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THE CALIBRATION OF STELLAR CONVECTION THEORIES

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SUMMARY

Any formula used to calculate the temperature gradient in a stellar convection zone must be calibrated, for example by evolving $1 M_{\odot}$ stellar models to fit the present age, luminosity and effective temperature of the Sun. When this procedure is followed for various convection theories, including those of Öpik and Böhm-Vitense, the corresponding models become almost indistinguishable. In particular, they predict the same depth, around 150 000 km, for the solar convective zone.

I. INTRODUCTION

The ionization of hydrogen and helium produces convective zones in cooler stars like the Sun. Since a fully predictive theory of convection is lacking (Spiegel 1971, 1972) workers on stellar structure are forced to rely on simple formalisms. The aim of this paper is to clarify the procedures that are followed.

In a main sequence stellar model most of the convective zone is adiabatically stratified. Between the adiabatic region and the radiative regions are thin transition regions upon whose properties the rest of the stellar model depends. The overall structure of the star depends only on the integrated properties of the transition zones and is insensitive to details of their stratification. Therefore one can get by with quite crude descriptions of convection, provided they are appropriately calibrated. In red giants, however, where the transition zones are much more extensive, the gross structure of a stellar model is correspondingly more sensitive to details, and a more accurate description of convection is required.

Of the various attempts to describe convection, that most often used in calculating the structure of stars grew out of the mixing-length approach devised by Taylor (1915, 1932), Schmidt (1925) and Prandtl (1925): in this theory fluid elements are supposed to move through a mixing length l and then to lose their identity. The application of these ideas to astronomy is principally due to Biermann (1932, 1945), who considered the adiabatic motion of bubble-like elements with diameters of order l . Taking l comparable with the observed dimensions of photospheric granules, he estimated the depth of the solar convection zone to be about 100 000 km (Biermann 1938). Prandtl originally suggested that the mixing length should be some fraction of the depth of the convecting layer but it is now usually believed that, in a compressible fluid, elements are unlikely to maintain their identity over distances much greater than the density scale height (Öpik 1938; Biermann 1943; Schwarzschild 1961). Vitense (1953) found it most convenient

to adopt a mixing length proportional to the pressure scale height. The effect of lateral heat exchange was discussed by Woolley (1941), but Öpik (1950) was the first to produce a formula for the heat flux accounting for radiative losses that has been used to compute entire stellar models. His theory was based on a cellular model of convection, suggested by the solar granulation. Later, Biermann's description was generalized to allow for lateral radiative losses by Böhm-Vitense (Vitense 1953; Böhm-Vitense 1958) and her formalism has been widely followed since.

We confine our discussion to local theories of convection, which provide expressions for both the radiative and convective fluxes at a given level in terms of mean conditions at that level only. These descriptions can be extended to cover 'non-local' theories for which the convective flux depends on properties in the vicinity of the level. We do not consider global theories, where the convective flux at a particular level depends on the properties of the entire convective zone. The prescriptions of Böhm-Vitense, Öpik and Biermann are examples of local theories, and yield formulae for the heat flux that depend on the convective cell height or mixing length l .

In practice it is usually assumed that

$$l = \alpha H, \quad (1.1)$$

where H is the pressure scale height and α is a constant. The formulae are calibrated by determining the value of α that is consistent with astronomical observations. This calibration may be effected by comparing solar models evolved with different values of α . The mass, radius and luminosity of the Sun are known; its composition is somewhat uncertain. Model convective envelopes with the correct mass, radius and luminosity and any plausible values of the abundances Y and Z of helium and heavy elements can be computed for any value of the mixing length by integrating inwards from the surface (Baker & Temesváry 1966), but in general such an envelope cannot be matched to an interior that has evolved for about 5×10^9 yr on the main sequence. For an assumed Z , say, the constraint that the Sun is now in the correct position on the Hertzsprung–Russell diagram fixes both Y and α (*cf.* Schwarzschild 1958). (This discussion does not concern itself with the solar neutrino problem.)

Each convection theory requires its own calibration. Moreover, a published calibration is useless unless the formula used to determine the convective heat flux is precisely stated. (In different papers both Biermann and Böhm-Vitense have adopted different values for constants of order unity.) The computations presented in this paper use the Böhm-Vitense theory in the form recommended by Böhm-Vitense (1958) and summarized by Baker & Temesváry (1966), and the Öpik theory as presented by Öpik (1950) and summarized by Mullan (1971a). Using Böhm-Vitense's (1958) formalism we find that for $Z = 0.02$ we must choose $Y = 0.245$ and $\alpha = 1.1$, and that the depth of the convection zone is about 150 000 km.

Recently Mullan (1971a) has computed model convection zones using Öpik's cellular description with $\alpha = 1$. Both Öpik's and Böhm-Vitense's theories assume plausible, though different, values of various constants of order unity and, taken at face value, the convective transport predicted by Öpik is less than that of Böhm-Vitense for the same l under the conditions prevalent throughout most of the solar convection zone. The comparatively low efficiency of Öpik's theory yields a convection zone with solar surface properties which is only about 10 000 km deep

when $l = H$. Some interesting consequences of such a shallow convection zone have been discussed by Mullan (1971b, 1972a, b).

The choice of mixing length adopted by Mullan is not necessarily consistent with the evolutionary history of the Sun. In this paper we apply to Öpik's theory the calibration procedure outlined above to determine the value of α . We find that stellar models of one solar mass do not evolve to the present solar luminosity and effective temperature if $\alpha = 1$; that Öpik's theory produces a consistent solar model (with $Z = 0.02$) when $\alpha = 2.4$; and that the model then has a convection zone about 150 000 km deep. Thus, if a proper calibration procedure is followed, both Öpik's and Böhm-Vitense's theories yield essentially the same results.

This concordance is not fortuitous: the bubble and cell descriptions are very similar. A local theory of convection relates the heat flux (measured by a dimensionless parameter N) to the mean superadiabatic temperature gradient (measured by a dimensionless parameter S), at any level. The prescriptions used in computing stellar models ignore the influence of Reynolds stresses on the mean structure and assume that the convective heat flux at any level depends on only a single length scale l . On simple physical grounds any such theory should tend asymptotically to the power laws,

$$N-1 = \begin{cases} AS^2, & S < (B/A)^{2/3} \\ BS^{1/2}, & S > (B/A)^{2/3} \end{cases} \quad (1.2)$$

for suitably chosen constants A and B , as we show in the next section. Both Öpik's and Böhm-Vitense's theories satisfy this criterion, and provide smooth, but different connections in the region between the two asymptotic limits. In stellar models this region is very thin, so the precise structure of the connection has little astrophysical significance. We have computed models using equation (1.2), which is simply two straight lines on a log-log plot, with A and B chosen to agree with the asymptotic limits of either Öpik's or Böhm-Vitense's formulae. The convection zones in these models are barely distinguishable from those computed using the detailed formulae.

This result clarifies the calibration procedure. The power laws (1.2) contain two free parameters, A and B . In principle these might be determined from a complete theory of convection. Since such a theory is lacking, we fix one of these parameters (B) by demanding that the Sun evolve to its present state. This leaves a one-parameter family of models. Inspection of these models shows that $N \propto S^2$ over only a very thin region of the convective zone and that for a given A the corresponding value of B adjusts itself slightly so that the depth of the convective zone remains virtually the same. In fact, in a solar model with $Z = 0.02$ the convection zone is adequately described by setting $\alpha = 1$ and $N = 1 + 0.20 S^{1/2}$ throughout. In Biermann's application of Prandtl's theory, $N-1 \propto S^{1/2}$ for all S . Of course radiative losses are important near the photosphere, but this calibration procedure provides no justification for departing from Biermann's original approach.

2. PROCEDURES FOR DESCRIBING CONVECTION IN STARS

In the local theories of convection used for stars the heat transport at each level depends on a single characteristic length l , which represents the dominant scale of coherent motion. In specific theories l is a cell height, a bubble diameter or a mean free path (mixing length). In the limits when convection is either very efficient

or very inefficient the asymptotic functional forms for the convective heat flux are independent of the precise details of the theory. Indeed, these forms can be obtained by elementary order of magnitude arguments.

The first step in the development of a mixing length theory is to assume that fluctuations are small. The equations of motion can be decomposed into mean (horizontally averaged) and fluctuating parts. The mean envelope is assumed to be determined by the equation of hydrostatic support, from which the Reynolds stresses are normally omitted, the mean energy conservation equation, which expresses the fact that the energy transported across a level surface is independent of height, and an equation of state. The equations for the fluctuations are linearized in fluctuations of thermodynamic state variables, and products of such fluctuations and the velocity. They may be written

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p' + \mathbf{g}\rho', \quad (2.1)$$

$$\rho \frac{D\rho'}{Dt} + \text{div } \rho\mathbf{u} = 0, \quad (2.2)$$

$$\rho c_p \frac{DT'}{Dt} - \delta \frac{Dp'}{Dt} = \rho c_p \beta w + \text{div} (K\nabla T') + \text{div} (K'\nabla T), \quad (2.3)$$

$$\frac{\rho'}{\rho} = -\delta \frac{T'}{T} + \epsilon \frac{p'}{p}, \quad (2.4)$$

where primes denote fluctuations and all unprimed variables except the velocity are assumed to be mean quantities. Here t is time, \mathbf{u} is the velocity and w its vertical component, $D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material time derivative, \mathbf{g} is the gravitational acceleration and p , T , ρ are pressure, temperature and density. The superadiabatic gradient β is given by

$$\beta = -\left[\frac{dT}{dr} - \left(\frac{\partial T}{\partial p} \right)_{\text{ad}} \frac{dp}{dr} \right], \quad (2.5)$$

where r is radius and the suffix ad implies that the thermodynamic derivative is taken at constant specific entropy; $\delta = -(\partial \ln \rho / \partial \ln T)_p$, $\epsilon = (\partial \ln \rho / \partial \ln p)_T$ and $K = \rho c_p \kappa$, where the thermal diffusivity

$$\kappa = \frac{4acT^3}{3\chi\rho^2c_p}. \quad (2.6)$$

with c_p the specific heat at constant pressure, χ the Rosseland mean opacity, a the radiation density constant and c the velocity of light. The fluctuations are assumed to be optically thick, though it is straightforward to generalize to arbitrary optical thickness, using a local treatment of radiative transfer such as the Eddington approximation (e.g. Unno & Spiegel 1966); molecular and radiative viscosity and perturbations in gravity have been ignored.

The second step is to ignore pressure fluctuations entirely, arguing that the motion is sufficiently slow for pressure readjustment to take place. Within this approximation the average convective heat flux may be written

$$F_c = \rho c_p \overline{wT'}, \quad (2.7)$$

where the overbar denotes a horizontal average, and the fluctuation equations (2.1)–(2.4) simplify considerably. It is the purpose of a convection theory to express F_c in terms of the horizontally averaged properties of the atmosphere.

Consider first the dynamics of the buoyant fluid. The vertical velocity w can be estimated by equating the work done by the buoyancy forces as the fluid rises through a height l either to the kinetic energy gained by the fluid or to the degradation of kinetic energy by the non-linear inertial terms. Since pressure fluctuations are ignored

$$\rho w^2 \simeq |\rho'| g l \simeq g \rho (\delta/T) l |T'|, \quad (2.8)$$

where g is the magnitude of \mathbf{g} . In each specific theory this relation incorporates a constant of order unity which depends on the assumed geometry of the flow and the precise argument leading to (2.8). The quantities appearing in equation (2.8) are characteristic or average values over the trajectory of a convective element. In a local theory these are equated to the values at the level at which F_c is to be evaluated; this is valid provided l is much less than the scale heights of the mean quantities.

The treatment of the thermal energy equation is generally more involved but becomes quite simple in the two extremes, when convection is either very efficient or very inefficient. In the former limit thermal diffusion may be ignored in (2.3). The temperature fluctuation generated by vertical motion through the superadiabatic environment is destroyed only by turbulent mixing, which is considered to operate either continuously via a turbulent eddy diffusivity wl or instantaneously after a fluid element has travelled through a distance l . In either case

$$|T'| \simeq \beta l; \quad (2.9)$$

once again there is a factor of order unity depending on the particular theory. Combining (2.8) and (2.9) gives

$$w^2 \simeq g(\delta/T) \beta l^2. \quad (2.10)$$

In the opposite extreme almost all the temperature gained or lost by rising or falling through the superadiabatically-stratified environment is destroyed by thermal diffusion. The assumption that l is much less than the scale heights of the mean quantities implies that the second of the two diffusion terms in equation (2.3) is small compared with the first. Thus there is an approximate balance between the remaining two terms on the right-hand side of that equation and

$$T' \simeq \beta w l^2 / \kappa, \quad (2.11)$$

subject to yet another factor of order unity. Comparing (2.11) with (2.9) we see that the characteristic length l in (2.9) is replaced by $w l^2 / \kappa$, the distance traversed in the thermal decay time l^2 / κ . From (2.8) and (2.11) we now have

$$|w| \simeq g(\delta/T) \beta l^3 / \kappa. \quad (2.12)$$

To calculate the convective heat flux we note that the temperature fluctuation is positive for rising fluid and negative for falling fluid, so all the motion contributes positively to F_c . Hence

$$F_c \simeq \rho c_p w T', \quad (2.13)$$

though once again there is an opportunity to introduce a factor of order unity to account for incomplete correlation between w and T' and for the possibility that

there is almost stationary fluid between rising and falling elements. Using the estimates (2.9) and (2.10), we obtain from (2.13)

$$F_c \simeq \rho c_p [g(\delta/T) \beta l^2]^{1/2} \beta l = S^{1/2} \rho c_p \kappa \beta, \quad (2.14)$$

where the dimensionless parameter

$$S = \frac{g(\delta/T) \beta l^4}{\kappa^2} \quad (2.15)$$

equals the product of the Prandtl number and a locally-defined Rayleigh number based on l , and is the square of the ratio of the thermal diffusion time to the free-fall time under reduced gravity ($l\beta\delta/T$) g . This is essentially the simple result of Prandtl (1932) that was first applied by Biermann (1932) to stellar convection zones. When the estimates (2.11) and (2.12) are used instead, (2.13) becomes

$$F_c \simeq \rho c_p [g(\delta/T) \beta l^3/\kappa]^2 \beta l^2/\kappa = S^2 \rho c_p \kappa \beta. \quad (2.16)$$

In obtaining these results we have assumed that fluctuations are small, that spatial variations are characterized by a single length scale l , and that mean quantities do not vary significantly over that length scale. In addition, pressure fluctuations were ignored. The assumption that l is small compared with the pressure scale height $H \equiv -(d \ln p/dr)^{-1}$ when applied to the dynamics of convection leads to the Boussinesq approximation (Spiegel & Veronis 1960; Malkus 1964). The Boussinesq equations for the fluctuations are (2.3) and (2.4) with p' and K' omitted, (2.1) unmodified, and $\text{div } \mathbf{u} = 0$, with the mean quantities regarded as constants. The fluctuations in the pressure gradient, retained in the Boussinesq approximation but neglected in the treatment above, are responsible for coupling the vertical and horizontal motion. Their gross effect, which may be estimated by taking the divergence of the momentum equation and using $\text{div } \mathbf{u} = 0$, is merely to modify the apparent inertia of a convective element and thereby to alter slightly the constant that would appear in (2.8) without changing the functional form. Thus the two asymptotic limits (2.14) and (2.16) are essentially Boussinesq results, based on plausible order of magnitude reasoning. In the Boussinesq limit (2.14) can be supported also by dimensional arguments. In practice astronomical calibrations yield values of l that are of the same order as H , and the conditions under which the Boussinesq approximation is justified are violated.

The total heat flux is the sum of the convective flux and the radiative flux:

$$F = F_c + F_r, \quad (2.17)$$

where

$$F_r = -\rho c_p \kappa \frac{dT}{dr} \equiv \rho c_p \kappa \beta + F_{r, \text{ad}}, \quad (2.18)$$

$F_{r, \text{ad}}$ being the contribution to the radiative flux due to the adiabatic temperature gradient. It is convenient to introduce as a dimensionless measure of the efficacy of convection the parameter

$$N = \frac{F - F_{r, \text{ad}}}{F_r - F_{r, \text{ad}}}, \quad (2.19)$$

this is simply the total excess heat flux (over that transported by radiation along the adiabatic gradient) measured in units of the excess radiative flux. In the absence

of convection $N = 1$. The two asymptotic laws (2.16) and (2.14) can now be written

$$N - 1 = AS^2 \quad (2.20)$$

and

$$N - 1 = BS^{1/2}, \quad (2.21)$$

where A and B are positive constants depending on geometry and the specific theory. The formula (2.20) holds when the thermal diffusion time is short compared with the reduced free-fall time ($S \ll (B/A)^{2/3}$); (2.21) applies when $S \gg (B/A)^{2/3}$. If the convective elements are optically thin, a law of the type (2.20) holds, except that the definition of S must be modified to allow for the fact that the thermal decay time is now $(\rho\chi)^{-2}/\kappa$ instead of l^2/κ . We shall not go into details because in any case we cannot determine the constant A .

Particular theories of convection must approach the asymptotic limits (2.20) and (2.21) for small and large S . They serve to provide values for the constants A and B and to provide a smooth connection between the two regimes. Two theories, due to Öpik (1950) and Vitense (1953), have been used to calculate the structure of convective zones.

Öpik (1950) described convection in terms of cylindrical cells of depth l and radius $\sqrt{2}l$. His results are summarized by the formula

$$N - 1 = \left(\frac{f}{1 + 4f^2} \right)^{3/2} S^{1/2}, \quad (2.22)$$

where f is a heat transfer parameter given by the unique real solution with $f > f_t$ of the cubic equation

$$f^3 - 2(f_t + 8S^{-1})f^2 + f_t^2f - 4S^{-1} = 0. \quad (2.23)$$

Here $f_t \simeq 0.10$ (Öpik 1950) is a turbulent transfer coefficient, determined experimentally by micro-meteorological observations in Tartu with a special hand-anemometer (Öpik 1967). In the limits of either very small or very large S , (2.22) and (2.23) reduce to (2.20) and (2.21) with the constants

$$A = 1.95 \times 10^{-3}, \quad B = 2.98 \times 10^{-2}. \quad (2.24)$$

The widely used theory of Böhm-Vitense (Vitense 1953; Böhm-Vitense 1958) generalizes Biermann's (1932) treatment of rising and falling bubbles which lose their identity after traversing a distance equal to their diameter l . This leads to the expression

$$N - 1 = \frac{729}{16} S^{-1} \left[\left(1 + \frac{2}{81} S \right)^{1/2} - 1 \right]^3. \quad (2.25)$$

The geometric constants in this formula are derived from Böhm-Vitense (1958) and differ slightly from those in Vitense (1953); others have added salt or pepper according to their taste. Once again, (2.25) reduces to the asymptotic forms (2.20) and (2.21) with

$$A = 8.57 \times 10^{-5}, \quad B = 1.77 \times 10^{-1}. \quad (2.26)$$

Fig. 1 shows $(N - 1)/B$ plotted logarithmically as a function of S from equations (2.22) and (2.23), and (2.25). The curves are therefore scaled so as to give the same asymptotic behaviour as $S \rightarrow \infty$; for small values of S the Öpik formula provides a relatively greater heat flux. The range of S in Fig. 1 has been chosen solely to display the transition between the two asymptotic regimes. In a typical

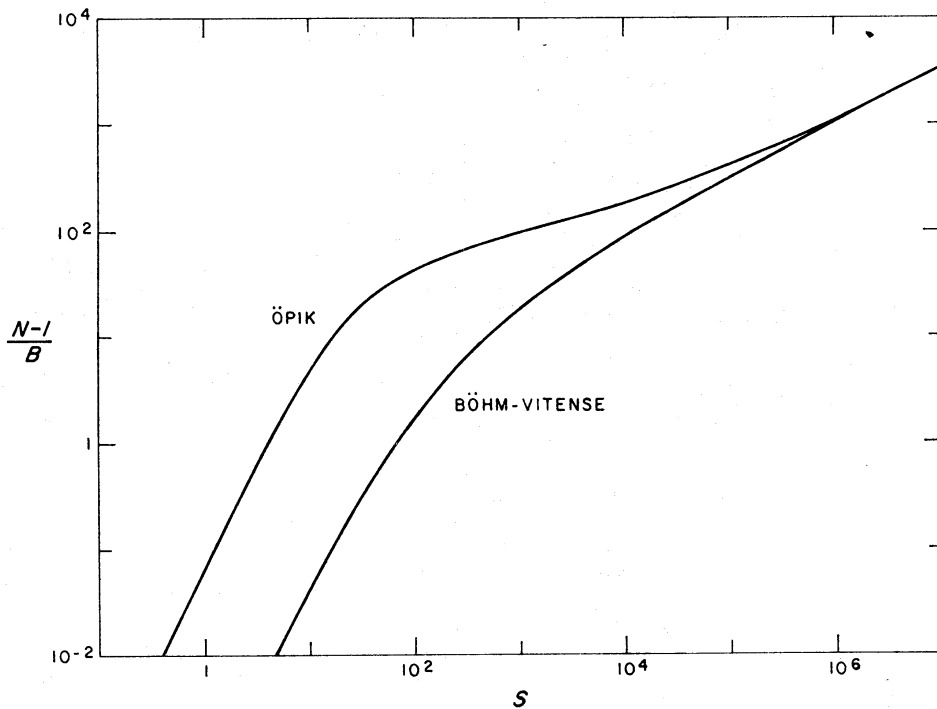


FIG. 1. Measures of the convective heat flux predicted by Öpik's and Böhm-Vitense's theories.

solar model based on Böhm-Vitense's theory, for example, $S = 100$ at a depth of about 45 km below the level at which the optical depth τ is $2/3$, increases to 10^4 at 100 km and 10^6 at 250 km; it reaches a maximum of about 10^{15} at a depth of 7000 km. So any deviations from the asymptotic laws apply only to an extremely thin layer in the convection zone. This is true also of solar models constructed with Öpik's theory; the transition between the asymptotic limits occurs over a wider range of S (see Fig. 1) but this is offset by a more rapid rise of S with depth. For the purpose of constructing solar type stellar models the principal difference between these two theories is just that between the values of A and B .

The distinction between the two theories arises not so much from the apparently different physical pictures as from the different assumptions in guessing constants of order unity in the approximations to the equations of motion, and from the assumed shape of the convective eddy or cell. For example, Öpik's cell is $2\sqrt{2}$ times wider than it is high whereas Böhm-Vitense assumes convective elements whose linear dimensions are similar in all directions. The estimates of the momentum and energy balances are crude and therefore uncertain; for example, Henyey, Vardya & Bodenheimer (1965) derive a heat transfer between convective elements and their surroundings that is a factor of about 10 below Vitense's value for a different but no less plausible temperature distribution. Indeed, if for example the ratio of cell width to height in Öpik's formulation is reduced by a factor 3, and the fluctuating heat transfer in Böhm-Vitense's formulation is reduced by a factor 12, the two theories have the same asymptotic limits as $S \rightarrow 0$ and $S \rightarrow \infty$, when Öpik's cell height is 2.44 Böhm-Vitense mixing lengths.

To close these theories the length scale l must be specified. This is usually taken to be a multiple of either the pressure or the density scale height. The pressure scale height H is more commonly used, mainly for computational convenience. Then $l = \alpha H$, where α is a constant of order unity. Thus the heat flux

predicted by any particular theory depends on the single arbitrary constant α . The asymptotic laws (2.20) and (2.21) can be re-expressed, to demonstrate explicitly their dependence on α , in terms of

$$S_H \equiv \alpha^{-4} S = \frac{g(\delta/T) \beta H^4}{\kappa^2}, \quad (2.27)$$

giving

$$N-1 \sim A\alpha^8 S_H^2 \equiv A_H S_H^2 \quad \text{as } S_H \rightarrow 0 \quad (2.28)$$

and

$$N-1 \sim B\alpha^2 S_H^{1/2} \equiv B_H S_H^{1/2} \quad \text{as } S_H \rightarrow \infty. \quad (2.29)$$

These equations are not in a form that is convenient for computing stellar models. What is needed is the mean temperature gradient as a function of the total flux F . Conventionally this is expressed in terms of the derivatives

$$\nabla \equiv \frac{d \ln T}{d \ln p}, \quad \nabla_{\text{ad}} \equiv \left(\frac{\partial \ln T}{\partial \ln p} \right)_{\text{ad}}, \quad \nabla_r \equiv \frac{FH}{\rho c_p \kappa T}. \quad (2.30)$$

Then equation (2.19) becomes

$$\frac{\nabla_r - \nabla_{\text{ad}}}{\nabla - \nabla_{\text{ad}}} = N, \quad (2.31)$$

and

$$S_H = \frac{g\delta H^3}{\kappa^2} (\nabla - \nabla_{\text{ad}}) \equiv S_0 (\nabla - \nabla_{\text{ad}}), \quad (2.32)$$

which defines S_0 . Substituting from (2.31) and (2.32) into (2.28) and (2.29) yields cubic equations which must be solved for ∇ in terms of S_0 , ∇_{ad} and ∇_r . Böhm-Vitense's formula (2.25) also leads to a cubic equation in ∇ , but the equation resulting from Öpik's formulation is somewhat more complicated. It can be shown that all these equations for ∇ have unique solutions in the allowable range (∇_{ad} , ∇_r).

3. METHOD OF CALIBRATION

Before implementing the theory the constant α must be determined. Unfortunately, laboratory experiments can provide no direct information about convection in a layer whose depth is much larger than the scale height, so the theory must be calibrated by comparison with astronomical observations. This is done either by constructing a solar model or by fitting a theoretical main sequence to a cluster diagram. The calibration depends on assuming a composition for the zero age main sequence, and is made uncertain by inaccuracies in opacities and nuclear reaction rates.

It is normal to start with a homogeneous zero age main sequence solar model and to evolve it to its present age (Fowler 1972) of about 4.8×10^9 yr. The appropriate abundances by mass, Y and Z , of helium and heavy elements for the zero age model are somewhat uncertain, and can therefore be adjusted, together with α , so that the model evolves to the present observed values of the effective temperature T_e and luminosity L . This method has been followed by various authors (e.g. Schwarzschild, Howard & Härm 1957; Schwarzschild 1958; Sears 1959, 1964; Demarque & Percy 1964; Weymann & Sears 1965) using different descriptions of convection. In practice, the luminosity is insensitive to the precise choice of α ; the density and temperature at the centre, where energy is generated, are hardly affected by changes in the structure of a convective envelope which contains only

a negligible proportion of the mass. So the value of L is used to determine a relation between Y and Z , leaving T_e to fix α .

The absolute abundances Z and Y are less certain. Some authors (e.g. Sears 1964; Bahcall, Bahcall & Shaviv 1968; Torres-Peimbert, Simpson & Ulrich 1969; Bahcall & Ulrich 1971) choose the value of X/Z , where $X = 1 - Y - Z$ is the hydrogen abundance, determined from spectroscopic observations (Gaustad 1964; Lambert 1967a, b, 1968; Lambert & Warner 1968a, b, c; Warner 1968). On the other hand, Iben (1968, 1969), for example, prefers to choose Y to be consistent with the abundance determinations from high mass stars. Since these procedures yield somewhat different values of Z we have chosen to present results for two values of Z (0.02 and 0.04) for which the Cox & Stewart (1969) opacity tables are available.

The procedure of trying to fit a computed main sequence to an observed cluster diagram or mass luminosity relation provides abundances that are not inconsistent with the results of the above methods together with a crude but independent check on α (Demarque & Larson 1964; Copeland, Jensen & Jørgensen 1970; Moss 1972).

4. SOLAR MODELS

We have calibrated local theories of convection principally by constructing solar models. These have been evolved from homogeneous ZAMS configurations using the evolution program of Eggleton (1971), appropriately modified to accommodate the different theories of convection. Nuclear reaction rates were computed from the cross-sections quoted by Bahcall & Sears (1972) and the ${}^7\text{Be}$ electron capture rate derived by Bahcall & Moeller (1969), and the opacities used were those of Cox & Stewart (1969). In addition, to obtain better resolution for the convection zones of particular models, we have used a program for computing stellar envelopes, written in collaboration with Professor N. H. Baker, which is descended from that described by Baker & Temesváry (1966).

For a given composition, models were evolved through 5×10^9 yr in about 30 steps and the value of X was adjusted until $L = L_\odot$ at $t \simeq 4.75 \times 10^9$ yr. For $Z = 0.02$, this yielded $X = 0.735$, $Y = 0.245$; for $Z = 0.04$, $X = 0.680$, $Y = 0.280$. No further adjustments of composition were necessary, though the lifetime t of a $1-M_\odot$ model is quite sensitive to composition variations: for $Z = 0.02$, $X = 0.735$ it was found that $(\partial \ln t / \partial \ln X)|_{L=L_\odot} \simeq 15$. The mixing length $l = \alpha H$ was then adjusted to produce an effective temperature $T_e = 5800$ K when $L = L_\odot$. Independently of the convection theory or the mixing length, it was found that $(\partial \ln T_e / \partial \ln \alpha)|_{L=L_\odot} \simeq 0.11$ for $Z = 0.02$ and 0.04 . The lifetime t is relatively insensitive to variations in the mixing length, for $(\partial \ln t / \partial \ln \alpha)|_{L=L_\odot} \simeq -0.05$, when $Z = 0.02$.

Fig. 2 shows the positions of various models on the zero-age main sequence. Although these positions depend both on the mixing length and on the convection theory used, two features are apparent. All the zero age models in the vicinity of the solar models lie close to a line with slope $d \log L / d \log T_e \simeq 0.22$. Moreover, for a given composition, all the models that evolved to the present Sun, regardless of the theory used, began near the same point. For $Z = 0.02$, this was at $\log (L/L_\odot) \simeq -0.1543$, $\log T_e \simeq 3.7526$; for $Z = 0.04$, $\log (L/L_\odot) \simeq -0.1540$, $\log T_e \simeq 3.7544$. The process of finding the correct mixing length was greatly accelerated by choosing ZAMS models in the right position. (A different evolution program might require a displacement in this position, though the same principle would still hold.)

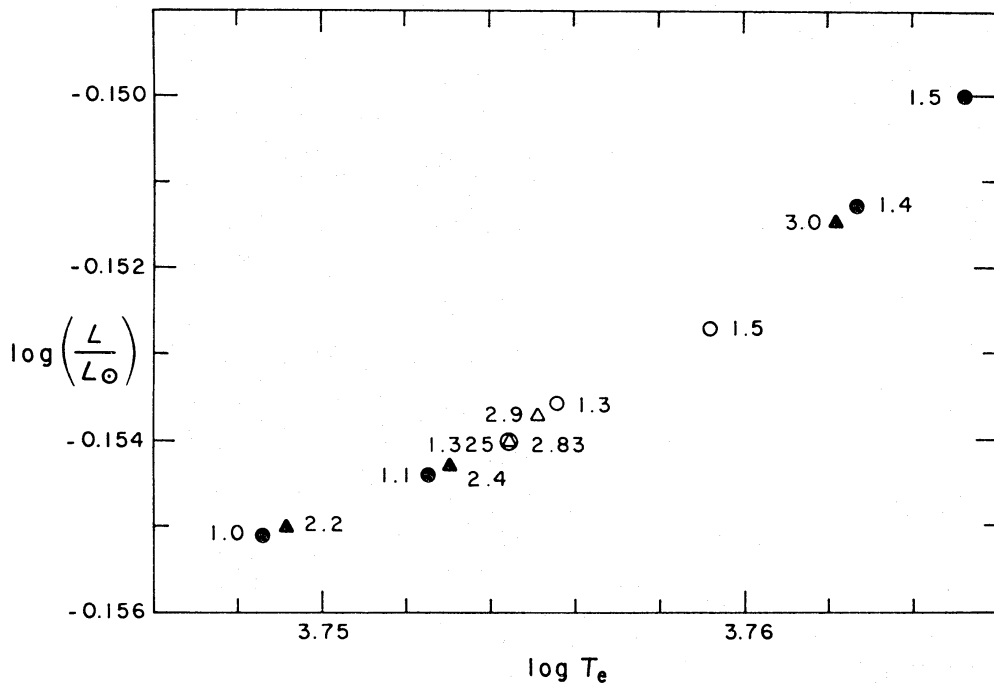


FIG. 2. Positions of various $1 M_{\odot}$ stellar models on the zero age main sequence. Models computed with Böhm-Vitense's (1958) convection formulae are represented by circles and those using Öpik's (1950) by triangles; filled symbols are for models with $Z = 0.02$, $X = 0.735$ and open symbols for $Z = 0.04$, $X = 0.680$. The number by each symbol denotes the value of $\alpha = l/H$.

Indeed, since the age at which $L = L_{\odot}$ is not quite independent of α , the value of X would have had to be corrected had it been determined initially using a value of α that was wrong by a large amount.

This procedure was carried out for the normal forms (2.22), (2.23) and (2.25) of both the Öpik and the Böhm-Vitense theories for $Z = 0.02$, 0.04 , and also for the asymptotic formulae (1.2) using the values (2.24) and (2.26) for A and B . The results are summarized in Table I. For intermediate compositions, corresponding values can be obtained by interpolation (which is no worse than repeating the calibrations with a linearly interpolated opacity table). Because the different theories yield different relations between N and S , different values of α are required to build solar models. For the commonly used Böhm-Vitense theory (in its 1958 form) we require $\alpha = 1.10$ for $Z = 0.02$ and $\alpha = 1.325$ for $Z = 0.04$. For the Öpik theory the corresponding values are 2.40 and 2.83 respectively. In solar models it is mainly the differences between the values of B in (2.24) and (2.26) that must be compensated for by changing α . The real differences between the theories are shown by the values of A_H , B_H defined in (2.28) and (2.29). For large S , when convection is efficient, the values of B_H differ by only 20 per cent. On the other hand, A_H changes by a factor of 10^4 . However, the superadiabatic gradient increases so rapidly that this difference can affect only a very narrow layer; when the asymptotic form of Öpik's theory was used with a mesh spacing of 0.015 in $\log p$ at the top of the convection zone, no mesh point was found to lie on the lower branch with $N \propto S^2$. So the values of A_H for Öpik's theory are hardly relevant and the asymptotic form becomes precisely equivalent to Prandtl's original formulation.

TABLE I
Calibrated solar models

Model	Böhm-Vitense	Böhm-Vitense asymptotic	Öpik	Öpik asymptotic
$Z = 0.02$				
α	1.10	1.06	2.40	2.50
A_H	1.84×10^{-4}	1.37×10^{-4}	2.15	2.98
B_H	0.214	0.199	0.172	0.186
D (km)	1.49×10^5	1.50×10^5	1.52×10^5	1.48×10^5
K	3.52×10^{-3}	3.51×10^{-3}	3.53×10^{-3}	3.48×10^{-3}
$Z = 0.04$				
α	1.325	1.265	2.83	3.00
A_H	8.14×10^{-4}	5.62×10^{-4}	8.04	12.81
B_H	0.310	0.283	0.239	0.268
D (km)	1.86×10^5	1.86×10^5	1.86×10^5	1.86×10^5
K	4.45×10^{-3}	4.45×10^{-3}	4.40×10^{-3}	4.39×10^{-3}

Table I also shows the depth D of the various convective zones. For fixed Z the variations in this depth are insignificant. We find that $D = 150\,000$ km for $Z = 0.02$ and $186\,000$ km for $Z = 0.04$. The lower part of the convection zone is adiabatically stratified and its state can be described by the specific entropy or the related parameter K defined by

$$p = KT^{(1/\nabla_{\text{ad}})}. \quad (4.1)$$

For a given composition, the value of K is independent of the convection theory used. The values of D and K are therefore determined by the calibration process, which fixes X and α .

These results confirm that the gross properties of the solar convection zone do not depend on the particular choice of convection theory, provided that the theory is calibrated so as to yield the appropriate adiabat. However, the detailed structure of the narrow superadiabatic region does depend on the relationship between N and S . In Fig. 3 we plot the superadiabatic gradient ($\nabla - \nabla_{\text{ad}}$) as a function of depth for the calibrated Böhm-Vitense and Öpik models. In the Böhm-Vitense formulation, where A_H is much smaller, radiative losses are important, F_c/F is smaller and $\nabla - \nabla_{\text{ad}}$ is correspondingly greater above a depth of about 100 km. This leads to a higher value of T at that depth. The difference is compensated at greater depths; $\nabla - \nabla_{\text{ad}}$ is smaller and B_H is larger than in the model based on Öpik's theory, and the two models become indistinguishable by a depth of about 2000 km.

Clearly the radiative interior cannot be sensitive to details of a superficial layer 2000 km deep, containing only $2 \times 10^{-8} M_\odot$. For any consistent model with a given value of Z the luminosity, effective temperature and depth of the convection zone will be virtually identical at all stages of evolution from the ZAMS to present solar values, as shown in Table I and Fig. 2. Since the initial homogeneous models are effectively the same it therefore suffices to evolve a single consistent model, based on a particular convection theory, to obtain a description of the Sun at present, and then to calibrate any other convection theories for the same value of Z by fitting new convective envelopes to the single radiative interior. All these models must then yield the unique depth of the convective zone that is compatible with the radiative interior. A reasonably satisfactory calibration of the solar model can be carried out using even the convective zero solution (Schwarzschild *et al.* 1957; Schwarzschild 1958).

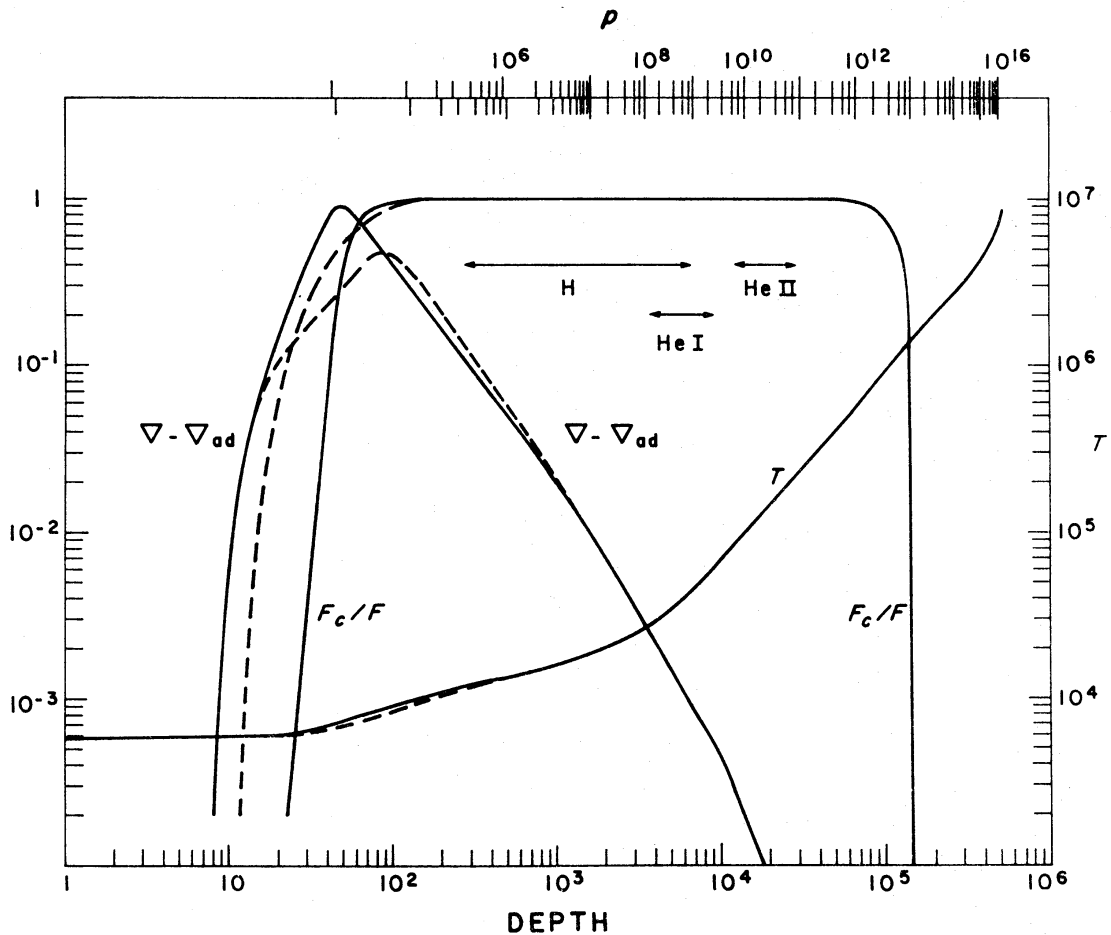


FIG. 3. The superadiabatic gradient $\nabla - \nabla_{ad}$, proportion of heat flux carried by convection F_c/F and temperature T (in K) as functions of depth (in km) and pressure (in dyne cm^{-2}) in two calibrated model solar envelopes. The models have composition $Z = 0.02$, $X = 0.735$ and represent the Sun at the present time. The solid curves and the lower pressure scale were computed with Böhm-Vitense's (1958) convection formulae, and the dashed curves and the upper pressure scale with Öpik's (1950). The left ordinate scale refers to $\nabla - \nabla_{ad}$ and F_c/F , the right to T . The extents of the ionization zones, from 10 to 90 per cent ionization, are indicated by the arrows.

Of course, the envelope can be fitted to a given interior only if the convection theory yields an adiabat that matches smoothly with the radiative zone. Thus a formalism (such as assuming that l is proportional to the distance from the boundary of the convective zone) that leads to a boundary layer at the interface requires an independent calibration.

The radiative interior is not so insensitive that any convective envelope can be used. If the wrong mixing length is taken to evolve a model until its luminosity $L = L_\odot$, with an inaccurate value of T_e , the proportional changes in the radius of the radiative zone and the thickness of the convective zone are comparable with the change in $\ln \alpha$. Thus, for Böhm-Vitense theory with $\alpha = 1.4$ (instead of 1.1) the radius of the radiative interior becomes 500 000 km. Any attempt to fit this model to the correct solar radius of 700 000 km would obviously require a convective zone 200 000 km deep, an increase of 30 per cent over the values in Table I.

Mullan (1971a) computed a solar envelope based on Öpik's theory with $\alpha = 1$ by integrating inwards from the observed surface boundary conditions, and found

that $D \simeq 10\,000$ km. For the same mixing length our program gives a depth of about $15\,000$ km for $Z = 0.02$. However, this choice of α is incompatible with the Sun's evolution. When $L = L_{\odot}$ the effective temperature would be about 6 per cent too high and the radius therefore 12 per cent too low. As we have seen, when appropriately calibrated, the theories of Öpik and Böhm-Vitense yield the same depth for the convective zone.

This depth depends only on the somewhat arbitrary choice of Z but is likely to be between $100\,000$ and $200\,000$ km. We can distinguish the principal factors that might affect the results given in Table I. There are differences between various programs used to follow stellar evolution which might allow a discrepancy of 5 per cent in the luminosity after 4.8×10^9 yr. This is accommodated by altering the abundance X (for fixed Z) and then adjusting the value of α , thereby changing α and D by about 5 per cent. The models are of course subject to uncertainties in the opacity and the nuclear reaction rates, which might alter the radiative core. (However, model envelopes based on the earlier opacity tables of Cox & Stewart (1965) yield values of D and K similar to those of Table I, provided $\alpha \simeq 1.5$.) But apparently the choice of a particular local formula for the convective heat flux is of little significance. For instance, some authors omit the factor δ in the equation of state (2.4), even though c_p is calculated accurately. This is conceptually incorrect but makes little difference to the calibrated models. Since δ appears only in the combination δl^4 , in S , its omission is equivalent to modifying the functional form of l . In the transition zone of the Sun, $\delta \approx 2$ and omitting this factor merely increases the calibrated mixing length by about 20 per cent. Indeed, the functional form for $N(S)$ hardly matters, provided it is monotonic increasing. As an extreme example, we have computed models using the asymptotic formula (2.20) throughout the entire convection zone. In order to obtain a consistent model with $Z = 0.02$ we need $A_H = 9.62 \times 10^{-28}$ which gives, for example, $\alpha = 1.35 \times 10^{-3}$ if A is chosen to be 8.57×10^{-5} ; nevertheless we find that $D = 150\,000$ km and $K = 3.49 \times 10^{-3}$, just as in Table I. If convection were assumed to be so efficient that the entire convective zone could be regarded as adiabatically stratified (Schwarzschild 1958), the radius could change only by an amount of order 2000 km (the depth of the significantly superadiabatic region), which is negligible.

The values of D and K might be different if a 'non-local' theory of convection (Ezer, Stein & Cameron 1963; Hofmeister & Weigert 1964; Böhm & Stückl 1967) which required a significant superadiabatic gradient at the base of the convective zone were used. Finally, any modification to the physics of stellar evolution that could explain a low solar neutrino flux (for example, spasmodic mixing of the central region) would also affect the calibration process.

5. OTHER STELLAR MODELS

We have computed model envelopes, using both the Öpik and the Böhm-Vitense formulae, for stars on the ZAMS and for red giants, though we have not attempted to evolve any non-solar models. The ZAMS models correspond to masses and luminosities in the ranges $0.8 < M/M_{\odot} < 1.6$, $0.5 < L/L_{\odot} < 5$, with parameters taken from Allen (1975). Using the calibration determined from solar models, there was no significant difference between the sequence of envelopes with $Z = 0.02$, $X = 0.735$ calculated using Öpik's theory (with $\alpha = 2.4$) and that obtained with Böhm-Vitense's theory ($\alpha = 1.1$). We infer that models calibrated by fitting the observed

slope of the low mass end of the ZAMS would yield the same values of α . Other authors (Iben 1963; Demarque & Larson 1964; Copeland *et al.* 1970; Moss 1972) have used this procedure to calibrate their descriptions of convection and obtained results consistent with the evolution of a solar model.

In the solar and stellar models discussed so far, convection is so efficient that radiative losses affect only a small proportion of the convective zone. With the Böhm-Vitense formulation, $S < (B/A)^{2/3}$ only in the topmost 40 km of the Sun's convective zone; with Öpik's theory the corresponding distance is less than 20 km. Thus the procedure described above can calibrate only the value of B . The value of A is undetermined, and free to vary over a wide range: if after calibrating the asymptotic Böhm-Vitense theory of equation (1.2) A is reduced by a factor of 10, the depth of the convective zone is increased by only 5 per cent; in the corresponding Öpik theory A is effectively infinite. So the various refinements to local convection theory, when applied to solar or near-solar models, offer no real improvement over Biermann's original theory, which corresponds to the case with A infinite and $B_H = 0.186$ ($B = 0.177$, $\alpha = 1.02$) for $Z = 0.02$, or to $B_H = 0.268$ for $Z = 0.04$, when appropriately calibrated.

Red giants are different: their entire convection zones can be non-adiabatically stratified (e.g. Schwarzschild 1975) rendering them more sensitive to details of the convection theory. This has been noticed by other authors who have attempted to calibrate convection theories or to assess the effects of varying l/H on red giants (e.g. Kippenhahn, Temesváry & Biermann 1958; Demarque & Geisler 1963; Hofmeister & Weigert 1964; Henyey *et al.* 1965; Auman & Bodenheimer 1967; Hofmeister 1967a, b). We computed three model envelopes for a star of $7 M_\odot$ with $Z = 0.02$ and values of L and T_e taken from Hofmeister, Kippenhahn & Weigert (1964). Even though the maximum convective flux exceeded 98 per cent, $\nabla - \nabla_{\text{ad}}$ was always of the same order as ∇ . Furthermore, the envelopes had extensive regions in which radiative losses from convective elements were important and the convection formula was essentially on the lower asymptotic branch (2.20). In particular, in the Böhm-Vitense model with $L = 4740 L_\odot$, the most extreme of our cases, $S < (B/A)^{2/3}$ throughout the lower 95 per cent by radius of the convection zone. On the assumption that B_H is the same for red giants as it is for the Sun, one might expect, therefore, a calibration of A_H to be possible. Models computed using the Öpik and Böhm-Vitense theories, calibrated from the Sun, predicted rather different convective fluxes in these regions of low S , and yielded convection zones that differed in depth by about 10 per cent (see Table II). However, the convective fluxes there were always a negligible fraction of the total, so that $\nabla \simeq \nabla_r$, and consequently the gross structures of the models were similar.

The question we must answer is: how sensitive are red giant evolutionary tracks to the convection theory adopted? Henyey, Vardya & Bodenheimer (1965) introduced into Böhm-Vitense's formalism three separate parameters: α and ν ,

TABLE II
Model envelopes for red giants

T_e (K)	4250	5320	5620
R/R_\odot	115.6	80.6	73.3
L/L_\odot	3860	4600	4740
D ($\times 10^6$ km)			
Öpik theory	15.5	4.78	4.06
Böhm-Vitense theory	17.4	4.21	3.69

which enter into the relation between velocity and the superadiabatic gradient, and y , which adjusts the radiative losses.* These parameters are related to A_H and B_H by the proportionalities $A_H \propto y^3(\alpha^4/\nu)^2$ and $B_H \propto (\alpha^4/\nu)^{1/2}$. Hence if y is held constant, varying α and ν at constant α^4/ν influences only the curve linking the two asymptotic branches (2.28) and (2.29); from Figs 2 and 7 of Henyey *et al.* we find that at constant luminosity and effective temperature $(\partial \ln \nu / \partial \ln \alpha)_y \simeq 3.9$, which confirms that the details of this transition curve are not significant. The parameter A_H can be adjusted independently of B_H by adjusting y , but Henyey *et al.* point out that this has little effect compared with varying α . The principal difference between the Böhm-Vitense and Öpik theories after calibration for the Sun lies in the factor of 10^4 in A_H , corresponding to a factor of about 20 in y . From Fig. 5 of Henyey *et al.* it appears that this would change the effective temperature at a given luminosity by only about 5 per cent, which is less than the observational uncertainty. At present, therefore, there is little prospect of using red giants to calibrate descriptions of convection with $S \lesssim 1$.

It might seem that red giant envelopes could be adequately described if A_H could be calibrated. This is not necessarily true. Even in a local convection theory of the type outlined in Section 2, the final formulae still depend on the assumption $l \propto H$, and any other assumption would lead to different results. For example, Hofmeister & Weigert (1964) showed that models using Böhm-Vitense's (1958) formulae with the mixing length equal to the density scale height are similar (though not identical) to those obtained by setting $\alpha = 1.5$ for the Sun but $\alpha = 2.1$ for red giants. Alternatively, one could obtain similar models by keeping α constant but adopting a different value for A_H . Local theories, moreover, provide a description of convection that is far from complete, and any simple non-local theory that appears to work in the Sun may not necessarily be valid in other stars.

6. DISCUSSION

We have argued that local theories of convection as used in stellar structure calculations depend on two free parameters and differ only in details that cannot be tested by observations. Moreover, all correctly calibrated solar models have the same radiative interior, regardless of the formulae used to compute the convective flux. The differences between various local theories of convection matter only where they predict a significantly superadiabatic stratification. In solar models this region is only about 1000 km thick. Fitting a different convective envelope onto the same interior might halve or double the thickness of the transition zone but the radius of the model would change by only 0.1 per cent and the mass by much less. Our calculations confirm that the adjustments necessary to restore the mass, radius and luminosity to solar values have little effect on its overall structure. Thus a main-sequence evolutionary calculation with any correctly calibrated local theory determines the structure of the radiative interior, the adiabatic part of the convection zone and, in particular, its depth. Any other convection theory can be calibrated by fitting a new convective envelope with identical composition to the same interior (*cf.* Spruit 1974). Though it might seem to someone interested primarily in stellar atmospheres that the adiabat is fixed by choosing a convection theory, in practice the convection theory must be calibrated by the adiabat. The

* Henyey *et al.* took some cognizance of turbulent pressure in their calculations and defined α to be the ratio of l to the scale height of total pressure.

envelope and the interior are parts of the same star. Any change in the physics used in calculating stellar structure (e.g. altering the opacity formula or the cross-sections for nuclear reactions) of course requires a recalibration of convection theories.

Much of the interest in constructing a convection theory that accounts for radiative losses has been motivated by a desire to construct a model solar atmosphere in agreement with the observed limb darkening, and to predict surface temperature and velocity fluctuations both in solar-type stars and in early-type main sequence stars (e.g. Woolley 1941; Vitense 1953; Spiegel 1960; Mullan 1971a; Travis & Matsushima 1973; Edmonds 1974; Spruit 1974). In principle, these observations too could provide a method of determining the parameter A . What is needed is an accurate description of convection around an optical depth $\tau = 2/3$, where all difficulties occur together. There are limits even to what one can attempt, and in the present state of convection theory it seems rash to rely upon results that depend on subtle details in models of the top of the convective zone. Qualitative agreement between mixing length theory and observations is the most that can be hoped for.

Since we cannot meaningfully distinguish between the details of different convection formulae which connect the two asymptotic limits of equation (1.2) in slightly different ways, it hardly seems necessary for an astrophysicist to follow the details of any particular convection model. He need only choose the formula joining those limits that is most convenient for computation. At first sight it would seem easiest to use whichever of the two simple formulae (2.20) and (2.21) produces the lower value of N for a given S_H . Each of these formulae, however, requires the solution of a cubic equation for ∇ . In practice, therefore, it is actually more straightforward to adopt the Böhm-Vitense formula (2.25), which yields only a single cubic equation. Equation (2.25) can then be regarded as a convenient interpolation formula, though it must be borne in mind that an uncalibrated factor of order unity should multiply S wherever it occurs. When this factor has been calibrated, the choice of constants in (2.25) leads to an estimate of A .

We have considered only local theories of convection. Mildly non-local modifications, such as taking the mixing length to be the shorter of a scale height and the distance from the edge of the zone, appear to have little effect (Hofmeister & Weigert 1964; Böhm & Stückl 1967). The consequence of including Reynolds stresses (*cf.* Henyey *et al.* 1965) might be more profound. The assumption that l/H is constant cannot be checked by experiment. However, an astronomically-calibrated mixing-length theory, with l proportional to a local scale height, does lead to broad agreement with observations. Discrepancies of detail still exist, but the mixing length procedure will be used until a demonstrably better theory is produced.

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