_k (g'^{ij} g'_{ij}) \partial_l (g'^{pq} g'_{pq}) \\
&= 4G'^{\gamma\bar{\delta}} (G^{\alpha\bar{\beta}} D_\alpha G'_{\gamma\bar{\beta}}) (G^{\mu\bar{\nu}} D_{\bar{\nu}} G'_{\mu\bar{\delta}}) \\
&\quad + 4G'^{\gamma\bar{\delta}} (G^{\alpha\bar{\beta}} D_\alpha \tilde{G}'_{\gamma\bar{\beta}}) (G^{\mu\bar{\nu}} D_{\bar{\nu}} \tilde{G}'_{\mu\bar{\delta}}) \\
&\quad + \frac{1}{2} \operatorname{Re} \left\{ G'^{\gamma\bar{\delta}} G^{\alpha\bar{\beta}} (a_{\gamma AB} + \sqrt{-1} b_{\gamma AB}) (G^{\mu\bar{\nu}} (a_{\delta\mu\nu} + a_{\nu MD} \right. \\
&\quad \left. - \sqrt{-1} b_{\delta\mu\nu} - \sqrt{-1} b_{\nu MD}) - 4\bar{E}_\delta) \right\} \\
&\quad + \frac{1}{4} G'^{\gamma\bar{\delta}} G^{\alpha\bar{\beta}} (a_{\gamma\alpha\beta} + \sqrt{-1} b_{\gamma\alpha\beta}) G^{\mu\bar{\nu}} (a_{\delta\mu\nu} - \sqrt{-1} b_{\delta\mu\nu}) \\
&\quad + \frac{1}{4} G'^{\gamma\bar{\delta}} G^{\alpha\bar{\beta}} (a_{\gamma AB} + \sqrt{-1} b_{\gamma AB}) G^{\mu\bar{\nu}} (a_{\delta MN} - \sqrt{-1} b_{\delta MN}) \\
&\quad - 2G'^{\gamma\bar{\delta}} E_\gamma \bar{E}_\delta.
\end{aligned}$$

We can now apply the Cauchy-Schwartz inequality twice to obtain:

$$\begin{aligned}
(3.23) \quad & G'^{\gamma\bar{\delta}} (G^{\alpha\bar{\beta}} D_\alpha G'_{\gamma\bar{\beta}}) (G^{\mu\bar{\nu}} D_{\bar{\nu}} G'_{\mu\bar{\delta}}) \\
&= \sum_{\gamma, \alpha, \mu} G'^{\gamma\bar{\gamma}} (D_\alpha G'_{\gamma\bar{\alpha}}) (D_{\bar{\mu}} G'_{\mu\bar{\gamma}}) \\
&\leq \sum_{\alpha, \mu} \left(\sum_{\gamma} G'^{\gamma\bar{\gamma}} |D_\alpha G'_{\gamma\bar{\alpha}}|^2 \right)^{1/2} \left(\sum_{\gamma} G'^{\gamma\bar{\gamma}} |D_{\bar{\mu}} G'_{\gamma\bar{\mu}}|^2 \right)^{1/2} \\
&= \left(\sum_{\alpha} \left(\sum_{\gamma} G'^{\gamma\bar{\gamma}} |D_\alpha G'_{\gamma\bar{\alpha}}|^2 \right)^{1/2} \right)^2 \\
&= \left(\sum_{\alpha} \sqrt{G'_{\alpha\bar{\alpha}}} \left(\sum_{\gamma} G'^{\gamma\bar{\gamma}} G'^{\alpha\bar{\alpha}} |D_\alpha G'_{\gamma\bar{\alpha}}|^2 \right)^{1/2} \right)^2 \\
&\leq \left(\sum_{\alpha} G'_{\alpha\bar{\alpha}} \right) \sum_{\gamma, \alpha} G'^{\gamma\bar{\gamma}} G'^{\alpha\bar{\alpha}} |D_\alpha G'_{\gamma\bar{\alpha}}|^2 \\
&\leq 4(\operatorname{tr}_g g') \sum_{\gamma, \alpha, \mu} g'^{\gamma\bar{\gamma}} g'^{\alpha\bar{\alpha}} g'^{\mu\bar{\mu}} |D_\mu G'_{\gamma\bar{\alpha}}|^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(3.24) \quad & G'^{\gamma\bar{\delta}} (G^{\alpha\bar{\beta}} D_\alpha \tilde{G}'_{\gamma\bar{\beta}}) (G^{\mu\bar{\nu}} D_{\bar{\nu}} \tilde{G}'_{\mu\bar{\delta}}) \\
&\leq 4(\operatorname{tr}_g g') \sum_{\gamma, \alpha, \mu} g'^{\gamma\bar{\gamma}} g'^{\alpha\bar{\alpha}} g'^{\mu\bar{\mu}} |D_\mu \tilde{G}'_{\gamma\bar{\alpha}}|^2.
\end{aligned}$$

But

$$\begin{aligned}
& 4 \sum_{\gamma, \alpha, \mu} g'^{\gamma\gamma} g'^{\alpha\alpha} g^{\mu\mu} |D_\mu G'_{\gamma\bar{\alpha}}|^2 \\
&= \frac{1}{4} \sum_{\gamma, \alpha, \mu} g'^{\gamma\gamma} g'^{\alpha\alpha} g^{\mu\mu} \{ (\partial_\mu g'_{\gamma\alpha} - J_\mu^i g'_{j\alpha} (\nabla_i J_\gamma^j) - J_\mu^i J_\gamma^j (\nabla_i g'_{j\alpha}))^2 \\
&\quad + (J_\mu^i \nabla_i g'_{\gamma\alpha} + g'_{j\alpha} \nabla_\mu J_\gamma^j + J_\gamma^j \nabla_\mu g'_{j\alpha})^2 \} \\
&= \frac{1}{4} \sum_{\gamma, \alpha, \mu} (g'^{\gamma\gamma} g'^{\alpha\alpha} g^{\mu\mu} (\nabla_\mu g'_{\gamma\alpha})^2 + g'^{\gamma\gamma} g'_{\alpha\alpha} g^{MM} (\nabla_M J_\gamma^\alpha)^2 \\
&\quad + g'^{CC} g'^{\alpha\alpha} g^{MM} (\nabla_M g'_{C\alpha})^2 + g'^{\gamma\gamma} g'^{\alpha\alpha} g^{MM} (\nabla_M g'_{\gamma\alpha})^2 \\
&\quad + g'^{\gamma\gamma} g'_{\alpha\alpha} g^{\mu\mu} (\nabla_\mu J_\gamma^\alpha)^2 + g'^{CC} g'^{\alpha\alpha} g^{\mu\mu} (\nabla_\mu g'_{C\alpha})^2 \\
&\quad - 2g'^{\gamma\gamma} g^{\mu\mu} (\nabla_\mu g'_{\gamma\alpha}) J_\mu^i (\nabla_i J_\gamma^\alpha) - 2g'^{\gamma\gamma} g'^{\alpha\alpha} g^{\mu\mu} J_\mu^i J_\gamma^j (\nabla_\mu g'_{\gamma\alpha}) (\nabla_i g'_{j\alpha}) \\
&\quad + 2g'^{\gamma\gamma} J_\gamma^j g^{MM} (\nabla_M J_\gamma^\alpha) (\nabla_M g'_{j\alpha}) + 2g'^{\gamma\gamma} g^{\mu\mu} J_\mu^i (\nabla_i g'_{\gamma\alpha}) (\nabla_\mu J_\gamma^\alpha) \\
&\quad + 2g'^{\gamma\gamma} g'^{\alpha\alpha} g^{\mu\mu} J_\mu^i J_\gamma^j (\nabla_i g'_{\gamma\alpha}) (\nabla_\mu g'_{j\alpha}) + 2g'^{\gamma\gamma} g^{\mu\mu} J_\gamma^j (\nabla_\mu J_\gamma^\alpha) (\nabla_\mu g'_{j\alpha}) \}.
\end{aligned}$$

A short calculation shows that

$$\sum_{\gamma, \alpha, \mu} g'^{\gamma\gamma} g^{\mu\mu} (\nabla_\mu g'_{\gamma\alpha}) J_\mu^i (\nabla_i J_\gamma^\alpha) + g'^{\gamma\gamma} g'^{\alpha\alpha} g^{\mu\mu} J_\mu^i J_\gamma^j (\nabla_\mu g'_{\gamma\alpha}) (\nabla_i g'_{j\alpha}) = 0,$$

and similarly that

$$\sum_{\gamma, \alpha, \mu} g'^{\gamma\gamma} g^{\mu\mu} J_\mu^i (\nabla_i g'_{\gamma\alpha}) (\nabla_\mu J_\gamma^\alpha) + g'^{\gamma\gamma} g'^{\alpha\alpha} g^{\mu\mu} J_\mu^i J_\gamma^j (\nabla_i g'_{\gamma\alpha}) (\nabla_\mu g'_{j\alpha}) = 0.$$

Hence

$$\begin{aligned}
(3.25) \quad & 4 \sum_{\gamma, \alpha, \mu} g'^{\gamma\gamma} g'^{\alpha\alpha} g^{\mu\mu} |D_\mu G'_{\gamma\bar{\alpha}}|^2 \\
&= \frac{1}{4} \sum_{\gamma, \alpha=1}^2 \sum_{k=1}^4 \left(g'^{\gamma\gamma} g'^{\alpha\alpha} g^{kk} (\nabla_k g'_{\gamma\alpha})^2 + g'^{CC} g'^{\alpha\alpha} g^{kk} (\nabla_k g'_{C\alpha})^2 \right. \\
&\quad \left. + 2g'^{\gamma\gamma} J_\gamma^j g^{kk} (\nabla_k J_\gamma^\alpha) (\nabla_k g'_{j\alpha}) + g'^{\gamma\gamma} g'_{\alpha\alpha} g^{kk} (\nabla_k J_\gamma^\alpha)^2 \right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(3.26) \quad & 4 \sum_{\gamma, \alpha, \mu} g'^{\gamma\gamma} g'^{\alpha\alpha} g^{\mu\mu} |D_\mu \tilde{G}'_{\gamma\bar{\alpha}}|^2 \\
&= \frac{1}{4} \sum_{\gamma, \alpha=1}^2 \sum_{k=1}^4 \left(g'^{CC} g'^{AA} g^{kk} (\nabla_k g'_{CA})^2 + g'^{\gamma\gamma} g'^{AA} g^{kk} (\nabla_k g'_{\gamma A})^2 \right. \\
&\quad \left. - 2g'^{CC} g^{kk} (\nabla_k J_C^A) (\nabla_k g'_{\gamma A}) + g'^{CC} g'_{AA} g^{kk} (\nabla_k J_C^A)^2 \right).
\end{aligned}$$

Now observe that, at p ,

$$\nabla_k J_C^A = -\nabla_k J_\gamma^\alpha,$$

and that

$$\nabla_k g'_{\gamma A} + \nabla_k g'_{C\alpha} = J'_{\alpha i} (\nabla_k J_C^i) + J'_{Ci} (\nabla_k J_\alpha^i).$$

Using these two simple equalities we obtain at the point p , by combining (3.25) and (3.26),

$$\begin{aligned} & 4 \sum_{\gamma, \alpha, \mu} g'^{\gamma\gamma} g'^{\alpha\alpha} g'^{\mu\mu} |D_\mu G'_{\gamma\bar{\alpha}}|^2 + 4 \sum_{\gamma, \alpha, \mu} g'^{\gamma\gamma} g'^{\alpha\alpha} g'^{\mu\mu} |D_\mu \tilde{G}'_{\gamma\bar{\alpha}}|^2 \\ &= \frac{1}{4} g'^{kl} g'^{ij} g'^{pq} (\nabla_p g'_{ik}) (\nabla_q g'_{jl}) + \frac{1}{4} \sum g'^{\gamma\gamma} g'_{\alpha\alpha} g'^{kk} (\nabla_k J_\gamma^\alpha)^2 \\ & \quad + \frac{1}{4} \sum g'^{CC} g'_{AA} g'^{kk} (\nabla_k J_C^A)^2 \\ & \quad - \frac{1}{2} \sum g'^{CC} g'^{kk} (\nabla_k J_C^A) (J'_{\alpha i} \nabla_k J_C^i + J'_{Ci} \nabla_k J_\alpha^i). \end{aligned}$$

Using this, together with (3.22), (3.23) and (3.24), we obtain the estimate of Lemma 3.2. q.e.d.

Returning to the proof of the theorem, we can now combine equations (3.8) with (3.13) and (3.14) together with Lemma 3.2 to obtain

$$\begin{aligned} & \Delta'(\log \operatorname{tr}_g g') \\ & \geq \frac{1}{\operatorname{tr}_g g'} \{ 2\Delta F + 2g'^{kl} g'^{ij} g'^{rs} (\nabla_k J_{ir}) (\mathcal{P}_{js}^{ab} g'_{bq} \nabla_a J_l^q - \frac{1}{2} \alpha_{sjq} J_l^q \\ & \quad - \mathcal{P}_{sj}^{ab} g'_{bq} \nabla_a J_l^q) + g'^{kl} g'^{ij} g'^{rs} (\nabla_r J_{ik}) J'_{ab} (J_l^b \nabla_s J_j^a + J_j^a \nabla_s J_l^b) \\ & \quad - 2g'^{kl} g'^{ij} g'^{rs} (\nabla_i J_{kr} + \nabla_k J_{ir}) g'_{ql} \nabla_s J_j^q \\ & \quad + 2g'^{ij} (\nabla_i J_q^r + \nabla_q J_i^r) (\nabla_r J_j^q) \\ & \quad + g'^{kl} (\nabla_k J_l^r) (\frac{1}{2} \beta_{irq} J^{iq} + 2g'^{ij} \mathcal{Q}_{ir}^{ab} g'_{bq} \nabla_a J_j^q \\ & \quad - g'^{ij} g'_{is} \nabla_j J_r^s - g'^{ij} J_r^s J'_{pj} \nabla_s J_i^p) \\ & \quad + J_r^k g'^{ij} \nabla_i \nabla_j J_k^r - 2g'^{ij} g'^{kl} g'_{qj} J_r^q \nabla_i \nabla_l J_k^r \\ & \quad - J_r^q g'_{qj} g'^{kl} g'^{ij} \nabla_k \nabla_l J_i^r - 2R + 2g'^{kl} g'_{ij} g'^{pj} R_{lpk}^i \\ & \quad - C'(\operatorname{tr}_g g')(\operatorname{tr}_g g') \}. \end{aligned}$$

From the Calabi-Yau equation and the arithmetic-geometric means inequality we have

$$\operatorname{tr}_g g' \geq 4 \exp\left(\frac{\inf_M F}{2}\right) > 0.$$

Hence, recalling from section 2 that $\|J\|_{C^2}$ can be bounded in terms of $\|\operatorname{Rm}(g)\|_{C^2}$, we have:

$$(3.27) \quad \Delta'(\log \operatorname{tr}_g g') \geq -\frac{A}{2} \operatorname{tr}_g g - A,$$

for a constant A depending only on $\|\operatorname{Rm}(g)\|_{C^2}$, $\sup |F|$ and the lower bound of ΔF . We now apply the maximum principle to the quantity

$(\log \operatorname{tr}_g g' - A\phi_1)$. Suppose that the maximum is achieved at a point x_0 on M . Then at x_0 we have

$$\Delta'(\log \operatorname{tr}_g g' - A\phi_1) \leq 0.$$

Recall that $\Delta'\phi_1 = 4 - \operatorname{tr}_{g'} g$. At x_0 we see that from (3.27)

$$\begin{aligned} 0 &\geq \Delta'(\log \operatorname{tr}_g g' - A\phi_1) \\ &\geq \frac{A}{2} \operatorname{tr}_{g'} g - 5A, \end{aligned}$$

so that $(\operatorname{tr}_{g'} g)(x_0) \leq 10$. On the other hand, from (2.8), we have that

$$(\operatorname{tr}_g g')(x_0) \leq 10e^{F(x_0)},$$

and thus at any point x we have

$$\log((\operatorname{tr}_g g')(x)) - A\phi_1(x) \leq \log 10e^{F(x_0)} - A\phi_1(x_0).$$

The theorem follows after exponentiating.

q.e.d.

4. Hölder estimate on the metric

In this section we will prove a Hölder estimate on g' given a uniform estimate of g' , using a modification of the method of Evans [Ev] and Krylov [Kr] (see also the estimate of Trudinger [Tr2] and the exposition of Siu [Si]).

Theorem 4.1. *Suppose that g' satisfies the equation (1.2) and there exists a constant C_0 with*

$$C_0^{-1}g \leq g' \leq C_0g.$$

Then there exist positive constants C and α depending only on g , C_0 and $\|F\|_{C^2(g)}$ such that

$$\|\operatorname{tr}_g g'\|_{C^\alpha(g)} \leq C.$$

Proof. We will work locally and fix a coordinate system (x^1, \dots, x^4) with the same properties as the one in the proof of Lemma 3.2, with the point p corresponding to $x = 0$. We will show that, with the notation of section 3,

$$(4.1) \quad |G^{\alpha\bar{\beta}} G'_{\alpha\bar{\beta}}(y) - G^{\alpha\bar{\beta}} G'_{\alpha\bar{\beta}}(x)| \leq C'R^\alpha,$$

for all $x, y \in B_R(0)$ and $0 < R < R_0/2$ for some positive constants α , R_0 and C' , where $B_R(0)$ is the ball of radius R centred at 0. This will prove the theorem, since a short calculation shows

$$G^{\alpha\bar{\beta}} G'_{\alpha\bar{\beta}} = \frac{1}{2} g'^{ij} g'_{ij} + O(R).$$

To prove (4.1), first note that, by a straightforward calculation,

$$(\det G')^2 = \frac{1}{16} \det g' + \eta,$$

where $\eta = \eta(x)$ is a function of the form

$$\eta = \sum_{a,b,c,d=1}^4 h_{abcd} G'_a G'_b G'_c G'_d,$$

where $G'_1 = G'_{1\bar{1}}$, $G'_2 = G'_{2\bar{2}}$, $G'_3 = G'_{1\bar{2}} + G'_{2\bar{1}}$ and $G'_4 = \sqrt{-1}(G'_{1\bar{2}} - G'_{2\bar{1}})$, and where the h_{abcd} are smooth functions depending only on J which vanish at 0. Note that here, and in the sequel, we are shrinking R_0 whenever necessary. Writing

$$K = \frac{1}{16} e^{2F} \det g,$$

and using the equation (3.1), we see that

$$(4.2) \quad 2 \log \det G' = \log(K + \eta).$$

Define a function Φ on the space of positive definite Hermitian matrices by $\Phi(A_{\alpha\bar{\beta}}) = 2 \log \det(A_{\alpha\bar{\beta}})$. Since Φ is concave, the tangent plane to the graph of Φ at a point $G'_{\alpha\bar{\beta}}(y)$ lies above the graph of Φ and so

$$2G'^{\alpha\bar{\beta}}(y)(G'_{\alpha\bar{\beta}}(y) - G'_{\alpha\bar{\beta}}(x)) \leq \Phi(G'_{\alpha\bar{\beta}}(y)) - \Phi(G'_{\alpha\bar{\beta}}(x)),$$

for x, y in $\bar{B}_{2R}(0)$. From (4.2),

$$(4.3) \quad \begin{aligned} 2G'^{\alpha\bar{\beta}}(y)(G'_{\alpha\bar{\beta}}(y) - G'_{\alpha\bar{\beta}}(x)) &\leq \log \left(1 + \frac{K(y) - K(x) + \eta(y) - \eta(x)}{K(x) + \eta(x)} \right) \\ &\leq \log(1 + C_1 R) \\ &\leq C_1 R. \end{aligned}$$

We now need the following elementary linear algebra lemma.

Lemma 4.1. *Let $S(\lambda, \Lambda)$ be the set of 2×2 positive definite Hermitian matrices with eigenvalues between λ and Λ , with $0 < \lambda \leq \Lambda$. Then there exist a finite number of bases of unit vectors $\{(V_\nu^{(1)}, V_\nu^{(2)})\}_{\nu=1}^N$ and constants $0 < \lambda^* < \Lambda^*$ depending only on λ and Λ such that any A in $S(\lambda, \Lambda)$ can be written*

$$A = \sum_{\nu=1}^N \beta_\nu (V_\nu^{(1)} \otimes \bar{V}_\nu^{(1)} + V_\nu^{(2)} \otimes \bar{V}_\nu^{(2)})$$

with $\lambda^* \leq \beta_\nu \leq \Lambda^*$.

Proof. This lemma can be proved by a straightforward modification of the argument in [MoWa]. q.e.d.

Using this lemma we see that

$$G'^{\alpha\bar{\beta}}(y) = \sum_{\nu=1}^N \beta_\nu(y) (V_\nu^{(1)} \otimes \bar{V}_\nu^{(1)} + V_\nu^{(2)} \otimes \bar{V}_\nu^{(2)}),$$

for $\lambda^* < \beta_\nu < \Lambda^*$ where the $V_\nu^{(i)}$ and λ^* and Λ^* depend only on the constant C_0 . Define

$$w_\nu = \left((V_\nu^{(1)})^\alpha \overline{(V_\nu^{(1)})^\beta} + (V_\nu^{(2)})^\alpha \overline{(V_\nu^{(2)})^\beta} \right) G'_{\alpha\bar{\beta}},$$

where $(V_\nu^{(i)})^\alpha$ is the α -component of the vector $V_\nu^{(i)}$. We can then rewrite (4.3) as

$$(4.4) \quad \sum_{\nu=1}^N \beta_\nu(y)(w_\nu(y) - w_\nu(x)) \leq C_1 R, \quad \text{for } x, y \in \overline{B}_{2R}(0).$$

We will now use the concavity of Φ again, this time to derive a differential inequality for w_ν . For each ν , apply the operator $\sum_{i=1}^2 D_\nu^{(i)} D_{\bar{\nu}}^{(i)} = \sum_{i=1}^2 (V_\nu^{(i)})^\gamma \overline{(V_\nu^{(i)})^\delta} D_\gamma D_{\bar{\delta}}$ to (4.2) to obtain:

$$(4.5) \quad \begin{aligned} & \sum_{i=1}^2 \left(2G'^{\alpha\bar{\beta}} D_\nu^{(i)} D_{\bar{\nu}}^{(i)} G'_{\alpha\bar{\beta}} - 2G'^{\alpha\bar{\gamma}} G'^{\sigma\bar{\beta}} (D_\nu^{(i)} G'_{\sigma\bar{\gamma}}) (D_{\bar{\nu}}^{(i)} G'_{\alpha\bar{\beta}}) \right) \\ &= \sum_{i=1}^2 \left(\frac{D_\nu^{(i)} D_{\bar{\nu}}^{(i)} (K + \eta)}{K + \eta} - \frac{|D_\nu^{(i)} (K + \eta)|^2}{(K + \eta)^2} \right). \end{aligned}$$

Apply Lemma 3.3 twice to the first term on the left hand side and the first term on the right hand side of (4.5). Making use of the good second term on the left hand side of (4.5) we see that there is a second order elliptic operator $L_\nu = a^{ij} \partial_i \partial_j$ with $C_2^{-1} |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq C_2 |\xi|^2$ such that

$$(4.6) \quad L_\nu w_\nu \geq -C_3.$$

From the inequalities (4.4) and (4.6) we make the following claim.

Claim. There exist positive constants \hat{C} and δ such that

$$\text{osc}_{B_R(0)} w_\nu \leq \hat{C} R^\delta, \quad \text{for } 0 < R < R_0/2.$$

Of course, given this claim, we are finished, since we can then write

$$G^{\alpha\bar{\beta}} G'_{\alpha\bar{\beta}} = \sum_{\nu=1}^N \hat{\beta}_\nu w_\nu,$$

with $\hat{\beta}_\nu$ smooth bounded functions depending only on g and J and satisfying $C_4^{-1} < \hat{\beta}_\nu < C_4$. This gives (4.1) and Theorem 4.1 follows.

Proof of Claim. Although this proof can easily be extracted from [Tr2] (see also [Si]), we will include a sketch of the argument here for the convenience of the reader. The key tool is the following Harnack inequality

[Tr1]: if $u \geq 0$ satisfies $Lu = a^{ij}\partial_i\partial_j u \leq C_3$ with $C_2^{-1}|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq C_2|\xi|^2$ on $B_{2R}(0)$ then there exists $p > 0$ such that

$$\left(\frac{1}{R^4} \int_{B_R(0)} u^p\right)^{1/p} \leq C_H \left(\inf_{B_R(0)} u + R\right),$$

where the constant C_H depends only on C_2 and C_3 .

Set $M_{s\nu} = \sup_{B_{sR}(0)} w_\nu$ and $m_{s\nu} = \inf_{B_{sR}(0)} w_\nu$ for $s = 1, 2$ and apply the Harnack inequality to $(M_{2\nu} - w_\nu)$ to obtain, for fixed l ,

$$\begin{aligned} (4.7) \quad & \left(\frac{1}{R^4} \int_{B_R(0)} \left(\sum_{\nu \neq l} (M_{2\nu} - w_\nu)\right)^p\right)^{1/p} \\ & \leq N^{1/p} \sum_{\nu \neq l} \left(\frac{1}{R^4} \int_{B_R(0)} (M_{2\nu} - w_\nu)^p\right)^{1/p} \\ & \leq C_5 \left(\sum_{\nu \neq l} (M_{2\nu} - M_{1\nu}) + R\right) \\ & \leq C_5(\omega(2R) - \omega(R) + R), \end{aligned}$$

where $\omega(sR) = \sum_{\nu=1}^N \text{osc}_{B_{sR}(0)} w_\nu = \sum_{\nu=1}^N (M_{s\nu} - m_{s\nu})$. From (4.4) we have

$$\beta_l(w_l(y) - w_l(x)) \leq C_1 R + \sum_{\nu \neq l} \beta_\nu(w_\nu(x) - w_\nu(y)),$$

for $x, y \in \overline{B_{2R}(0)}$. Choosing $x \rightarrow m_{2l}$ and integrating over $y \in B_R(0)$ gives

$$\begin{aligned} (4.8) \quad & \left(\frac{1}{R^4} \int_{B_R(0)} (w_l - m_{2l})^p\right)^{1/p} \\ & \leq C_6 R + C_6 \left(\frac{1}{R^4} \int_{B_R(0)} \left(\sum_{\nu \neq l} (M_{2\nu} - w_\nu)^p\right)\right)^{1/p} \\ & \leq C_7(\omega(2R) - \omega(R) + R). \end{aligned}$$

Now apply the Harnack inequality to $(M_{2l} - w_l)$ to obtain

$$\begin{aligned} (4.9) \quad & \left(\frac{1}{R^4} \int_{B_R(0)} (M_{2l} - w_l)^p\right)^{1/p} \leq C_8(M_{2l} - M_{1l} + R) \\ & \leq C_8(\omega(2R) - \omega(R) + R). \end{aligned}$$

Combining (4.8) and (4.9) we see that

$$M_{2l} - m_{2l} \leq C_9(\omega(2R) - \omega(R) + R),$$

and summing in l gives

$$\omega(2R) \leq C_{10}(\omega(2R) - \omega(R) + R),$$

from which it follows that

$$\omega(R) \leq \left(\frac{C_{10} - 1}{C_{10}} \right) \omega(2R) + R,$$

and the claim follows by a well-known argument (see [GiTr], Chapter 8). q.e.d.

5. Higher order estimates

In this section we will prove estimates on g' and all of its derivatives given a Hölder estimate

$$(5.1) \quad \|\mathrm{tr}_g g'\|_{C^\alpha(g)} \leq C,$$

with $0 < \alpha < 1$ and an estimate $g' \geq C^{-1}g$. In light of Theorem 3.1 and Theorem 4.1 this will complete the proof of Theorem 1.

Consider the normalized almost-Kähler potential ϕ_0 defined by (see Section 2)

$$\Delta\phi_0 = \mathrm{tr}_g g' - 4, \quad \int_M \phi_0 \frac{\omega^2}{2} = 0.$$

From (5.1), by the elliptic Schauder estimates for the Laplacian we have

$$(5.2) \quad \|\phi_0\|_{C^{2+\alpha}(g)} \leq C_0.$$

Recall that the 1-form a_0 satisfies

$$\omega' = \omega - \frac{1}{2}d(Jd\phi_0) + da_0.$$

Without loss of generality, we can assume that a_0 is L^2 orthogonal to the harmonic 1-forms. Then since a_0 satisfies the uniformly elliptic system (2.7) for $s = 0$ and is orthogonal to its kernel we can use (5.2) and the Schauder elliptic estimates to obtain

$$\|a_0\|_{C^{2+\alpha}(g)} \leq C_1.$$

It follows that $\|g'\|_{C^\alpha(g)} \leq C_2$. Differentiating the Calabi-Yau equation (3.1), we see that

$$(5.3) \quad g'^{ij} \partial_i \partial_j (\partial_k \phi_0) + \{\text{lower order terms}\} = g'^{ij} \partial_k g_{ij} + 2\partial_k F,$$

where the lower order terms may contain up to two derivatives of ϕ_0 or a_0 . Since the coefficients of this elliptic equation are in C^α we can apply the standard Schauder estimates again to obtain

$$\|\phi_0\|_{C^{3+\alpha}(g)} \leq C_3.$$

From (2.7) we then obtain

$$\|a_0\|_{C^{3+\alpha}(g)} \leq C_4.$$

The rest of the higher order estimates follow from (5.3), (2.7) and a bootstrapping argument. This completes the proof of Theorem 1.

6. Proof of Theorem 2: the case $b^+(M) = 1$

For this section we assume $b^+(M) = 1$. Consider the equation

$$(6.1) \quad \omega'_t{}^2 = e^{tF+c_t}\omega^2,$$

where c_t is the constant given by $c_t = \log(\int_M \omega^2 / \int_M e^{tF} \omega^2)$, and where ω'_t is required to be cohomologous to ω and compatible with J . Let

$$T = \{t' \in [0, 1] \mid \exists \text{ smooth } \omega'_{t'} \text{ solving (6.1) for } t \in [0, t']\}.$$

Clearly $0 \in T$. We will show that T is both open and closed in $[0, 1]$. This will prove Theorem 2. Note that if ω'_t is in C^α then by the estimates of section 5 it is smooth.

We show now that T is open. Fix t_0 in T . We will show that (6.1) can be solved for t in an open neighbourhood containing t_0 . Fix $\tilde{\omega} = \omega'_{t_0}$. Then solving (6.1) near t_0 is equivalent to solving

$$\log \frac{\omega'_t{}^2}{\tilde{\omega}^2} - (t - t_0)F - (c_t - c_{t_0}) = 0,$$

for t close to t_0 .

Let $\Lambda^{k,s+\alpha}$ be the space of k -forms in $C^{s+\alpha}$, and let $W^\alpha \subset \Lambda_{\tilde{\omega}}^+$ be the space of self-dual two forms γ in C^α satisfying $\int_M \exp\left(\frac{2\gamma \wedge \tilde{\omega}}{\tilde{\omega}^2}\right) \tilde{\omega}^2 = \int_M \tilde{\omega}^2$. Then define a map

$$\Phi : \Lambda^{1,1+\alpha} \times \mathbb{R} \rightarrow W^\alpha,$$

by

$$\Phi(b, t) = \left(\log \frac{(\tilde{\omega} + db)^2}{\tilde{\omega}^2} - (t - t_0)F - \hat{c} \right) \frac{\tilde{\omega}}{2} + \mathcal{P}db,$$

where

$$\hat{c}(b, t) = \log \left(\int_M e^{-(t-t_0)F} (\tilde{\omega} + db)^2 \right) - \log \int_M \omega^2.$$

Note that if we can find $b = b(t)$ solving $\Phi(b, t) = 0$ for t close to t_0 , then this would imply $\hat{c} = c_t - c_{t_0}$ and complete the openness argument.

Since $b^+(M) = 1$, the space $\mathcal{H}_{\tilde{\omega}}^+$ of harmonic self-dual 2-forms with respect to $\tilde{\omega}$ is spanned by $\tilde{\omega}$. Notice that the tangent space to W^α at $\Phi(0, t_0)$ is equal to $(\mathcal{H}_{\tilde{\omega}}^+)^{\perp} \cap \Lambda^{2,\alpha}$, where $(\mathcal{H}_{\tilde{\omega}}^+)^{\perp} \subset \Lambda_{\tilde{\omega}}^+$ is the space of self-dual 2-forms which are $L^2(\tilde{\omega})$ orthogonal to $\mathcal{H}_{\tilde{\omega}}^+$. Then the derivative of Φ in the b -variable at $(0, t_0)$ is a map

$$(D_1\Phi)_{(0,t_0)} : \Lambda^{1,1+\alpha} \rightarrow (\mathcal{H}_{\tilde{\omega}}^+)^{\perp} \cap \Lambda^{2,\alpha}$$

given by

$$(D_1\Phi)_{(0,t_0)}(\beta) = d_{\tilde{\omega}}^+ \beta.$$

It is well known (see [DoKr], for example) that this map is surjective and hence openness follows by the implicit function theorem.

We now need to prove closedness under the assumption that the Nijenhuis tensor is small in the L^1 sense. Note that from the discussion in section 2, since $|\nabla N(J)|$ is uniformly bounded in terms of the curvature of g , if the Nijenhuis tensor is small in the L^1 norm, it is small in the C^0 norm, and hence also in the L^p norm for any $p > 1$. It will be convenient (and sufficient) for us to prove Theorem 2 under the assumption that $N(J)$ is small in some L^p norm, where p will be a fixed constant strictly larger than 2.

We will use the following lemma.

Lemma 6.1. *Let $\omega' = \omega - \frac{1}{2}d(Jd\phi_1) + da_1$ be a solution of the Calabi-Yau equation (1.2). Suppose that for some constants $p > 2$ and B ,*

$$(6.2) \quad \left(\int_M \left| \frac{da_1 \wedge \omega}{\omega^2} \right|^p \omega^2 \right)^{1/p} \leq B.$$

Then there exists a constant C' depending only on g, p, B and $\sup_M |F|$ such that

$$\sup_M \phi_1 - \inf_M \phi_1 \leq C'.$$

Proof. This is a modification of Yau's well-known Moser iteration argument. For ease of notation, write $\phi = \phi_1$. Assume that $\int_M \phi \omega^2 = 0$. For $l \geq 0$,

$$\begin{aligned} C_0 \int_M |\phi|^{l+1} \omega^2 &\geq \int_M \phi |\phi|^l (\omega^2 - \omega'^2) \\ &= \frac{1}{2} \int_M \phi |\phi|^l d(Jd\phi) \wedge (\omega + \omega') - \int_M \phi |\phi|^l da_1 \wedge \omega \\ &= -\frac{(l+1)}{2} \int_M |\phi|^l d\phi \wedge Jd\phi \wedge (\omega + \omega') \\ &\quad - \int_M \phi |\phi|^l da_1 \wedge \omega \\ &= -\frac{(l+1)}{2(l/2+1)^2} \int_M d(\phi |\phi|^{l/2}) \wedge Jd(\phi |\phi|^{l/2}) \wedge (\omega + \omega') \\ &\quad - \int_M \phi |\phi|^l da_1 \wedge \omega \\ &\geq \frac{(l+1)}{4(l/2+1)^2} \int_M |\nabla(\phi |\phi|^{l/2})|^2 \omega^2 \\ &\quad - \left(\int_M |\phi|^{q(l+1)} \omega^2 \right)^{1/q} \left(\int_M \left| \frac{da_1 \wedge \omega}{\omega^2} \right|^p \omega^2 \right)^{1/p}, \end{aligned}$$

for q satisfying $1/p + 1/q = 1$. Setting $l = 0$ we see that since $q < 2$,

$$\int_M |\nabla\phi|^2 \leq C_1 \left(\left(\int_M |\phi|^q \right)^{1/q} + \int_M |\phi| \right) \leq C_2 \left(\left(\int_M |\phi|^2 \right)^{1/2} + 1 \right),$$

where we have omitted the volume form ω^2 . Since $\int_M \phi = 0$ we obtain $\|\phi\|_{L^2} \leq C_3$ from the Poincaré inequality.

We have for general l ,

$$(6.3) \quad \int_M |\nabla(\phi|\phi|^{l/2})|^2 \leq C_4(l+2) \max \left\{ 1, \int_M |\phi|^{l+2}, \left(\int_M |\phi|^{q(l+1)} \right)^{1/q} \right\}.$$

The Sobolev inequality gives

$$\left(\int_M |u|^4 \right)^{1/2} \leq C_5 \left(\int_M |\nabla u|^2 + \int_M u^2 \right),$$

for functions u on M . Set $r = l+2 \geq 2$. Applying the Sobolev inequality to $u = \phi|\phi|^{l/2}$, making use of (6.3) and raising to the power $1/r$ gives

$$\|\phi\|_{L^{2r}} \leq C_6^{1/r} r^{1/r} \max\{1, \|\phi\|_{L^r}, \|\phi\|_{L^{q(r-1)}}^{(r-1)/r}\}.$$

Setting $r = 2$ we obtain $\|\phi\|_{L^4} \leq C_7$. For general r we use that fact that $\|\phi\|_{L^a} \leq C_8\|\phi\|_{L^b}$ whenever $a \leq b$ to see that

$$\|\phi\|_{L^{2r}} \leq C_9^{1/r} r^{1/r} \max\{1, \|\phi\|_{L^{qr}}\}.$$

By successively replacing r by σr for $\sigma = 2/q > 1$ we see that for all $k = 0, 1, 2, \dots$,

$$\|\phi\|_{L^{2r\sigma^k}} \leq C_9^{(\frac{1}{r} \sum_{i=0}^k \sigma^{-i})} r^{(\frac{1}{r} \sum_{i=0}^k \sigma^{-i})} \sigma^{(\frac{1}{r} \sum_{i=1}^k i\sigma^{-i})} \max\{1, \|\phi\|_{L^{qr}}\}.$$

Set $r = 2$ and let $k \rightarrow \infty$. This gives a bound

$$\|\phi\|_{C^0} \leq C_{10} \max\{1, \|\phi\|_{L^{2q}}\},$$

which is uniformly bounded since $q < 2$. This completes the proof of the lemma. q.e.d.

Remark 6.1. In a private discussion, Donaldson made the following surprising observation: the almost-Kähler potential $\phi_{1/2}$ is uniformly bounded, without any assumption on $N(J)$. This can be proved using a Moser iteration argument and equation (2.5).

It is now not difficult to complete the proof of Theorem 2. We suppose that we have a solution of (6.1) for $t \in [0, t_0)$ for some $t_0 \in [0, 1)$. We require uniform estimates on ω'_t and all its derivatives and by Theorem 1, it is sufficient to obtain a uniform estimate of ϕ_1 . We have the following claim.

Claim. Let $p > 2$. There exists $\epsilon > 0$ depending only on p, g and $\|F\|_{C^2(g)}$ such that if $\|N(J)\|_{L^p} < \epsilon$ then for $t \in [0, t_0)$, $a_1 = a_1(t)$ satisfies

$$(6.4) \quad \left(\int_M \left| \frac{da_1 \wedge \omega}{\omega^2} \right|^p \omega^2 \right)^{1/p} < 1.$$

Proof of Claim. At $t = 0$ we have $da_1 = 0$. Suppose that the claim is false. Then there is a first time $t' \in [0, t_0)$ with

$$\left(\int_M \left| \frac{da_1 \wedge \omega}{\omega^2} \right|^p \omega^2 \right)^{1/p} = 1.$$

Then it follows from Lemma 6.1 that we have a Hölder estimate on ω' at $t = t'$. Now the L^p *a priori* estimates for the elliptic system (2.7) with $s = 1$ give

$$\left(\int_M \left| \frac{da_1 \wedge \omega}{\omega^2} \right|^p \omega^2 \right)^{1/p} \leq K \|N(J)\|_{L^p},$$

for some uniform constant K . Picking $\epsilon = 1/2K$ gives a contradiction and proves the claim. q.e.d.

Then the first part of Theorem 2 follows from this claim and the previous lemma.

7. Proof of Theorem 2: the case $b^+(M) > 1$

Suppose that $b^+(M) = r + 1$. We begin with the openness argument. For convenience, assume that ω has been scaled so that $\int_M \omega^2 = 1$. We wish to solve the equation

$$(7.1) \quad \omega_t^2 = e^{tF + c_t} \omega^2,$$

with $c_t = -\log(\int_M e^{tF} \omega^2)$, for ω'_t satisfying $\int_M \omega'_t \wedge \omega > 0$ and $[\omega'_t] \in H_\omega^+$. As in section 6, we suppose that there is a solution $\tilde{\omega} = \omega'_{t_0}$ at $t = t_0$ and show that (7.1) can be solved for t close to t_0 .

Let χ_1, \dots, χ_r and $\tilde{\chi}_1, \dots, \tilde{\chi}_r$ be self-dual harmonic 2-forms with respect to ω and $\tilde{\omega}$ respectively such that $\{\omega, \chi_1, \dots, \chi_r\}$ and $\{\tilde{\omega}, \tilde{\chi}_1, \dots, \tilde{\chi}_r\}$ are L^2 orthonormal bases for \mathcal{H}_ω^+ and $\mathcal{H}_{\tilde{\omega}}^+$. Let $\Lambda^{1,1+\alpha}$ and W^α be as in section 6. Consider the operator $\Phi : \Lambda^{1,1+\alpha} \times \mathbb{R}^r \times \mathbb{R} \rightarrow W^\alpha$ given by

$$\begin{aligned} \Phi(b, \underline{s}, t) &= \left(\log \frac{(\tilde{\omega} + \sum_{i=1}^r s_i \chi_i + db)^2}{\tilde{\omega}^2} - (t - t_0)F - \hat{c} \right) \frac{\tilde{\omega}}{2} \\ &\quad + \mathcal{P} \left(\sum_{i=1}^r s_i \chi_i + db \right), \end{aligned}$$

where

$$\hat{c}(b, \underline{s}, t) = \log \left(\int_M e^{-(t-t_0)F} (\tilde{\omega} + \sum_{i=1}^r s_i \chi_i + db)^2 \right).$$

We have a solution $\Phi(0, \underline{0}, t_0) = 0$ and if we can find b and \underline{s} depending on t solving $\Phi(b, \underline{s}, t) = 0$ for t near t_0 , then after rescaling we would have our desired solution. Write $\Pi_{\langle \tilde{\chi}_1, \dots, \tilde{\chi}_r \rangle}$ for the $L^2(\tilde{\omega})$ projection onto the space spanned by $\tilde{\chi}_1, \dots, \tilde{\chi}_r$. Define $\Psi_1 : \Lambda^{1,1+\alpha} \times \mathbb{R}^r \times \mathbb{R} \rightarrow W^\alpha \cap (\langle \tilde{\chi}_1, \dots, \tilde{\chi}_r \rangle)^\perp$ by

$$\Psi_1(b, \underline{s}, t) = (1 - \Pi_{\langle \tilde{\chi}_1, \dots, \tilde{\chi}_r \rangle}) \Phi(b, \underline{s}, t).$$

The derivative $(D_1 \Psi_1)_{(0, \underline{0}, t_0)} : \Lambda^{1,1+\alpha} \rightarrow (\mathcal{H}_{\tilde{\omega}}^+)^\perp \cap \Lambda^{2,\alpha}$ is given by

$$(D_1 \Psi_1)_{(0, \underline{0}, t_0)}(\beta) = d_{\tilde{\omega}}^+ \beta.$$

This map is surjective and so by the implicit function theorem, given (\underline{s}, t) near $(\underline{0}, t_0) \in \mathbb{R}^r \times \mathbb{R}$ there exists $b = b(\underline{s}, t)$ solving $\Psi_1 = 0$. Now define a map $\Psi_2 : \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}^r$ in a neighbourhood of $(\underline{0}, t_0)$ by

$$\Psi_2(\underline{s}, t) = \Pi_{\langle \tilde{\chi}_1, \dots, \tilde{\chi}_r \rangle} \Phi(b(\underline{s}, t), \underline{s}, t),$$

where we are identifying \mathbb{R}^r and the space spanned by the $\tilde{\chi}_i$. Calculate

$$\begin{aligned} ((D_1 \Psi_2)_{(\underline{0}, t_0)})_{ij} &= \int_M \left\langle \frac{1}{2} (1 + *\tilde{\omega}) \left(\chi_j + d \left(\frac{\partial b}{\partial s_j}(\underline{0}, t_0) \right) \right), \tilde{\chi}_i \right\rangle_{\tilde{\omega}} \frac{\tilde{\omega}^2}{2} \\ &= \int_M \chi_j \wedge \tilde{\chi}_i, \end{aligned}$$

which is invertible. Applying the implicit function theorem again we find $\underline{s} = \underline{s}(t)$ solving $\Psi_2(\underline{s}(t), t) = 0$ and hence

$$\Phi(b(\underline{s}(t), t), \underline{s}(t), t) = 0,$$

for t close to t_0 . This completes the proof of openness.

We now turn to the question of closedness. As discussed in section 6, we may assume that that $N(J)$ is small in the L^p sense for any fixed p . Assume that we have a solution of (7.1) with $\int_M \omega'_t \wedge \omega > 0$ and $[\omega'_t] \in H_{\tilde{\omega}}^+$ on some maximal interval $[0, t_0)$. Write $\omega' = \omega'_t$ and define s_i by $[\omega'] = [\omega] + \sum_{i=0}^r s_i [\chi_i]$ for $\chi_0 = \omega$ and χ_1, \dots, χ_r as above. Notice that by squaring both sides of this equation we see that the s_i are bounded. Define ϕ_0 and ϕ_1 by

$$(7.2) \quad \frac{1}{4} \Delta \phi_0 = \frac{\omega \wedge \omega'}{\omega^2} - \int_M \omega' \wedge \omega$$

$$(7.3) \quad \frac{1}{4} \Delta' \phi_1 = \int_M \omega' \wedge \omega - \frac{\omega \wedge \omega'}{\omega'^2},$$

where we recall that $\int_M \omega^2 = \int_M \omega'^2 = 1$. Let us first assume that ϕ_1 is uniformly bounded. Then since $\int_M \omega' \wedge \omega = 1 + s_0 > 0$ is uniformly bounded from above, Theorem 3.1 still holds with essentially the same

proof. Notice that the bound on $\text{tr}_g g'$ implies a uniform positive lower bound for $\int_M \omega' \wedge \omega$. No changes are necessary for section 4. For the higher order estimates, we argue as follows. Define a_0 by $d^*a_0 = 0$ and

$$\omega' = \omega + \sum_{i=0}^r s_i \chi_i - \frac{1}{2} d(Jd\phi_0) + da_0.$$

Then a_0 satisfies the equations

$$\begin{aligned} da_0 \wedge \omega &= - \sum_{i=1}^r s_i \chi_i \wedge \omega \\ \mathcal{P}da_0 &= - \sum_{i=1}^r s_i \mathcal{P}\chi_i + \frac{1}{4} (\partial_i J_j^k - \partial_j J_i^k) (\partial_k \phi_0) dx^i \wedge dx^j, \end{aligned}$$

and the arguments of section 5 follow in just the same way as before.

We will now show that ϕ_1 can be bounded if $N(J)$ is small in the L^p norm for $p > 2$. Define a_1 by

$$\omega' = \omega + \sum_{i=0}^r s_i \chi_i - \frac{1}{2} d(Jd\phi_1) + da_1,$$

and $d_1^*a_1 = 0$ where we are using the subscript 1 to denote the metric ω' . For ease of notation, set

$$\zeta = \frac{1}{4} (\partial_i J_j^k - \partial_j J_i^k) (\partial_k \phi_1) dx^i \wedge dx^j.$$

The 1-form a_1 satisfies

$$\begin{aligned} da_1 \wedge \omega' &= -s_0 \omega'^2 - \sum s_i \chi_i \wedge \omega' \\ \mathcal{P}da_1 &= \zeta - \mathcal{P}(\sum s_i \chi_i), \end{aligned}$$

where, here and from now on, we are always summing i from 0 to r . This equation can be rewritten as

$$d_1^+ a_1 = \zeta - s_0 \omega' - \frac{1}{2} (1 + *_1) \sum s_i \chi_i.$$

Write Π for the $L^2(\omega')$ projection onto the space $\mathcal{H}_{\omega'}^+$ of self-dual harmonic forms with respect to ω' . Then we see that

$$(7.4) \quad (1 - \Pi)\zeta = d_1^+ a_1 + (1 - \Pi) \frac{1}{2} (1 + *_1) \sum s_i \chi_i$$

$$(7.5) \quad \Pi\zeta = s_0 \omega' + \Pi \sum s_i \chi_i = \Pi \sum \tilde{s}_i \chi_i,$$

for $\tilde{s}_0 = s_0(2 + s_0)$ and $\tilde{s}_i = s_i(1 + s_0)$ for $i \geq 1$. Now Lemma 6.1 holds as before if we replace (6.2) by the inequality

$$\left(\int_M \left| \frac{da_1 \wedge \omega}{\omega^2} \right|^p \omega^2 \right)^{\frac{1}{p}} + \left(\int_M \left| \frac{\sum s_i \chi_i \wedge \omega}{\omega^2} \right|^p \omega^2 \right)^{\frac{1}{p}} \leq B,$$

where we are making use of the fact that $(da_1 + \sum s_i \chi_i) \wedge \omega' = -s_0 \omega'^2$ is bounded. We can now replace the inequality (6.4) in the Claim by

$$(7.6) \quad \left(\int_M \left| \frac{da_1 \wedge \omega}{\omega^2} \right|^p \omega^2 \right)^{\frac{1}{p}} + \left(\int_M \left| \frac{\sum s_i \chi_i \wedge \omega}{\omega^2} \right|^p \omega^2 \right)^{\frac{1}{p}} < 1.$$

Indeed, arguing for a contradiction as in the proof of the claim, we suppose that we have equality in (7.6). Then ω' and ω are uniformly equivalent and we can essentially ignore the fact that they define different norms. Writing C for a uniform constant which may change from inequality to inequality we have $\|\zeta\|_{L^p} \leq C\|N(J)\|_{L^p}$ from which it follows that $\|\Pi\zeta\|_{L^2} \leq C\|N(J)\|_{L^p}$. Then we see from (7.5) that $|\tilde{s}_i| \leq C\|N(J)\|_{L^p}$. Hence $|s_i| \leq C\|N(J)\|_{L^p}$ and

$$(7.7) \quad \left\| \sum s_i \chi_i \right\|_{L^p} \leq C\|N(J)\|_{L^p}.$$

But we also have $\|\Pi\zeta\|_{L^p} \leq C\|N(J)\|_{L^p}$ and hence $\|(1 - \Pi)\zeta\|_{L^p} \leq C\|N(J)\|_{L^p}$. Then from (7.4) and the elliptic L^p estimates we have

$$(7.8) \quad \|da_1\|_{L^p} \leq C\|N(J)\|_{L^p}.$$

Choosing ϵ sufficiently small, we obtain the contradiction from (7.7) and (7.8). q.e.d.

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