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The Canonical Class and the C^∞ Properties of Kähler Surfaces

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Abstract. We give a self contained proof that for Kähler surfaces with non-negative Kodaira dimension, the canonical class of the minimal model and the (-1)-curves are oriented diffeomorphism invariants up to sign. This includes the case $p_g = 0$. It implies that the Kodaira dimension is determined by the underlying differentiable manifold. We then reprove that the multiplicities of the elliptic fibration are determined by the underlying oriented manifold, and that the plurigenera of a surface are oriented diffeomorphism invariants. We also compute the Seiberg Witten invariants of all Kähler surfaces of non-negative Kodaira dimension. The proof uses a set up of Seiberg Witten theory that replaces generic metrics by the construction of a localised Euler class of an infinite dimensional bundle with a Fredholm section. This makes the techniques of excess intersection available in gauge theory.

Contents

A compact complex surface X with non-negative Kodaira dimension κ has a unique minimal model $X_{\min}$. The pullback of the canonical line bundle of the minimal model $\omega_{\min}$ is in some ways the most basic birational invariant of the surface, if only because it is the polarisation $\mathcal{O}(1)$ of the canonical model $\mathrm{Proj}(\oplus H^0(nK))$. It was conjectured by Friedman and Morgan that the cohomology class, $K_{\min} = c_1(\omega_{\min}) \in H^2(X, \mathbb{Z})$ is determined by the underlying oriented smooth manifold if $\kappa(X) \geq 0$ [FM1, Conj. 3]. Recently, Kronheimer, Mrowka and

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Tian, Yau proved this for minimal surfaces of general type with $p_g > 0$ [Ste]. While completing this manuscript, Friedman and Morgan posted a proof for the case $p_g = 0$ [FM3]. In the case of elliptic surfaces it was already known to be true by the joint effort of many people, as it is a direct consequence of the invariance of the multiplicities of the elliptic fibration.

The difference between minimal and non minimal surfaces is measured by the (-1)-curves. If $p_g > 0$, it is not hard to show using a little Donaldson theory that the invariance of $\pm K_{\min}$ implies that the homology classes of the (-1)-curves can be characterised up to sign as the ones which are represented by (-1)-*spheres*, i.e., smoothly embedded spheres with self intersection (-1) (the ex (-1)-curve conjecture [FM1, Conj 2,3, Prop. 4]).

Theorem 1. *If X is a Kähler surface of non-negative Kodaira dimension then*

1. *The class $K_{\min} \in H^2(X, \mathbb{Z})$ is determined by the underlying smooth oriented manifold up to sign,*
2. *every (-1)-sphere in X is $\mathbb{Z}$-homologous to a (-1)-curve up to sign.*

Corollary 2. *If a Kähler surface X has non-negative Kodaira dimension then every smooth sphere S with $S^2 \geq 0$ is $\mathbb{Z}$-homologous to 0.*

Corollary 3. *A Kähler surface is rational or ruled if and only if it contains a smooth sphere $S \neq 0 \in H^2(X, \mathbb{Z})$ with $S^2 \geq 0$.*

Corollary 4. *The Kodaira dimension of a Kähler surface is determined by the underlying differentiable manifold.*

The proof of Theorem 1 is based on fundamental work of Witten and Seiberg [Wit], who introduced a new set of non linear equations, the monopole equations. Using these equations allow one to define Seiberg Witten (SW) invariants, new oriented diffeomorphism invariants, similar in spirit to the Donaldson invariants, but much easier to handle both in practice and in theory. The simplest SW invariants are just the signed number of solutions to the monopole equations for generic values of the parameters (metric and some canonical perturbation). The monopole equations and the SW invariants, once specialised to the Kähler case, give exactly the right information to apply the method in [Br2] to prove the invariance of $K_{\min}$. Previously this required many strong and technical assumptions and relied on formidable technical machinery [KM1].

From the point of view of classification of surfaces, it is rather satisfactory that the nefness of $K_{\min}$ is what makes the proof work for Kodaira dimension $\kappa \geq 0$, what makes it fail for the rational and ruled case, and that the various levels of nefness (nef and big, nef but not big, torsion) is what makes for the difference in the different Kodaira dimensions. If $p_g = 0$, the higher plurigenera, and in particular P_2, play an essential role.

While proving the invariance of $K_{\min}$, we have to prove the invariance of (-1)-curves as well. This leads directly to the differentiable characterisation Corollary 3 of rational and ruled surfaces which are characterised algebraically by the existence of a smooth rational curve l with $l^2 \geq 0$ [BPV, Prop. V.4.3]. The invariance of the Kodaira dimension (the ex Van de Ven conjecture [VdV]) and the invariance of the plurigenera for surfaces of general type is then an immediate consequence of the invariance of $\pm K_{\min}$. The Van de Ven Conjecture had already been proved using

Donaldson theory (see [FM2] for all surfaces but rational surfaces and surfaces of general type with $p_g = 0$, and Friedman Qin [FQ] and Pidstrigatch [P-T],[Pi2] for the remaining case, see also [OT1] for an easy proof of the remaining case with Seiberg Witten theory).

To prove Theorem 1 we get away with a simple but useful ad hoc computation of the SW-invariants of classes "close to K_X" (Corollaries 31 and 32). Using an elegant argument of Stefan Bauer (Proposition 41), this is also enough to give yet another proof that for elliptic surfaces with finite cyclic fundamental group, the multiplicities of the elliptic fibration are determined by the underlying oriented manifold. The oriented homotopy type determines the multiplicities for other elliptic surfaces (see the first two chapters of [FM2], in particular Theorem S.7. Although these chapters consist of "classical" homotopy theory and algebraic geometry largely going back to Kodaira and Iitaka, this is now perhaps the most difficult and deepest part of the story). Together this implies:

Theorem 5. *Let $X \to C$ be an elliptic Kähler surface. Then the multiplicities of the elliptic fibration are determined by the underlying oriented smooth manifold. In particular, for Kähler elliptic surfaces, deformation type and oriented diffeomorphism type are the same notions.*

This theorem has been well established with Donaldson theory by the work of Bauer, Donaldson, Fintushel, Friedman, Iitaka, Kodaira, Kronheimer, Lisca, Morgan, Mrowka, O'Grady, Okonek, Pidstrigatch, Stern, Van de Ven and probably others. (See e.g., Chapter VII of [FM2] for a sample algebraic geometric, and e.g., [FS1]) for a sample cut and paste computation.)

Corollary 6. *The plurigenera of a Kähler surface are determined by the underlying oriented manifold.*

This corollary has been conjectured by Okonek and Van de Ven [OV]. Let me remark that it seems to be known that in the non-Kähler case, with the exception of the equivalence of deformation and diffeomorphism type of non Kähler elliptic surfaces, (where there can be a two to one discrepancy) all the previous statements are true as well, but seemingly for "classical" reasons like the homotopy type.

Inspired by results in the preprint of Friedman and Morgan, I realised how the results in this article give an easy proof of:

Corollary 7. *No Kähler surface of non-negative Kodaira dimension admits a metric of positive scalar curvature.*

For Kähler metrics the monopole equations reduce to the vortex equation which has been studied extensively by Bradlow [B1] and García Prada [Gar], and the moduli space of solutions can be completely described in algebraic geometric terms. However, Kähler metrics are not generic, and if we try to use this description to compute all the SW invariants of elliptic or ruled surfaces we encounter positive dimensional moduli spaces of solutions even if the virtual or expected dimension is zero. Following Pidstrigatch and Tyurin, we will define the SW invariant as a *localised Euler class* of an infinite rank bundle with a section with Fredholm derivative. Using this technique we will compute the SW invariants of elliptic surfaces and a SW blow up formula. The localised Euler class seems to be a useful and powerful notion which should be of independent interest.

In Section 1, we prove most of the corollaries and slightly abstract and generalise the relevant part of [Br2]. In Section 2 we introduce the localised Euler class. Logically it is needed for the definition of the SW invariants, but in practice it is largely independent of Sections 3, 4 and 5. In Section 3 we define the SW invariants. In Section 4 we study the monopole equations and SW invariants for Kähler manifolds. In Section 5 we then prove the main Theorem 1 and Corollary 7. Finally in Section 6 we compute the SW invariants of elliptic surfaces and prove a blow up formula.

While working on this article, a flood of information on the Seiberg Witten classes came in. The holomorphic interpretation of the monopole equations is already in Witten's paper [Wit], and it seems that several people have remarked that his work implies that the canonical class is invariant for minimal surfaces of general type with $p_g > 0$ because of the numerical connectedness of the canonical divisor. Kronheimer informed me that he, Fintushel, Mrowka,Stern and Taubes are working on a note containing among many other things the mentioned proof of the invariance of $K_{\min}$. The results and methods of the before mentioned paper [FM3] of Friedman and Morgan are rather similar to the present one. The main difference seems to be that they deal mostly with the case $p_g = 0$, and that they rely on chamber changing formulas and a detailed analysis of the chamber structure. They also use a stronger version of the blow up formula which allows them to prove a stronger version of Theorem 1.2: If a surface of Kodaira dimension $\kappa \geq 0$ has a connected sum decomposition $X \cong X'\#N$, where N is negative definite, then $H_2(N,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$ is spanned by (-1)-curves. We will indicate how this result follows from the present methods. Finally, Taubes shows that the results for Kähler surfaces are but the top of the iceberg. It seems that most results can be generalised to symplectic manifolds [Ta1],[Ta2].

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1. Preparation

We first prove the corollaries from the Main Theorems 1 and 5.

Proof. Corollary 2. Let S be a positive smooth sphere in a surface X with $\kappa(X) \geq 0$. Let $\tilde{X}$ be the blow up in $n = S^2+1$ points, then $H^2(\tilde{X},\mathbb{Z}) = H^2(X,\mathbb{Z}) \oplus \oplus_{i=1}^{n} \mathbb{Z}E_i$.

Now $e = S + E_1 + \cdots + E_n$ is represented by a (-1)-sphere. Hence there is a (-1)-curve E_0 on $\tilde{X}$ such that $e = \pm E_0 \in H_2(\tilde{X}, \mathbb{Z})$. Since (-1)-curves on a surface with $\kappa \geq 0$ are either equal or disjoint (cf. [BPV, prop. III.4.6]), either $n = 0$ and $S = \pm E_0$, or $n = 1$, $S = 0 \in H^2(X, \mathbb{Z})$, and $E_0 = E_1$, say. But the first possibility leads to the contradiction $E_0^2 \geq 0$. (Reducing non-negative spheres to (-1)-spheres is a well known trick, but I forgot where I read it precisely.)

Corollary 3 follows directly from Corollary 2.

Corollary 4. By the above, a Kähler surface is of Kodaira dimension $-\infty$ if it contains a non trivial (0)-sphere. Clearly all ruled surfaces contain one. To deal with $\mathbb{P}^2$, note that there is no surface with $b_+ = b_1 = 0$ [BPV, Thm. IV.2.6]. Thus diffeomorphisms between surfaces with $b_2 = 1$, $b_1 = 0$ are automatically orientation preserving. Then a surface diffeomorphic to $\mathbb{P}^2$ must contain a $(+1)$-sphere, and is therefore of Kodaira dimension $-\infty$. Since $b_2 = 1$ it must in fact be equal to $\mathbb{P}^2$. (Alternatively, use Yau's result that $\mathbb{P}^2$ is the only surface with the homotopy type of $\mathbb{P}^2$ [BPV, Theorem 1.1], but this is a deep theorem). We conclude that Kodaira dimension $-\infty$ can be characterised by just diffeomorphism type. Without loss of generality, we can therefore assume that $\kappa \geq 0$.

If $K_{\min}^2 > 0$, then X is of general type. If $K_{\min}^2 = 0$ and $K_{\min}$ is not torsion, then $\kappa(X) = 1$. Finally, if $K_{\min}$ is torsion, $\kappa(X) = 0$. This proves that Kodaira dimension is determined by the oriented diffeomorphism type. If X and Y are orientation reversing diffeomorphic, both are minimal: Otherwise, one of them would contain a positive sphere. Then necessarily either $K_X^2 = K_Y^2 = 0$, or both have $K_X^2, K_Y^2 > 0$, i.e., X and Y are of general type. Now copy the argument of [FM2, Lemma S.4]: For minimal surfaces with $\kappa = 0, 1$, the signature $\sigma = \dfrac{1}{3}(K^2 - 2e) \leq 0$. Thus $\sigma(X) = -\sigma(Y) = 0$, and $e(X) = e(Y) = 0$. In Kodaira dimension 0, this leaves only tori and hyperelliptic surfaces, which can fortunately be recognised by homotopy type [FM2, Lemma 2.7].

Corollary 6. Since $P_1 = p_g$ is an oriented topological invariant, we will henceforth assume that $n \geq 2$. We have to distinguish between the different Kodaira dimensions. For surfaces of general type (i.e., $\kappa = 2$), we argue as follows. The plurigenera P_n and $\chi(O_X)$ are birational invariants. Then by Ramanujan vanishing and Riemann Roch (cf. [BPV, corollary VII.5.6]) we have

$$P_n(X) = P_n(X_{\min}) = \tfrac{1}{2}n(n-1)K_{\min}^2 + \chi(\mathcal{O}_X) \tag{1}$$

Since $\chi(\mathcal{O}_X)$ is an oriented topological invariant the P_n are oriented diffeomorphism invariants in this case. For surfaces with Kodaira dimension 0 or 1 with a fundamental group that is not finite cyclic, we simply quote [FM2, S.7]. For surfaces with finite cyclic fundamental group, it follows from the invariance of the multiplicities and the canonical bundle formula which gives an explicit formula for $P_n(X)$ in terms of the multiplicities and $\chi(O_X)$. (See [FM2, Lemma I.3.18, Prop. I.3.22].) Finally, by definition, $P_n(X) = 0$ if $\kappa = -\infty$. $\square$

Here is an other easy corollary.

Corollary 8. *Every (-2)-sphere τ is orthogonal to $K_{\min}$. If there is a (-1)-curve E_1 such that $\tau \cdot E_1 \neq 0$, then there is a (-1)-curve E_2 such that $\tau = \pm E_1 \pm E_2 \in H_2(X, \mathbb{Z})$.*

Proof. Let R_τ be the reflection in τ. It is represented by a diffeomorphism with support in a neighborhood of τ. By the invariance of $K_{\min}$ up to sign, $R_\tau K_{\min} = K_{\min} + (\tau \cdot K_{\min})\tau = \pm K_{\min}$. But if $K_{\min} \neq 0 \in H^2(X, \mathbb{Q})$, then τ and $K_{\min}$ are independent, since $\tau^2 = -2$ and $K_{\min}^2 \geq 0$. Thus in either case $(\tau, K_{\min}) = 0$. Moreover if E_1 is a (-1)-curve then either $R_\tau E_1 = E_1$, $R_\tau E_1 = -E_1$, or there is a different (-1)-curve E_2 such that $R_\tau(E_1) = \pm E_2$. The first possibility gives $\tau \cdot E_1 = 0$, the second $(\tau \cdot E_1)^2 = 2$ i.e., is impossible, and the third $(\tau \cdot E_1) = \pm 1$. The statement follows. $\square$

It will be convenient to first prove the main Theorem 1 with (co)homology groups with $\mathbb{Q}$ coefficients, and later mop up to prove the theorem over $\mathbb{Z}$. Let X be a smooth oriented compact 4-manifold with $b_+ \geq 1$. Theorem 1 mod torsion is a formal consequence of the existence of a set of basic classes

$$\mathcal{K}(X) = \{K_1, K_2 \ldots\} \subset H^2(X, \mathbb{Z})$$

functorial under oriented diffeomorphism and having the following properties:

Properties $(*)$**.** If X is a Kähler surface of non-negative Kodaira dimension then

1. the K_i are of type $(1,1)$ i.e., represented by divisors,
2. if X is minimal, then for every Kähler form Φ, $\deg_\Phi(K_X) \geq |\deg_\Phi(K_i)|$,
3. if $\tilde{X} \xrightarrow{\sigma} X$ is the blow-up of a point $x \in X$, then $\sigma_*\left(\mathcal{K}(\tilde{X})\right) \subset \mathcal{K}(X)$.
4. every K_i is characteristic i.e., $K_i \equiv w_2(X) \pmod 2$,
5. $K_X \in \mathcal{K}$.

In case X is an algebraic surface we could replace item 2 by the weaker and more geometric requirement that $2g(H) - 2 \geq H^2 + |K_i \cdot H|$ for every very ample divisor H without changing the results. We will see later that Seiberg Witten theory will give property 2 for all Kähler surfaces with $\kappa \geq 0$, minimal or not. This should not be confused with a Thom conjecture type of statement, since our methods do not give information about the minimal genus for arbitrary smooth real surfaces in a homology class. It is also clearly impossible to have a degree inequality like property 2 for all Kähler forms if X is rational or ruled.

Recall that for algebraic surfaces, the Mori cone $\overline{\mathrm{NE}}(X) \subset H_2(X, \mathbb{R})$ is the closure of the cone generated by effective curves. It is dual to the nef (or Kähler) cone. In other words, the numerical equivalence class of a curve D lies in $\overline{\mathrm{NE}}(X)$ if and only if $H \cdot D \geq 0$ for all H ample. For a Kähler surface (X, Φ), it will be convenient to define the nef cone as the closure of the positive cone in $H^{1,1}(X) \subset H^2(X, \mathbb{R})$ spanned by all Kähler forms, and containing Φ. The Mori cone $\overline{\mathrm{NE}}$ is then just the dual cone in $H_2(X, \mathbb{R}) \cap H^{1,1^\vee}$ i.e.,

$$\overline{\mathrm{NE}} = \{C \in H^{1,1^\vee} \subset H_2(X, \mathbb{R}) \mid \int_C \omega \geq 0, \text{ for all Kähler forms } \omega \}.$$

(With this definition, a line bundle is nef iff for all $\epsilon > 0$, it admits a metric such that the curvature form F has $\dfrac{\sqrt{-1}}{2\pi} F \geq -\epsilon\Phi$. A class $\omega \in \overline{\mathrm{NE}}$ if there exists a sequence of closed positive currents of type $(1,1)$ converging to the dual of ω, i.e $\overline{\mathrm{NE}}$ is dual to N_{psef} in [Dem, Proposition 6.6]. We will freely identify homology and cohomology by Poincaré duality.

Lemma 9. *If a class $L \in H^{1,1}(X)$ satisfies $\deg_\Phi(K_X) \geq |\deg_\Phi(L)|$ for all Kähler forms Φ, then there is a unique decomposition of the canonical divisor $K_X = D_+ + D_-$ with D_+, $D_- \in \overline{\mathrm{NE}}(X)$ such that $L = D_+ - D_-$.*

Proof. Define $D_\pm = \dfrac{1}{2}(K_X \pm L)$. Then $K_X = D_+ + D_-$, $L = D_+ - D_-$, and $D_\pm \in \overline{\mathrm{NE}}$. □

The following simple lemma is a minor generalisation of the fact that the canonical divisor of a surface of general type is numerically connected [BPV, VII.6.1].

Lemma 10. *Let X be a minimal Kähler surface with $\kappa(X) \geq 0$. Suppose there is a decomposition $K_X = D_+ + D_-$ with D_+, $D_- \in \overline{\mathrm{NE}}(X) \hookrightarrow H^{1,1}(X)$. Then $D_+ \cdot D_- \geq 0$, with equality if and only if say $K_X \cdot D_+ = D_+^2 = 0$. More precisely, upon equality, we have the following identities in $H^2(X, \mathbb{R})$: $D_+ = 0$ if X is of general type, $D_+ = \lambda K_X$ with $0 \leq \lambda \leq 1$ if $\kappa(X) = 1$, and finally $D_+ = D_- = 0$ if $\kappa(X) = 0$.*

Proof. First assume that $D_+^2 \leq 0$. Since K_X is nef, $D_+ \cdot D_- = (K_X - D_+) \cdot D_+ \geq -D_+^2 \geq 0$, with equality if and only if $K_X \cdot D_+ = D_+^2 = 0$. If $D_+^2 > 0$ and $D_-^2 > 0$, then using the Kähler form Φ, we can write $D_+ = \alpha\Phi + C_+$ and $D_- = \beta\Phi + C_-$ with α, $\beta > 0$ and $C_\pm \in \Phi^\perp$. By the Hodge index theorem,

$$D_+ \cdot D_- = \alpha\beta\Phi^2 + C_+ \cdot C_- \geq \alpha\beta\Phi^2 - \sqrt{-C_+^2}\sqrt{-C_-^2} > 0.$$

The statement for surfaces of general type follows directly from Hodge index and the fact that $K_X^2 > 0$. If $\kappa(X) = 1$, then K_X is a generator of the unique isotropic subspace of $K_X^\perp$, so $D_+ = \lambda K_X$, and $D_- = (1 - \lambda)K_X$. Since K_X, D_+ and $D_- \in \overline{\mathrm{NE}}(X)$, λ is bounded by $0 \leq \lambda \leq 1$. Finally if $\kappa(X) = 0$, K_X is numerically trivial, and D_+ and D_- must be zero as well. □

Lemma 11. *Let X be a surface of non-negative Kodaira dimension with (-1)-curves $E_1, \ldots E_m$. Assume that $\mathcal{K}$ has properties $(*)$. Then $K_i^2 \leq K_X^2$ for all $K_i \in \mathcal{K}(X)$, and upon equality*

$$K_i = \lambda K_{\min} + \sum_{j=1}^m \pm E_j \in H^2(X, \mathbb{Q})$$

where $\lambda = \pm 1$ if X is of general type, λ is a rational number with $|\lambda| \leq 1$ if $\kappa(X) = 1$, and where $\lambda = 0$ if $\kappa(X) = 0$.

Proof. By property (3), and (4), $K_i = K_{i,\min} + \sum_j (2a_{ij} + 1)E_j$, with $K_{i,\min} \in \mathcal{K}(X_{\min})$. Thus

$$K_i^2 \leq K_{i,\min}^2 - \#(-1)\text{-curves},$$

with equality if and only if $a_{ij} = 0$, or -1 for all i, j. Since $K_X^2 = K_{\min}^2 - \#(-1)$-curves, we can assume that X is minimal. Using property (1), (2) and Lemma 9, we can write $K_X = D_+ + D_-$ and $K_i = D_+ - D_-$, with $D_\pm \in \overline{\mathrm{NE}}(X)$. Then by Lemma 10, $K_i^2 = K_X^2 - 4D_+ \cdot D_- \leq K_X^2$ with equality under the stated condition. □

We can now prove the main theorem mod torsion assuming the existence of suitable basic classes.

Proposition 12. *Assume that for all smooth oriented 4-manifolds M with $b_+ \geq 1$ there is a set of basic classes $\mathcal{K}(M) = \{K_1, K_2, \dots\} \subset H^2(M, \mathbb{Z})$ functorial under oriented diffeomorphism, having properties $(*)$. Then Theorem 1 holds with $\mathbb{Q}$ coefficients.*

Proof. In this proof all cohomology classes will be rational classes, and X is a Kähler surface with $\kappa(X) \geq 0$. Using Lemma 11 we will first reduce the invariance of $K_{\min}$ up to sign and torsion (part $1 \otimes \mathbb{Q}$ of Theorem 1) to showing that (-1)-spheres are represented by (-1)-curves up to sign and torsion (part $2 \otimes \mathbb{Q}$).

Since $K_X \in \mathcal{K}$, there is a nonempty subset $\mathcal{K}_0 = \{K_j\} \subset \mathcal{K}$ with $K_j^2 = K_X^2 = 2e(X) + 3\sigma(X)$. By assumption, the subspace $H^2(X_{\min}, \mathbb{Q}) \subset H^2(X, \mathbb{Q})$ is the orthogonal complement of the (-1)-spheres. Now consider the projection $K_{j,\min}$ of K_j to $H^2(X_{\min}, \mathbb{Q})$. By Lemma 11 we know that $K_{j,\min} = \lambda K_{\min}$, and there are only 3 possibilities.

If $K_{j,\min}^2 > 0$, then X is of general type, and $K_{\min} = \pm K_{j,\min}$. If $K_{j,\min} = 0$ for all j, then X is of Kodaira dimension 0 and $K_{\min} = 0 \in H^2(X, \mathbb{Q})$. If $K_{j,\min}^2 = 0$ but not all $K_{j,\min} = 0$, then $\kappa(X) = 1$, and if j_0 is chosen such that $K_{j_0,\min} \neq 0$ has maximal divisibility then $K_{\min} = \pm K_{j_0,\min}$.

Now let e be the class of a (-1)-sphere in $H^2(X, \mathbb{Q})$. Without loss of generality, we can assume that $K_X \cdot e < 0$. Consider R_e the reflection generated by a (-1)-sphere e. It is represented by an orientation preserving diffeomorphism. Since $\mathcal{K}$ is invariant under oriented diffeomorphisms, the characterisation of basic classes with square K_X^2 tells us that

$$R_e K_X = K_{\min} + \sum E_i + 2(K_X \cdot e)e \tag{2}$$

$$= \lambda K_{\min} + \sum \pm E_i \tag{3}$$

with $|\lambda| \leq 1$. Since $\kappa(X) \geq 0$, we know that (-1)-curves are orthogonal or equal. Hence taking intersection with E_i we find that $(E_i \cdot e)(e \cdot K_X) = 0$ or 1. Since $K_X \cdot e \equiv e^2$ is odd, e is either orthogonal to all (-1) curves (i.e., $e \in H^2(X_{\min}, \mathbb{Q})$) or there is a (-1)-curve, say E_1, such that $K_X \cdot e = E_1 \cdot e = -1$. However, $e \in H^2(X_{\min})$ implies that $e = \frac{\lambda - 1}{2K_X \cdot e} K_{\min}$, which is impossible because $K_{\min}^2 \geq 0$. Thus, after renumbering the (-1)-curves, (2) and (3) can be rewritten to

$$e = \tfrac{1}{2}(1 - \lambda) K_{\min} + \sum_{i=1}^{N} E_i \tag{4}$$

with $N = \dfrac{1}{4}(1 - \lambda)^2 K_{\min}^2 + 1$.

Now reflect e in $E_1^\perp$. $R_{E_1} e$ is also a (-1)-sphere, so it has a representation as in Equation (4), except possibly for an overall sign because we cannot assume that $K_X \cdot R_{E_1} e < 0$:

$$\begin{aligned} R_{E_1} e &= \tfrac{1}{2}(1 - \lambda) K_{\min} - E_1 + \sum_{i=2}^{N} E_i \\ &= \pm\big(\tfrac{1}{2}(1 - \mu) K_{\min} + \sum_{j=1}^{M} E_{i_j}\big). \end{aligned}$$

Upon comparison, we see that the sign is minus, that $N = M = 1$, and that $0 \le 1 - \lambda = \mu - 1 \le 0$ unless $K_{\min} = 0$. In other words $e = E_1 \in H^2(X, \mathbb{Q})$. $\square$

2. The Localised Euler Class of a Banach Bundle.

This section is needed for the technical definition of the Seiberg Witten invariants. However we will actually avoid using the full definition in Section 5 when we prove the main Theorem 1 and Theorem 5 so some readers may want to skip to Section 3. The results in this section are used in an essential way in Section 6.

Consider an infinite dimensional bundle E over an infinite dimensional manifold M with a section s with Fredholm derivative. In practice this situation occurs whenever we have system of PDE's which are elliptic when considered modulo some gauge group action. The zero set $Z(s)$ is then the moduli space of solutions modulo gauge, and the index of the derivative is the virtual dimension. The localised Euler class of the pair (E, s) is a homology class with closed support on the zero set of the section. Its dimension is the index of the derivative. When the section is transversal, the class is just the fundamental class of the zero set with the proper orientation. The class is well behaved in one parameter families and therefore defines the "right" fundamental cycle even if the section is no longer transversal.

Its construction was pioneered by Pidstrigatch and Pidstrigatch Tjurin [Pi1], [P-T, §2]. Unfortunately their construction is not quite in the generality we will need it, and we will therefore set it up in fairly large generality here. The construction is modeled on Fulton's intersection theory and in the complex case it makes the machinery of excess intersection theory available. Unfortunately, although the construction is quite simple in principle, the whole thing has turned a bit technical. On first reading it is best to ignore the difference between Čech and singular homology, and continue to Proposition 14, the construction of the Euler class in the proof of Proposition 14 and Proposition 15.

We first make some algebraic topological preparations. For any pair of topological spaces $A \subset X$, homology with closed support and with local coefficients ξ is defined as

$$H_i^{cl}(X, A; \xi) = \lim_{\substack{\leftarrow \\ K}} H_i(X, A \cup (X - K); \xi)$$

where we take the limit over all compacta $K \subset X - \overset{\circ}{A}$. The groups H_*^{cl} are functorial under proper maps. Unfortunately this "homology theory" suffers the same tautness problems that singular homology has. To be able to work with well behaved cap products we will have to complete it. The following works well enough for our purposes but is a bit clumsy.

Suppose that X is *locally modelable* i.e., is locally compact Hausdorff and has local models which are each subsets of some $\mathbb{R}^n$. Obviously, locally compact subsets of locally modelable spaces are locally modelable. In particular, a locally closed subset of a locally modelable space is locally modelable. If X is locally modelable then for every compact subset $K \subset X - \overset{\circ}{A}$ there is a neighborhood $U_K \supset K$ in X which embeds in $\mathbb{R}^N$. We now define

$$\check{H}_i^{cl}(X, A, \xi) = \lim_{\substack{\leftarrow \\ K}} \check{H}_i(U_K, A \cap U_K \cup (U_K - K); \xi)$$

where for every pair (Y, B) in a manifold M, Čech homology is defined as

$$\check{H}_i(Y,B) = \lim_{\leftarrow}\{H_i(V,W),\ (V,W) \text{ neighborhoods of } (Y,B) \text{ in } M\}$$

This definition depends neither on the choice of U_K, nor on the embedding $U_K \hookrightarrow \mathbb{R}^N$, since two embeddings are dominated by the diagonal embedding, and $\check{H}_*(Y,B)$ does not depend on M but only on (Y,B) (cf. [Dol, VIII.13.16]).

Fortunately, we do not usually have to bother with Čech homology. Suppose in addition that X is locally contractible, e.g., locally a sub analytic set (cf. [GM, §I.1.7], and the fact that Whitney stratified spaces admit a triangulation). Then X is locally an Euclidean neighborhood retract (ENR) by [Dol, IV 8.12] and since in a Hausdorff space a finite union of ENR's is an ENR by [Dol, IV 8.10] we can assume that U_K is an ENR. Now assume that A is open. Then by [Dol, Prop. VIII 13.17],

$$\check{H}_*(U_K, U_K \cap A \cup (U_K - K)) \cong H_*(U_K, U_K \cap A \cup (U_K - K)) \cong H_*(X, A \cup X - K).$$

Thus, in this case $\check{H}^{cl}_*(X,A) = H^{cl}_*(X,A)$. If A is closed and locally contractible then one should be able to organise things such that $U_K \cap A$ is an ENR and the same conclusion would hold.

Lemma 13. *Let X be a locally modelable space, and Z a locally compact (e.g., locally closed) subspace, then there are cap products*

$$\check{H}^i(X, X-Z, \xi) \otimes \check{H}^{cl}_j(X, \xi') \xrightarrow{\cap} \check{H}^{cl}_{j-i}(Z, \xi \otimes \xi')$$

with the following properties.

1. *If Y is locally embeddable, $f\colon Y \to X$ is proper, $\sigma' \in \check{H}^{cl}_j(Y, Y - f^{-1}(Z), \xi')$, and $c \in \check{H}^i(X, X-Z, \xi)$, then the push-pull formula holds:*
$$f_*(f^*c \cap \sigma') = c \cap f_*\sigma'.$$
2. *If $Z \overset{i}{\hookrightarrow} W$ is proper and W is locally compact, we can increase supports, i.e., for $c \in \check{H}^i(X, X-Z, \xi)$ and $\sigma \in \check{H}^{cl}_j(X, \xi')$ we have*
$$c|_{(X,X-W)} \cap \sigma = i_*(c \cap \sigma).$$

Proof. For every $c \in \check{H}^i(X, X-Z)$ and $\sigma \in \check{H}^{cl}_j(X)$, we have to construct a class $c \cap \sigma \in \check{H}_{i-j}(Z, Z-K)$ for a cofinal family of compacta $\{K\}$. Since Z is locally compact, every compactum K is contained in a compactum $L \subset Z$ with $L \Supset K$ (i.e., $L \supset \overset{\circ}{L} \supset K$). Likewise there exists a compactum $L' \Supset L$. By excision it suffices to construct a class in $\check{H}_{i-j}(L, L-K)$.

Let $U_{L'}$ be a neighborhood of L' in X which embeds in $\mathbb{R}^N$. Let V_L, $W_{L-K} \subset V_L$, and $V_K \subset V_L$ be neighborhoods of respectively L, $L-K$ and K in $\mathbb{R}^N$. Let $U_L = V_L \cap \overset{\circ}{L}{}'$, then $U_L \subset U_{L'}$. Shrinking V_K, we can assume that $V_K \cap L' = V_K \cap L$. After replacing V_L by $(V_L - L') \cup W_{L-K} \cup V_K$, we can then assume that $V_L \cap (L' - K) = W_{L-K} \cap (L' - K)$.

We have a restriction map $\check{H}^i(X, X-Z) \to \check{H}^i(U_L, U_L - L')$. After shrinking V_L if necessary, $c|_{(U_L, U_L - L')}$ comes from a class $c_L \in H^i(V_L, V_L - L')$. By definition there is map

$$\check{H}^{cl}_j(X) \to \check{H}_j(U_L, U_L - K) \to H_j(V_L, V_L - K).$$

Let $\sigma_L \in H_j(V_L, V_L - K)$ be the image of σ. Now our task is to construct a class $c_L \cap \sigma_L \in H_{i-j}(V_L, W_{L-K})$ possibly after shrinking V_L and W_{L-K} even further.

By our choice of neighborhoods, we can write $V_L - K = (V_L - L') \cup (W_{L-K} - K)$. Then the standard cap product [Dol, VII Def. 12.1] gives a map

$$H^i(V_L, V_L - L') \otimes H_j(V_L, V_L - K) \xrightarrow{\cap} H_{j-i}(V_L, W_{L-K} - K)$$

so composing with the map $H_{j-i}(V_L, W_{L-K} - K) \to H_{j-i}(V_L, W_{L-K})$ we get a class $c_L \cap \sigma_L \in H_{j-i}(V_L, W_{L-K})$ as required. This construction defines our class for a cofinal family of neighborhoods (V_L, W_{L-k}) so we can take the limit. Moreover if $K' \supset K$, choices for K' will work a fortiori for K, so we can pass to the limit over K.

To prove the first property, note that since f is proper, $f^{-1}Z$ is locally compact. Choose compacta $K \Subset L \Subset L' \subset Z$ giving compacta $f^{-1}K \Subset f^{-1}L \Subset f^{-1}L'$. Note further that compacta of the form $f^{-1}K$ are a cofinal family of compacta in $f^{-1}(Z)$. Embed neighborhoods $U_{L'} \subset V_{L'} \subset \mathbb{R}^N$ and $U_{f^{-1}L'} \subset \mathbb{R}^M$. Now we carry out the construction above with the diagonal embedding of $U_{f^{-1}L'}$ in $\mathbb{R}^{N+M}$. Let $V_{f^{-1}L'}$ be a neighborhood of $U_{f^{-1}L'} \in \mathbb{R}^{N+M}$. We can assume that $V_{f^{-1}L'} \to V_{L'}$ under the projection π to $\mathbb{R}^N$. We can also assume that $c|_{(U_L, U_L - L')}$ comes from a class $c_L \in H^i(V_L, V_L - L')$. Finally let $\sigma_{f^{-1}L'}$ be the image of σ in $H_j(V_{f^{-1}L'}, \pi^{-1}W_{K-L})$. Then the first property follows from the identity

$$\pi_*(\pi^* c_L \cap \sigma'_{f^{-1}L'}) = c_L \cap \pi_* \sigma'_{f^{-1}L'}$$

in $H_j(V_L, W_{K-L})$. The second property is left to reader. $\square$

A smooth manifold X of dimension n, has an orientation system $or(X)$. It is the sheafification of the presheaf $U \to H^n(X, X - U)$. Equivalently, we can define $or(X)$ as the sheaf $R^d\pi_*(X \times X, X \times X - \Delta, \mathbb{Z})$ on X, where Δ is the diagonal of $X \times X$, π the projection on the first coordinate, and $R^d\pi_*$ the parametrised version of the d^{th} cohomology.

Likewise, for a real vector bundle E of rank r there is an orientation system $or(E)$, the sheafification of $H_q(E|_U, E|_U - U)$. We have $or(X) = or(TX)^\vee$, as can be seen immediately from the alternative description of $or(X)$ and excision.

A manifold X has a unique fundamental class $[X] \in H_n^{cl}(X, or(X))$ in singular or Čech homology such that for small U,

$$[X]|_{\bar{U}} \in H_d(X, X - U, H^d(X, X - U)) = \mathrm{Hom}(H^d(X, X - U), H^d(X, X - U))$$

is identified with the identity (cf [Spa, p. 357]).

Similarly, a bundle $E \xrightarrow{\pi} X$ has a canonical Thom class

$$\Phi_E \in \check{H}^r(E, E - X, \pi^* or(E))$$

[Spa, p. 283]. In turn for every section s in E with zero set $Z(s)$, the Thom class defines a localised cohomological Euler class

$$e(E, s) = s^*\Phi_E \in \check{H}^r(X, X - Z(s), or(E)).$$

Let M be a Banach manifold, E a real Banach vector bundle on M and s a section of E with zero set $Z(s)$. The zero section s_0 defines an exact sequence

$$0 \to TM \xrightarrow{Ts_0} TE|_M \to E \to 0$$

This gives a canonical map $Ds\colon TM|_{Z(s)} \to E|_{Z(s)}$ defined by the diagonal arrow in the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & TM|_{Z(s)} \arrow[d, "Ts_0"] \\
TM|_{Z(s)} \arrow[r, "Ts"] \arrow[dr, "Ds"'] & TE|_{Z(s)} \arrow[d] \\
 & E|_{Z(s)}. \arrow[d] \\
 & 0
\end{tikzcd}$$

If D is a connection on E then $D(s)$ extends Ds from $Z(s)$ to M (hence the notation), but in general connections need not exist on Banach manifolds.

To state the homotopy property of the localised Euler class we introduce one more notion. For a topological space X with a family of closed subsets $\{X_\alpha\}_{\alpha \in A}$, we define the *confined homology* as

$$H_j^{\mathrm{c}f}(X) = \lim_{\substack{\leftarrow \\ \alpha \in A}} H_j(X, X - X_\alpha).$$

There are three situations we have in mind: $X_\alpha = X$, then confined homology is just homology; the family is the set of compacta, then confined homology is homology with closed support; and finally infinite dimensional configuration spaces are usually filtered by some norm that controls "bubbling". For example in Donaldson theory the moduli space of ASD connections with curvature bounded in the L^4 norm is compact. From the point of view of Proposition 14 it is then natural to filter the space $\mathcal{B}^*$ of all irreducible L^2_2 connections mod gauge by the family of subsets $\{\mathcal{B}^{\leq C}\}_{C \in \mathbb{R}^+}$, where $\mathcal{B}^{\leq C}$ the subset of connections with L^4 norm of the curvature bounded by C.

Proposition 14. *Let M be a smooth Banach manifold, E a banach bundle over M and s a section in E. Assume that*

1. *The map Ds is a section in the bundle* $\mathrm{Fred}^d(TM|_{Z(s)}, E|_{Z(s)})$ *of Fredholm maps of index d. We say that $Z(s)$ has virtual dimension d, and that Ds is Fredholm of index d.*
2. *The real line bundle* $\det(\mathrm{Ind}(Ds))$ *is trivialised over $Z(s)$.*

Then these data define a Čech homology class with closed support

$$\mathbb{Z}(M, E, s) = \mathbb{Z}(s) \in \check{H}_d^{\mathrm{cl}}(Z(s), \mathbb{Z})$$

with the following properties.

1. *The class $\mathbb{Z}(s) = [Z(s)]$ if $Z(s)$ is smooth of dimension d and carries the natural orientation defined by the trivialisation of* $\det(\mathrm{Ind} Ds)$.
2. *Let $\{M_\alpha\}_{\alpha \in A}$ be a family of closed subsets of M such that $M_\alpha \cap Z(s)$ is compact for all $\alpha \in A$. Then there is a natural map $i_*\colon \check{H}_j^{\mathrm{cl}}(Z(s), \mathbb{Z}) \to H_j^{\mathrm{c}f}(M, \mathbb{Z})$. Now if s_t with $t \in [0,1]$ is a one parameter family of sections*

such that $Z(s_\bullet)\cap M_\alpha\times[0,1]$ *is compact for all* $a\in A$*, then*

$$i_{0*}\mathbb{Z}(s_0)=i_{1*}\mathbb{Z}(s_1))\in H^{\mathrm{c}f}_j(M,\mathbb{Z}).$$

For every exact sequence

$$0\to E'\to E\to E''\to 0$$

defined over a neighborhood of $Z(s)$*, let* s'' *be the induced section in* E''*, and* s' *the induced section of* $E'|_{Z(s'')}$ *with zero set* $Z(s)$*. Then*

3. *If* E' *has finite rank,*

$$\mathbb{Z}(s)=e(E'|_{Z(s'')},s')\cap\mathbb{Z}(s'').$$

4. *If* $Ds''|_{Z(s)}$ *is surjective, then* $Z(s'')$ *is smooth in a neighborhood of* $Z(s)$*,* $Ds'\colon TZ(s'')|_{Z(s)}\to E'|_{Z(s)}$ *is Fredholm with* $\mathrm{Ind}Ds'=\mathrm{Ind}Ds$*, and*

$$\mathbb{Z}(E,s)=\mathbb{Z}(E'|_{Z(s'')},s').$$

Proof. If M (hence E) is a finite dimensional manifold of dimension $N+d$ then E is a real vector bundle of rank N with an isomorphism $\det(E)=\det(TM)$ over $Z(s)$. Let $[M]\in H^{cl}_{N+d}(M,or(M))$ be the fundamental class, and Φ_E the twisted Thom class of E in $H^N(E,E-M,or(E))$. Define

$$\mathbb{Z}(s)=e(E,s)\cap[M]\in\check H^{\mathrm{cl}}_d(Z,or(E)\otimes or(M))=\check H^{\mathrm{cl}}_d(Z(s),\mathbb{Z})$$

i.e., $\mathbb{Z}(s)$ is the Poincaré dual of the localised cohomological Euler class. In the last step we used the chosen trivialisation of $or(E)\otimes or(M)=or(\det TM^\vee\otimes\det E)=or(\det(\mathrm{Ind}(Ds)))$ given by the trivialisation of the index.

In the infinite dimensional case we proceed similarly but we have to go through a limiting process and use that we know what to do when the section is regular. For each compactum $K\subset Z$ we have to construct a class $\mathbb{Z}_K\in\check H_d(Z,Z-K)$ such that for $K'\supset K$ the class $\mathbb{Z}_{K'}|_K=\mathbb{Z}_K$ under the restriction map $\check H_d(Z,Z-K')\to\check H_d(Z,Z-K)$.

Over a neighborhood U of K in M we can find a subbundle F in E of finite rank N such that $\mathrm{Im}(Ds)|_K+F|_K=E|_K$. Such a bundle certainly exists: We can choose a finite number of sections $s_1,\dots s_N$ such that the s_i span $\mathrm{Coker}(Ds_x)$ for every $x\in K$, and possibly after perturbing we can assume that the s_i are linearly independent in a neighborhood of K. (Remember that $K\hookrightarrow\mathbb{R}^M$ and that E has infinite rank, so there is plenty of freedom.) Let $\tilde E$ be the quotient bundle E/F defined over U, and $\tilde s$ the induced section with zero set $M_f=Z(\tilde s)$ (f is for finite).

Clearly the map $TM|_{Z(s)}\xrightarrow{Ds}E|_{Z(s)}\to\tilde E$ is surjective. Since the canonical map $D\tilde s$ on M_f restricts to this composition on $Z(s)$, $D\tilde s$ is surjective on M_f possibly after shrinking U. Hence M_f is a smooth manifold. Let $T=\ker(TM|_{M_f}\to\tilde E)$. There is a canonical identification $T\cong TM_f$. Now T is a bundle of rank $N+d$ since

$$\mathrm{Ind}(Ds)|_K=T-F. \tag{5}$$

Thus M_f has dimension $N+d$.

On M_f, the section s in E lifts to a section s_f of the subbundle F. Clearly $Z(s_f)=Z(s)\cap U$. Define

$$\mathbb{Z}_K=e(F|_{M_f},s_f)\cap[M_f]\in\check H_d(Z(s),Z(s)-K;\mathbb{Z})).$$

Here we have used the restriction map

$$\check{H}^{\mathrm{cl}}_d(Z(s)\cap U; or(F)\otimes or(M_f)) \to \check{H}_d(Z(s), Z(s)-K; or(F)\otimes or(M_f)),$$

the identification $or(\det(\mathrm{Ind}(Ds))) = or(F)\otimes or(M_f)$ and the chosen trivialisation of $\det(\mathrm{Ind}(Ds))$ as in the finite dimensional case.

This construction does not depend on the choices. If F_1 and F_2 are two choices of subbundles of E then there is third bundle G containing F_1+F_2. We can therefore assume that we are dealing with a subbundle $F'\subset F$. Then using primes to denote objects we get out of the construction above using F' instead of F, we have the sections s_f in F over M_f, s'_f in F' over M'_f and a section $s''_f = s_f \bmod F'$ in F/F' over M_f cutting out M'_f. They satisfy the identity

$$\begin{aligned}\mathbb{Z}_K &= e(F|_{M_f} s_f)\cap[M_f]\\ &= e(F'|_{M_f}, s'_f)\cap e(F/F'|_{M'_f}, s''_f)\cap[M_f]\\ &= e(F'|_{M_f}, s'_f)\cap[M'_f] = \mathbb{Z}'_K,\end{aligned}$$

where in the third step we have used the identification

$$or(M'_f) = or(M_f)\otimes or(F/F')|_{M_f}.$$

In particular, if $K'\supset K$ all choices on K' work a fortiori for K, so we can pass to the limit.

The relation $\mathbb{Z}(s) = [Z(s)]$ for regular sections (property 1), and the compatibility with Euler classes of finite rank bundles (property 3) are now clear from the construction. The stability property 4 also follows from the construction. For every compactum K, we can choose the finite rank subbundle F as a subbundle of E'. Then $\tilde{E}\twoheadrightarrow E''$. Now one checks that by a diagram chase that

$$Z(\tilde{E},\tilde{s}) = Z(E'/F|_{Z(E'',s'')}, s' \bmod F)$$

and that

$$\begin{aligned}TZ(\tilde{E},\tilde{s}) &= \mathrm{Ker}(TM\to\tilde{E})\\ &= \mathrm{Ker}(\mathrm{Ker}(TM\to E'')\to E'/F)\\ &= \mathrm{Ker}(TZ(s'')\to E'/F) = TZ(E'/F).\end{aligned}$$

In particular, the orientations agree. Thus we see that

$$\begin{aligned}\mathbb{Z}_K(E,s) &= e(F,s_f)\cap[Z(\tilde{E},\tilde{s})]\\ &= e(F,s_f)\cap[Z(E'/F|_{Z(E'',s'')}, s' \bmod F)] = \mathbb{Z}_K(E'|_{Z(s'')}, s').\end{aligned}$$

It only remains to pass to the limit over K.

To see that $\check{H}^{\mathrm{cl}}_j(Z(s))$ maps to $H^{cf}_j(M)$ note that for every compact subset $K_\alpha = M_\alpha\cap Z(s)$, we constructed a finite dimensional submanifold $M_f\supset K_\alpha$. Then we have maps

$$\begin{aligned}\check{H}^{cl}_j(Z(s)) \to &\check{H}_j(Z(s), Z(s)-K_\alpha) = \check{H}_j(Z(s)\cap M_f, (Z(s)\cap M_f)-K_\alpha)\\ &\to H_j(M_f, M_f - M_\alpha)\to H_j(M, M-M_\alpha).\end{aligned}$$

Again this map is independent of choices, and we can pass to the limit.

The homotopy property of $\mathbb{Z}$ is a formal consequence of the compatibility with finite dimensional Euler classes. Consider the trivial bundle $\mathbb{R}$ over the interval $[-1,2]$ with the one parameter family of sections $\theta-\tau$ where $\theta\colon[-1,2]\to\mathbb{R}$ is the inclusion and $0 \le \tau \le 1$. Then clearly $e(\mathbb{R}, \theta) = e(\mathbb{R}, \theta - 1) \in H^1([-1,2], \{-1,2\})$ is the canonical generator. Consider $M \times [-1,2]$. Let $\pi: M \times [-1,2] \to M$ be the projection and $S: M \times [-1,2] \to \pi^* E$ an extension of our one parameter family of sections, e.g., $S_t = s_0$ for $t \le 0$ and $S_t = s_1$ for $t \ge 1$. The bundle $\pi^* E \oplus \mathbb{R}$ has a one parameter family of sections $(S, \theta - \tau)$. Now

$$\begin{aligned}\mathbb{Z}(s_0) &\overset{4}{=} \pi_* \mathbb{Z}(\pi^* E \oplus \mathbb{R}; (S, \theta)) \\ &\overset{3}{=} \pi_* e(\mathbb{R}, \theta) \cap \mathbb{Z}(\pi^* E; S) \\ &= \pi_* e(\mathbb{R}, \theta - 1) \cap \mathbb{Z}(\pi^* E; S) \\ &= \pi_* \mathbb{Z}(\pi^* E \oplus \mathbb{R}; (S, \theta - 1)) = \mathbb{Z}(s_1)\end{aligned}$$

□

Now consider the case of a complex manifold with a holomorphic bundle.

Proposition 15. (Compare [P-T, Prop. III.2.4].) *Let M be a complex Banach manifold, E a holomorphic vector bundle and s a holomorphic section with zero set $Z(s)$. Assume that Ds is a section of* $\mathrm{Fred}^d_{\mathbb{C}}(TM|_{Z(s)}, E|_{Z(s)})$. *We say that $Z(s)$ has complex virtual dimension d, and that Ds is Fredholm of complex index d. Then the localised Euler class $\mathbb{Z}(s) = [Z(s)] \in H^{cl}_{2d}(Z(s), \mathbb{Z})$, if $Z(s)$ is a local complete intersection of dimension d, and more generally*

$$\mathbb{Z}(s) = [c(\mathrm{Ind}(Ds))^{-1} c_*(Z(s))]_{2d}. \tag{6}$$

Here, $c_*(Z(s))$ is the total homological Chern class which will be defined later by equation (9)*. It coincides with the Poncaré dual of the total cohomological Chern class of the tangent bundle if $Z(s)$ is smooth.*

Remark 16. If $Z(s)$ is smooth we can even get away with an almost complex manifold M and the assumption that Ds is complex linear.

Remark 17. The definition of $c_*(Z(s))$ is analogous to the definition of the homological Chern classes in [Ful, Example 4.2.6]. I have tacitly removed M and E from the notation for it. I strongly believe that $c_*(Z(s))$ is independent of the embedding but I did not prove this. There is one case where independence of $c_*(Z(s))$ on the embedding can be proved completely analogous to [Ful, Example 4.2.6] by simply replacing algebraic arguments by complex analytic ones: If for every $K \subset Z(s)$ compact, there exists a *holomorphic* finite rank sub bundle $F \hookrightarrow E$ defined over a neighborhood of K such that $F|_K + \mathrm{Im}(Ds)|_K = E|_K$. Then a neighborhood U_K of K in $Z(s)$ sits in a complex rather than almost complex finite dimensional manifold M_f. Such a bundle should typically exist if $Z(s)$ has the structure of a quasi projective variety, and $\mathrm{Coker}\, DS$ has the interpretation of a coherent sheaf as in [Pi1, §5, §6].

Proof. We will use Mac Pherson's graph construction, that is we consider the limit $\lambda \to \infty$ of the map $(\lambda s : 1)$ in $\mathbb{P}(E \oplus \mathcal{O})$ or finite dimensional approximations thereof. We use the notations of the proof of Proposition 14.

For a compactum $K \subset Z(s)$ we choose the finite rank bundle F as follows. It is a complex bundle, and in every point of $Z(s)$ there are sections of F which restricted to a neighborhood are holomorphic sections of E and which span locally a subbundle $F^{\text{hol}} \hookrightarrow F$, such that

$$Ds\colon TE|_{Z(s)} \twoheadrightarrow E/F^{\text{hol}}|_{Z(s)}$$

is a surjection. We do not assume that F is a holomorphic subbundle, because I do not see a reason why such a bundle should exist. However since F is a complex bundle, both the quotient bundle $\tilde{E} = E/F$ and the tangent bundle

$$TM_f|_{Z(s)} = T|_{Z(s)} = \operatorname{Ker}(TM|_{Z(s)} \xrightarrow{Ds} E|_{Z(s)} \to \tilde{E}|_{Z(s)})$$

are complex bundles. We extend this complex structure on TM_f over all of M_f, possibly after shrinking M_f, making it into an almost complex manifold of complex dimension $d+N$.

Consider the space $\mathbb{P}(F \oplus \mathcal{O}) \xrightarrow{\pi} M_f$. Then the total space of F can naturally be identified with an open subspace of $\mathbb{P}(F \oplus \mathcal{O})$. The image of the zero section will still be called the zero section, and the complement of F the divisor at infinity. The divisor at infinity can be identified with $\mathbb{P}F$.

Let Q be the universal quotient bundle. The bundle Q has sections $\overline{(0,1)}$, and $\overline{(\lambda s_f, 1)}$, cutting out the zero section and the graph of λs_f, respectively. Equivalently, we can cut out the graph of λs_f by $\overline{(s_f, 1/\lambda)}$. Then clearly as $\lambda \to \infty$ the graph degenerates to a set contained in the zero set of $\overline{(s_f, 0)}$.

Now $Z(\overline{(s_f,0)})$ has two "irreducible components". One component $\tilde{M}_f \hookrightarrow \mathbb{P}F$ is the closure of the image of $(s_f\colon 0)\colon M_f - Z(s) \to \mathbb{P}F|_{M_f - Z(s)} \subset \mathbb{P}(F \oplus \mathcal{O})$. It will be called the strict transform. The other component is just $\mathbb{P}(F \oplus \mathcal{O})|_{Z(s)}$. Let $\mathcal{E}_f = \tilde{M}_f \cap \mathbb{P}(F \oplus \mathcal{O})|_{Z(s)}$. It will be called the exceptional divisor.

I claim that

$$\check{H}^{cl}_{2d+2N-1+i}(\mathcal{E}_f) = 0 \text{ for } i \geq 0. \tag{7}$$

Accepting this claim we see from the exact sequence

$$\check{H}^{\mathrm{cl}}_{2d+2N}(\mathcal{E}_f) \to \check{H}^{\mathrm{cl}}_{2d+2N}(\tilde{M}_f) \to H^{cl}_{2d+2N}(\tilde{M}_f - \mathcal{E}_f) \to \check{H}^{\mathrm{cl}}_{2d+2N-1}(\mathcal{E}_f)$$

that $\tilde{M}_f$ carries a unique fundamental class $[\tilde{M}_f]$ restricting to $[\tilde{M}_f - \mathcal{E}_f]$. Now consider $C' = \mathbb{Z}(\overline{(s_f,0)}) - [\tilde{M}_f] \in \check{H}^{\mathrm{cl}}_{2d+2N}(Z(\overline{(s_f,0)}))$. Then C' comes from a unique class $C \in \check{H}^{\mathrm{cl}}_{2d+2N}(\mathbb{P}(F \oplus \mathcal{O})|_{Z(s)})$ because of the sequence.

$$0 \to \check{H}^{cl}_{2d+2N}(\mathbb{P}(F \oplus \mathcal{O})|_{Z(s)}) \to \check{H}^{cl}_{2d+2N}(Z(\overline{(s_f,0)})) \to H^{cl}_{2d+2N}(\tilde{M}_f - \mathcal{E}_f).$$

Now note that Q restricted to the zero section is canonically isomorphic to F. We therefore have the following chain of equivalences

$$\begin{aligned}
\mathbb{Z}(s)_K &= e(F, s_f) \cap [M_f] \\
&= \pi_* e(\pi^* F, \lambda s_f) \cap e(Q, \overline{(0,1)}) \cap [\mathbb{P}(F \oplus \mathcal{O})] \\
&= \pi_* \Big(e(Q, \overline{(\lambda s_f, 1)}) \cup e(Q, \overline{(0,1)}) \Big) \cap [\mathbb{P}(F \oplus \mathcal{O})] \\
&= \pi_* e(Q, \overline{(0,1)}) \cap e(Q, \overline{(s_f, 1/\lambda)}) \cap [\mathbb{P}(F \oplus \mathcal{O})] \\
&= \pi_* e(Q, \overline{(0,1)}) \cap \Big(e(Q, \overline{(s_f, 0)}) \cap [\mathbb{P}(F \oplus \mathcal{O})] \Big) \\
&= \pi_* e(Q, \overline{(0,1)}) \cap \mathbb{Z}(\overline{(s_f, 0)}).
\end{aligned}$$

If we accept the claim (7) for a moment and we note that the support of $\tilde M_f$ and $e(Q,\overline{(0,1)})$ are disjoint we see further that

$$\mathbb{Z}(s)_K = \pi_* e(Q,\overline{(0,1)}) \cap C' = \pi_* e(Q) \cap C,$$

where in the last expression we can drop supports because $\mathbb{P}(F \oplus \mathcal{O})|_{Z(s)} \to Z(s)$ is proper. If we use that $e(Q) = c_{\mathrm{top}}(Q)$, this can be rewritten further to

$$\begin{aligned}\mathbb{Z}(s)_K &= [\pi_* c(Q) \cap C)]_{2d} \\ &= [c(F)\pi_* \left((1-h)^{-1} \cap C\right)]_{2d} \\ &= [c(F-T)\,(c(T)s_*(Z(s), M_f))]_{2d}\end{aligned}$$

where we used the notation $h = c_1(\mathcal{O}_{\mathbb{P}(F\oplus\mathcal{O})}(+1))$ and

$$s_*(Z(s), M_f) \overset{\text{def}}{=} \pi_* \left((1-h)^{-1} C\right) \tag{8}$$

for the total homological Segre class of the normal cone. (This terminology will be justified in a minute.) But $c(F-T) = c(\mathrm{Ind} Ds)^{-1}$ and since $T = TM_f$,

$$c_*(Z(s)) \overset{\text{def}}{=} c(T)s_*(Z(s), M_f) \tag{9}$$

is exactly the analogue of the homological chern classes of [Ful, example 4.2.6].

We show that $c_*(Z(s))$ does not depend on the choice of F. Again it suffices to treat the case that $F' \subset F$. We use primes whenever an object is associated to F'. The independence follows directly from a formula for the Segre classes which expresses how they behave under the extension $M'_f \subset M_f$ in terms of the normal bundle F/F' of $M'_f \subset M_f$.

$$s_*(Z(s), M_f) = c(F/F')^{-1} s_*(Z(s), M'_f). \tag{10}$$

Assuming (10), we see that

$$\begin{aligned}c_*(Z(s)) &= c(T)s_*(Z(s), M_f) \\ &= c(T)c(F/F')^{-1} s_*(Z(s), M'_f) = c(T')s_*(Z(s), M'_f).\end{aligned}$$

In particular we can take the limit over K.

Formula (10) is well known for integrable complex manifolds [Ful, example 4.1.5], and we will follow the proof closely. There are two terms in the class C occurring in the definition (8) of the Segre class, which we treat separately.

Note that there is a regular section σ of $F/F'(1)$ on $\mathbb{P}(F \oplus \mathcal{O})|_{M_f}$ cutting out $\mathbb{P}(F' \oplus \mathcal{O})|_{M_f}$. Therefore

$$\begin{aligned}[\mathbb{P}(F' \oplus \mathcal{O})|_{M'_f}] &= e(F/F', s_f \bmod F') \cap [\mathbb{P}(F' \oplus \mathcal{O})|_{M_f}] \\ &= e(F/F', s_f \bmod F') \cap e(F/F'(1), \sigma) \cap [\mathbb{P}(F \oplus \mathcal{O})|_{M_f}].\end{aligned}$$

Since on $\mathbb{P}(F' \oplus \mathcal{O})|_{M_f}$ there is an exact sequence

$$0 \to Q' \to Q \to F/F' \to 0,$$

we have $e(Q', \overline{(s'_f, 0)}) \cup e(F/F', s_f \bmod F') = e(Q, \overline{(s_f, 0)})$. Then the above implies that

$$\begin{aligned}\mathbb{Z}(Q', \overline{(s'_f, 0)}) &= e(Q', \overline{(s'_f, 0)}) \cap [\mathbb{P}(F' \oplus \mathcal{O})|_{M'_f}] \\ &= e(Q, \overline{(s_f, 0)}) \cap e(F/F'(1), \sigma) \cap [\mathbb{P}(F \oplus \mathcal{O})|_{M_f}] \\ &= e(F/F'(1), \sigma) \cap \mathbb{Z}(Q, \overline{(s_f, 0)}).\end{aligned}$$

As for the other term, on $\tilde{M}_f$ there is a smooth section in $\mathcal{O}(-1)$ given by $(s_f, 0)$ which is an isomorphism $\mathcal{O} \cong \mathcal{O}(-1)$ on $\tilde{M}_f - \mathcal{E}$. It follows that

$$[\tilde{M}'_f - \mathcal{E}] = e(F/F', s_f \bmod F') \cap [\tilde{M}_f - \mathcal{E}] = e(F/F'(1), \sigma) \cap [\tilde{M}_f - \mathcal{E}].$$

Then we have the equality

$$[\tilde{M}'_f] = e(F/F'(1), \sigma) \cap [\tilde{M}_f].$$

because both left and right hand side are supported on $\tilde{M}'_f - \mathcal{E} \cup \mathbb{P}(F' \oplus \mathcal{O})|_{Z(s)} \cap \tilde{M}_f$, i.e., the closure of $[\tilde{M}'_f - \mathcal{E}]$, and both cycles restrict to $[\tilde{M}'_f - \mathcal{E}]$.

For the computation of the Segre class we can forget about the support given by σ and use

$$e(F/F'(1)) = c_{\mathrm{top}}(F/F'(1)) = \sum c_{\mathrm{top}-j}(F/F')h^j.$$

Thus we finally get the expression

$$\begin{aligned}s_*(Z(s), M'_f) &= \pi_* \left(\sum_{ij} h^{i+j} c_{\mathrm{top}-j}(F/F') \cap (\mathbb{Z}(Q, \overline{(s_f, 0)}) - [\tilde{M}_f]) \right) \\ &= c(F/F') s_*(Z(s), M_f),\end{aligned}$$

which we set out to prove.

It remains to prove the claim (7). We first turn to the case that $Z(s)$ is smooth but possibly of the wrong dimension. Smoothness of $Z(s)$ implies that $\operatorname{Im} Ds|_T \subset F$ has constant rank over $Z(s)$ because $\ker Ds|_T = \ker Ds = TZ(s)$. Then $\operatorname{Im} Ds|_T$ is just the normal bundle $\mathcal{N}$ of $Z(s)$ in M_f. Now let us identify the limit set $(s_f : 1/\lambda)(M_f)$ when $\lambda \to \infty$. If we have a smooth path $\gamma: I \to M_f$ with $\gamma(0) = x_0 \in Z(s)$, then we see that $\lim_{t \to 0}(s_f : 0)(\gamma(t)) = (Ds_f(\frac{d}{dt}|_{t=0}\gamma) : 0)$. Therefore $\tilde{M}_f$ is just the blowup $\hat{M}_f$ of $Z(s)$ in M_f. This makes sense even though M_f is only an almost complex manifold since the normal bundle $\mathcal{N}$ has a complex structure. The blow up is obtained abstractly by identifying a tubular neighborhood N_ϵ of $Z(s)$ with the normal bundle, and replacing N_ϵ with $I = \{(l, x) \in \mathbb{P}\mathcal{N} \times N_\epsilon \mid l \ni x\}$. It is an almost complex manifold, so certainly carries a fundamental class $[\tilde{M}_f]$. It is also clear that $\mathcal{E}_f = \mathbb{P}\mathcal{N}$ is a submanifold of real codimension 2, and certainly satisfies the claim (7).

Let $\mathcal{O}(\mathcal{E}_f)$ be the smooth complex line bundle on the blow-up $\hat{M}_f$ defined by the exceptional divisor $\mathcal{E}_f$, and let $z \in A^0(\mathcal{O}(E))$ be a section cutting out $\mathcal{E}_f = \mathbb{P}\mathcal{N}$ with the proper orientation, i.e., $\mathbb{Z}(\mathcal{O}(\mathcal{E}_f), z) = [\mathcal{E}_f]$. On $\hat{M}_f$ the pulled back section is of the form $s_f = z\hat{s}_f$ with $\hat{s}$ nowhere vanishing. Therefore the limit set of $(s_f : 1/\lambda)(\hat{M}_f)$ in $\mathbb{P}(F \oplus \mathcal{O})|_{\hat{M}_f}$ as $\lambda \to \infty$ is just $(\hat{s} : 0)(\hat{M}_f) \cup D$ where $D \subset \mathbb{P}(F \oplus \mathcal{O})|_{\mathcal{E}_f}$ is the $\mathbb{P}^1$ bundle joining the zero section $(0 : 1)|_{\mathcal{E}_f}$ and the section

$(\hat{s}_f\colon 0)$. Then down on M_f the limit set of $(s_f\colon 1/\lambda)(M_f)$ is just $\tilde{M}_f \cup C\mathcal{E}_f$, where $C\mathcal{E}_f$ is cone bundle over $Z(s)$ joining $\mathcal{E}_f \subset \tilde{M}_f$ and the zero section.

Now $C\mathcal{E}_f$ represents the homology class C. Thus,

$$s_*(Z(s), M_f) = \pi_*(1-h)^{-1}C\mathcal{E}_f = \pi_*(1-h)^{-1}\mathcal{E}_f = \pi_*(1-h)^{-1}\mathbb{P}\mathcal{N} = s(\mathcal{N}) \cap [Z(s)].$$

Therefore, if $Z(s)$ is smooth, we find the expected formula

$$c_*(Z(s)) = c(TM_f)s(\mathcal{N}) \cap [Z(s)] = c(TZ(s)) \cap [Z(s)].$$

Note that in deriving this formula we have not really used the holomorphicity of s. It was sufficient that M has an almost complex structure and that Ds is complex linear. Replacing manifolds by stratified spaces the proof carries over essentially verbatim if $Z(s)$ is a local complete intersection since this condition implies that $Ds|_T$ has constant rank, and that we have a well defined normal bundle.

In proving the claim (7) in the general case we use holomorphicity more strongly. We first blow up $Z(s)$ in M to get a new infinite dimensional analytic space $\hat{M}$. That this is possible follows from the local analysis of the normal cone in [P-T, §III.1].

Locally on M, the exceptional divisor $\mathcal{E} \subset \hat{M}$ can be described as follows. Locally on M we have an exact sequence of holomorphic bundles

$$0 \to F^{\mathrm{hol}} \to E \to E''^{\mathrm{hol}} \to 0,$$

such that $TM|_{Z(s)} \twoheadrightarrow E''^{\mathrm{hol}}|_{Z(s)}$ is surjective, i.e., locally F^{hol} can take the role of F. Further, locally we can split the sequence since F^{hol} has finite rank. Let the holomorphic subbundle $\tilde{E} \subset E$ be a lift of E''^{hol}. We write $s = s_f^{\mathrm{hol}} \oplus \tilde{s}$ corresponding to the decomposition $E = F^{\mathrm{hol}} \oplus \tilde{E}$. Let M_f^{hol} be the integrable finite dimensional complex manifold $Z(s''^{\mathrm{hol}}) = Z(\tilde{s})$ and $\mathcal{E}_f^{\mathrm{hol}}$ the exceptional divisor of the blow up of $Z(s)$ in M_f^{hol}. Then locally $\mathcal{E} \cong \mathcal{E}_f^{\mathrm{hol}} \times_{Z(s)} \mathbb{P}\tilde{E}$. Moreover $\mathcal{E}_f^{\mathrm{hol}}$ is naturally embedded in $\mathbb{P}(F^{\mathrm{hol}} \oplus \mathcal{O})|_{Z(s)} \subset \mathbb{P}(E \oplus \mathcal{O})|_{Z(s)}$. If we are a little more careful and choose $\tilde{E}$ such that $\mathbb{P}\tilde{E}|_{Z(s)} \subset \mathcal{E}$ then $\mathcal{E} = \mathrm{Join}(\mathcal{E}_f^{\mathrm{hol}}, \mathbb{P}\tilde{E}|_{Z(s)}) \subset \mathbb{P}E|_{Z(s)}$.

Let $z \in H^0(\mathcal{O}(\mathcal{E}))$ be a section vanishing exactly along $\mathcal{E}$. On $\hat{M}$ we can decompose the section as $s = z^n\hat{s}$. Therefore, just as in the previous finite dimensional case, $(s\colon 1/\lambda)(\hat{M}) \to \mathbb{P}(E \oplus \mathcal{O})$ degenerates to $(\hat{s}\colon 0)(\hat{M}) \cup nD$ where D is the $\mathbb{P}^1$ bundle over $\mathcal{E}$ joining the zero section $(0\colon 1)|_{\mathcal{E}}$ and $(\hat{s}_f\colon 0)|_{\mathcal{E}}$. Down on M, this means that $(s\colon 1/\lambda)(M) \subset \mathbb{P}(E \oplus \mathcal{O})$ degenerates to $\tilde{M} \cup C\mathcal{E}$ where $\tilde{M} \subset \mathbb{P}E$ is isomorphic to $\hat{M}$ with $\tilde{M} \cap \mathbb{P}(E \oplus \mathcal{O})|_{Z(s)} \cong \mathcal{E}$, and $C\mathcal{E}$ is the cone bundle over $Z(s)$ joining the zero section and $\mathcal{E}$.

Now we finally come to our claim (7). The set $\mathcal{E}_f$ has the description $\mathcal{E}_f = \mathbb{P}(F \oplus \mathcal{O}) \cap \mathcal{E}$. At the very beginning we have chosen F such that $F \supset F^{\mathrm{hol}}$. Locally we define $\tilde{F} = F \cap \tilde{E}$, then locally $F = F^{\mathrm{hol}} \oplus \tilde{F}$ and locally $\mathcal{E}_f = \mathrm{Join}(\mathcal{E}_f^{\mathrm{hol}}, \mathbb{P}\tilde{F}|_{Z(s)})$. Thus $\mathcal{E}_f$ is a stratified space of real dimension $2d + 2N - 2$, and we are done. □

Remark 18. In the complex case we have obviously defined a class containing more information about the section. Let

$$\widehat{\mathbb{Z}}(s) = c(\mathrm{Ind}(Ds))^{-1}c_*(Z(s)).$$

3. Seiberg Witten Classes

We will collect a few facts about Seiberg Witten basic classes in a formulation suitable for arbitrary Kähler surfaces. In the usual formulation, these classes are the support of a certain function on the set of Spin^c-structures. However in the presence of 2-torsion, Spin^c-structures cause endless confusion which is why I have chosen to base my exposition on SC-structures [Kar]. This notion catches the essence of Spin^c-structures, the existence of spinors. It is well suited to the Kähler case and is equivalent to that of a Spin^c-structure in dimension 4. For more details see [Kar].

Let X be a closed oriented manifold of dimension $2n$. Choose a Riemannian metric g with Levi-Civita connection ∇^g, and Clifford algebra bundle $C(X,g) = C(T^\vee X, g)$. There is a natural isomorphism of bundles $c\colon \wedge^* T^\vee X \to C(X,g)$ given by anti-symmetrisation. It induces a connection and metric on $C(X,g)$ also denoted ∇^g and g.

An *SC-structure* is a smooth complex vector bundle W of rank 2^n together with an algebra bundle isomorphism $\rho\colon C(X,g) \to \mathcal{E}nd(W)$. In other words an SC-structure is an irreducible module of the Clifford algebra bundle. A section $\phi \in A^0(W)$ is called a (smooth) spinor. An SC-structure exists if and only if $w_2(X)$ can be lifted to the integers [Kar, §3.4]. Existence will be clear in the case of Kähler surfaces.

An SC-structure admits an invariant hermitian metric, i.e., one such that Clifford multiplication by 1-forms is skew hermitian (sh). The chirality operator $\Gamma = (\sqrt{-1})^n c(\mathrm{Vol}_g)$ has square 1, and is hermitian. Thus, Γ has an orthogonal eigenbundle decomposition $W = W^+ \oplus W^-$ with eigenvalue ± 1, the positive and negative spinors of the SC-structure. A one form $\omega \in A^1(X)$ defines an skew hermitian map $c(\omega)\colon W^\pm \to W^\mp$ which is an isomorphism away from the zero set of ω.

In this paragraph we assume $\dim(X) = 4$. Then $T_X^\vee \cong \mathcal{H}om(W^+, W^-)^{\mathrm{sh}}$. Let $L_W = \det W^+$. Then $L_W \cong \det W^-$, by the isomorphism induced from Clifford multiplication by a generic 1-form, which is an isomorphism outside codimension 4. Thus W is a $\mathrm{Spin}^c(4)$-bundle if we identify

$$\mathrm{Spin}^c(4) = \{(U_1, U_2) \in U(2) \times U(2) \mid \det(U_1) = \det(U_2)\}.$$

We recover the usual definition $\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\mathbb{Z}/2/Z} U(1)$ from the isomorphism $\mathrm{Spin}(4) = SU(2) \times SU(2)$. In any case by chasing around the cohomology sequences of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathrm{Spin}^c(4) & \longrightarrow & \mathrm{SO}(4) \times U(1) & \longrightarrow 1 \\
 & & \| & & \uparrow & & \uparrow & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathrm{Spin}(4) & \longrightarrow & \mathrm{SO}(4) & \longrightarrow 1
\end{array}$$

we see that $L_W + w_2(X) \equiv 0 \pmod 2$, and that this is the only obstruction to lifting the $SO(4) \times U(1)$ bundle to $\mathrm{Spin}^c(4)$. If $H^2(X, \mathbb{Z})$ has no 2-torsion, the line bundle $L \equiv w_2(X)$ determines such a lift completely, and it is common to speak of the Spin^c-structure L.

An *hermitian SC-structure* is a pair $(W, \langle , \rangle)$ of an SC-structure W together with a non-degenerate invariant hermitian metric $\langle , \rangle$. A *unitary SC-structure* $(W, \langle , \rangle, \nabla)$, is an hermitian SC-structure together with a unitary Clifford connection ∇, i.e., a unitary connection such that for all vector fields X, spinors $\phi \in A^0(S)$, and 1-forms ω we have

$$\nabla_X(\omega \cdot \phi) = (\nabla^g_X \omega) \cdot \phi + \omega \cdot \nabla_X \phi.$$

The *Dirac operator* $\not\partial$ of a unitary SC-structure is the composition

$$A^0(W) \xrightarrow{\nabla} A^1(W) \xrightarrow{\cdot} A^0(W).$$

It is an elliptic self adjoint first order differential operator, and it maps positive spinors to negative ones and vice versa (i.e., $\not\partial\colon A^0(W^\pm) \to A^0(W^\mp)$). Since ρ is parallel, ∇ respects the decomposition $W = W^+ \oplus W^-$. Thus ∇ induces a connection on L_W with curvature F.

Much of the usefulness of SC-structures is a consequence of the following easy lemma.

Lemma 19. *The set of isomorphism classes $\mathcal{SC}$ of SC-structures is an $H^2(X,\mathbb{Z})$ torsor, i.e., if $\mathcal{SC} \neq \emptyset$ and we fix an SC-structure W_0, then for every SC-structure W_1, there exits a unique line bundle $\mathcal{L}$ such that $W_1 = W_0 \otimes \mathcal{L}$. Every SC-structure W admits a unitary SC-structure $(W, \langle,\rangle, \nabla)$. If we fix one unitary SC-structure $(W_0, \langle,\rangle_0, \nabla_0)$, there is a unique triple $(\mathcal{L}, h, d)$ of a smooth line bundle $\mathcal{L}$, with hermitian metric h and unitary connection d, such that*

$$(W, \langle,\rangle, \nabla) \cong (W_0, \langle,\rangle_0, \nabla_0) \otimes (\mathcal{L}, h, d). \tag{11}$$

Proof. Clearly if W_0 is an SC-structure, so is $W_0 \otimes \mathcal{L}$ for every line bundle $\mathcal{L}$. Conversely, the bundle of Clifford linear homomorphisms $\mathcal{L}(W_0, W) = \mathcal{H}om_C(W_0, W)$ has rank 1, and the natural map $W_0 \otimes \mathcal{L}(W_0, W) \to W$ is an isomorphism.

A partition of unity reduces the existence of a unitary SC-structure Clifford module structures to a local question. But a local example is obtained by lifting the Levi-Civita connection on the oriented frame bundle to the Spin covering. In any case existence will be clear for Kähler surfaces. It follows directly from the definition of a Clifford module that the natural connection and metric on $\mathcal{H}om(W_0, W)$ leaves $\mathcal{L}(W_0, W)$ invariant. Hence there is an induced metric and connection (h, d) on $\mathcal{L}(W_0, W)$, which has property (11). Conversely, if $(W, \langle,\rangle, \nabla)$ is defined by Equation (11), then

$$(\mathcal{L}, h, d) = \mathcal{H}om_C\left[(W_0, \langle,\rangle_0, \nabla_0)\,;\, (W_0, \langle,\rangle_0, \nabla_0) \otimes (\mathcal{L}, h, d)\right]$$

which proves uniqueness. $\square$

If a base SC-structure is chosen, the line bundle $\mathcal{L}$ will be called the twisting line bundle.

There is a natural gauge group $\mathcal{G}^{\mathbb{C}}$ acting on a unitary SC-structure, the group of all smooth invertible Clifford linear endomorphisms. $\mathcal{G}^{\mathbb{C}}$ can be canonically identified with $C^\infty(X, \mathbb{C}^*)$. In the representation (11), $\mathcal{G}^{\mathbb{C}} = C^\infty(X, \mathbb{C}^*)$ acts in the usual way on the set of metrics and unitary connections on the twisting line bundle $\mathcal{L}$. Since every hermitian metric on a line bundle is gauge equivalent, so is every Clifford invariant metric on a hermitian SC-structure. Thus, up to gauge we can fix an invariant metric and we are left with a residual gauge group $\mathcal{G} = C^\infty(X, \mathrm{U}(1))$.

The set of Clifford connections $\mathcal{A}$ on a fixed hermitian SC-structure $(W, \langle,\rangle)$ (i.e., unitary SC-structures) is an affine space $\nabla_0 + \sqrt{-1}A^1_{\mathbb{R}}(X)$. Using the representation (11) and harmonic representatives, one shows that the set of connections mod gauge is

$$\mathcal{B} = \mathcal{A}/\mathcal{G} \cong \sqrt{-1}A^1_{\mathbb{R}}(X)/d\log C^\infty(X, \mathrm{U}(1)) \cong H^1_{DR}(X)/H^1(X,\mathbb{Z}) \oplus \ker d^*$$

Let $\mathcal{Q}^* = \mathcal{A} \times A^0(W^+)^*$, where the star denotes non trivial spinors. We set

$$\mathcal{P}^* = \mathcal{Q}^*/\mathcal{G}.$$

It is a $\mathbb{CP}^\infty \times \mathbb{R}^+$ bundle over $\mathcal{B}$. Thus $\mathcal{P}^*$ has the homotopy type of $(S^1)^{b_1(X)} \times \mathbb{CP}^\infty$. We have the following natural description of the hyperplane class x. Fix a point and consider the based gauge group $\mathcal{G}_0$. Then $\mathcal{Q}^*/\mathcal{G}_0$ is a principal $U(1)$ bundle over $\mathcal{P}^*$ and $x = -c_1(\mathcal{Q}^*/\mathcal{G}_0)$.

There is an alternative description of $\mathcal{B}$ and $\mathcal{P}^*$ that will be useful. Let $\mathcal{A}^{\mathbb{C}}$ be the set of all Clifford connections, and $\mathcal{H}$ the set of all hermitian metrics on $\mathcal{L}$. Let

$$\mathcal{A}^{\mathrm{uni}} = \{(\nabla, <,>),\ \nabla \text{ is } <,> \text{-unitary}\} \subset \mathcal{A}^{\mathbb{C}} \times \mathcal{H}$$

be the set of unitary SC-structures. Fix a metric $<,>_0$ and a $<,>_0$-unitary connection ∇_0. The representation $\nabla = \nabla_0 + a$, models $\mathcal{A}^{\mathbb{C}}$ on $A^1_{\mathbb{C}}(X)$, and the representation $<,>= e^f <,>_0$ models $\mathcal{H}$ on $A^0_{\mathbb{R}}(X)$. A pair $(\nabla, <,>) \in \mathcal{A}^{\mathrm{uni}}$ if and only if $a + \bar{a} = df$. In particular a is determined by f and $\mathrm{Im}(a)$, so $\mathcal{A}^{\mathrm{uni}}$ is modeled on $A^0_{\mathbb{R}}(X) \times A^1_{\mathbb{R}}(X)$.

Now the diagonal action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{A}^{\mathbb{C}} \times \mathcal{H}$ leaves $\mathcal{A}^{\mathrm{uni}}$ invariant. Our alternative description of $\mathcal{B}$ and $\mathcal{P}^*$ is

$$\mathcal{P}^* = (\mathcal{A}^{\mathrm{uni}} \times A^0(W^+)^*)/\mathcal{G}^{\mathbb{C}} \to \mathcal{B} = \mathcal{A}^{\mathrm{uni}}/\mathcal{G}^{\mathbb{C}} \tag{12}$$

Finally, to do decent gauge theory we have to complete to Banach spaces and -manifolds. Seiberg Witten theory works fine with an L^p_1 completion of $\mathcal{A}$, $\mathcal{A}^{\mathbb{C}}$, and $A^0(W^+)$ and an L^p_2 completion of $\mathcal{G}$, $\mathcal{G}^{\mathbb{C}}$ and $\mathcal{H}$ if $p > \dim X$. In this range $L^p_1 \hookrightarrow C^0$, and therefore the two possible L_p descriptions of $\mathcal{P}^*$ and $\mathcal{B}$ coincide. On the other hand, the Sobolev range does not seem optimal: with more care and work one can probably use all p-completions with $2 - \dim(X)/p > 0$. We will suppress completions from the notation, explicitly mentioning completions if necessary.

From now on we assume $\dim X = 4$. Fix an SC-structure W and choose an invariant hermitian metric $\langle , \rangle$. Choose a Riemannian metric g and a real 2 form ϵ The anti-symmetrisation map gives an isomorphism $c\colon \Lambda^+ \cong \mathrm{End}^{sh}_0(W^+)$ between the real self-dual forms and the traceless skew hermitian endomorphisms of W^+. This special phenomenon allows us (or rather Seiberg and Witten) to write down the monopole equations [Wit]:

$$\partial\!\!\!/\phi = 0 \qquad \phi \in A^0(W^+) \tag{13}$$

$$c(F^+) = 2\pi\phi\,\langle\phi, -\rangle - \pi|\phi|^2 - 2\pi\sqrt{-1}c(\varepsilon^+). \tag{14}$$

Let $\mathcal{M} = \mathcal{M}(W, g, \epsilon) \subset \mathcal{P}^*$ be the space of solutions modulo gauge.

As a technical remark, note that we use the conventions of [BGV], and that in their conventions the Weitzenböck (Lichnerowitz) formula restricted to W^+ reads

$$\partial\!\!\!/^2 = \nabla^*\nabla + s/4 + c(F^+/2)$$

where s is the scalar curvature ([BGV, th. 3.52] and the observation that the twisting curvature of an SC-structure is $1/\operatorname{rank}(W^+)$ times the curvature on $\det(W^+)$). The sign difference in the $c(F^+)$ term in [KM2, Lemma 2] explains the relative change of sign with respect to [KM2, formula $(*)$] in the Seiberg Witten equations.

It is chosen in such a way that the Weitzenböck formula gives C^0 control on the harmonic positive spinor ϕ.

A basic property of the monopole equation noted by Witten, which follows from the Weitzenböck formula [KM2, Lemma 2] or a variational description [Wit, Section 3], is the following

Proposition 20. *If the metric has non negative scalar curvature s, and $4\pi|\epsilon| \leq s$, then all solutions of the monopole equations have $\phi = 0$.*

Alternatively, we can define $\mathcal{M}$ as the zero of a Fredholm section in an infinite dimensional vector bundle. Let

$$\mathsf{A}^0(W^-) = \mathcal{Q}^* \times_{\mathcal{G}} A^0(W^-)$$

where $A^0(W^-)$ is completed in L^p. When in the second monopole Equation (14) we bring everything to one side, the monopole equations define a section s in in

$$E = \mathsf{A}^0(W^-) \oplus A^+(X),$$

where the second summand is considered as a trivial bundle (and is also completed in L^p).

To see that it is actually a Fredholm section we linearise the equations, assuming that (∇, ϕ) is a solution, and $(\nabla + \varepsilon a, \phi + \varepsilon \psi)$ with $a \in \sqrt{-1}A^1_{\mathbb{R}}(X)$ and $\psi \in A^0(W^+)$ is a solution up to order 1 in ε. We get (cf. [Wit, Eq. 2.4])

$$\partial\!\!\!/\psi + a \cdot \phi = 0$$
$$c^{-1}(2\pi(\phi\,\langle\psi, -\rangle + \psi\,\langle\phi, -\rangle - \mathrm{Re}\,\langle\phi, \psi\rangle) - d^+ a = 0.$$

The tangent space of the $\mathcal{G}$-orbit of (∇, ϕ) is $\{(a, \psi) = (-du, u\phi),\ u \in \sqrt{-1}A^0_{\mathbb{R}}(X)\}$. Thus the Zariski tangent space of $\mathcal{M}$ in (∇, ϕ) is the first cohomology of the Fredholm complex

$$0 \to \sqrt{-1}A^0_{\mathbb{R}}(X) \to \sqrt{-1}A^1_{\mathbb{R}}(X) \oplus A^0(W^+) \to \sqrt{-1}A^+_{\mathbb{R}}(X) \to 0 \oplus A^0(W^-),$$

where the maps are given by the left hand side of the linearised equations. The virtual dimension $d(W) = (-1 + b_1 - b^+) + 2\mathrm{Ind}_{\mathbb{C}}(\partial\!\!\!/)$. By the Atiyah Singer index formula and a little rewriting this is

$$d(W) = \mathrm{vdim}_{\mathbb{R}}(\mathcal{M}) = \tfrac{1}{4}(L_W^2 - (2e(X) + 3\sigma(X))), \tag{15}$$

where $e(X)$ is the topological Euler characteristic, and $\sigma(X)$ the signature [Wit, Eq. 2.5].

The crucial property that makes Seiberg Witten theory so much easier than Donaldson theory is

Proposition 21. [KM2, Corollary 3],[Wit, §3] *The moduli space $\mathcal{M}$ is compact. For fixed $c > 0$ there are only finitely many SC-structures W with $d(\mathcal{M}(W)) \geq -c$ and $\mathcal{M}(W, g, \epsilon) \neq \emptyset$.*

Note that for generic pairs (g, ϵ), moduli spaces of negative virtual dimension are empty, but I do not see an a priori reason why moduli spaces of arbitrary negative virtual dimension should not exist for special pairs. In fact for generic pairs the moduli space is smooth of dimension $d(W)$ [KM2]. However we will not use this fact.

A pair (g, ϵ) is *admissible* if L_W admits no connection with $F^+ = -2\pi\sqrt{-1}\epsilon^+$, where as usual $+$ means taking the self dual part. Admissible metrics and forms exist if $b_+ \geq 1$, since a pair is certainly admissible if $c_1(L_W) \notin \epsilon^{\mathrm{harm}} + H_g^-$ where H_g^- is the space of g-anti-self-dual closed forms, and "harm" means projection to the harmonic part. Note that no use of Sard-Smale is made to define admissibility.

By a transversality argument [Don], or a slightly modified version of Lemma 22 below, the admissible pairs form a connected set if $b_+ \geq 2$. We say that a metric g is admissible if $(g, 0)$ is. Even if $b_+ = 1$, all metrics are admissible when $L_W^2 \geq 0$, and L_W is not torsion, but otherwise we have to be a little bit more careful.

If $b_+ = 1$, the choice of an orientation o_+ of H^+ is the choice of a connected component in $\{\omega^2 > 0\} \subset H^2(X, \mathbb{R})$. It will be called the forward timelike cone. For every metric g, let ω_g be the unique harmonic self dual form in the o_+-forward timelike cone with $\int \omega^2 = 1$. For a pair (g, ϵ) and an SC-structure W define the *discriminant*

$$\Delta_W(g, \epsilon, o_+) = \int (c_1(L_W) - \epsilon)\omega_g \tag{16}$$

Clearly the discriminant depends only on the period $(\omega_g, \epsilon^{+\,\mathrm{harm}})$ and the choice of orientation o_+. It depends on o_+ only through its sign, and we will often drop it from notation. A pair (g, ϵ) is admissible if the discriminant $\Delta_W(g, \epsilon) \neq 0$, because it means precisely that $c_1(L_W) \notin \epsilon^{\mathrm{harm}} + H_g^-$.

Lemma 22. *If $b_+ = 1$ a pair (g, ϵ) is admissible if and only if the discriminant $\Delta_W(g, \epsilon) \neq 0$. There are exactly two connected components of admissible pairs labeled by the sign of the discriminant.*

Proof. Suppose two pairs (g_i, ϵ_i), $i = 0, 1$, have discriminants Δ_i of equal sign. Connect the pairs by a path (g_t, ϵ_t) in the space of all pairs. Let $(\omega_t, \epsilon_t^{+,\mathrm{harm}})$ be the corresponding path of periods. Then the discriminant

$$\Delta_t = \int (c_1(L_W) - \epsilon_t^{+,\mathrm{harm}})\omega_t$$

is continuous in t but may change its sign. However if we modify the path by setting

$$\epsilon'_t = \epsilon_t + (\Delta_t - (1-t)\Delta_0 - t\Delta_1)\omega_t$$

then using $\Delta_W(g, \epsilon + \delta) = \Delta_W(g, \epsilon) - \int \delta \wedge \omega_g$ and $\int \omega_g^2 = 1$ we see that

$$\Delta'_t = \Delta_W(g_t, \epsilon'_t) = (1-t)\Delta_0 + t\Delta_1.$$

In particular Δ'_t does not change sign, so (g_t, ϵ'_t) is a path of admissible pairs.

Conversely, if $c_1(L_W) \in \epsilon^{\mathrm{harm}} + H^-$, then any connection ∇ with induced Chern form ϵ^{harm} determines a "reducible" solution $(\nabla, 0) \in \mathcal{P} - \mathcal{P}^*$ of the monopole equations. $\square$

The index bundle $\mathrm{Ind}(Ds)$ of the deformation complex can be deformed by compact operators (over a compact space!) into the sum of the index of the signature complex and the index of the complex dirac operator. Thus, the determinant line bundle $\det(\mathrm{Ind}(Ds)$ of the index is naturally oriented by choosing an orientation o for $\det H^1(X, \mathbb{R})^\vee \otimes H^+(X, \mathbb{R})$. We will in fact assume that an orientation for both H^+ and H^1 is chosen. Suppose further that the pair (g, ϵ) is admissible (i.e., $\mathcal{M}((W, g, \epsilon) \subset \mathcal{P}^*)$, then the identification of the monopole equations as a section s in the bundle E and Proposition 14 in Section 2 gives us a homology class $\mathbb{M} = \mathbb{Z}(E, s) \in H_{d(W)}(\mathcal{P}^*)$, i.e., a homology class of the proper virtual dimension even if $\mathcal{M}$ is not smooth, not reduced and not of the proper dimension (note that in our case the moduli space $\mathcal{M} = Z(s)$ is compact, and homology with closed support is just ordinary homology). In case $\mathcal{M}$ is smooth and has the proper dimension, it is just the fundamental class. The class $\mathbb{M}$ depends only on the connected component of (g, ϵ) in the space of admissible pairs, by the homotopy property of the localised Euler class, Proposition 14.2. In particular $\mathcal{M}$ is independent of the admissible pair if $b_+ \geq 2$, and depends only on the sign of the discriminant if $b_+ = 1$.

Definition 23. Let X be a smooth oriented 4-manifold with $b_+ \geq 1$. Choose an orientation o of $\det H^1(X, \mathbb{R})^\vee \otimes \det H^+(X, \mathbb{R})$, and if $b_+ = 1$, an orientation o_+ of H^+. If $b_+ \geq 2$, the *SW-multiplicity* is the map

$$\begin{aligned} n_o \colon \mathcal{SC} &\to \Lambda^* H^1(X, \mathbb{Z})[t] \cong H_*(\mathcal{P}^*, \mathbb{Z}) \\ W &\mapsto \mathbb{M}(W, g, \epsilon, o) \end{aligned}$$

where (g, ϵ) is any W-admissible pair. If $b_+ = 1$ the *SW-multiplicities* $n_{o,o_+,+}$ and $n_{o,o_+,-}$ are defined similarly but with pairs $(g_\pm, \epsilon_\pm)$ having positive respectively negative discriminant Δ.

We will usually suppress the dependence of the SW-multiplicity (ies) on the orientations o and o_+. The choice of orientation o only determines the sign of the multilicity: $n_o = -n_{-o}$ and $n_{o,o_+,\pm} = -n_{-o,o_+,\pm}$. The orientation o_+ determines which invariant is n_+ and which is n_- since $n_{o,o_+,+} = n_{o,-o_+,-}$.

All known examples with $b_+ \geq 2$ have non trivial multiplicities only when the virtual dimension $d(W) = 0$. However for surfaces with $p_g = 0$ it is easy to give examples with one of $n_\pm$ non trivial for $d(W) > 0$. We will use such an invariant in fact. If $b_1 \neq 0$, the H^1 part of the multiplicity becomes essential.

Remark 24. Since $H_i(\mathcal{P}^*) = 0$ for $i < 0$, a moduli space of negative virtual dimension never defines a nontrivial class. Thus, if for a class $L \in H^2(X, \mathbb{Z})$ there exists an SC-structure W with $L = c_1(L_W)$ and the multiplicity $n(W) \neq 0$ (respectively one of $n_\pm(W) \neq 0$), then $L^2 \geq 3e(X) + 2\sigma(X)$ (cf. Equation (15)).

Remark 25. In the case $b_+ = 1$ we can alternatively consider the multiplicity as depending not on a sign of Δ but on a chamber structure in

$$\Gamma = \{(\omega, \epsilon) \in H^2(X, \mathbb{R})^2 \mid \omega^2 = 1,\ \omega_0 > 0\}$$

where a chamber is defined by walls which are in turn defined by all classes $L \equiv w_2(X)$ through Equation (16). This is particularly useful when we consider structures with $L_W^2 \geq 0$, L_W is not torsion. Then all pairs $(g, 0)$ are admissible and have discriminant of equal sign, because the forward timelike cone is strictly on one side of the hyperplane $L_W^\perp \subset H^2(X, \mathbb{R})$. Thus, for this subset we have a preferred chamber.

We will say that $L \in H^2(X, G)$ has non trivial multiplicity if there is an SC-structure W such that $L = c_1(L_W)$ and W has non trivial multiplicity. This includes the the fact that L has a lift $\hat{L}$ to $H^2(X, \mathbb{Z})$ with $\hat{L} \equiv w_2(X) \in H^2(X, \mathbb{Z}/2\mathbb{Z})$. If $b_+ = 1$ we will further qualify which multiplicity is non trivial (i.e., n_+ or n_-) or which chamber is chosen. We will simply write $n(L) \neq 0$ or $n_+(L) \neq 0$ etc.

A final and important piece of general theory is the following blow-up formula [Ste],[FS1, §8], [FS2]. We will give a proof valid for Kähler surfaces in Section 6.

Theorem 26. *Let X be a closed oriented 4-manifold with $b_+ \geq 1$. An SC-structure $\tilde{W}$ on $X\#\bar{\mathbb{P}}^2$ can be decomposed as $\tilde{W} = W\#W_k^{\bar{\mathbb{P}}^2}$, with determinant lines $L_{\tilde{W}} = L_W + (2k+1)E$. If the multiplicity $n_{(\pm)}(\tilde{W}) \neq 0$ then $d(\tilde{W}) = d(W) - k(k+1) \geq 0$, and the multiplicity $n_{(\pm)}(W) \neq 0$. Moreover if $L_{W_{\bar{\mathbb{P}}^2}} = \pm E$ (i.e., $E \cdot L_{\tilde{W}} = \pm 1$), then $n_{(\pm)}(\tilde{W}) = n_{(\pm)}(W)$ under the identification $H^1(X,\mathbb{Z}) \cong H^1(\tilde{X},\mathbb{Z})$.*

Here, $n_{(\pm)} = n$ if $b_+ > 1$, and if $b_+ = 1$, it is understood that we compare say $n_+(W\#W_k^{\bar{\mathbb{P}}^2})$ with $n_+(W)$.

4. Seiberg Witten Classes of Kähler Surfaces

From now on, (X, Φ) denotes a Kähler surface. Then X has a natural base SC-structure

$$W_0 = \Lambda^{0,*}X$$

with Clifford multiplication given by

$$c(\omega^{10} + \omega^{01}) = \sqrt{2}\left(-i(\omega^{10}) + \varepsilon(\omega^{01})\right),$$

where i is contraction and ε is exterior multiplication. The metric and connection induced by the Kähler structure on $\Lambda^{0*}X$ define a unitary SC-structure on W_0. For an arbitrary SC-structure $W = W(\mathcal{L})$ the spinor bundles are of form

$$W^+ = (\Lambda^{00} \oplus \Lambda^{02}) \otimes \mathcal{L}, \qquad W^- = \Lambda^{01}(\mathcal{L}).$$

and $L_W = \det(W^+) = -K \otimes \mathcal{L}^2$ (cf. Lemma 19). We call $\mathcal{L}$ the twisting line bundle.

We now turn to the monopole equations (see also [Wit, Section 4]). In the decomposition of W^+, a positive spinor will be written $\phi = (\alpha, \beta)$. The Dirac equation is then [BGV, Propos. 3.67]

$$\partial\!\!\!/\phi = \sqrt{2}(\bar{\partial}\alpha + \bar{\partial}^*\beta) = 0.$$

Since X is Kähler, we can locally choose holomorphic geodesic coordinates (z_1, z_2). A basis of the self dual forms is then the Kähler form $\Phi = \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$, $dz_1 \wedge dz_2$ and $d\bar{z}_1 \wedge d\bar{z}_2$. Let h be an hermitian metric on $\mathcal{L}$. Choose a unit generator e for $\mathcal{L}$. Then an orthonormal basis for W^+ is e and $\frac{1}{2}ed\bar{z}_1 \wedge d\bar{z}_2$.

Using the definition of Clifford Multiplication we compute:

$$\begin{aligned} c(\Phi)e &= \tfrac{\sqrt{-1}}{2}(-i(dz_1)\varepsilon(d\bar{z}_1) + \varepsilon(d\bar{z}_1)i(dz_1) - i(dz_2)\varepsilon(d\bar{z}_2) + \varepsilon(d\bar{z}_2)i(dz_2))e \\ &= -2\sqrt{-1}e. \end{aligned}$$

In exactly the same way we compute $c(\Phi)$, $\frac{1}{2}ed\bar{z}_1 \wedge d\bar{z}_2$, and the action of $c(dz_1 \wedge dz_2)$ and $c(d\bar{z}_1 \wedge d\bar{z}_2)$ on e and $\frac{1}{2}ed\bar{z}_1 \wedge d\bar{z}_2$. The result in matrix form is given by

$$c(\Phi) = \begin{pmatrix} -2\sqrt{-1} & 0 \\ 0 & 2\sqrt{-1} \end{pmatrix} \quad c(dz_1 \wedge dz_2) = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \quad c(d\bar{z}_1 \wedge d\bar{z}_2) = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}.$$

On the other hand, writing $\alpha = \alpha_e e$, and $\beta = \frac{1}{2}\beta_{\dot{1}\dot{2}} ed\bar{z}_1 \wedge d\bar{z}_2$,

$$(\alpha + \beta)\langle \alpha + \beta, - \rangle = \begin{pmatrix} |\alpha_e|^2 & \alpha_e \bar{\beta}_{\dot{1}\dot{2}} \\ \bar{\alpha}_e \beta_{\dot{1}\dot{2}} & |\beta_{\dot{1}\dot{2}}|^2 \end{pmatrix}.$$

Thus, if we define $\alpha^* = h(\alpha, -)$, $\beta^* = h(\beta, -)$ and take the trace free part, we get the healthy global expression

$$(2\pi(\alpha+\beta)\langle\alpha+\beta,-\rangle)_0 = -2\pi\sqrt{-1}c\left(\tfrac{1}{2}(|\beta|_h^2 - |\alpha|_h^2)\Phi + \sqrt{-1}(-\alpha\beta^* + \beta\alpha^*)\right).$$

Plug all this in the monopole equations (13),(14). Writing $c_1(F) = \dfrac{-1}{2\pi i}F$, and using that $\Lambda\Phi = 2$ the monopole equation for a Kähler metric and perturbation $\epsilon = \lambda\Phi$ can be rewritten to

$$\bar\partial\alpha + \bar\partial^*\beta = 0 \tag{17}$$

$$F^{02} = 2\pi\beta\alpha^* \tag{18}$$

$$F^{20} = -2\pi\alpha\beta^* \tag{19}$$

$$\Lambda c_1(F)^{11} = (|\beta|^2 - |\alpha|^2) + 2\lambda. \tag{20}$$

Note that F is the curvature on L_W, but that these are equations for a unitary connection $d = \partial + \bar\partial$ on $\mathcal{L}$ and sections $\alpha \in A^{00}(\mathcal{L})$, and $\beta \in A^{02}(\mathcal{L})$ through the identity $F = -F(K) + 2F(\mathcal{L}, d)$. Here $F(K)$ is the curvature of the canonical line bundle, i.e., minus the Ricci form.

In terms of the twisting bundle the virtual (real) dimension of the moduli space reads

$$d(\mathcal{L}) = d(\Lambda^{0*}(\mathcal{L})) = \tfrac{1}{4}(L^2 - K^2) = \mathcal{L}\cdot(\mathcal{L} - K). \tag{21}$$

To give a more precise description of the moduli space of solutions we lean heavily on the work on the abelian vortex equation by Steve Bradlow [B1], Oscar García-Prada [Gar], and earlier in a different guise by Kazdan Warner [KW]. See also [B2] and [OT2].

Proposition 27. *A necessary condition for the existence of solutions to the monopole equations* (17) *to* (20)*, is that* $(\mathcal{L}, \bar\partial)$ *is a holomorphic line bundle, and that*

$$-\deg_\Phi(K) \le \deg_\Phi(L) < \int(\lambda\Phi^2), \text{ or} \tag{22}$$

$$\int \lambda\Phi^2 < \deg_\Phi(L) \le \deg_\Phi(K), \text{ or} \tag{23}$$

$$\int \lambda\Phi^2 = \deg_\Phi(L). \tag{24}$$

In particular, $L_W = -K \otimes \mathcal{L}^2$ *has a natural holomorphic structure. In case* (22)*, the moduli space* $\mathcal{M} = \mathcal{M}(\mathcal{L}, \Phi, \lambda)$ *of solutions can be identified as a real analytic space with the moduli space of pairs of a holomorphic structures* $\bar\partial$ *on* $\mathcal{L}$*, and a divisor* $\alpha \in |(\mathcal{L}, \bar\partial)|$*. In particular, the Zariski tangent space in* $(\bar\partial, \alpha)$ *is canonically identified with* $H^0(\mathcal{L}|_{Z(\alpha)})$*. In case* (23)*, the moduli space* $\mathcal{M}$ *of solutions can be identified with the moduli space of pairs of a holomorphic structure* $\bar\partial$ *on* $\mathcal{L}$*, and an element* $\beta \in \mathbb{P}H^2(\mathcal{L}) = \underline{|K \otimes \mathcal{L}^\vee|^\vee}$*. In particular, the Zariski tangent space at* $(\bar\partial, \beta)$ *is isomorphic to* $\overline{H^0(K \otimes \mathcal{L}^\vee|_{Z(\bar\beta)})}$*. In case* (24)*, the "moduli space"* $\mathcal{M} \subset \mathcal{P} - \mathcal{P}^*$ *(i.e.,* $\alpha = \beta = 0$*) can be identified with the space of holomorphic structures* $\bar\partial$ *on* $\mathcal{L}$*.*

Proof. Combining (17) and (18) yields

$$\bar\partial\bar\partial^*\beta = -\bar\partial^2\alpha = -F^{02}\alpha = -2\pi|\alpha|^2\beta. \tag{25}$$

(Strictly speaking, this is an equation in L^p_{-1}.) Integrating both sides against $\langle\beta,-\rangle$ immediately gives that $\alpha\beta = 0$ and $\bar\partial\beta = \bar\partial\alpha = 0$. Thus, $F^{02} = F^{20} = 0$, since $F^{02} = 2F^{02}(\mathcal{L}, d)$, $\bar\partial$ is a holomorphic structure on $\mathcal{L}$, and either $0 \neq \alpha \in H^0(\mathcal{L})$ and $\beta = 0$ or $0 \neq \beta \in H^2(\mathcal{L})$ and $\alpha = 0$, or $\alpha = \beta = 0$. Note that if for example $\alpha \neq 0$, then $\beta = 0$ is cut out transversely by Equation (25). The last monopole Equation (20) gives the condition

$$\deg(L) = -\deg(K) + 2\deg(\mathcal{L}) = \tfrac{1}{2}\int \Lambda c_1(F)\Phi^2 = \tfrac{1}{2}\int (|\beta|^2 - |\alpha|^2 + 2\lambda)\Phi^2$$

which fixes the global L_2 norm of α and β, and determines whether $\alpha \neq 0$ or $\beta \neq 0$ or $\alpha = \beta = 0$.

Finally, we deal with Equation (20). If $\alpha \neq 0$ and $\beta = 0$, then it is essentially the abelian vortex equation.

It is slightly more convenient to use our alternative description (12) of $\mathcal{P}^*$, and solve for a pair $(d_{\mathcal{L}}, h)$ where $h = e^f h_0$ is a hermitian metric and $d_{\mathcal{L}} = \partial + \bar\partial = d_0 + a$ is a h-unitary connection on $\mathcal{L}$, and mod out the full gauge group $\mathcal{G}^{\mathbb{C}}$ of all complex nowhere vanishing functions. For an h-unitary connection, we have $\partial h(s,t) = h(\partial s, t) + h(s, \bar\partial t)$ for all sections $s, t \in A^0(\mathcal{L})$. Thus, $d_{\mathcal{L}}$ is determined by $\bar\partial$ and h, or equivalently, a^{01} and f. Then if we let $\mathcal{A}^{01}$ be the space of $\bar\partial$-operators on $\mathcal{L}$ modeled on $A^{01}(X)$ through $\bar\partial = \bar\partial_0 + a^{01}$ with the complex gauge group $\mathcal{G}^{\mathbb{C}}$ acting by conjugation, we see that

$$\mathcal{P}^* \cong \{(\bar\partial, h, \alpha, \beta) \in \mathcal{A}^{01} \times \mathcal{H}_{\mathcal{L}} \times (A^{00}(\mathcal{L}) \oplus A^{02}(\mathcal{L}))^*\}/\mathcal{G}^{\mathbb{C}}.$$

To be precise, we take $d_{\mathcal{L}}$ and $\bar\partial$ in L^p_1, and $\mathcal{G}^{\mathbb{C}}$ and f in L^p_2 with $p > 4$. The sections α and β, being disguised spinors, are as before in L^p_1.

Expressed in a^{01} and f, Equation (20) becomes

$$\Delta f = 2\pi(|\beta|^2_{h_0} - |\alpha|^2_{h_0})e^f - 2\sqrt{-1}\Lambda(\partial_0 a^{01} - \bar\partial_0 \bar{a}^{01}) + \mu \tag{26}$$

where $\mu = 2\pi(2\lambda + \Lambda c_1(F(K)) - 2\Lambda c_1(\mathcal{L}, \nabla_0))$ (compare [B1, Lemma 4.1]).

If β is small in L^p_1, hence in C^0, we can solve for f in Equation (26) with the solution depending real analytically on (a^{01}, α) by the analytical Lemma 33. More invariantly, if β is small, there is a unique metric $h(\bar\partial, \alpha, \beta) = h_0 e^{f(\bar\partial - \bar\partial_0, \alpha, \beta)}$ solving the last monopole Equation (20).

In geometric terms, this has the following consequence. Let

$$\mathcal{P}^{01*} = \mathcal{A}^{01} \times (A^{00}(\mathcal{L}) \oplus A^{02}(\mathcal{L}))^*/\mathcal{G}^{\mathbb{C}}.$$

Clearly, there is a projection $\mathcal{P}^* \to \mathcal{P}^{01*}$ forgetting h. What we have done is showing that there is section

$$\begin{aligned} \mathcal{P}^{01*} &\to \mathcal{P}^* \\ (\bar\partial, \alpha, \beta) &\to (\bar\partial, \alpha, \beta, h(\bar\partial, \alpha, \beta)) \end{aligned}$$

in a neighborhood of $\beta = 0$ whose image is cut out as a real analytic space by the last monopole Equation (20).

Now we will cut out $\mathcal{M} = Z(E, s)$ in three steps rather than in one. In each step we define a quotient bundle $E \to E'' \to 0$ with kernel E'. We check that the quotient section s'' is transversal near $\mathcal{M}$, so near $\mathcal{M}$, the zero set $Z(s'')$ is a smooth manifold. On $Z(s')$ we have an induced section s' in $E'|_{Z(s'')}$ and we have $\mathcal{M} = Z(E'|_{Z(s'')}, s')$. If everything was real analytic then this is an identification as real analytic spaces. Moreover, if s is Fredholm, then $\mathrm{Ind} Ds = \mathrm{Ind} Ds'$. This is exactly the procedure needed to apply the localised Euler class machinery (see Proposition 14), which is what we will do in Section 6.

Let

$$\mathcal{Q}^* = \mathcal{A}^{01} \times \mathcal{H}_{\mathcal{L}} \times (A^{00}(\mathcal{L}) \oplus A^{02}(\mathcal{L}))^*$$

and define

$$\mathsf{A}^{pq}(\mathcal{L}) = \mathcal{Q}^* \times_{\mathcal{G}^{\mathbb{C}}} A^{pq}(\mathcal{L}) \tag{27}$$

in a suitable completion which may vary and which we will indicate. Then the bundle E over $\mathcal{P}^*$ in which the monopole equations define a section s can be identified with

$$E = \mathsf{A}^{01}(\mathcal{L}) \oplus A^{02}(X) \oplus A^0_{\mathbb{R}}(X)$$

(all completed in L^p_1) with decomposition of the section $s = (s_{01}, s_{02}, s_\Phi)$ corresponding to equation (17), (18) and (20). On the other hand

$$T\mathcal{P}^* = \left(A^{01}(X) \oplus A^0_{\mathbb{R}}(X) \oplus \mathsf{A}^{00}(\mathcal{L}) \oplus \mathsf{A}^{02}(\mathcal{L})\right) / A^{00}(X)$$

(completed in respectively L^p_1, L^p_2, L^p_1, L^p_1 and L^p_2).

For the first step, define $E_1 = \mathsf{A}^{01}(\mathcal{L}) \oplus A^{02}(X)$ (both completed in L^p_1). Then we have the exact sequence

$$0 \to E_1 \to E \to A^0_{\mathbb{R}}(X) \to 0$$

with the projection s_Φ of s defining a section in $A^0_{\mathbb{R}}(X)$. It is just the vortex Equation (20). We have maps

$$A^0_{\mathbb{R}}(X) \xrightarrow{i} T\mathcal{P}^*|_{Z(s_\Phi)} \xrightarrow{Ds_\Phi} A^0_{\mathbb{R}}(X)$$

(first $A^0_{\mathbb{R}}$ completed in L^p_2 second in L^p). In a point $(\bar\partial, \alpha, 0, h(\bar\partial, \alpha, 0)$ the composition is given by

$$\delta f \mapsto (\Delta + 2\pi|\alpha|^2_h)\delta f. \tag{28}$$

which is surjective if $\alpha \neq 0$. Hence near $\beta = 0$, the map Ds_Φ is surjective, and $Z(s_\Phi)$ is smooth. Thus near $\beta = 0$, in particular near $\mathcal{M}$, the solutions of the vortex equation $Z(s_\Phi)$ can be identified with the image of the section $\mathcal{P}^{01*} \to \mathcal{P}^*$ even as a real analytic space.

Now $\mathcal{M}$ is cut out on $Z(s_\Phi)$ by the section $s_1 = (s_{01}, s_{02})$ in $E_1|_{Z(s_\Phi)}$. But there is a bundle $\tilde E_1$ over $\mathcal{P}^{01}$, defined similarly to E_1 with section $\tilde s_1$ defined similarly by the monopole equations (17) and (18) and we identify $\mathcal{M}$ with $\mathcal{M}^{01} = Z(\tilde s_1)$.

For step two consider the following exact sequence over a neighborhood of $\mathcal{M}^{01}$ in $\mathcal{P}^{01*}$

$$0 \to \tilde E_2 \to \tilde E_1 \xrightarrow{\delta} \mathsf{A}^{02}(\mathcal{L}) \to 0$$

(the last term completed in L^p_{-1}) where the operator δ in $(\bar\partial, \alpha, \beta)$ is given by

$$\delta\colon (\xi, \omega) \mapsto \bar\partial\xi - \alpha\omega.$$

Then δ is really surjective near $\mathcal{M}$. In a point $(\bar\partial, \alpha, 0)$ with $\bar\partial^2 = 0$ and $\bar\partial\alpha = 0$, the space $\operatorname{Coker}\delta$ is a dolbeault representative of the hyper cohomology group

$$\mathbb{H}^3(0 \to \mathcal{O} \xrightarrow{\alpha} (\mathcal{L}, \bar\partial) \to 0) \cong H^2((\mathcal{L}, \bar\partial)|_{Z(\alpha)}.$$

Since $Z(\alpha)$ is a complex curve we conclude that $\operatorname{Coker}\delta = 0$.

The induced section s_1'' in A^{02} is given by the composition

$$(\bar\partial,\alpha,\beta)\mapsto(\bar\partial\alpha+\bar\partial^*\beta,\bar\partial^2-2\pi\alpha^*\beta)\mapsto(\Delta+2\pi|\alpha|^2_{h(\bar\partial,\alpha,\beta)})\beta.$$

Since we have assumed that $\alpha\neq 0$ we conclude that

$$Z(s_1'')=\{(\bar\partial,\alpha,\beta)\in\mathcal{P}^{01*},\beta=0\}. \tag{29}$$

Now the derivative $Ds_1''\colon T\mathcal{P}^{01}|_{Z(s_1'')}\to\mathsf{A}^{02}(\mathcal{L})|_{Z(s_1'')}$ can be identified with a map

$$\operatorname{Coker}\big(0\to A^{00}(X)\to\mathcal{A}^{01}(X)\oplus\mathsf{A}^{00}(\mathcal{L})\oplus\mathsf{A}^{02}(\mathcal{L})\big)\to\mathsf{A}^{02}(\mathcal{L}).$$

In $(\bar\partial,\alpha,0)$ it is given by

$$(\delta a^{01},\delta\alpha,\delta\beta)\to(\Delta+2\pi|\alpha|^2)\delta\beta,$$

and we see that Ds_1'' is surjective. Moreover, Equation (29) is an identification as real analytic spaces. Then $\mathcal{M}\cong Z(s_2)$ where $\tilde s_2$ is the induced section $\tilde s_1'$ in $\tilde E_2|_{Z(s_1'')}$.

In step three we introduce the space

$$\mathcal{P}^{\mathrm{BN}*}=\mathcal{A}^{01}(X)\times A^{00}(\mathcal{L})^*/\mathcal{G}^{\mathbb{C}}$$

(BN for Brill Noether) which we identify with $Z(s_1'')=\{\beta=0\}\subset\mathcal{P}^{01*}$. Let $E_2=\tilde E_2|_{\mathcal{P}^{\mathrm{BN}*}}$ and s_2 the section identified with $\tilde s_2$. Then $s_2\colon\mathcal{P}^{\mathrm{BN}*}\to E_2$ is given by

$$s_2\colon(\bar\partial,\alpha)\mapsto(\bar\partial\alpha,\bar\partial^2)$$

and we finally find our identification as real analytic spaces

$$\mathcal{M}\cong Z(s_2)=\mathcal{M}^{\mathrm{BN}}=\{(\bar\partial,\alpha),\,\bar\partial^2=0,\,\bar\partial\alpha=0,\,\alpha\neq 0\}/\mathcal{G}^{\mathbb{C}}\subset\mathcal{P}^{\mathrm{BN}*}. \tag{30}$$

The space $\mathcal{M}^{\mathrm{BN}}$ is exactly the moduli space of holomorphic line bundles together with a non vanishing section carried by the same underlying smooth line bundle $\mathcal{L}$, i.e., all homologically equivalent effective divisors. This is the Brill Noether space.

For the Zariski tangent space, Equation (30) gives

$$\begin{aligned}T_{(\nabla,\alpha,0,h)}\mathcal{M}&=T_{(\bar\partial,\alpha)}\mathcal{M}^{\mathrm{BN}}\\&=\operatorname{Ker}\begin{pmatrix}\bar\partial&\\ \alpha&\bar\partial\end{pmatrix}\Big/\operatorname{Im}\begin{pmatrix}\alpha\\-\bar\partial\end{pmatrix}\\&=\mathbb{H}^1(0\to\mathcal{O}\xrightarrow{\alpha}\mathcal{L}\to 0)\\&=H^0(\mathcal{L}|_{Z(\alpha)}).\end{aligned}$$

It is easy to check that the linearised versions of equations (17), (18), (19), and (26) give the same result (as it should).

Case (23) is reduced to the previous case by Serre duality. In case (24) the metric h we look for is an (almost) Hermite-Einstein metric. $\square$

Remark 28. For future reference we note that $\mathcal{P}^{\mathrm{BN}*}$ and E_2 have a natural complex structure, and that s_2 is holomorphic! Thus $\mathcal{M}^{\mathrm{BN}}$ has a naturally the structure of a complex space. It need not be a complex manifold, we only know that Ds_2 can be identified with the map

$$\text{(31)}\quad \operatorname{Coker}\left(0 \to A^{00}(X) \xrightarrow{\binom{\bar\partial}{-\alpha}} A^{01}(X) \oplus \mathsf{A}^{00}(\mathcal{L})\right) \xrightarrow{Ds_2 = \begin{pmatrix}\bar\partial & \\ \alpha & \bar\partial\end{pmatrix}} \operatorname{Ker}\left(A^{02}(X) \oplus \mathsf{A}^{01}(\mathcal{L}) \xrightarrow{(-\alpha\ \bar\partial)} \mathsf{A}^{02}(\mathcal{L}) \to 0\right)$$

Thus, after a compact perturbation on a compact space,

$$\text{(32)}\qquad \operatorname{Ind}_{\mathbb{C}} Ds_2 = \mathbb{C}^{-\chi(\mathcal{O}_X)} + \operatorname{Ind}_{\mathbb{C}}\bar\partial$$

where $\bar\partial$ is the universal $\bar\partial$ operator on the complex

$$\text{(33)}\qquad 0 \to \mathsf{A}^{00}(\mathcal{L}) \xrightarrow{\bar\partial} \mathsf{A}^{01}(\mathcal{L}) \xrightarrow{\bar\partial} \mathsf{A}^{02}(\mathcal{L}) \to 0.$$

Also note that the orientation conventions for the signature on a Kähler manifold are set up such that the orientation index of the signature complex is $\det_{\mathbb{R}} \mathbb{C}^{\chi(\mathcal{O}_X)}$. Further, $\operatorname{Ind}_{\mathbb{C}}\bar\partial = \operatorname{Ind}_{\mathbb{C}}\not\partial$. Hence the standard complex orientation of $\operatorname{Ind}_{\mathbb{C}} Ds_2$ is the one compatible with the identification $(\operatorname{Ind}_{\mathbb{C}} Ds_2)_{\mathbb{R}} = \operatorname{Ind}_{\mathbb{R}} Ds$ needed for the definition of the SW-multiplicities.

Corollary 29. *Let X be Kähler surface. Suppose that a cohomology class $L \in H^2(X, \mathbb{Z})$ satisfies $n(L) \neq 0$. Then L is of type $(1,1)$. Moreover if $p_g > 0$, then for all Kähler forms Φ on X, the class L satisfies*

$$\text{(34)}\qquad \deg_\Phi(K_X) \geq \deg_\Phi(L) \geq -\deg_\Phi K_X.$$

If $p_g = 0$, and $n_-(L) \neq 0$ (resp. $n_+(L) \neq 0$), then

$$\deg_\Phi(L) \geq -\deg_\Phi(K_X) \quad \textit{(resp.}\ \deg_\Phi(L) \leq \deg_\Phi(K_X)).$$

Proof. First we consider the case $p_g > 0$. Under the conditions of the corollary, there is an SC-structure W with $L_W = L$ which admits at least one solution to the monopole equation for *every* admissible pair (g, ϵ). In particular W admits a solution for every Kähler metric and $\epsilon = \lambda\Phi$. Thus $L = L_W$ is of type $(1,1)$. Moreover the necessary condition for the existence of a solution of section *or* cosection type (i.e., Equation (22) *or* (23) in Proposition 27) gives precisely the required inequality (34) if we let λ tend to zero.

If $p_g = 0$, then L is automatically of type $(1,1)$ and say the condition $n_-(L) \neq 0$ means that there is an SC-structure W with $L_W = L$ such that for any Kähler metric, W admits solutions of section type (i.e., Equation (22)) if λ is sufficiently large. This gives a lower bound but no upper bound on $\deg_\Phi(L)$. □

Remark 30. Recall that an admissible metric is an admissible pair with $\varepsilon = 0$. If $p_g = 0$, and $L^2 \geq 0$ with L not torsion, then all metrics are admissible and have discriminant of equal sign σ. If in addition the preferred multiplicity $n_\sigma(L) \neq 0$, we still obtain the stronger inequality (34). In particular, on a del Pezzo surface, classes with $L^2 \geq 0$ and $n_\sigma(L) \neq 0$ do not exist.

We can now do our useful ad hoc computations of SW multiplicities for classes L "close to $\pm K_X$".

Corollary 31. *Let X be a Kähler surface with base SC-structure $W_0 = \Lambda^{0*}X$. Then $n(W_0) = 1$ if $p_g > 0$ and $n_-(W_0) = 1$ if $p_g = 0$. In particular, $n(-K_X) \neq 0$ resp. $n_-(-K_X) \neq 0$. Likewise, $n(W_0(K_X)) = \pm 1$ if $p_g > 0$ and $n_+(W_0(K_X)) = \pm 1$ if $p_g = 0$. In particular, $n(K_X) \neq 0$ resp. $n_+(K_X) \neq 0$. Moreover, W_0 is the only SC-structure W with $L_W = -K_X$ mod torsion and nontrivial multiplicity n respectively n_-. In particular, if there is an $L \in H^2(X, \mathbb{Z})$ such that $L = -K \in H^2(X, \mathbb{Q})$ and $n(L) \neq 0$ resp. $n_-(L) \neq 0$, then $L = -K \in H^2(X, \mathbb{Z})$.*

Proof. We will prove the statement for $-K_X$. Then we have to consider SC-structures $W = \Lambda^{0*}(\mathcal{L})$ with $c_1(\mathcal{L})$ torsion. Choose a Kähler metric and $\lambda \gg 0$. Then $\mathcal{M}(W) \cong \mathcal{M}^{\mathrm{BN}}(c_1(\mathcal{L}))$, the moduli space of line bundles with a section of topological type given by $c_1(\mathcal{L})$. But $\mathcal{M}^{BN}(\mathcal{L})$ is just a reduced point if $c_1(\mathcal{L}) = 0$, and empty if $c_1(\mathcal{L})$ is non trivial torsion. Thus $W_0 = \Lambda^{0*}X$ is unique among the SC-structures W with $L_W = -K_X$ mod torsion with $n(W) \neq 0$ (resp. $n_-(W) \neq 0$). In fact its multiplicity is 1. The case $+K_X$ can be dealt similarly with Serre duality. Its multiplicity is ± 1 because of the unpleasant orientation flips. $\square$

Corollary 32. *Let D be an effective divisor with $D \cdot (D - K) = 0$, $h^0(\mathcal{O}(D)) = 1$, $h^0(\mathcal{O}_D(D)) = 0$, and $h^0(\mathcal{L}(D)) = 0$ for every non trivial holomorphic line bundle $\mathcal{L} \in \mathrm{Pic}^0(X) - 0$. Then $n(-K_X + 2D) \neq 0$ if $p_g > 0$ and $n_-(-K_X + 2D) \neq 0$ if $p_g = 0$. Likewise, $n(K_X - 2D) \neq 0$ if $p_g > 0$ and $n_+(K_X - 2D) \neq 0$ if $p_g = 0$.*

Proof. This corollary is proved just as the previous one, and reduces to it if $D = 0$. The conditions of the corollary ensure that $\mathcal{M}^{\mathrm{BN}}([D])$ consists of one smooth point and that $\mathrm{vdim}(\Lambda^{0*}(D)) = 0$. $\square$

It remains to collect the relevant analysis from Steve Bradlow [B1, §4] and Kazdan Warner [KW].

Lemma 33. *Let X be a compact Riemannian manifold, and $\dim(X) < p < \infty$ a Sobolev weight. Then for every real non-negative function $0 \leq w_0 \in L^p$, with $\int w_0 > 0$ and real function $\mu_0 \in L^p$, with $\int \mu_0 > 0$, there exists a neighborhood $U_{(w_0,\mu_0)} \subset L^p \times L^p$ such that for all $(w, \mu) \in U_{(w_0,\mu_0)}$ the equation*

$$\Delta f = -we^f + \mu \tag{35}$$

has a unique L^p_2 solution depending analytically on w and μ. The solution is smooth if w and μ are smooth.

Proof. As in [B1, Lemma 4] make the substitution $f = \tilde{f} - g$ where g is the unique solution of $\Delta g = \int \mu - \mu$ to reduce to the case where μ is constant. Then apply [KW, Theorem 10.5(a)] to solve the equation for w_0, μ_0 (note that Kazdan-Warner's Laplacian is negative definite and that the proof works fine with $w \in L^p$ instead of C^∞). Since at a solution f_0 for (w_0, μ_0) we have

$$\delta\text{“eqn (35)”} = (\Delta + w_0 e^{f_0})\delta f$$

and $(\Delta + w_0 e^{f_0})$ is invertible, we conclude with the implicit function theorem that there continues to exist a solution for (w, μ) in a small neighborhood of (w_0, μ_0), and that this solution depends real analytically on (w, μ). Regularity follows from standard bootstrapping techniques. Uniqueness follows from the weak maximum principle ([GT, Theorem 8.1], cf. [KW, remark 10.12]). $\square$

5. Proof of the Main Theorems

We will first prove Theorem 1. Our first task is to define a suitable set $\mathcal{K}$ of basic classes.

Definition 34. Let X be a smooth oriented compact four manifold. If $b_+ \geq 2$ then the basic classes are defined by

$$\mathcal{K} = \{K \in H^2(X,\mathbb{Z}) \mid n(K) \neq 0\}.$$

If $b_+ = 1$ then $\mathcal{K} = \mathcal{K}_- \cup \mathcal{K}_+$ where

$$\mathcal{K}_- = \{K \in H^2(X,\mathbb{Z}) \mid n_-(K) \neq 0, \text{ and } \exists L \in H^2(X,\mathbb{Z}) \text{ with } n_-(L) \neq 0 \\ \text{such that } n_-(L - m(K+L)) \neq 0 \text{ for some } m \geq 1 \ \}.$$

The set $\mathcal{K}_+$ is defined similarly in terms of n_+. Here, we allow $m \in \mathbb{Q}$, but $m(K+L)$ must necessarily lift to a two divisible integral class.

These basic classes are rightfully *the* Seiberg Witten basic classes when $b_+ \geq 2$, but for $b_+ = 1$ the definition is geared towards the specific application we have in mind.

Proposition 35. *The classes $\mathcal{K}$ defined above have all properties* $(*)$ *of Section* 1.

Proof. It is clear that $\mathcal{K}$ is an oriented diffeomorphism invariant, and that the basic classes are characteristic.

For Kähler surfaces the classes are of type $(1,1)$ by Corollary 29.

The degree inequality $((*).2)$ (for all surfaces minimal or not) also follows from Corollary 29. This is immediate for $p_g > 0$. If $p_g = 0$ assume that $K \in \mathcal{K}_+$ say, the case $K \in \mathcal{K}_-$ being essentially the same. Now Corollary 29 gives the three inequalities

$$\begin{aligned} \deg K &\leq \deg K_X, \\ \deg L &\leq \deg K_X, \\ -m \deg K + (1-m)\deg L &\leq \deg K_X \end{aligned}$$

which together imply $-\deg K_X \leq \deg K \leq \deg K_X$.

The pushforward property under blow down $(*).3$ follows immediately from the blow up formula Theorem 26 or Proposition 43.

If $p_g > 0$ then $K_X \in \mathcal{K}$ by Corollary 31. Thus it remains to check that $K_X \in \mathcal{K}$ if $p_g = 0$. In fact we will check that $-K_X \in \mathcal{K}$.

We have already seen in Corollary 31 that $n_-(-K_X) \neq 0$. Define

$$L = -K_X + 2\sum E_i = -K_{\min} + \sum E_i.$$

Either directly from Corollary 32, or using the invariance under the reflection in the exceptional curves $E_1, \dots, E_n$ we see that $n_-(L) \neq 0$. Now denoting

$$\mathcal{L}_m = mK_{\min} + \sum E_i,$$

we have $L - m(-K_X + L) = -K_X + 2\mathcal{L}_m$, so we check that $n_-(-K_X + 2\mathcal{L}_m) \neq 0$. We will distinguish four cases.

If $\kappa(X) = 0$, then $K_{\min}$ is torsion and we can take $m = \operatorname{ord}(K_{\min})$, since $n_-(-K_X + 2\sum E_i) \neq 0$.

If $\kappa(X) = 1$, then $X_{\min}$ has a unique elliptic fibration $X_{\min} \xrightarrow{\pi} C$. By the canonical bundle formula, $K_{\min} = \pi^*\mathcal{L}_C(\pi^* K_C + \sum(p_i - 1)F_i)$, where $\mathcal{L}_C$ is a holomorphic line bundle on C of degree χ. Since $p_g = 0$ and $\chi \geq 0$, we have $0 \leq g \leq q \leq 1$, and we distinguish further between $g = 0$ and $g = 1$.

If $g = 0$, then $c_1(\pi^*\mathcal{L}_C(K_C)) = (\chi - 2)F$, where F is a general fibre, and there are at least $3 - \chi$ multiple fibers because $K_{\min} > 0$. Now the class

$$K_{\min} + \sum_{i=1}^{2-\chi} F_i = \sum_{j=3-\chi}^{n} (p_j - 1)F_j$$

is of the form $mK_{\min}$ with rational $m > 1$. Again by Corollary 32, we have

$$n_-(-K_X + 2\mathcal{L}_m) = n_-(-K_X + 2(\sum_{j=3-\chi}^{n} (p_j - 1)F_j + \sum E_i)) \neq 0.$$

If $g = 1$, then $\chi = 0$, and $K_C = 0$. In this case we can take $m = 1$, since $c_1(\mathcal{L}_C) = 0 \in H^2(X, \mathbb{Z})$ and by Corollary 32

$$n_-(-K_X + 2\mathcal{L}_1) = n_-(-K_X + 2(\sum(p_i - 1)F_i + \sum E_i)) \neq 0.$$

The most instructive case is when X is of general type. Then the irregularity $q = 0$ since $p_g = 0$ and $\chi(\mathcal{O}_X) > 0$. Take $m = 2$, then $\mathcal{M}^{\mathrm{BN}}(c_1(\mathcal{L}_2)) = |2K_{\min} + \sum E_i|$. By formula (1) (or directly by Ramanujan vanishing)

$$\dim_{\mathbb{C}} \mathcal{M}^{\mathrm{BN}}(\mathcal{L}_2) = P_2 - 1 = K_{\min}^2 = \tfrac{1}{2}\,\mathrm{vdim}_{\mathbb{R}}(\Lambda^{0*}(\mathcal{L}_2)).$$

Thus, the moduli space is again smooth of the proper dimension and we conclude that $n_-(-K_X + 2\mathcal{L}_2) \neq 0$. In fact $n_-(\Lambda^{0*}(\mathcal{L}_2)) = t^{K_{\min}^2}$ since the canonical line bundle $\mathcal{O}(1)$ on $\mathcal{P}^*$ corresponds to the $\mathcal{O}(1)$ on $\mathcal{M}^{\mathrm{BN}}$. This is because both measure the weight of the action of the constant gauge transformations on the spinors respectively sections. □

Remark 36. It is easy to give a definition of oriented diffeomorphism invariant basic classes for $b_+ = 1$ that satisfy all properties $(*)$ except the invariance under blow down (i.e., property $(*)$.3). A class K is then basic if there exists a metric g such that for all $\delta > 0$ there exists an ϵ with (g, ϵ) admissible, $\|\epsilon^{+,\mathrm{harm}}\| < \delta$ and $n(K, g, \epsilon) \neq 0$. The degree inequality for minimal surfaces then follows from Remark 25. But alas, if $K^2 < 0$ one cannot avoid the possibility that a sign of the discriminant Δ realisable with small ϵ on the blow up can only be realised for large ϵ on the blow down. In my original treatment I used this definition. I am grateful to Robert Friedman for pointing out this mistake.

Corollary 37. *Suppose that X is a surface with $\kappa(X) \geq 0$. Then for all $K \in \mathcal{K}$ we have*

$$K = \lambda K_{\min} + \sum \pm E_i \in H^2(X, \mathbb{Q}) \tag{36}$$

where $\lambda = 0$ if $\kappa(X) = 0$, $|\lambda| \leq 1$ if $\kappa(X) = 1$ and $\lambda = \pm 1$ if $\kappa(X) = 2$. In particular, all classes correspond to a moduli space of virtual dimension $d = 0$.

Proof. Since the virtual dimension $d(K) = \dfrac{1}{4}(K^2 - K_X^2)$ of the moduli space corresponding to a basic class K is non-negative (cf. Remark 24) this lemma is just Lemma 11 and the fact that $\mathcal{K}$ has all properties $(*)$ (Proposition 35). □

We now give the proof of the main Theorem 1.

With basic classes $\mathcal{K}$ having the properties $(*)$ available Proposition 12 implies that for every surface of $\kappa \geq 0$, the class $K_{\min}$ is invariant up to sign and torsion and every (-1)-sphere is represented by a (-1)-curve up to sign and torsion.

We first get rid of torsion in the (-1)-curve conjecture, i.e., part 2 of Theorem 1. Let e be a (-1)-sphere, giving a connected sum decomposition $X = X'\#\bar{\mathbb{P}}^2$. As we have used before, there is a diffeomorphism $R_e = \mathrm{id}\,\#\mathbb{C}$-conjugation representing the reflection in e.

I claim that for any SC-structure W on a 4-manifold

$$R_e^*(W) = W \otimes \mathcal{O}((c_1(L_W), e)e),$$

where $\mathcal{O}(e)$ is the line bundle corresponding to the Poincaré dual of e. In fact if we write $R_e^* W = W \otimes \mathcal{L}$, then $\mathcal{L} = \mathcal{H}om_C(W, R_e^* W)$, the bundle of Clifford module homomorphisms (cf. the proof of Lemma 19). Now we can just identify W and $R_e^* W$ on X', i.e., $\mathcal{L}$ is trivialised on X'. Thus

$$c_1(\mathcal{L}) \in \operatorname{Im} H^2(X, X - X', \mathbb{Z}) \cong H^2(\bar{\mathbb{P}}^2) \subset H^2(X, \mathbb{Z}).$$

Write $\mathcal{L} = \mathcal{O}(ae)$ for some integer a. Since

$$L_W + 2ae = L_{R_e^* W} = R_e^* L_W = L + 2(e, L_W)e$$

the claim is proved.

Going back to the Kähler case, we can assume that e is homologous to a (-1)-curve E up to torsion. Consider $W = R_e^* R_E^*(\Lambda^{0*}(X)) = \Lambda^{0*}(E - e)$. By oriented diffeomorphism invariance $n_{(-)}(W) \neq 0$ (in case $p_g = 0$ we have tacitly used the fact that $R_e^* R_E^*$ induces the identity on rational cohomology so in particular does not change the orientation of H^+). Moreover $c_1(L_W) = -K_X$ up to torsion. By Corollary 31, we conclude that $W = \Lambda^{0*}(X)$, so $e = E \in H^2(X, \mathbb{Z})$.

Now we know that $\pm K_{\min} \in H^2(X, \mathbb{Q})$ is determined by the oriented smooth manifold, and we want to find $\pm K_{\min}$ over the integers. Pick one of the classes, say $+K_{\min} \in H^2(X, \mathbb{Q})$. Choose a basis $E_1', \ldots, E_n'$ of the lattice in $H^2(X, \mathbb{Z})$ spanned by the (-1)-spheres. Then there is a (-1)-curve E_i with $E_i' = \pm E_i \in H^2(X, \mathbb{Z})$.

There is a class $K \in \mathcal{K}$ such that $K = K_{\min} + \sum E_i' \in H^2(X, \mathbb{Q})$, because the orbit of K_X under the group G generated by the reflections in the (-1)-spheres contains a class of this type. Conversely, in the G orbit of K there is a class $K' \in \mathcal{K}$ with $K' = K_X \in H^2(X, \mathbb{Q})$. Then necessarily $K' = K_X \in H^2(X, \mathbb{Z})$ by Corollary 31. Hence the basic class K is unique. Now we have the identity

$$K_{\min} = K + \sum (E_i', K) E_i' \in H^2(X, \mathbb{Z}).$$

This finally proves Theorem 1.

Remark 38. After reading [FM3] I realised the following. The blow up formula Theorem 26 can be generalised to connected sum decompositions $X = X'\#N$ with N negative definite and $H_1(N, \mathbb{Z}) = 0$. The latter condition is automatic for Kähler surfaces of non-negative Kodaira dimension by a beautiful observation of Kotschick [Kot] (an unramified covering $\tilde{N} \to N$ of degree d gives an unramified covering $\tilde{X} = dX'\#\tilde{N} \to X'\#N$ which is a Kähler surface of non-negative Kodaira dimension with a connected sum decomposition with a factor with $b_+ > 0$). Such smooth negative definite manifolds N have $H_2(N) = \oplus_{i=1}^n \mathbb{Z} n_i$ with $n_i^2 = -1$. SC-structures W_N on N are determined by $L_N = \sum (2a_i + 1) n_i$. Thus the reflections

R_{n_i} in $n_i^{\perp}$, act on the SC-structures on N. SC-structures on $X'\#N$ are of the form $W = W_{X'}\#W_N$. Now the blow up formula is as if $N = n\bar{\mathbb{P}}^2$: $W = W_{X'}\#W_N$ is an SW-structure on $X'\#N$ if and only if $W_{X'}$ is a SW-structure on X' and $d(W) \geq 0$. In particular the Seiberg Witten structures are invariant under the operation $R_{n_i}\colon W_{X'}\#W_N \to W_{X'}\#R_{n_i}W_N$, and $\mathcal{H}om_C(W, R_{n_i}W)$ has a trivialisation over X'. With these remarks the arguments for (-1)-spheres carry over directly to prove that for Kähler surfaces X with $\kappa(X) \geq 0$, with a connected sum decomposition $X = X'\#N$, $H_2(N) \subset H_2(X)$ is spanned by (-1)-curves.

Remark 39. An easy application of the techniques of the next section gives the following. If $\mathcal{L}$ is a holomorphic line bundle on a surface with $p_g = q = 0$ with $h^0(\mathcal{L}) \geq \chi(\mathcal{L}) \geq 1$, then $n_-(\Lambda^{0*}(\mathcal{L})) = t^{\frac{\mathcal{L}(\mathcal{L}-K_X)}{2}}$. If $p_g = q = 0$ and $\kappa(X) \geq 0$ we can apply this to $\mathcal{L}_2 = 2K_{\min} + \sum E_i$. Then by the Castelnuovo criterion and the above we conclude $n_-(-K_X + 2\mathcal{L}_2) \neq 0$. This gives an alternative way to prove that $-K_X \in \mathcal{K}$ for the case $p_g = q = 0$. Conversely, the degree inequality $(*).2$ cannot hold true for rational and ruled surfaces for Kähler forms Φ such that $\deg_\Phi(K_X) < 0$. Since in deriving the degree inequality we did not use that $\kappa(X) \geq 0$, we conclude that for $\kappa(X) = -\infty$ the set of the above defined basic classes $\mathcal{K} = \emptyset$. In particular we see that the following proposition is a rather direct analog of the classical Castelnuovo criterion.

Proposition 40. *A Kähler surface is rational if and only if $b_1 = 0$, and $\mathcal{K} = \emptyset$.*

Stefan Bauer showed me how to use the Seiberg Witten multiplicities and the basic classes to determine the multiplicities of an elliptic surface with finite fundamental group. They are all of type X_{pq} in the proposition below [FM2, Theorem II.2.3]. If an elliptic surface does not have finite cyclic fundamental group, the multiplicities can be read off from the topology [FM2, theorem II.2.5 and Corollary II.7.17].

Proposition 41. (Bauer) *Let X_{pq} be a minimal elliptic surface fibred over $\mathbb{P}^1$ with 2 multiple fibers of multiplicity p and q with $p \leq q$. Then the multiplicities p and q are determined by the underlying oriented differentiable manifold, unless $p_g = 0$, $p = 1$ and q arbitrary. The surfaces X_{1q} are all rational and diffeomorphic.*

Proof. Let F be the homology class of a general fibre, and F_p, F_q the fibers of multiplicity p respectively q. The ray of the fibre in $H^2(X_{pq}, \mathbb{Z})/\text{Torsion}$, is spanned by the primitive vector $\kappa = (\gcd(p,q)/pq)F$. If we fix a Kähler form Φ this is the primitive vector normalised so that $\kappa \cdot \Phi > 0$. The notation κ is traditional and should not be confused with the Kodaira dimension. Now we can write K_X modulo torsion in terms of κ:

$$K_X = (p_g - 1)F + (p-1)F_p + (q-1)F_q = \frac{(p_g+1)pq - p - q}{\gcd(p,q)}\kappa \in H^2(X, \mathbb{Z})/\text{Torsion}.$$

Let $d(p,q) = ((p_g+1)pq - p - q)\,/\gcd(p,q)$ be the oriented divisibility of K_X. If K_X is torsion we set $d(p,q) = 0$. The number $\gcd(p,q)$ is determined by the topology of the manifold, being the order of the fundamental group. We now show how to recover $d(p,q)$ from the underlying smooth oriented manifold (when possible).

The surfaces X_{pq} are rational if and only if the divisibility $d(p,q) < 0$, which is equivalent to $p_g = 0$, $p = 1$ and q arbitrary. Now by Proposition 40, X_{pq} is rational if and only if $\mathcal{K} = \emptyset$ (alternatively use the ex Van de Ven conjecture Corollary 4). Thus, from now on we can assume that X_{pq} has non-negative Kodaira dimension. Then $d(p,q) \geq 0$ and $p \geq 2$.

Let $\bar{\mathcal{K}} = \mathcal{K}/\text{Torsion}$. Since the Kodaira dimension is non negative, $\mathcal{K} \neq \emptyset$, and by Corollary 37 the basic classes in $\bar{\mathcal{K}}$ are on the ray $\mathbb{Z}\kappa \cap [-K_X, K_X]$, i.e., in between $-K_X$ and $+K_X$, and $\pm K_X$ are the extremal classes. Hence if $\bar{\mathcal{K}} = \{0\}$, then K_X is torsion, i.e., $d(p,q) = 0$. If $|\bar{\mathcal{K}}| \geq 2$ then K_X is not torsion and $d(p,q)$ is the unoriented divisibility of $\pm K_X$.

Now assume that $|\bar{\mathcal{K}}| \geq 3$. Choose one of $\pm K_X$, say $-K_X$. We will recover p from the unique basic class $K_1 \in \bar{\mathcal{K}}$ which is extremal but one, and such that $K_1 \in [-K_X, 0]$. Since $d(p,q)$, p_g and $\gcd(p,q)$ are known, this determines q as well.

First consider the case $p_g > 0$. I claim that $K_1 = -K_X + 2F_q$. Write $K_1 = -K_X + 2m\kappa$ with $m > 0$ and $m\kappa$ represented by the smallest effective divisor D such that $n(-K_X + 2D) \neq 0$. Now F_q is the smallest among *all* nontrivial effective divisors on the ray $\mathbb{Z}\kappa$, and $n(-K_X + 2F_q)) \neq 0$ by Corollary 32. Then since $F_q = (p/\gcd(p,q))\kappa$, we see that p is determined by the divisibility of $K_1 - (-K_X)$. If we choose $+K_X$ among $\pm K_X$, then the Serre dual version of Corollary 32 shows that $K_1' = K_X - 2F_q$ is the class extremal but one in $\mathcal{K} \cap [K_X, 0]$. Hence we recover the same value for p from the divisibility of $K_1' - K_X$. From the differential geometric point of view there is nothing that prefers K_X over $-K_X$.

In the case $p_g = 0$ we choose the unique orientation o_+ of H^+, such that $-K_X$ is in backward lightcone, i.e., the standard orientation o_Φ with the forward timelike cone of H^+ containing the Kähler class Φ. Now we can repeat the argument for the case $p_g > 0$ with n replaced by $n_- = n_{o_\Phi,-}$. If we choose $+K_X$ among $\pm K_X$, then the chosen orientation is $-o_\Phi$. We are then looking at the invariant $n_{-o_\Phi,-} = n_{o_\phi,+}$, and again the argument works as in the case $p_g > 0$.

If $1 \leq |\mathcal{K}| \leq 2$, then $-K_X + 2F_q > 0$, and $K_X \geq 0$ i.e., $0 \leq d(p,q) < 2p/\gcd(p,q)$. The few possibilities are listed in the following table

	(p,q)	$\gcd(p,q)$	$d(p,q)$	Type
$p_g = 0$	$(2,2)$	2	0	Enriques
	$(2,3)$	1	1	
	$(2,4)$	2	1	
	$(2,5)$	1	3	
	$(3,3)$	3	1	
	$(3,4)$	1	5	
$p_g = 1$	$(1,1)$	1	0	K3
	$(1,2)$	1	1	

Clearly, in this case the pair (p,q) is determined by the oriented differentiable manifold as well. □

We now give a proof of Corollary 7.

To prove that no surface with Kodaira dimension $\kappa \geq 0$ admits a metric with positive scalar curvature, first consider the case $p_g > 0$. Then the statement is clear, and one of Witten's basic observations. For 4-manifolds with positive scalar curvature $n(K) = 0$ for all $K \in H^2(X,\mathbb{Z})$, since for our metric with positive scalar curvature g and small perturbations ϵ, we have $\mathcal{M}(W,g,\epsilon) = \emptyset$ for all SC-structures

W by Proposition 20. On the other hand we just showed that $n(-K_X) \neq 0$ using a Kähler metric.

The same argument works if $p_g = 0$ and $K_X^2 \geq 0$: $n(-K_X, g, \epsilon)$ is independent of the metric g and of ϵ as long as ϵ is small, with the exception of the case $-K_X$ torsion in which case we have to choose ϵ in the forward light cone. But we can do better.

For the general case $p_g = 0$, we choose our metric of positive curvature g and a sufficiently small perturbation $\epsilon = \lambda\Phi$ with $0 < \lambda \ll 1$. Choose the standard Kähler orientation o_Φ of H^+.

Suppose that g has period $\omega_g = \omega_{\min} + \sum \eta_i E_i$ where $\omega_{\min}$ is the projection to the cohomology of the minimal model. Then since ω_g is in the interior of the forward light cone, and $K_{\min}$ is in the closure of the forward light cone, $\omega \cdot K_{\min} = \omega_{\min} \cdot K_{\min} \geq 0$ with equality if and only if $K_{\min}$ is torsion.

Let $\operatorname{sgn}(\eta) = 1$ if $\eta \geq 0$ and -1 otherwise. Define the class $K = -K_{\min} - \sum \operatorname{sgn}(\eta_i) E_i \in H^2(X, \mathbb{Z})$ then $n_-(K) = n_-(-K_X) \neq 0$. On the other hand we have

$$\omega_g \cdot K \leq 0 < \lambda \int \omega_g \Phi$$

so the discriminant $\Delta(K, g, \epsilon, o_\phi) < 0$. Thus

$$n(K, g, \lambda\Phi, o_\Phi) = n_-(K)$$

i.e., we compute n_- rather than n_+ with respect to the standard orientation o_Φ with the admissible pair (g, ϵ). But g has positive scalar curvature and λ is small so $n(K, g, \lambda\Phi, o_\Phi) = 0$, a contradiction just like before.

6. Some Computations of Seiberg Witten Multiplicities

In this section we will go beyond determining potential basic classes and compute the Seiberg Witten multiplicity of elliptic surfaces. We also prove an algebraic version of the blow up formula. It is here that our excess intersection formulas pay off. We first show how to go over to express the multiplicities in complex geometric terms. Then we use the special geometry of elliptic surfaces to compute them and finally we prove a blow up formula.

From now on we identify an SC-structure with the corresponding twisting line bundle $\mathcal{L}$. We will consider the solutions of the monopole equations of section type, i.e., corresponding to Equation (22), so if necessary we take a perturbation of the form $\lambda\phi$ with $\lambda \gg 0$.

Recall that the SW-multiplicity is essentially the localised Euler class $\mathbb{M}$ of a bundle E over the configuration space $\mathcal{P}^*$ with a section s given by the monopole equations (cf. Definition 23). The zero set of s is the moduli space of solutions $\mathcal{M}(c_1(\mathcal{L}))$ with virtual dimension $d(\mathcal{L}) = c_1(\mathcal{L}) \cdot (c_1(\mathcal{L}) - K_X)$ (cf. Equation 21). To determine this class we use the properties of the localised Euler class in Proposition 14 and work through the three step process in the last part of the proof of Proposition 27 (from (27) onwards). We use the notation introduced in this proof.

If we identify $\mathcal{M}$ with $\mathcal{M}^{\mathrm{BN}}$, the moduli space of holomorphic line bundles with a non trivial section, and the other incarnations of $\mathcal{M}$ in the last part of the proof of 27, then by Proposition 14 part 4 we have

(37)
$$\mathbb{M}(c_1(\mathcal{L})) = \mathbb{Z}(\mathcal{P}^*, E, s) = \mathbb{Z}(\mathcal{P}^{01*}, E_1, s_1) = \mathbb{Z}(\mathcal{P}^{\mathrm{BN}*}, E_2, s_2) \in H_{d(\mathcal{L})}(\mathcal{M}^{\mathrm{BN}}, \mathbb{Z}).$$

Now as we remarked in Remark 28, the section s_2 is holomorphic! Moreover, we have an identification $\mathrm{Ind}_{\mathbb{C}} Ds_2 = \mathrm{Ind}_{\mathbb{C}}\bar\partial + \mathbb{C}^{-\chi(\mathcal{O}_X)}$ where $\bar\partial$ is the universal $\bar\partial$ operator in the sequence

$$0 \to \mathsf{A}^{00}(\mathcal{L}) \xrightarrow{\bar\partial} \mathsf{A}^{01}(\mathcal{L}) \xrightarrow{\bar\partial} \mathsf{A}^{02}(\mathcal{L}) \to 0 \tag{38}$$

on $\mathcal{M}^{\mathrm{BN}}$. Therefore, formula (6) in Proposition 15 tells us that

$$\mathbb{M} = [c(\mathrm{Ind}_{\mathbb{C}}(\bar\partial))^{-1} \cap c_*(\mathcal{M}^{\mathrm{BN}})]_{d(\mathcal{L})}.$$

We will now rewrite $\mathrm{Ind}_{\mathbb{C}}(\bar\partial)$ in more useful holomorphic terms.

We spell out the definition of $\mathsf{A}^{0q}(\mathcal{L})$. Let $\mathcal{Q}^{\mathrm{BN}*} = \mathcal{A}^{01} \times A^{00}(\mathcal{L})^*$. Then we define

$$\mathsf{A}^{0q}(\mathcal{L}) = \mathcal{Q}^{\mathrm{BN}*} \times_{\mathcal{G}^{\mathbb{C}}} A^{0q}(\mathcal{L})$$

(with $A^{0q}(\mathcal{L})$ completed in L^p_{1-q} and $\mathcal{G}^{\mathbb{C}}$ in L^p_2). There is also a local version: Let $\Omega^{0q}(\mathcal{L})$ be the sheaf of smooth (or more precisely locally L^p_{1-q}) differential forms with values in $\mathcal{L}$ considered as an $\mathcal{O}(X)$ module. Then consider the following sheaf on $X \times \mathcal{P}^{\mathrm{BN}*}$:

$$\Omega^{0q}(\mathcal{L}) = (X \times \mathcal{Q}^{\mathrm{BN}*}) \times_{\mathcal{G}^{\mathbb{C}}} p_1^*\Omega^{0q}(\mathcal{L})$$

where a group element $g \in \mathcal{G}^{\mathbb{C}} = map(X, \mathbb{C}^*)$ acts on forms in a point $(x; \bar\partial, \alpha)$ as $g(x)$. It is clear from the definition that the projection to $\mathcal{P}^{\mathrm{BN}*}$ is given by $p_{2*}\Omega^{0q}(\mathcal{L}) = \mathsf{A}^{0q}(\mathcal{L})$, whereas the higher groups $R^i p_{2*}\Omega^{0q}(\mathcal{L})$ vanish for $i > 0$ by the usual fineness argument.

Consider the universal divisor

$$\Delta = \{(x; \bar\partial, \alpha) \mid \alpha(x) = 0\}/\mathcal{G}^{\mathbb{C}} \subset \mathcal{P}^{\mathrm{BN}*} \times X$$

Now I claim that on $X \times \mathcal{M}^{\mathrm{BN}}$ there is an exact sequence

$$0 \to \mathcal{O}(\Delta) \xrightarrow{i} p_2^*\Omega^{00}(\mathcal{L}) \xrightarrow{\bar\partial} p_2^*\Omega^{01}(\mathcal{L}) \xrightarrow{\bar\partial} p_2^*\Omega^{02}(\mathcal{L}) \to 0. \tag{39}$$

where $\bar\partial$ is the universal $\bar\partial$ operator. In fact this claim is equivalent to three statements two local and one global: that the first $\bar\partial$ has a locally free rank 1 kernel, that the sequence is exact in the middle and the end, and that the kernel of the first $\bar\partial$ has a section which vanishes along Δ.

First the local statements: Let a^{01} be a form of type $(0,1)$ on a two complex dimensional polydisk with coefficients in a ring of germs of holomorphic functions on a complex space S, i.e., "depending on S", and $\bar\partial_0$ the standard $\bar\partial$ operator acting on forms on the polydisk. We extend $\bar\partial_0$ linearly over the germs on S. For the local statements it is enough that if locally $\bar\partial = \bar\partial_0 + a^{01}$ with $\bar\partial_0 a^{01} = 0$ then $a^{01} = \bar\partial_0 f$. Of course f will also "depend on S". We can then gauge away a^{01} because $\bar\partial = \exp(-f)\bar\partial_0\exp(f)$, and the local statements are clear from the Poincaré lemma for $\bar\partial_0$. The proof for such a family $\bar\partial$ Poincaré lemma carries over verbatim from the usual one in [GH, page 5,25].

The global statement is a bit of a tautology. On $X \times \mathcal{P}^*$ there is a natural section in $p_1^*\Omega^{00}(\mathcal{L})$ given by

$$(\bar\partial, \alpha; x) \to (\bar\partial, \alpha, \alpha(x)).$$

On the support of $\mathcal{M}^{\mathrm{BN}} \times X$ as an analytic space, this section lies in the kernel of $\bar{\partial}$. Now *by definition* a section in the kernel of $\bar{\partial}$ depending holomorphically on $\mathcal{M}^{\mathrm{BN}}$ is holomorphic. Likewise the zero of this section is *by definition* the support of Δ.

Having proved the claim, we see that the sequence (39), is an p_{2*}-acyclic resolution of $\mathcal{O}(\Delta)$. Thus, the i^{th} cohomology of the complex of sheaves (38) is the sheaf $R^i p_{2*}(\mathcal{O}(\Delta)$. It follows that $\mathrm{Ind}(\bar{\partial}) = \mathrm{Ind}\,(Rp_*\mathcal{O}(\Delta))$ and our SW multiplicity class is finally given by

$$\mathbb{M}(c_1(\mathcal{L})) = [c(Rp_*\mathcal{O}(\Delta))^{-1} c_*(\mathcal{M}^{\mathrm{BN}})]_{d(\mathcal{L})}. \tag{40}$$

A more precise description of $\mathcal{M}^{\mathrm{BN}}$ and its homological Chern classes $c_*(\mathcal{M}^{\mathrm{BN}})$ depends on the surface. Here we will do the case of elliptic surfaces. The author has succeeded in treating ruled surfaces in a similar way.

Proposition 42. *Let $X \xrightarrow{\pi} C$ be a Kählerian elliptic surface of holomorphic Euler characteristic χ over a curve C of genus g, with multiple fibers $F_1, \dots F_r$ of multiplicity $p_1, \dots p_r$. Consider the line bundle $\mathcal{L} = \mathcal{O}(\pi^* D + \sum a_i F_i)$ where D is a divisor on C of degree d, and $0 \leq a_i < p_i$. Then the Seiberg Witten multiplicity is zero if $d < 0$, and if $d \geq 0$ it is given by*

$$n_{(-)}(\Lambda^{0*}(\mathcal{L})) = \begin{cases} (-1)^d \binom{\chi+2g-2}{d} & \text{if } \chi + g - 2 \geq 0 \\ \sum_{j=0}^{\max(g,d)} (-1)^j \binom{1-g-\chi+d-j}{d-j} \binom{g}{j} & \text{if } \chi + g - 2 < 0. \end{cases}$$

Note that if the topological Euler characteristic $e > 0$ (or equivalently $\chi > 0$) then $g = q = \dfrac{1}{2} b_1(X)$ [FM2, corollary II.2.4], so in this case $\chi + g - 2 = p_g - 1$. Note further that the second formula is just 1 if $p_g = q = 0$ (i.e., $e > 0$). This illustrates Remark 39.

If $p_g > 0$ and $q = g = 0$, so in particular $e = 12\chi > 0$, Witten proves this formula by choosing a general $\omega \in H^0(K_X)$ and using the perturbation $\epsilon = \omega + \bar{\omega}$. He then argues that the multiplicity $n(\mathcal{L})$ is the number of ways we can decompose a fixed canonical divisor K_0 as $K_0 = D_+ + D_-$ with $D_+ \in |(\mathcal{L}, \bar{\partial}_0)|$, and $D_- \in |K \otimes (\mathcal{L}, \bar{\partial}_0)^\vee|$, where $\bar{\partial}_0$ is the unique holomorphic structure that $\mathcal{L}$ admits [Wit, Eq. (4.23) e.v.].

To be honest, this is what I read out of it. Actually I think that the computations below are the mathematical version of (I paraphrase) "integrating over the bosonic and fermionic collective coordinates in the path integral" and "computing the Euler class of the bundle of the cokernel of the operator describing the linearised monopole equations over the moduli space (the bundle of antighost zero modes)" [Wit, above (4.11)]. In fact with hindsight, the latter seems a dual description of the localised Euler class in the case that the cokernel has constant rank.

Proof. We choose a Kähler metric and λ such that $\deg_\Phi(\mathcal{L}^{\otimes 2}(-K)) < \lambda \,\mathrm{Vol}(X)$. This means that if $\mathcal{L}$ has non zero multiplicity, it must carry a holomorphic structure with a section. In case $p_g = 0$ it also means we are looking at n_-. But $(\mathcal{L}, \bar{\partial})$ has a section if and only if D is an effective divisor on C. In fact a family of vertical line bundles with a section gives a family of effective divisors on C by pushforward of the line bundle, and conversely a family of effective divisors on C gives a family of vertical line bundles with a section by pull back and multiplication with a fixed section in $\mathcal{O}(B) = \mathcal{O}(\sum a_i F_i)$ (B for base locus). Thus there is a natural isomorphism

$$\mathcal{M}^{\mathrm{BN}} \cong \mathcal{M}_C^{\mathrm{BN}} = C^d$$

where C^d is the d^{th} symmetric power of C. The functorial isomorphism comes with an isomorphism $\mathcal{O}(\Delta_X) = \mathcal{O}(\pi^*\Delta_C + B)$.

Next we use Grothendieck Riemann Roch (an alias of the family index theorem). Let $q\colon C \times C^d \to C^d$ be the projection map. Then the projection

$$p\colon X \times \mathcal{M}^{\mathrm{BN}} \to \mathcal{M}^{\mathrm{BN}}$$

can be factored as $p = q \circ \pi \times \mathrm{id}$. Thus, writing $\pi \times \mathrm{id}$ as π,

$$\begin{aligned}
\mathrm{ch}(Rp_*\mathcal{O}(\Delta)) &= \mathrm{ch}\, Rq_* \left[\mathcal{O}(\Delta_C) \otimes R\pi_*\mathcal{O}(B)\right] \\
&= q_* \left[\mathrm{ch}\,\mathcal{O}(\Delta_C) \cdot \mathrm{ch}\,(R\pi_*\mathcal{O}(B)) \cdot \mathrm{td}(C)\right] \\
&= q_* \left[\mathrm{ch}\,\mathcal{O}(\Delta_C) \cdot \pi_* \left(\mathrm{ch}\,\mathcal{O}(B) \cdot \mathrm{td}(X)\right)\right] \\
&= q_* \left[\mathrm{ch}\,\mathcal{O}(\Delta_C) \cdot \pi_* \left(e^B \cdot (1 - K/2 + \chi(\mathcal{O}_X)(pt \times C^d))\right)\right] \\
&= \chi(\mathcal{O}_X) q_* \left[\mathrm{ch}(\mathcal{O}(\Delta_C)) \cdot (pt \times C^d)\right] \\
&= \mathrm{ch}(\mathcal{O}(1)^\chi),
\end{aligned}$$

where we have abbreviated the holomorphic Euler characteristic by χ. If we denote by x the Chern class of $\mathcal{O}(1)$, then our computation shows that

$$c_t(Rp_*(\mathcal{O}(\Delta)) = (1 + tx)^\chi,$$

at least over the rationals.

The Chern classes of the tangent bundle of C^d are computed in [ACGH, Eq. VII.5.4]. Denoting the pullback of the θ divisor on Pic^d to C^d by θ the result is

$$c_t(T_{C^d}) = (1 + tx)^{d+1-g} e^{-t\theta/(1+tx)}$$

Combining these two expressions with formula (40), and remembering that the virtual dimension is zero our multiplicity drops out:

$$\begin{aligned}
n(\Lambda^{0*}(\mathcal{L})) &= c(\mathrm{Ind} Rp_*\mathcal{O}(\Delta))^{-1} c(T_C^d) \cap [C^d] \\
&= [(1 + tx)^{d+1-g-\chi} e^{-t\theta/1+tx}]_{t^d}.
\end{aligned}$$

With the following identity of formal power series [ACGH, Eq. VIII.3.1]

$$[(1 + xt)^a f(-t/(1 + xt))]_{t^b} = [(1 - xt)^{b-a-1} f(-t)]_{t^b}$$

the expression becomes

$$n(\Lambda^{0*}(\mathcal{L})) = [(1 - tx)^{\chi+g-2} e^{-t\theta}]_{t^d} = \begin{cases} (-1)^d \sum_{j=0}^d \binom{\chi+g-2}{d-j} \frac{\theta^j x^{d-j}}{j!} & \text{if } \chi + g - 2 \geq 0 \\ \sum_{j=0}^d (-1)^j \frac{1-g-\chi+d-j}{d-j} \frac{\theta^j x^{d-j}}{j!} & \text{if } \chi + g - 2 < 0. \end{cases}$$

Now $\theta^j x^{d-j} \cap [C^d] = j!\binom{g}{j}$ [ACGH, below Eq. VIII.3.3]. The elementary identity $\sum_j \binom{a}{j}\binom{b}{c-j} = \binom{a+b}{c}$ then gives the answer as stated. □

As a second application of the methods developed we give a complex analytic version of the blow up formula.

Proposition 43. *Let (X, Φ) be a Kähler surface, and $\mathcal{L}$ a line bundle on X. Suppose that $\deg_\Phi(\mathcal{L}^{\otimes 2}(-K)) < \lambda \operatorname{Vol}(X)$. Let $\sigma: \tilde{X} \to X$ be the blow up of X in a point, with Kähler form $\tilde{\Phi}$, and let $\tilde{\mathcal{L}} = \mathcal{L}(aE)$ be a line bundle on $\tilde{X}$ with $a \geq 0$. Suppose that the cohomology class of $\tilde{\Phi}$ is close to Φ. Then there is a natural identification $\mathcal{M}(\tilde{\mathcal{L}}) = \mathcal{M}(\mathcal{L})$, and*

$$\mathbb{M}(\tilde{\mathcal{L}}) = [(1+x)^{a(a-1)/2} \widehat{\mathbb{M}}(\mathcal{L})]_{\dim_{\mathbb{R}} = \mathcal{L}\cdot(\mathcal{L}-K) - a(a-1)}.$$

Here $\widehat{\mathbb{M}}$ is the class defined in Remark 18, and x the class of the natural bundle $\mathcal{O}(1)$ over $\mathcal{M}$. In particular if $a = 0, 1$ then $n(\tilde{\mathcal{L}}) = n(\mathcal{L})$.

Of course, this proposition determines the multiplicity

$$n_{(-)}(\Lambda^{0*}(\mathcal{L}(aE))) = n_{(-)}(\Lambda^{0*}(\mathcal{L}(-aE))).$$

Since quite in general $n_+(\Lambda^{0*}(\mathcal{L})) = \pm n_-(\Lambda^{0*}(K \otimes \mathcal{L}^\vee)$ it determines the corresponding relation for n_+ up to sign, which is really all we need here.

Proof. The conditions on the degree imply that a solution of the monopole equations correspond to a holomorphic structure on $\mathcal{L}$ with a section. Since $\tilde{\Phi}$ is close to Φ we have (by definition of close) $\deg_{\tilde{\Phi}}(\tilde{L}) < \lambda \operatorname{Vol}(\tilde{X})$, hence solutions on the blowup also correspond to holomorphic structures on $\tilde{\mathcal{L}}$ with a section.

Now aE is contained in the base locus of the sections. Thus, similarly to what we did for elliptic surfaces, we get an identification of $\mathcal{M}(\mathcal{L})$ with $\mathcal{M}(\tilde{\mathcal{L}})$ by multiplication of the section with a section in $\mathcal{O}(aE)$, and the universal divisor on $\tilde{X} \times \mathcal{M}(\tilde{\mathcal{L}})$ is $\tilde{\Delta} = \Delta + aE$.

Let $\tilde{p}$ be the projection $\tilde{X} \times \mathcal{M}(\mathcal{L}) \to \mathcal{M}(\mathcal{L})$, and p the projection $X \times \mathcal{M}(\mathcal{L}) \to \mathcal{M}(\mathcal{L})$. Then the index computation becomes

$$R\tilde{p}_*(\tilde{\Delta}) = Rp_* \left[\mathcal{O}(\Delta) \otimes R\sigma_* \mathcal{O}(aE)\right].$$

By induction on a, one shows that

$$R\sigma_* \mathcal{O}(aE) = \mathcal{O} - \mathcal{O}_{pt}^{a(a-1)/2}.$$

Since $\mathcal{O}(\Delta|_{pt \times \mathcal{M}(\mathcal{L})}) = \mathcal{O}(1)$ this implies

$$c(R\tilde{p}_*(\tilde{\Delta})) = c(Rp_* \mathcal{O}(\Delta))/c(\mathcal{O}(1))^{a(a-1)/2}.$$

Formula (40) now gives us

$$\mathbb{M}(\tilde{L}) = [(1+x)^{a(a-1)/2} \left(c(Rp_*(\Delta))^{-1} c_*(\mathcal{M}(\mathcal{L})\right)]_{d(\tilde{L})}.$$

Since the real virtual dimension of $\mathcal{M}(\tilde{\mathcal{L}})$ is $d(\tilde{\mathcal{L}}) = \mathcal{L} \cdot (\mathcal{L} - K) - a(a-1)$ and the term in brackets is exactly $\widehat{\mathbb{M}}(\mathcal{L})$, we have proved the formula. □

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