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THE CALCULATION OF ANTENNA RADIATION PATTERNS BY A VECTOR THEORY USING DIGITAL COMPUTERS

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ABSTRACT

This document reviews the Kirchhoff-Huygens-Fresnel theory of diffraction, including Kottler boundary correction, and discusses the reduction of this formulation for the special case of metallic reflectors with unbounded conductivity. The formal transition from the scalar to the vector case is given in detail. Radial and transverse field components are identified, the entry of the near-field terms is discussed, and the physical quantities (sheet current, charge distribution, etc.) associated with the mathematical development are presented through a simple dimensional analysis. The analytic aspects of the problem are considered, and the points of departure from conventional analysis to numerical methods and digital computation are outlined.

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GLOSSARY OF NOTATION

Symbol	Meaning
∇	vector operator del
r, R	radial distances, subscripted
ψ (x, y, z, t)	scalar wave with space and time dependence
n	normal to surface or line
S, s	surface, subscripted
V, v	volume, subscripted
ε	an infinitesimal value
t	time
f	frequency
ω	angular frequency
x, y, z	Cartesian coordinates
k	wave number
c	velocity of light
$\overline{E}$, $\overline{H}$	electric and magnetic field vectors, subscripted
σ	electrical conductivity, subscripted
μ	magnetic permeability, subscripted
ϵ	inductive capacity, subscripted
$\overline{D}$, $\overline{B}$	dielectric displacement and magnetic flux density
η	surface charge density
$\overline{K}$	sheet current, subscripted

GLOSSARY OF NOTATION (Continued)

Symbol	Meaning
C	contour
$d\bar{l}$	differential length, a vector
M, L, T, Q	fundamental quantities: mass, length, time, and volume change density
$\bar{J}$	current density
Δ	delta, an increment indication
i, j	a constant equal to $\sqrt{-1}$ in complex variable theory
ϕ	a scalar field, generic symbol
$\rightarrow$	corresponds to, goes over to, or approaches (as required by context)
$\mid$	given that
$\Rightarrow$	implies that
$\hat{i}\,\hat{i}$, etc.	dyads or second order tensors when vector dot or cross product operation is omitted
$\hat{i}, \hat{j}, \hat{k}$	unit vectors in the x, y, z Cartesian coordinate directions respectively
$\hat{1}_r$	a unit vector in the radial direction, spherical coordinate system

THE CALCULATION OF ANTENNA RADIATION PATTERNS BY A VECTOR THEORY USING DIGITAL COMPUTERS

INTRODUCTION

This document is the first in a series of documents describing the development of a general purpose, modular computer program for the calculation of antenna radiation patterns. The program represents an outgrowth of several investigations of monopulse antennas and the inertialess displacement of monopulse antenna patterns in space, carried out during the period 1966-1967. In the course of these investigations it was found that reliable volumetric radiation pattern information was frequently unavailable and that direct measurement was both time-consuming and costly. Range and component imperfections added to the measurement problem. A transfer function was desired to bridge the gap between prime-feed and secondary radiation patterns.

Examination of the several approaches available for calculating antenna radiation patterns showed that the current distribution method (Ref. 1, p. 144) appeared to offer the best compromise between the extremes of complete rigor and the approximations of the simple geometrical optics approach. The current distribution method provides a vector solution to the problem. It leads to the satisfactory solution of real physical problems where the size of the scattering body is large compared to the operating wave length. The current distribution method supplies information on the amplitude, phase, and polarization of radiation patterns. It also yields radial as well as transverse field components and the electric and magnetic field integrals contain near-field terms proportional to $\frac{1}{r^2}$, permitting field computations in that region. Mutual coupling on the scattering surface and reaction on the prime radiator are not included.

This document contains some material which is not readily available in published texts. Its main purpose is to review the vector theory principles required in the subsequent computations, including the current distribution method and the Kirchhoff-Huygens-Fresnel theory, using a common notation and convention system. It provides several frequently omitted steps in the development of the theory, and details the arguments which lead from a scalar to a vector formulation. The purpose of this first document is to provide a theoretical foundation for the analytical details and computational methods to be described in subsequent documents.

Kirchhoff's scalar theory is first reviewed and a vector analogue is written. The analogue is manipulated by means of vector identities until a convenient form is obtained for calculating the scattered fields. Kottler's correction term for open (finite) surfaces, is derived from boundary conditions, and several integrals of the general formulation are set to zero after an examination of the boundary conditions established by the assumption of perfectly conducting surfaces. Transverse and radial components of the final version of the scattered field equations are identified. The origin of induction terms (fields proportional to $\frac{1}{r^2}$) is discussed. Physical quantities such as sheet current, and charge distribution are associated with each of the integrals comprising the general solution. Finally, a brief discussion of analysis and numerical/computational methods is presented.

Subsequent documents will present details relating to the source polarization vector, field interaction at a boundary, feed rotation matrices, feed displacement vectors, surface normals, differential areas, and radii of curvature. Composite surfaces and distorted surfaces will also be considered, together with dual reflector systems. The function of various subroutines in a modular-type computer program will be discussed, and direct as well as numerical integration techniques will be presented. Examples of radiation patterns obtained with IBM 7094 and IBM 360 Mod 91 computers and SC 4020 plotters will be appended in these latter reports.

KIRCHHOFF'S DIFFRACTION THEORY

The integral theorem attributed to Kirchhoff is a straightforward mathematical description of the physical process described earlier by the Huygens-Fresnel principle. A detailed discussion can be found in the literature. The derivation of the scalar Kirchhoff integral theorem involves several assumptions and the limitations of the theory, which become evident in the derivation, are worth noting at the outset as these cause some difficulty later in the application of the vector formulation to real physical problems.

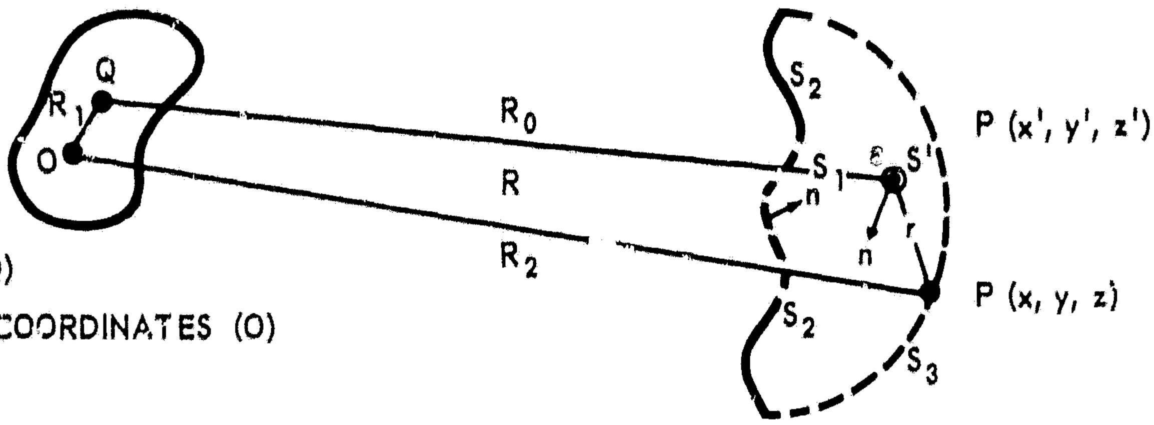
A monochromatic scalar wave (ψ) is assumed to originate from a region containing sources as shown in Figure 1.

In a vacuum, the space dependent part of ψ satisfies the time-independent <u>scalar</u> Helmholtz equation.

$$(\nabla^2 + k^2)\psi = 0$$

where

$\psi(x, y, z, t) = \psi(x, y, z)\, e^{-i\omega t}$ on surface $S = S_1 + S_2 + S_3$



SOURCES (Q)
ORIGIN OF COORDINATES (O)
NORMAL (n)
INFINITESIMAL RADIUS (ε)
VOLUME (V)
SURFACES (S_1, S_2, S_3, S')

S_2 IS CLOSED AT INFINITY

Figure 1. Diffracting System[1]

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

when $\psi = \psi(r)$ alone.

$k = \omega / c$ = wave number

ω = angular frequency of radiation

c = velocity of light

It is assumed that ψ has continuous first- and second-order partial derivatives inside and on surface S. If ψ' is any other function which satisfies the same continuity requirements, Green's Theorem[2] holds and

$$\int_V (\psi \nabla^2 \psi' - \psi' \nabla^2 \psi)\, dv = \oint_S \left(\psi \frac{\partial \psi'}{\partial n} - \psi' \frac{\partial \psi}{\partial n} \right) ds = 0$$

[1]Ref. 2, p. 375 and Ref. 3, p. 280.

[2]Ref. 4, p. 82, and Ref. 5, p. 22.

if $(\nabla^2 + k^2)\,\psi' = 0$ also. Numerous functions satisfy the wave equation, but $\psi' = \frac{e^{ikr}}{r}$, or $\frac{e^{-ikr}}{r}$, conforms to the physics of the problem and satisfies the radiation condition.

If $\psi' = \frac{e^{ikr}}{r}$, then ψ' is not analytic due to the singularity at $r = 0$, and the continuity conditions for Green's Theorem are violated. The situation is corrected by surrounding the singularity with an infinitesimally small sphere S' of radius ε for which $n \equiv r$. Instead of $\oint_s = 0$, $\oint_s + \oint_{s'} = 0$ because of the singularity.

The Kirchhoff Integral Theorem follows immediately.

$$\oint_s = -\oint_{s'} = -\oint_{s' \text{ as } \epsilon \to 0} \left[\psi \frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right) - \left(\frac{e^{ikr}}{r}\right)\frac{\partial \psi}{\partial r}\right] ds'$$

$$= -\int_{\Omega \text{ Solid Angle}} \left[\psi \frac{e^{ik\epsilon}}{\epsilon}\left(ik - \frac{1}{\epsilon}\right) - \frac{e^{ik\epsilon}}{\epsilon}\frac{\partial \psi}{\partial \epsilon}\right] \epsilon^2 \, d\Omega$$

$$= +\int_{\Omega \text{ Solid Angle}} \psi \, d\Omega = 4\pi\, \psi(x', y', z')$$

$$\boxed{\psi(x', y', z') = \frac{1}{4\pi}\oint_s \left[\psi \frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right) - \left(\frac{e^{ikr}}{r}\right)\frac{\partial \psi}{\partial n}\right] dS}$$

It is noteworthy that the theorem was derived without any explicit reference to frequency or to the curvature of S. The description of the boundaries S_1 and S_2 was indefinite. In particular, the physical aspects of the boundaries were not discussed. It is possible to construct a vector analogue directly from the Kirchhoff Integral Theorem above, however, it is instructive to follow the historical developments in optics to obtain a better insight into the physics of the problem before making this step.

The Huygens-Fresnel principle (1818) included periodicity in both space and time, and represented the first significant explanation of diffraction phenomena. Heuristic arguments were invoked in the original derivations based upon this principle which was, in fact, detached from physical reality. Kirchhoff's Integral Theorem (1882), as derived above, embodies the basic ideas of the Huygens-Fresnel principle and actually reduces to the latter for many special cases. The Kirchhoff Integral Theorem is superior since it is more inclusive and simpler. Many texts on optics now proceed with the derivation of the so called Fresnel-Kirchhoff diffraction formula,

$$\psi(x', y', z') = \frac{-iA}{2\lambda}\iint \frac{e^{ik(r+s)}}{rs}\left[\cos(n, r) - \cos(n, s)\right] ds ,$$

Where A is a constant, r is the distance from a source to an aperture point, s is the distance from aperture point to observer and ds is on the aperture alone. The physical picture is that of Figure 1. Good discussions on the limitations and assumptions of the diffraction formula can be found in the literature, and these are binding on the Kirchhoff Integral Theorem and its vector analogue. They are worth reviewing.

In the derivation of the Kirchhoff diffraction formula, S_2 is usually taken to be a planar, opaque screen extending to infinity. If the problem is one in optics, S_1 is large in terms of the operating wavelength. Kirchhoff originally assumed that

(1) $\psi = 0$ and $\frac{\partial \psi}{\partial n} = 0$ on the interior of S_2;

(2) the field at S_1 is identical with the incident field, and is unaffected by the presence of the opaque boundary.

Fundamental analytic errors are involved in this procedure! If $\psi = \frac{\partial \psi}{\partial n} = 0$ over S_2, then there is a discontinuity about the contour C, which is the boundary between S_1 and S_2, and Green's Theorem is violated. The electromagnetic field cannot be represented by a single scalar wave function, but is characterized by a set of such functions which gives the components of the electric and magnetic fields, $\bar{E}$ and $\bar{H}$. Finally, Maxwell's equations should be satisfied, but the solution is defective in this respect also, at this stage of the development, since the field equation for $\bar{E}$ cannot be obtained from the curl of the expression for $\bar{H}$, and conversely. The computation of ψ behind S_2, by Kirchhoff's methods, can lead to non-zero values if the wavelength is sufficiently large. This is contrary to the initial assumption. In spite of these difficulties, the classical Kirchhoff theory leads to satisfactory solutions for many diffraction problems.[1]

The success of the Kirchhoff approach in physical optics problems is due to the fact that the size and radius of curvature of the aperture S_1 is large in terms of wavelengths, polarization is usually ignored, only power density is recorded (on film, usually), phase is ignored, and most of the diffracted radiation is thrown forward so that the assumption $\psi = \frac{\partial \psi}{\partial n} = 0$ is largely satisfied. The adaptation of the theory to radio-frequencies must, therefore, be made with discretion and the original assumptions of the theory should be reviewed upon application to physical problems.

VECTOR EXTENSION OF THE KIRCHHOFF INTEGRAL THEOREM

The formal vector extension of the Kirchhoff Integral Theorem is obtained directly from the scalar theorem

$$\psi(x', y', z') = \frac{1}{4\pi} \oint_S \left[\psi \frac{\partial}{\partial n}\left(\frac{e^{ikr}}{r}\right) - \left(\frac{e^{ikr}}{r}\right)\frac{\partial \psi}{\partial n}\right] dS .$$

Let

$$G = \frac{1}{4\pi} \frac{e^{ikr}}{r} .$$

Then

[1]Ref. 6, pp. 461-464

$$\psi(x', y', z') = \oint_s \left[G \frac{\partial \psi}{\partial n} - \psi \frac{\partial G}{\partial n}\right] dS = \oint_s [G\bar{n} \cdot \nabla\psi - \psi\bar{n} \cdot \nabla G] \, dS$$

which is still a scalar formulation. If $\psi \rightarrow \bar{E}$,

$$\bar{E}(x', y', z') = \oint_s [G(\bar{n} \cdot \nabla)\bar{E} - \bar{E}(\bar{n} \cdot \nabla)G] \, dS$$

which is the vector formulation[1] for the electric field. $\bar{E}$ is a complex vector, representing the illumination of the aperture whose diffraction pattern is to be calculated.

The vector transformations which lead to the field equations[2] are straightforward, with the exception of a single dyad operation. A detailed derivation is included in Appendix A as the manipulations are somewhat tedious. The result is:

$$\bar{E}(x', y', z') = -\oint_s [i\omega\mu \, (\bar{n} \times \bar{H})G + (\bar{n} \times \bar{E}) \times \nabla G + (\bar{n} \cdot \bar{E}) \nabla G] \, dS \, ,$$

(closed)

for the electric field and by similar process, or from the principle of duality

$$\bar{H}(x', y', z') = +\oint_s [i\omega \varepsilon (\bar{n} \times \bar{E})G - (\bar{n} \times \bar{H}) \times \nabla G - (\bar{n} \cdot \bar{H}) \nabla G] \, dS$$

(closed)

for the magnetic field.

DISCONTINUOUS SURFACE DISTRIBUTIONS

The application of Green's Theorem in the derivation of the scalar Kirchhoff Integral Theorem makes it necessary that the illumination ψ be continuous and have continuous first- and second-order partial derivatives. All of the derivations

[1]Ref. 6, p. 464 and Ref. 3, p. 283

[2]Ref. 6, p. 469, equations (29) and (31)

up to this point of the development have tacitly assumed that this was so, and that the integration was over a closed surface as in Figure 1. The formal vector extension must now be modified since $\bar{E}$ and $\bar{H}$ will be discontinuous at the edge of reflecting surfaces. Kottler has given a contour distribution consistent with the requirements of the problem.[1] A discontinuity in the tangential components of $\bar{E}$ and $\bar{H}$ in passing from a physical surface to a void implies an abrupt change in the surface current density. This can be accounted for, according to Kottler, by an accumulation of charge on the contour.

Boundary conditions at the surface are[2]:

(1) $\bar{n} \cdot (\bar{B}_2 - \bar{B}_1) = 0$

(2) $\bar{n} \cdot (\bar{D}_2 - \bar{D}_1) = \eta$ (reserving ω for angular frequency)

(3) $\bar{n} \times (\bar{E}_2 - \bar{E}_1) = 0$

(4) $\bar{n} \times (\bar{H}_2 - \bar{H}_1) = \bar{K}$.

The subscripts identify the opposite sides of a boundary as shown in Figure 2.

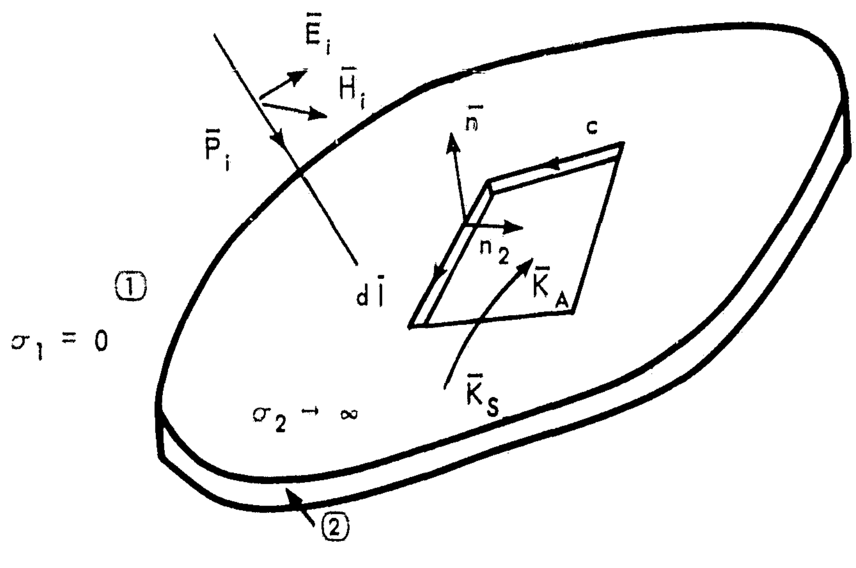


Figure 2. Contour at Edge of Reflecting Surface

The figure shows region A as a void, and region S as a perfectly conducting surface, however, the dual of this could be assumed and would lead to the same

[1] Ref. 6, p. 468

[2] Ref. 6, p. 35 η (surface charge density) and $\bar{K}$ (sheet current) can exist only when one σ (conductivity) is unbounded.

basic arguments. Following Kottler,[1] the discontinuity in the sheet current equals the line charge accumulation on the contour.

$$\bar{n}_2 \cdot (\bar{K}_A - \bar{K}_S) = i\omega\nu$$

The $\bar{K}_A$ and $\bar{K}_S$ are obtainable via boundary relationships:

$$\bar{K}_A = \bar{n} \times (\bar{H}_2 - \bar{H}_1) = \bar{n} \times (\bar{H}_i - \bar{H}_i) = 0 ,$$

and

$$\bar{K}_S = \bar{n} \times (\bar{H}_2 - \bar{H}_1) = \bar{n} \times (0 - 2\bar{H}_i) = -2\bar{n} \times \bar{H}_i .$$

Then

$$\bar{n}_2 \cdot (\bar{K}_A - \bar{K}_S) = \bar{n}_2 \cdot (2\bar{n} \times \bar{H}_i) = 2\bar{H}_i \quad \bar{n}_2 \times \bar{n} = 2\bar{H}_i \cdot d\bar{l}_1 = \bar{H}_1 \cdot d\bar{l}_1 ,$$

where

$$\bar{n}_2 \times \bar{n} = d\bar{l}_1$$

is in the direction of $d\bar{l}$.

Since the physical phenomena involves line charges, the radial vector ∇G is annexed directly to $\nu = \frac{\bar{H}_1 \cdot d\bar{l}_1}{i\omega}$, recalling the previous results for surface charge distributions. A factor $\frac{1}{\epsilon}$ is also required for the development of an electric field in the present system of units. The total effect is obtained from a line integral along Contour C. The complete Kottler field correction terms are given in Stratton[2] as

[1]Ref. 7, p. 484

[2]Ref. 6, p. 468

$$\bar{E}(x', y', z') = -\frac{1}{i\omega\varepsilon} \oint_c \nabla G \, \bar{H}_1 \cdot d\bar{l}$$

and

$$\bar{H}(x', y', z') = \frac{1}{i\omega\mu} \oint_c \nabla G \, \bar{E}_1 \cdot d\bar{l} .$$

The latter integral is obtained by similar process or directly by the principle of duality.

CALCULATION OF THE SCATTERED FIELD

The results of the vector extension of Kirchhoff's Integral Theorem are now recalled and joined to the Kottler contour integrals for discontinuous surface distributions. The surface integrals are taken over an open surface (S_1) and the hypothetical illuminating fields $\bar{E}$ and $\bar{H}$ are given subscripts (1) to emphasize the fact that the incident fields $\bar{E}_i$ and $\bar{H}_i$ have interacted with the metallic surface. Further notational changes are now introduced so that engineering forms will be available and comparison with available examples on applications will be facilitated. Using

$$\psi = \frac{e^{-jkr}}{r} ,$$

the field equations satisfying Maxwell's equations

$$\nabla \times \bar{E} + \frac{\partial \bar{B}}{\partial t} = 0$$

and

$$\nabla \times \bar{H} - \frac{\partial \bar{D}}{\partial t} = \bar{J}$$

become:

$$\overline{E}(x', y', z') = -\frac{1}{j\omega\epsilon}\frac{1}{4\pi}\oint_c \nabla\psi\,\overline{H}_1 \cdot d\overline{l}$$

(1)
(Kottler)

$$-\frac{1}{4\pi}\int_{s_1}\left[j\omega\mu(\overline{n}\times\overline{H}_1)\psi + (\overline{n}\times\overline{E}_1)\times\nabla\psi + (\overline{n}\cdot\overline{E}_1)\nabla\psi\right] dS,$$

(2) (3) (4)
(Kirchhoff)

$$\overline{H}(x', y', z') = \frac{1}{j\omega\mu}\frac{1}{4\pi}\oint_c \nabla\psi\,\overline{E}_1 \cdot d\overline{l}$$

(5)
(Kottler)

$$+\frac{1}{4\pi}\int_{s_1}\left[j\omega\epsilon(\overline{n}\times\overline{E}_1)\psi - (\overline{n}\times\overline{H}_1)\times\nabla\psi - (\overline{n}\cdot\overline{H}_1)\nabla\psi\right] dS .$$

(6) (7) (8)
(Kirchhoff)

If the scattered field is to be calculated for metallic surfaces with unbounded conductivity, then boundary conditions can be examined for the surface integrals and some simplification results. From the preceding discussion the fields behind the physical surface vanish ($\overline{E}_2 = \overline{H}_2 = 0$). Also, $\epsilon_1 = \epsilon_2 = 1$, $\mu_1 = \mu_2 = 1$, $\sigma_1 = 0$, $\sigma_2 \to \infty$ in the constitutive equations[1] $\overline{D} = \epsilon_r\epsilon_0\,\overline{E}$, $\overline{B} = \mu_r\mu_0\,\overline{H}$, $\overline{J} = \sigma\overline{E}$.

Then, for this special case, the general equations can be simplified since

$$\overline{n}\times(\overline{E}_2 - \overline{E}_1) = -\overline{n}\times\overline{E}_1 = 0$$

and

$$\overline{n}\cdot(\mu_2\overline{H}_2 - \mu_2\overline{H}_1) = -\mu_1\overline{n}\cdot\overline{H}_1 = 0$$

Surface integrals (3), (6), and (8) can be set to zero. An examination of the line integral (5) shows that $\overline{E}_1 \cdot d\overline{l}$ refers to the tangential component of the total

[1]Ref. 6, p. 10

field in medium (1). The third boundary condition, $\bar{n} \times (\bar{E}_2 - \bar{E}_1) = \underline{0}$ states that the tangential electric fields are continuous at the boundary. Since $\bar{E}_2 = 0$, the tangential electric field in medium (1) must vanish. That is $\bar{E}_1 \neq 0$, but $\bar{n} \times \bar{E}_1 = 0$ and $\bar{E}_1 \cdot d\bar{l} = 0$. By this argument integral ⑤ must also vanish.[1]

An examination of the operator (∇) provides additional insight to the computational aspects of the problem. Since

$$\psi = \frac{e^{-jkr}}{r}$$

is a function of (r) alone, take

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{1}_1 = \frac{\partial\psi}{\partial r}\hat{1}_r \,.^2$$

Then

$$\nabla\psi = \nabla\left(\frac{e^{-jkr}}{r}\right) = -\left(jk + \frac{1}{r}\right)\psi\,\hat{1}_r$$

for the near-field case and

$$\nabla\psi = -jk\psi\,\hat{1}_r$$

for the far-field case. Vector $\nabla\psi$ is a local radial vector on dS. The radial and transverse components of $\bar{E}(x', y', z')$ and $\bar{H}(x', y', z')$ can be identified prior to calculation for the far-field case since the Cartesian components of all the $\hat{1}_r$ vectors are equal when the field point is taken at infinity, or at very large distances.

For the distant $\bar{E}(x', y', z')$ field, integrals ① and ④ must be radial vectors due to the fact that the direction of $\nabla\psi$ is a unique radial vector. In general, integral ② will contain transverse vector components $(\hat{1}_\theta, \hat{1}_\phi)$ and a radial $(\hat{1}_r)$ component. As the point of the observer (the point at which the pattern is

[1]Ref. 1, p. 147
[2]Ref. 4, p. 97; Ref. 8, Table II

calculated) moves toward infinity the $\frac{1}{r^2}$ terms become negligibly small with respect to the terms that diminish as $\frac{1}{r}$. The summation of the radial components, proportional to $\frac{1}{r}$ for integrals ①, ②, and ④ can be expected to vanish also, leaving only transverse electric fields in the limit as $r \to \infty$.

For the $\overline{H}(x', y', z')$ field, only integral ⑦ contributes to the solution. It is known that the cross-product of two vectors is a vector which is orthogonal to the given vectors. The direction of vector $\nabla\psi$ is a unique radial vector for the far-field case, and $(\overline{n} \times \overline{H}) \times \nabla\psi$ is, therefore, orthogonal to $\hat{1}_r$, or transverse. Only transverse magnetic fields remain in the limit as $r \to \infty$.

The general formulation, therefore, reduces to:

$$\overline{E}(x', y', z') = \frac{1}{4\pi j\omega\varepsilon} \oint_c \nabla\psi \overline{H}_1 \cdot d\overline{l} - \frac{1}{4\pi} \int_{s_1} [j\omega\mu (\overline{n} \times \overline{H}_1)\psi + (\overline{n} \cdot \overline{E}_1)\nabla\psi] \, dS ,$$

(Kottler) (Kirchhoff)

$$\overline{H}(x', y', z') = -\frac{1}{4\pi} \int_{s_1} (\overline{n} \times \overline{H}_1) \times \nabla\psi \, dS ;$$

(Kirchhoff)

which is applicable to near- and far-field problems for perfectly conducting surfaces.

ANALYSIS AND NUMERICAL METHODS

The application of the field equations for $\overline{E}(x', y', z')$ and $\overline{H}(x', y', z')$ to a variety of antenna configurations presents several problems. These will be discussed in detail in succeeding documents, and only the general approach will be outlined here. If the reflectors which are to be considered can be described by a parametric or other suitable representation, then the normals to the surface $(\overline{n})$ and the differential areas (dS) can frequently be derived in a straightforward manner. The illuminating fields at the surface $(\overline{E}_i, \overline{H}_i)$ are calculated next. A polarization vector is needed for the source, and six degrees of freedom (three due to translation, three due to rotation) can be visualized for the placement and orientation of that source. Space divergence from source to surface (proportional to $\frac{1}{r}$) is included.

The incident fields are allowed to interact with the surface according to the boundary conditions, local normal $(\bar{n})$, etc. and $\bar{E}$, and $\bar{H}$, are established for the evaluation of the integrals for $\bar{E}(x', y', z')$. Conventional analysis (vector analysis, complex algebra, differential geometry) appears adequate up to this point, but the evaluation of the integrals for arbitrary antenna configurations is directed toward numerical methods.

The difficulties associated with the integration phase of the problem are already evident.[1]

SUMMARY

This document provides a theoretical basis for the diffraction-pattern calculations performed by the Antenna Systems Branch, Advanced Development Division of the Goddard Space Flight Center. The vector Kirchhoff theory, with Kottler correction for finite (open) surfaces, was presented without reference to particular geometric forms or antenna configurations. Subsequent reports will present detailed derivations of subroutines for $\bar{n}$, dS, etc. When using ellipsoidal, paraboloidal, spheroidal, hyperboloidal, conical and other surfaces. Composite and distorted surfaces will be considered, together with dual reflector systems. The relation of the various subroutines to a modular-type computer program will be discussed in these reports. Direct and numerical integration techniques will be presented. Examples of radiation patterns obtained with the IBM 7094 and IBM 360 Mod. 91 computers will be appended in the form of a series of plots from an SC-4020.

ACKNOWLEDGMENTS

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[1]Ref. 9 and Ref. 10

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APPENDIX A

VECTOR TRANSFORMATIONS FOR THE KIRCHHOFF INTEGRAL

$$\overline{E}(x', y', z') = \left[G(\overline{n} \cdot \nabla)\overline{E} - \overline{E}(\overline{n} \cdot \nabla)G\right] dS$$

$$(\overline{n} \cdot \nabla) G\overline{E} = G(\overline{n} \cdot \nabla)\overline{E} + \overline{E}(\overline{n} \cdot \nabla)G$$

$$[\ ] = (\overline{n} \cdot \nabla) G\overline{E} - 2\overline{E}(\overline{n} \cdot \nabla)G$$

Using

$$\overline{A} \times (\overline{B} \times \overline{C}) = (\overline{A} \cdot \overline{C})\overline{B} - (\overline{A} \cdot \overline{B})\overline{C},$$

$$\overline{n} \times (\overline{E} \times \nabla G) = (\overline{n} \cdot \nabla G)\overline{E} - (\overline{n} \cdot \overline{E})\nabla G$$

$$\nabla G \times (\overline{n} \times \overline{E}) = (\nabla G \cdot \overline{E})\overline{n} - (\nabla G \cdot \overline{n})\overline{E}.$$

Then

$$[\ ] = (\overline{n} \cdot \nabla) G\overline{E} - \overline{n} \times (\overline{E} \times \nabla G) - (\overline{n} \cdot \overline{E})\nabla G + \nabla G \times (\overline{n} \times \overline{E}) - (\nabla G \cdot \overline{E})\overline{n}.$$

Using

$$\nabla \times (\phi \overline{A}) = \phi \nabla \times \overline{A} + \nabla\phi \times \overline{A},$$

$$\nabla \times (G\overline{E}) = G\nabla \times \overline{E} + \nabla G \times \overline{E}.$$

Using

$$\nabla \cdot (\phi A) = \phi\nabla \cdot \bar{A} + \nabla\phi \cdot \bar{A} ,$$

$$\nabla \cdot (G\bar{E}) = G\nabla \cdot \bar{E} + \nabla G \cdot \bar{E} ,$$

and

$$G\nabla \cdot \bar{E} = 0$$

in the absence of <u>free</u> charge.

Then

$$[\ ] = (\bar{n} \cdot \nabla)\, G\bar{E} + \bar{n}x\left(\nabla\, x(G\bar{E})\right) - \bar{n}\left(\nabla \cdot (G\bar{E})\right)$$

$$- \bar{n}x\left(G(\nabla\, x\, \bar{E})\right) - (\bar{n} \cdot \bar{E})\nabla G + \nabla G x(\bar{n}\, x\, \bar{E}) .$$

Using the integral relationships (Ref. 6, Appendix II)

(1) $$\int_v (\nabla \cdot \bar{A})\, dv = \oint_s \bar{A} \cdot \bar{n}\, dS$$

(2) $$\int_v (\nabla\, x\, \bar{A})\, dv = \oint_s (\bar{n}\, x\, \bar{A})\, dS$$

(3) $$\int_v \nabla\phi\, dv = \oint_s \phi\bar{n}\, dS$$

and making the correspondences

(1) $\bar{A} \rightarrow (\nabla G\bar{E})$, (2) $\bar{A} \rightarrow \nabla x\, (G\bar{E})$, (3) $\phi \rightarrow \nabla \cdot (G\bar{E})$

obtain Jackson's (Ref. 3, p. 284) result

(1) $$\oint_s (\nabla G\bar{E}) \cdot \bar{n}\, dS = \int_v \nabla \cdot (\nabla G\bar{E})\, dv$$

(2) $$\oint_s \bar{n}\text{x} \left(\nabla \text{ x } (G\bar{E})\right)\, dS = \int_v \nabla \text{ x } (\nabla \text{ x } G\bar{E})\, dv$$

(3) $$\oint_s \bar{n} \left(\nabla \cdot (G\bar{E})\right) dS = \int_v \nabla\left(\nabla \cdot (G\bar{E})\right)\, dv \text{ .}$$

Apparently Jackson uses $(\bar{n} \cdot \nabla)G\bar{E} = \bar{n} \cdot (\nabla G\bar{E})$ where the latter is a dyadic form (second-order tensor). Relationships in Morse and Feshbach (Ref. 11, p. 65) and Coburn (Ref. 12, p. 46, p. 113) show that this is admissible. In general, the distributive rule of vector multiplication holds so that

$$\bar{n} \cdot (\nabla\bar{F}) = \hat{i} \cdot (\hat{i}\hat{i}) \frac{\partial F_x}{\partial x} n_x + \hat{i} \cdot (\hat{i}\hat{j}) \frac{\partial F_y}{\partial x} n_x + \hat{i} \cdot (\hat{i}\hat{k}) \frac{\partial F_x}{\partial x} n_x$$

$$+ \hat{j} \cdot (\hat{j}\hat{i}) \frac{\partial F_x}{\partial y} n_y + \hat{j} \cdot (\hat{j}\hat{j}) \frac{\partial F_y}{\partial y} n_y + \hat{j} \cdot (\hat{j}\hat{k}) \frac{\partial F_z}{\partial y} n_y$$

$$+ \hat{k} \cdot (\hat{k}\hat{i}) \frac{\partial F_x}{\partial z} n_z + \hat{k} \cdot (\hat{k}\hat{j}) \frac{\partial F_y}{\partial z} n_z + \hat{k} \cdot (\hat{k}\hat{k}) \frac{\partial F_z}{\partial z} n_z$$

and

$$\bar{n} \cdot (\nabla\bar{F}) = (\bar{n} \cdot \nabla)\bar{F} .$$

Then

$$\oint_s \left[(\bar{n} \cdot \nabla)G\bar{E} + \bar{n}\text{x}\left(\nabla\text{x}(G\bar{E})\right) - \bar{n}\left(\nabla \cdot (G\bar{E})\right)\right] dS =$$

$$\int_v \left[\nabla^2 (G\bar{E}) + \nabla \text{ x } \nabla \text{ x } (G\bar{E}) - \nabla\left(\nabla \cdot (G\bar{E})\right)\right] dv = 0$$

since

$$\nabla \text{ x } \nabla \text{ x } \bar{V} \equiv \nabla(\nabla \cdot \bar{V}) - \nabla^2\bar{V} .$$

The vector extension may be rearranged to read as follows:

$$\bar{E}(x', y', z') = -\oint_s \left[\bar{n} \times \left(G(\nabla \times \bar{E})\right) + (\bar{n} \cdot \bar{E})\nabla G + (\bar{n} \times \bar{E}) \times \nabla G \right] dS .$$
(closed)

The texts of Stratton and Jackson use $G = \frac{e^{+ikr}}{r}$ and the time-varying term $e^{-i\omega t}$ consistently. Silver uses $G = \frac{e^{-jkr}}{r}$ and $e^{+j\omega t}$. The present development follows the former so that[1]

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}(t)}{\partial t} = -\mu \frac{\partial \bar{H} e^{-i\omega t}}{\partial t} = i\omega\mu\bar{H}$$

$$k^2 = \omega^2 \mu \varepsilon , \qquad c = (\mu_0 \varepsilon_0)^{-1/2} .$$

Then

$$\underline{E}(x', y', z') = -\oint_s \left[i\omega\mu (\bar{n} \times \bar{H})\, G + (\bar{n} \times \bar{E}) \times \nabla G + (\bar{n} \cdot \bar{E})\, \nabla G \right] dS$$
(closed)

which is part of Stratton's equation (29).[2]

[1]Ref. 6, p. 23
[2]Ref. 6, p. 469

APPENDIX B

UNITS, DIMENSIONS, AND PHYSICAL QUANTITIES

The M.K.S. system of units (Ref. 6, Appendix I) will be used. Units and dimensions for the field equations are tabulated below.

Table 1

Units and Dimensions

Quantity	Symbol	Dimensions	M.K.S. Unit
Charge	q	Q	Coulomb
Current	I	$T^{-1} Q$	Ampere
Charge density	ρ	$L^{-3}Q$	Coulomb/cubic meter
Current density	$\bar{J}$	$L^{-2}T^{-1}Q$	Ampere/square meter
Conductivity	σ	$M^{-1}L^{-3}T\,Q^2$	Mho/Meter
Electric potential	ϕ	$ML^2T^{-2}\,Q^{-1}$	Volt
Electric field intensity	$\bar{E}$	$MLT^{-2}\,Q^{-1}$	Volt/meter
Dielectric displacement	$\bar{D}$	$L^{-2}\,Q$	Coulomb/square meter
Inductive capacity	ϵ	$M^{-1}L^{-3}T^2Q^2$	Farad/meter
Magnetic flux	ϕ	$ML^2T^{-1}Q^{-1}$	Weber
Flux density	$\bar{B}$	$MT^{-1}Q^{-1}$	Weber/square meter
Magnetomotive force	m.m.f.	$T^{-1}Q$	Ampere-turn
Magnetic field intensity	$\bar{H}$	$L^{-1}T^{-1}Q$	Ampere-turn/meter
Permeability	μ	MLQ^{-2}	Henry/meter

The eight integrals of the complete formulation for $\bar{E}(x', y', z')$ and $\bar{H}(x', y', z')$ are examined to verify that the dimensions are correct throughout. Certain factors are then singled out and associated with physical quantities for the case when $\sigma_1 \to 0$, $\sigma_2 = \infty$.

Integral ①

$$-\frac{1}{i\omega\epsilon}\frac{1}{4\pi}\oint_c \nabla\psi\bar{H}_1 \cdot d\bar{l} \Rightarrow (T)(ML^3T^{-2}Q^{-2})(L^{-2})(L^{-1}T^{-1}Q)(L) \Rightarrow MLT^{-2}Q^{-1}$$

where $\left(\frac{\bar{H}_1 \cdot d\bar{l}}{\omega}\right) \Rightarrow (T)(L^{-1}T^{-1}Q)(L) \Rightarrow Q \Rightarrow$ electric charge.

Integral ②

$$-\frac{1}{4\pi}\int_{s_1} j\omega\mu(\bar{n} \times \bar{H}_1)\psi dS \Rightarrow (T^{-1})(MLQ^{-2})(L^{-1}T^{-1}Q)(L^{-1})(L^2) \Rightarrow MLT^{-2}Q^{-1}$$

where $(\bar{n} \times \bar{H}_1) \Rightarrow L^{-1}T^{-1}Q \Rightarrow$ electric sheet current $\bar{K}$ (Ref. 6, p. 37).

$\bar{K}$ is defined as

$$\bar{J}\Delta l \Big|_{\Delta l \to 0,\ \bar{J} \to \infty,\ \sigma_2 \to \infty} .$$

Integral ③

$$-\frac{1}{4\pi}\int_{s_1}(\bar{n} \times \bar{E}_1) \times \nabla\psi dS \Rightarrow (MLT^{-2}Q^{-1})(L^{-2})(L^2) \Rightarrow MLT^{-2}Q^{-1}$$

where $(\bar{n} \times \bar{E}_1) \Rightarrow MLT^{-2}Q^{-1} \Rightarrow$ magnetic sheet current $\bar{K}_m$ (Ref. 1, p. 68).

Integral (4)

$$\frac{-1}{4\pi} \int_{s_1} (\bar{n} \cdot \bar{E}_1) \nabla\psi dS \Rightarrow (MLT^{-2}Q^{-1})(L^{-2})(L^2) \Rightarrow MLT^{-2}Q^{-1}$$

where $(\bar{n} \cdot \bar{E}_1) \Rightarrow MLT^{-2}Q^{-1}$, but $(\bar{n} \cdot \bar{D}) = \varepsilon(\bar{n} \cdot \bar{E}) \Rightarrow L^{-2}Q \Rightarrow$ electric surface-charge density η (Ref. 1, p. 67), and $\sigma_2 \to \infty$.

Integral (5)

$$\frac{1}{j\omega\mu} \frac{1}{4\pi} \oint_c \nabla\psi \bar{E}_1 \cdot d\bar{l} \Rightarrow (T)(M^{-1}L^{-1}Q^2)(L^{-2})(MLT^{-2}Q^{-1})(L) \Rightarrow L^{-1}T^{-1}Q$$

where $\frac{(\bar{E}_1 \cdot d\bar{l})}{\omega} \Rightarrow (T^1)(MLT^{-2}Q^{-1})(L) \Rightarrow ML^2T^{-1}Q^{-1} \Rightarrow$ magnetic charge.

Integral (6)

$$\frac{1}{4\pi} \int_{s_1} j\omega\varepsilon(\bar{n} \times \bar{E}_1) \psi dS \Rightarrow (T^{-1})(M^{-1}L^{-3}T^2Q^2)(MLT^{-2}Q^{-1})(L^{-1})(L^2) \Rightarrow L^{-1}T^{-1}Q$$

where $(\bar{n} \times \bar{E}_1) \Rightarrow MLT^{-2}Q^{-1} \Rightarrow$ magnetic sheet current $\bar{K}_m$ as before.

Integral (7)

$$-\frac{1}{4\pi} \int_{s_1} (\bar{n} \times \bar{H}_1) \times \nabla\psi dS \Rightarrow (L^{-1}T^{-1}Q)(L^{-2})(L^2) \Rightarrow L^{-1}T^{-1}Q$$

where $(\bar{n} \times \bar{H}_1) \Rightarrow L^{-1}T^{-1}Q \Rightarrow$ electric sheet current $\bar{K}$ as before.

Integral (8)

$$-\frac{1}{4\pi} \int_{s_1} (\bar{n} \cdot \bar{H}_1) \nabla\psi dS \Rightarrow (L^{-1}T^{-1}Q)(L^{-2})(L^2) \Rightarrow L^{-1}T^{-1}Q$$

where $(\bar{n} \cdot \bar{H}_1) \Rightarrow L^{-1} T^{-1} Q$, but $(\bar{n} \cdot \bar{B}_1) = \mu (\bar{n} \cdot \bar{H}_1) \Rightarrow (MLQ^{-2}) (L^{-1} T^{-1} Q) \Rightarrow MT^{-1} Q^{-1} \Rightarrow$ magnetic flux density.