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Moshe A. Milevsky
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The Calculus of Retirement Income

Financial Models for Pension Annuities and Life Insurance

This book introduces and develops—from a unique financial perspective—the basic actuarial models that underlie the pricing of life-contingent pension annuities and life insurance. The ideas and techniques are then applied to the real-world problem of generating sustainable retirement income toward the end of the human life cycle. The roles of lifetime income, longevity insurance, and systematic withdrawal plans are investigated within a parsimonious framework. The underlying technology and terminology of the book are based on continuous-time financial economics, merging analytic laws of mortality with the dynamics of equity markets and interest rates. Nonetheless, the text requires only a minimal background in mathematics, and it emphasizes examples and applications rather than theorems and proofs. *The Calculus of Retirement Income* is an ideal textbook for an applied course on wealth management and retirement planning, and it can serve also as a reference for quantitatively inclined financial planners. This book is accompanied by material on the Web site ⟨www.ifid.ca/CRI⟩.

Moshe A. Milevsky is Associate Professor of Finance at the Schulich School of Business, York University, and the Executive Director of the IFID Centre in Toronto, Canada. He was elected Fellow of the Fields Institute in 2002. Professor Milevsky is co-founding editor of the *Journal of Pension Economics and Finance* (published by Cambridge University Press) and has authored more than thirty scholarly articles in addition to three books. His writing for popular media received a Canadian National Magazine Award in 2004. He has lectured widely on the topics of retirement income planning, insurance, and investments in North America, South America, and Europe, and he is a frequent guest on North American television and radio.

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Financial Models for Pension Annuities and Life Insurance

MOSHE A. MILEVSKY
Schulich School of Business

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