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THE CAR WITH N TRAILERS: CHARACTERISATION OF THE SINGULAR CONFIGURATIONS

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Abstract. In this paper we study the problem of the car with n trailers. It was proved in previous works ([9], [12]) that when each trailer is perpendicular with the previous one the degree of nonholonomy is F_{n+3} (the $(n+3)$-th term of the Fibonacci's sequence) and that when no two consecutive trailers are perpendicular this degree is $n+2$. We compute here by induction the degree of non holonomy in every state and obtain a partition of the singular set by this degree of non-holonomy. We give also for each area a set of vector fields in the Lie Algebra of the control system wich makes a basis of the tangent space.

1. Introduction

A car with n trailers is a nonholonomic system; it is, indeed, subject to non integrable constraints, the rolling without sliding of the wheels. The configuration of the system is given by two positions coordinates and $n+1$ angles. There are only two inputs, namely one tangential velocity and one angular velocity which represent the action on the steering wheel and on the accelerator of the car.

The problem of finding control laws was intensively treated in many papers throughout the literature: for instance by using sinusoids (see the works of Murray, Sastry and alii [11], [15]) or from the point of view of differentially flat systems (introduced by Fliess and alii [3]).

In general the study of such systems (to prove controllability, to find control laws, ...) involves tools from nonlinear control theory and differential geometry. In particular an important concept for such problems is the degree of nonholonomy, which expresses the level of Lie-bracketing needed to generate the tangent space at each configuration. This degree comes up for instance in estimation of the complexity required to steer the system from a point to another (see [7], [2]).

Laumond ([6]) has presented a kinematic model for the car with n trailers in 1991 and has proved the controllability for this model. He has also proved that the degree of nonholonomy of the system is bounded toward the top by 2^{n+1}. Sørdalen has afterwards proved in [12] that, when no two consecutive trailers are perpendicular, this degree is equal to $n+2$. The system

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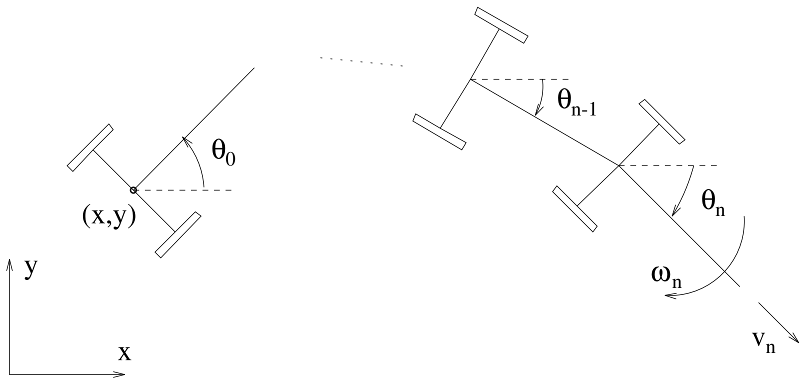


Figure 1. Model of the car with n trailers.

configurations corresponding to such cases are called the regular points of the configuration space (see [1], [8]).

More recently, it has been proved that the degree of nonholonomy is bounded by the $n+3$-th Fibonacci number ([13]) and that this bound is a maximum ([9],[10]) which is reached if and only if each trailer (exept the last one) is perpendicular to the previous one.

To close definitively the problem, we still have to study the non regular points for which the maximum degree of nonholonomy is not reached. For the car with 2, 3 and 4 trailers, a complete classification of the singularities has already been done in [4]. The goal of our paper is to extend this classification to any number of trailers. Let us note that some results given here have already been presented without proof in [5].

In Section 2 of this report, we are going to write equations, give definitions and notations and construct an induction procedure. Section 3 groups together the main result of this paper, Theorem 3.1, and some conclusion on the form of the singular locus and on the degree of nonholonomy. Section 4 is devoted to the demonstration of Theorem 3.1, but the proof of some technical lemmas are relegated to the appendix.

2. Equations and notations

2.1. Control system

In this paper we are going to use the same representation as Fliess [3] and Sørdalen [12] for the car with n trailers. A car in this context will be represented by two driving wheels connected by an axle. The state is parametrised by $q = (x, y, \theta_0, \ldots, \theta_n)^T$ where:

- (x, y) are the coordinates of the last trailer,
- θ_n is the orientation angle of the car with respect to the x-axis,
- θ_i, for $0 \leq i \leq n-1$, is the orientation angle of the trailer $(n-i)$ with respect to the x-axis.

The kinematic model of a car with two degrees of freedom pulling n trailers can be given by:

$$\begin{array}{lcl} \dot{x} & = & \cos\theta_0 \; v_0, \\ \dot{y} & = & \sin\theta_0 \; v_0, \\ \dot{\theta}_0 & = & \frac{1}{R_1}\sin(\theta_1-\theta_0) \; v_1, \\ & \vdots & \\ \dot{\theta}_i & = & \frac{1}{R_{i+1}}\sin(\theta_{i+1}-\theta_i) \; v_{i+1}, \\ & \vdots & \\ \dot{\theta}_{n-1} & = & \frac{1}{R_n}\sin(\theta_n-\theta_{n-1}) \; v_n, \\ \dot{\theta}_n & = & \omega_n. \end{array}$$

where R_i is the distance from the trailer $(n-i)$ to the trailer $(n-i+1)$, ω_n is the angular velocity of the car and v_n is the tangential velocity of the car. v_n and ω_n are the two inputs of the system.

The tangential velocity v_i of trailer $n-i$ is given by:

$$v_i = \prod_{j=i+1}^{n} \cos(\theta_j - \theta_{j-1}) v_n.$$

Let us denote:

$$f_i^n = \prod_{j=i+1}^{n} \cos(\theta_j - \theta_{j-1}),$$

$$v_i = f_i^n v_n, \quad i = 0, \cdots, n-1.$$

The motion of the system is then characterized by the equation:

$$\dot{q} = \omega_n X_1^n(q) + v_n X_2^n(q)$$

with

$$\left\{ \begin{array}{lcl} X_1^n & = & \frac{\partial}{\partial\theta_n} \\ X_2^n & = & \cos\theta_0 \; f_0^n \frac{\partial}{\partial x} + \sin\theta_0 \; f_0^n \frac{\partial}{\partial y} + \frac{\sin(\theta_1-\theta_0)}{R_1} f_1^n \frac{\partial}{\partial\theta_0} + \cdots \\ & & \qquad \cdots + \frac{\sin(\theta_n-\theta_{n-1})}{R_n} \frac{\partial}{\partial\theta_{n-1}} \end{array} \right. \tag{2.1}$$

We will suppose that the distance R_i doesn't depend on i and to simplify we shall, from now on, consider it equal to 1 (we will come back to this hypothesis in Subsection 3.4).

2.2. Characterization of the singular locus

We are going now to define the singular locus of the control system $\{X_1^n, X_2^n\}$, and give a characterization of this locus easy to use.

In this section n is fixed and we write X_1 and X_2 instead of X_1^n and X_2^n.

Let $\mathcal{L}_1(X_1, X_2)$ be the set of linear combinations with real coefficients of X_1 and X_2. We define recursively the distribution $\mathcal{L}_k = \mathcal{L}_k(X_1, X_2)$ by:

$$\mathcal{L}_k = \mathcal{L}_{k-1} + \sum_{i+j=k} [\mathcal{L}_i, \mathcal{L}_j]$$

where $[\mathcal{L}_i, \mathcal{L}_j]$ denotes the set of all brackets $[V, W]$ for $V \in \mathcal{L}_i$ and $W \in \mathcal{L}_j$.

Thus $\mathcal{L}_k$ is the set of linear combinations of iterated Lie brackets of X_1 and X_2 of length $\leq k$. The union $\mathcal{L}$ of all $\mathcal{L}_k$ is a Lie subalgebra of the Lie algebra of vector fields on $\mathbf{R}^2 \times (\mathbf{S}^1)^{n+1}$. It is called the Control Lie Algebra of the system $\{X_1, X_2\}$.

Let us introduce some notations. For $s \geq 1$ denote by $\underline{i} = (i_1, \ldots, i_s)$ a sequence of s elements in $\{1, 2\}$ and by $\mathcal{A}_s$ the sets of these sequences such that $i_1 = 1$ if $s > 1$, that is:

$$\begin{array}{l} \mathcal{A}_1 = \{(1), (2)\}, \\ \mathcal{A}_s = \{\underline{i} = (1, i_2, \ldots, i_s) \mid i_j = 1 \text{ or } 2\} \text{ if } s > 1. \end{array} \tag{2.2}$$

The functions $u(\underline{i})$ and $d(\underline{i})$ indicate respectively the number of occurences of 1 and the number of occurences of 2 in the sequence $\underline{i} = (i_1, \ldots, i_s)$, and the length of the sequence is $|\underline{i}| = s$. Obviously we have $u(\underline{i}) + d(\underline{i}) = |\underline{i}|$.

The vector field $[[\ldots[X_{i_1}, X_{i_2}], \ldots, X_{i_{s-1}}], X_{i_s}]$ will be denoted by $[X_{\underline{i}}]$ or $[X_{i_1}, \ldots, X_{i_s}]$ and its value in q by $[X_{\underline{i}}]_q$. By using the Jacobi identity, we can write a bracket of length $\leq k$ (*i.e.* belonging to $\mathcal{L}_k$) as a sum of $[X_{\underline{i}}]$ with $|\underline{i}| \leq k$.

Moreover, because of skew symetry of Lie bracket, $[X_{\underline{i}}] = 0$ if $i_1 = i_2$ and $[X_2, X_{i_1}, \ldots, X_{i_s}] = -[X_{i_1}, X_2, \ldots, X_{i_s}]$. Then $\mathcal{L}_k$ is generated by the brackets $[X_{\underline{i}}]$ such that $\underline{i} \in \mathcal{A}_s$, for $1 \leq s \leq k$.

For a given state q, let $L_k(q)$ be the subspace of $T_q(\mathbf{R}^2 \times (\mathbf{S}^1)^{n+1})$ wich consists of the values at q taken by the vector fields belonging to $\mathcal{L}_k$. We have an increasing sequence of dimensions:

$$2 = \dim L_1(q) \leq \cdots \leq \dim L_k(q) \leq \cdots \leq n + 3. \tag{2.3}$$

If this sequence stays the same in an open neighbourhood of q, the state q is called a regular point of the control system; otherwise, q is called a singular point of the control system (see [1]). Thus the sequence (2.3) at any state q allows to characterize regular and singular points, *i.e.*, the singular locus.

To determinate the sequence (2.3), we define, for $i \in \{1, n+3\}$:

$$\left\{ \begin{array}{l} \beta_i^n(q) = \min\{k \mid \dim L_k(q) \geq i\} \\ d_i^n(q) = \min\{d \mid \dim \operatorname{span}\langle\, [X_{\underline{j}}]_q,\ |\underline{j}| \leq \beta_i^n(q),\ d(\underline{j}) \leq d\, \rangle \geq i\} \end{array} \right. \tag{2.4}$$

In other words, the fact that $k = \beta_i^n(q)$ is equivalent to:

$$\left\{ \begin{array}{lcl} \dim L_k(q) & \geq & i \\ \dim L_{k-1}(q) & < & i \end{array} \right. \tag{2.5}$$

The sequence (2.3) can be deduced from the $\beta_i^n(q)$'s, $i = 1, \ldots, n+3$, by:

- if $\exists i \in \{1, n+3\}$ such that $k = \beta_i^n(q)$, then $\dim L_k(q)$ is strictly greater than $\dim L_{k-1}(q)$ and equal to the greatest j such that $\beta_j^n(q) = k$,
- otherwise $\dim L_k(q) = \dim L_{k-1}(q)$.

Thus the functions $\beta_i^n(q)$, $i = 1, \ldots, n+3$, characterize completely the singular locus. Hence the paper is devoted to the calculation of these functions. Notice that the functions $d_i^n(q)$ are not useful in the characterization of the singular locus, they are only a tool for the computation of the functions $\beta_i^n(q)$.

To simplify we will omitt the dependence on q in $\beta_i^n(q)$ and $d_i^n(q)$. According to its definition, β_i^n increases with respect to i, for i lesser than $\dim L(q)$ (when i is strictly greater than this dimension, β_i^n is equal to $-\infty$). We will prove in this paper (in Theorem 3.1) that this sequence is strictly increasing with respect to i for $2 \le i \le n+3$, which means that, for any k, $\dim L_k(q) - \dim L_{k-1}(q) \le 1$. In other words, we will prove that, for $2 \le i \le n+3$, $k = \beta_i^n$ is equivalent to (compare with (2.5)):

$$\left\{ \begin{array}{lcl} \dim L_k(q) & = & i \\ \dim L_{k-1}(q) & = & i-1 \end{array} \right. \tag{2.6}$$

We can yet calculate the first values of these sequences.

- $L_1(q)$ is two dimensionnal for all q, and β_1^n and β_2^n are equal to 1. To span a two dimensionnal linear space we need both X_1 and X_2 whereas for a one dimensionnal linear space X_1 is sufficient. Then $d_1^n = 0$ and $d_2^n = 1$.
- $L_2(q)$ is generated by the family X_1, X_2, $[X_1, X_2]$ which is three dimensional for all q (it is clear from Formula (2.1)), so $\beta_3^n = 2$. Moreover it is not possible to find another three dimensionnal family of vector fields which contains "a fewer number of 2", then $d_3^n = 1$.

Finally, for all state q:

$$\begin{array}{ll} \beta_1^n = 1 & d_1^n = 0 \\ \beta_2^n = 1 & d_2^n = 1 \\ \beta_3^n = 2 & d_3^n = 1. \end{array} \tag{2.7}$$

2.3. Induction procedure

For $q \in \mathbf{R}^2 \times (\mathbf{S}^1)^{n+1}$ and $1 \le p < n$, we will denote by q^p the projection of q on the first $(n+3-p)$ coordinates, that is $q^p = (x, y, \theta_0, \ldots, \theta_{n-p})^T$.

Let us consider now the system of a car with $n-p$ trailers. The states q' belong to $\mathbf{R}^2 \times (\mathbf{S}^1)^{n-p+1}$ and the control system $\{X_1^{n-p}, X_2^{n-p}\}$ is given by Formulas (2.1) with $n-p$ instead of n, that is:

$$\left\{ \begin{array}{l} X_1^{n-p} = \frac{\partial}{\partial \theta_{n-p}} \\ X_2^{n-p} = \cos\theta_0 \; f_0^{n-p} \frac{\partial}{\partial x} + \sin\theta_0 \; f_0^{n-p} \frac{\partial}{\partial y} + s_1 f_1^{n-p} \frac{\partial}{\partial \theta_0} + \cdots + s_{n-p} \frac{\partial}{\partial \theta_{n-p-1}} \end{array} \right.$$

where $c_m = \cos(\theta_m - \theta_{m-1})$, $s_m = \sin(\theta_m - \theta_{m-1})$ and $f_i^{n-p} = c_{i+1} \cdots c_{n-p}$.

Hence for any q' we have the sequences $\beta_j^{n-p}(q')$ and $d_j^{n-p}(q')$, $j = 1, \ldots, n-p+3$. The dimensions of the spaces $L_k(X_1^{n-p}, X_2^{n-p})(q')$, $k \ge 1$, are characterized by the sequence $\beta_j^{n-p}(q')$.

On the other hand, X_1^{n-p} and X_2^{n-p} can be seen as vector fields on $\mathbf{R}^2 \times (\mathbf{S}^1)^{n+1}$ whose last p coordinates are zero and which values at q depends only on the projection q^p. We can then consider $\mathcal{L}(X_1^{n-p}, X_2^{n-p})$ as a subalgebra of $\mathcal{L}(X_1^n, X_2^n)$.

REMARK 2.1. The motion of a point $q' \in \mathbf{R}^2 \times (\mathbf{S}^1)^{n-p+1}$ is characterized by an equation $\dot{q}' = u_1 X_1^{n-p}(q') + u_2 X_2^{n-p}(q')$, and so depends only on the state q'. On the other hand, the motion of the point q^p is given by the projection on the first $n-p$ coordinates of the motion equation of q: then the motion of q^p depends on q and not only on q^p.

According to Formulas (2.1), we can write X_2^n as:

$$X_2^n = s_n X_1^{n-1} + c_n s_{n-1} X_1^{n-2} + \cdots + c_n \cdots c_{n-p+2} s_{n-p+1} X_1^{n-p} + \\ + c_n \cdots c_{n-p+2} c_{n-p+1} X_2^{n-p} \tag{2.8}$$

where $c_m = \cos(\theta_m - \theta_{m-1})$ and $s_m = \sin(\theta_m - \theta_{m-1})$.

With this relation, for $1 \leq p \leq n-1$, we will be able to express a vector field in $\mathcal{L}(X_1^n, X_2^n)$ in function of $X_1^n, \ldots, X_1^{n-p+1}$ and of vector fields in $\mathcal{L}(X_1^{n-p}, X_2^{n-p})$.

For instance, for $p = 1$, we have:

$$\begin{aligned} X_2^n &= s_n X_1^{n-1} + c_n X_2^{n-1} \\ [X_1^n, X_2^n] &= c_n X_1^{n-1} - s_n X_2^{n-1} \\ [X_1^n, X_2^n, X_2^n] &= -X_1^{n-1} + [X_1^{n-1}, X_2^{n-1}]. \end{aligned} \tag{2.9}$$

Hence $\mathcal{L}(X_1^n, X_2^n)$ is equal to $\mathcal{L}(X_1^{n-1}, X_2^{n-1}) \oplus \langle X_1^n \rangle$, where $\langle X_1^n \rangle$ is the subalgebra generated by X_1^n. Formula (2.8) allows to describe the projection of $\mathcal{L}(X_1^n, X_2^n)$ on $\mathcal{L}(X_1^{n-1}, X_2^{n-1})$. We will see for instance that the projection of $\mathcal{L}_k(X_1^n, X_2^n)$ is $\mathcal{L}_{k-1}(X_1^{n-1}, X_2^{n-1})$.

The induction will be done in the following way: we will assume that the functions $\beta_j^{n-p}(q^p)$ $(j = 1, \ldots, n-p+3)$ are known for any $p < n$ and, by using the relation (2.8), we will calculate the dimensions of the $L_k(X_1^n, X_2^n)(q)$, and so the $\beta_i^n(q)$'s $(i = 1, \ldots, n+3)$, in function of the $\beta_j^{n-p}(q^p)$'s.

From now on the dependence on q or q^p will be omitted if there is no possible confusion; for example we will write β_i^n instead of $\beta_i^n(q)$ and β_j^{n-p} instead of $\beta_j^{n-p}(q^p)$.

3. Singular configurations

3.1. Exposition of the result

In this chapter we present the main result of this paper, Theorem 3.1, which gives the recursion formulas satisfied by the sequence of functions β_i^n. The proof of the theorem is given in Section 4.

Let us introduce a sequence a_p by:

$$\begin{cases} a_1 = \frac{\pi}{2}, \\ a_p = \arctan \sin a_{p-1}. \end{cases} \tag{3.1}$$

This sequence is clearly positive and decreasing, that is: $0 < a_p < \frac{\pi}{2}$ for $p > 1$. Notice that the recursion relationship is odd, that is if we define an other sequence a_p^- by the same recursion relationship and the initial value $a_1^- = -\frac{\pi}{2}$, we have $a_p^- = -a_p$.

Let us define also some brackets $[A^n(\beta, d)]$ ($\beta > d$ or $\beta = d = 1$) by:

$$\left\{\begin{array}{lcl} [A^n(1,0)] & = & X_1^n \\ [A^n(1,1)] & = & X_2^n \\ [A^n(\beta,d)] & = & [X_1^n, \underbrace{X_2^n, \ldots, X_2^n}_{d}, \underbrace{X_1^n, \ldots, X_1^n}_{\beta-d-1}] \quad \textit{for } \beta > 2 \end{array}\right. \tag{3.2}$$

From this definition we can see that the bracket $[A^n(\beta, d)]$ is of length β and that X_2^n occurs d times in it.

With these notations, we have:

THEOREM 3.1.
$\forall q \in \mathbf{R}^2 \times (\mathbf{S}^1)^{n+1}$, for $2 \le i \le n+3$, β_i^n is streactly increasing with respect to i, and we have $d_i^n = \beta_{i-1}^{n-1}$.

We can calculate the functions $\beta_i^n(q)$ by the following induction formulas, for $i \in \{3, n+3\}$:

1. *If $\theta_n - \theta_{n-1} = \pm\frac{\pi}{2}$, then:*
$$\beta_i^n = \beta_{i-1}^{n-1} + d_{i-1}^{n-1}.$$
2. *If $\exists p \in [1, n-2]$ and $\epsilon = \pm 1$ such that $\theta_k - \theta_{k-1} = \epsilon a_{k-p}$ for every $k \in \{p+1, n\}$, then:*
$$\beta_i^n = 2\beta_{i-1}^{n-1} - d_{i-1}^{n-1}.$$
3. *Otherwise,*
$$\beta_i^n = \beta_{i-1}^{n-1} + 1.$$

Moreover, a basis $\mathcal{B}^n = \{B_i^n, i = 1, \ldots, n+3\}$ of $T_q(\mathbf{R}^2 \times (\mathbf{S}^1)^{n+1})$ is given by:

$$B_i^n = [A^n(\beta_i^n, d_i^n)]_q \tag{3.3}$$

3.2. FORM OF THE SINGULAR LOCUS

Let us study the sequence $\beta^n = (\beta_i^n)_{i=2,\ldots,n+3}$ (we remove β_1^n because it is always equal to β_2^n). The level sets of this sequence give a partition of the configuration space. For example, Figure 2 shows us the partition obtained for $n = 3$. Since each area is a cylinder with respect to the direction $\theta_1 - \theta_0$, we have just shown the projection of these cells on a plane $\theta_1 - \theta_0 = constant$. The complement of the four lines are the regular points of the system and corresponds to the values $(1, 2, 3, 4, 5)$ of the sequence β^3.

For $n \ge 3$, let q^2 be the projection of q on the first $n+1$ coordinates. Theorem 3.1 allows us to calculate the values of $\beta^n(q)$ in function of $\beta^{n-2}(q^2)$. We illustrate it in Figure 3, where we represent the set of points $q = (q^2, \theta_{n-1}, \theta_n)$ wich have the same projection q^2.

Let us consider now a point q such that $\theta_k - \theta_{k-1} \neq \pm\frac{\pi}{2}$ for $k = 2, \ldots, n$. It is clear (for instance by looking at Figures 2 and 3) that there exists a neighbourhood of q in which the sequence β^n is constant. Conversaly, if there is an integer k, $2 \leq k \leq n$, such that $\theta_k - \theta_{k-1} = \pm\frac{\pi}{2}$, such a neighbourhood doesn't exist. Then we have determined the regular and singular points (see also [12] and [1]).

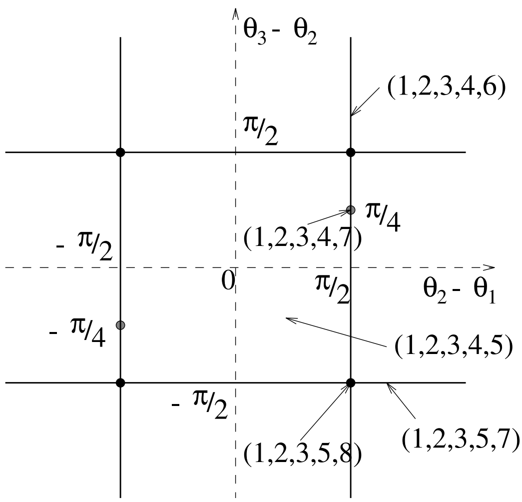
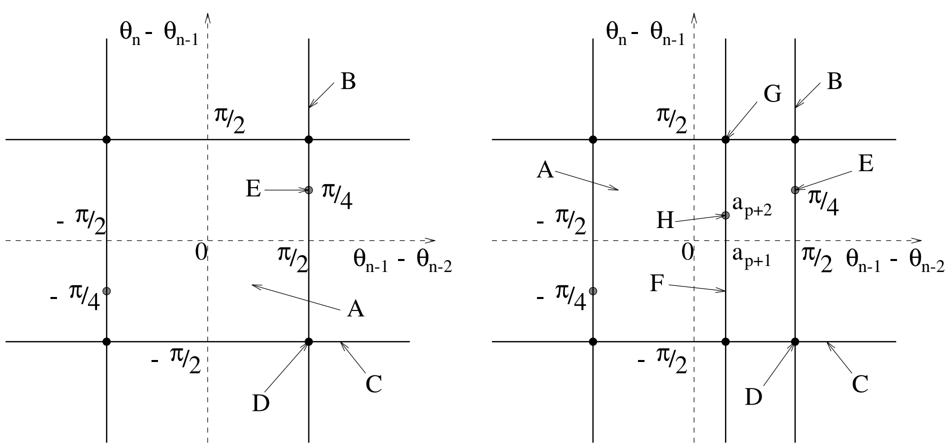


FIGURE 2. Partition of the configuration space ($n = 3$) by the values of the sequence $(\beta_2^3, \beta_3^3, \beta_4^3, \beta_5^3, \beta_6^3)$.



q^2 in case 3 $\qquad\qquad$ q^2 in case 1 or 2

A : $\beta_i^n = \beta_{i-2}^{n-2} + 2$ $\qquad$ B : $\beta_i^n = \beta_{i-2}^{n-2} + d_{i-2}^{n-2} + 1$

C : $\beta_i^n = 2\beta_{i-2}^{n-2} + 1$ $\qquad$ D : $\beta_i^n = 2\beta_{i-2}^{n-2} + d_{i-2}^{n-2}$

E : $\beta_i^n = \beta_{i-2}^{n-2} + 2d_{i-2}^{n-2}$ $\qquad$ F : $\beta_i^n = 2\beta_{i-2}^{n-2} - d_{i-2}^{n-2} + 1$

G : $\beta_i^n = 3\beta_{i-2}^{n-2} - d_{i-2}^{n-2}$ $\qquad$ H : $\beta_i^n = 3\beta_{i-2}^{n-2} - 2d_{i-2}^{n-2}$

FIGURE 3. Cells of the subset $(q^2, \theta_{n-1}, \theta_n)$.

THEOREM 3.2. *The singular locus of the system is the set of the points for which there exists $k \in [2, n]$ such that $\theta_k - \theta_{k-1} = \pm\frac{\pi}{2}$.*

3.3. Application to the degree of nonholonomy

The degree of nonholonomy of the system at a point q is the degree from wich the sequence (2.3) is constant, that is the degree r such that:

$$L_{r-1}(q) \subsetneq L_r(q) = L_{r+1}(q) = \cdots = L(q).$$

With our notations, this degree r is given by the greatest β_i^n, for $i \leq n+3$. Thus Theorem 3.1 implies that the degree of nonholonomy of the system is equal to β_{n+3}^n at any point q, and then the rank of the Control Lie Algebra at any point is $n+3$.

Let us recall the Chow theorem (also called the Lie Algebra Rank Condition): if the rank of the Control Lie Algebra at any point q of the configuration space is equal to the dimension of the tangent space in this point, the system is controllable (see for instance [14]).

This condition is satisfied here, therefore the system is controllable. We are meeting a classic result, wich was first proved by Laumond in 1990 ([6]).

By using Theorem 3.1, we can study the function β_{n+3}^n and find some other results about the degree of nonholonomy (these results were already proved in [12],[13], and [10]):

THEOREM 3.3.

(i) *At a regular point, that is a point such that $\theta_k - \theta_{k-1} \neq \pm\frac{\pi}{2}$ for every $k \in [2, n]$, the degree of nonholonomy of the system is $n+2$.*

(ii) *The maximum of the degree of nonholonomy is the $(n+3)$-th Fibonacci number F_{n+3} (recall that the Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$), and this maximum is obtained if and only if all the trailers are perpendicular (except the last one), that is if $\theta_k - \theta_{k-1} = \pm\frac{\pi}{2}$ for every $k \in [2, n]$.*

PROOF.

This theorem is obtained by applying the recursion formulas of Theorem 3.1 and by using the values for $n = 0$ given by Formula (2.7): $\beta_3^n = 2 = F_3$ and $d_3^n = 1 = F_2$.

∎

Hence we see that $n+2 \leq \beta_{n+3}^n \leq F_{n+3}$. Moreover, for $n \leq 4$, β_{n+3}^n can take all the values from $n+2$ to F_{n+3}, but this property is no more true for $n > 4$.

3.4. Case where the distances between the trailers are not all equals

We have assumed (see Subsection 2.1) that the distance R_i between the trailer $(n-i)$ and the trailer $(n-i+1)$ is independent on i and equal to 1. If we remove this hypothesis, the result is the same as Theorem 3.1, except that we have to replace the case 2 by:

2 bis. *If* $\exists p \in [1, n-1]$ *such that* $\theta_k - \theta_{k-1} = \epsilon a_{k-p}(p)$ *for every* $k \in \{p+1, n\}$, *then:*

$$\beta_i^n = 2\beta_{i-1}^{n-1} - d_{i-1}^{n-1}$$

where the sequence $a_{k-p}(p)$ *is defined by:*

$$\begin{cases} a_1(p) = \frac{\pi}{2}, \\ a_{k-p}(p) = \arctan(\frac{R_k}{R_{k-1}} \sin a_{k-p-1}(p)), \ \ for\ k > p+1. \end{cases}$$

The proof of this result is similar than the one of Theorem 3.1, but requires more notations. Therefore we don't give it in this paper. The difference with the case $R_i = 1$ is that the sequence of angles wich gives singularities depends on p. For instance, if $R_{p+2} \neq R_{p+1} = R_p$, $a_2(p-1)$ is equal to $\frac{\pi}{4}$ whereas $a_2(p)$ is not.

4. Proof of Theorem 3.1

In this chapter, $n \geq 1$ is fixed. The proof is organized as follows: in a first time, we study the relationships between the Lie Algebra for the n-trailers system and the Lie Algebras for the systems with less than n trailers. In a second time we use these relationships to establish the induction formulas for the functions $\beta_i^n(q)$. The main point of the proof is the first part, that is Lemma 4.1. This kind of proof is inspired by [10].

4.1. Preliminary result

We have seen in Subsection 2.3 that, for $2 < m < n$, a vector field in $\mathcal{L}(X_1^n, X_2^n)$ can be decomposed in a linear combination (with functions as coefficients) of $X_1^n, \ldots, X_1^{m+1}$ and of vector fields in $\mathcal{L}(X_1^m, X_2^m)$. Lemma 4.1 gives such a decomposition and allows to conclude, in some particular cases, on the nullity or non nullity of the decomposition coefficients.

We will denote, for $m \leq n$:

$$\begin{array}{rcl} \varphi_m & = & \theta_m - \theta_{m-1} \\ c_m & = & \cos \varphi_m \\ s_m & = & \sin \varphi_m \\ t_m & = & \sin \varphi_m - \cos \varphi_m \sin \varphi_{m-1}. \end{array} \tag{4.1}$$

Lemma 4.1. *Let* p, $1 \leq p \leq n-1$ *and* $\underline{i} \in \mathcal{A}_{|\underline{i}|}$, $\underline{i} \neq (1)$ *(the sets* $\mathcal{A}_s$ *are defined in Formula 2.2). Then there exist functions* $h_k(\varphi_{n-p+1}, \ldots, \varphi_n)$, $n-p+1 \leq k \leq n-1$ *and* $f_{\underline{l}}(\varphi_{n-p+1}, \ldots, \varphi_n)$ *in* $C^\infty(\mathbf{S}^p)$ *depending on* $\underline{i}$ *such that:*

$$[X_{\underline{i}}^n] = \sum_{k=n-p}^{n-1} h_k\, X_1^k + \sum_{s=1}^{d} \sum_{\underline{l} \in \mathcal{A}_s} f_{\underline{l}}\, [X_{\underline{l}}^{n-p}]$$

where $d = \max\{1, d(\underline{i}) - p + 1\}$.

Moreover, if $|\underline{i}| \geq p+1$, *then:*

1. *we have:*
$$f_{\underline{l}} = \sum_{\underline{b} \in I_{\underline{l}}} (c_{n-p+1})^{b_0-b_1} (t_{n-p+2})^{b_1-b_2} \cdots (t_n)^{b_{p-1}-b_p} g_{\underline{b},\underline{l}} \tag{4.2}$$
where the functions $g_{\underline{b},\underline{l}}(\varphi_{n-p+1}, \ldots, \varphi_n) \in C^\infty(\mathbf{S}^p)$ depend on $\underline{b}, \underline{l}$ and $\underline{i}$ and the set $I_{\underline{l}} \subset \mathbf{Z}^{p+1}$ satisfies:
$$I_{\underline{l}} \subset \{\underline{b} = (b_0, \ldots, b_p) \mid d(\underline{l}) = b_0 \geq \cdots \geq b_p \geq 0,\ \sum_{j=1}^{p} b_j \leq |\underline{i}| - |\underline{l}|\};$$
2. *if $\underline{b} \in I_{\underline{l}}$ is such that $b_p = 0$, then $\sum_{j=1}^{p-1} b_j < d(\underline{i}) - |\underline{l}|$;*
3. *if we denote by $I_{\underline{l}}^+$ the following subset of $I_{\underline{l}}$:*
$$I_{\underline{l}}^+ = \{\underline{b} \in I_{\underline{l}} \mid \sum_{j=1}^{p} b_j = |\underline{i}| - |\underline{l}|\},$$
for every $\underline{b} \in I_{\underline{l}}^+$, there exist an integer $\alpha_{\underline{b}} > 0$ and a function $G(\varphi_{n-p+1}, \ldots, \varphi_n)$ which depends only on $|\underline{i}|$, $|\underline{l}|$, $d(\underline{i})$, $d(\underline{l})$ and $\underline{b}$ such that:
$$g_{\underline{b},\underline{l}} = \alpha_{\underline{b}}\ G_{\underline{b},|\underline{i}|,|\underline{l}|,d(\underline{i}),d(\underline{l})}$$
4. *if $[X_{\underline{i}}^n] = [A^n(\beta + (p-1)\delta + r, \beta + (p-1)\delta)]$, with $\beta > \delta \geq r \geq 1$, and if $[X_{\underline{l}}^{n-p}]$ is such that $\underline{l} \in \mathcal{A}_\beta$ and $d(\underline{l}) = \delta$, the sequence $(\delta, \ldots, \delta, r)$ belongs to $I_{\underline{l}}^+$ (the definition of the bracket $[A^n(\beta, d)]$ is given by (3.2));*
5. *if $p = 1$, $|\underline{i}| \geq 3$, and $d(\underline{i}) = |\underline{i}| - 1$, then the coefficient of $[X_{\underline{l}}^{n-1}]$ such that $\underline{l} \in \mathcal{A}_{d(\underline{i})}$ and $d(\underline{l}) = d(\underline{i}) - 1$ is:*
$$f_{\underline{l}}(\varphi_n) = (c_n)^{d(\underline{i})-2}.$$

PROOF. The proof is quite long and technical, so it is done in the appendix (where the lemma is divided in four parts: Lemmae 5.1, 5.2, 5.3 and 5.4).

■

The point 3 implies that functions $G(\varphi_{n-p+1}, \ldots, \varphi_n)$ doesn't depend on the sequences $\underline{i}$ and $\underline{l}$ but only on the length $|\underline{i}|$, $|\underline{l}|$ and on the "number of 2" in these sequences (namely $d(\underline{i})$ and $d(\underline{l})$). The form of the sequences $\underline{i}$ and $\underline{l}$ acts only on the integer $\alpha_{\underline{b}}$, and then not on the sign of $g_{\underline{b},\underline{l}}$. The exact form of $G(\varphi_{n-p+1}, \ldots, \varphi_n)$ is given in Lemma 5.3 but it is not useful here.

REMARK 4.2. It appears from this lemma that the terms c_{n-p+1}, t_{n-p+2}, $\ldots$, t_n have a particular part in the decomposition (4.2) (it will be confirmed in what follows). Thus it is interesting to notice that all of these quantities are zero if there exists $\epsilon = \pm 1$ such that $\theta_k - \theta_{k-1} = \epsilon a_{k-p}$ for $k = p+1, \ldots, n$. In this case, the function $f_{\underline{l}}$ can be non zero only if there exists $\underline{b} \in I_{\underline{l}}$ such that $b_0 = \cdots = b_p$ (we set $0^0 = 1$).

4.2. PROOF

4.2.1. PLAN OF THE PROOF. We are not going to prove directly Theorem 3.1 but the following proposition, which implies the theorem.

PROPOSITION 4.3. *Let $n \geq 1$. Then, for every state q we have:*

- *β_i^n increases strictly with respect to i (for $i > 1$),*
- *$d_i^n = \beta_{i-1}^{n-1}$,*

- *- if* $\exists\, 1 \le p \le n-1$ *and* $\epsilon = \pm 1$ *such that* $\varphi_{n-p+1} = \epsilon a_1, \dots, \varphi_n = \epsilon a_p$ *(*a_p *is defined by (3.1)), then:*

$$\begin{cases} \beta_i^n = \beta_{i-1}^{n-1} + 1 & \textit{for } i = 3, \dots, p+2 \\ \beta_i^n = \beta_{i-p}^{n-p} + p\, d_{i-p}^{n-p} & \textit{for } i = p+2, \dots, n+3 \end{cases}$$

- otherwise, $\beta_i^n = \beta_{i-1}^{n-1} + 1$ *for* $i = 3, \dots, n+3$,
- $\{B_i^n = [A^n(\beta_i^n, d_i^n)]_q, i = 1, \dots, n+3\}$ *is a basis of* $T_q(\mathbf{R}^2 \times (\mathbf{S}^1)^{n+1})$ *(the definition of the brackets* $[A^n(\beta, d)]$ *is given by (3.2)),*
- *every vector* $[X_{\underline{i}}^n]$ *such that* $\underline{i} \in \mathcal{A}_{\beta_i^n}$ *and* $d(\underline{i}) = d_i^n$ *has a positive coordinate on the basis vector* B_i^n *(recall that, if* $\underline{i} \in \mathcal{A}_s$ *and* $s > 1$*, then* $i_1 = 1$ *(see (2.2)).*

The first four points of this proposition are equivalent to Theorem 3.1 (the induction formulas for β_i^n are the same but expressed in a different way). The last point of the proposition is an induction hypothesis required for the proof and then is omitted in the theorem.

The proof will be done by induction on n. We assume that Proposition 4.3 is true for every $m < n$, and we will prove that it is true also for n by proceeding as follows:

- for each q and i we have "candidates" for β_i^n and d_i^n (given by the proposition);
- in Lemma 4.5 and Corollary 4.8, we prove that these "candidates" are less than β_i^n and d_i^n;
- with Lemma 4.9, we establish that there exists a basis of $T_q(\mathbf{R}^2 \times (\mathbf{S}^1)^{n+1})$ formed by vectors the length (and number of X_2^n) of which are equal to the "candidates", for $i = 1$ to $n+3$;
- by using Lemma 4.7 we prove that β_i^n and d_i^n are indeed equal to the "candidates";
- Lemma 4.10 allows us to establish the last point of Proposition 4.3 and so to conclude.

The form of Proposition 4.3 implies that we have to distinguish several possibilities for the state q:

- $\exists\, \epsilon = \pm 1$ such that $\varphi_n = \epsilon a_1$,
- $\exists\, p \ge 2$ *and* $\epsilon = \pm 1$ such that $\varphi_{n-p+1} = \epsilon a_1, \dots, \varphi_{n-1} = \epsilon a_{p-1}$ and $\varphi_n = \epsilon a_p$,
- $\exists\, p \ge 2$ *and* $\epsilon = \pm 1$ such that $\varphi_{n-p+1} = \epsilon a_1, \dots, \varphi_{n-1} = \epsilon a_{p-1}$ and $\varphi_n \ne \epsilon a_p$ and $\ne \pm a_1$,
- such a $p \ge 2$ doesn't exist and $\varphi_n \ne \pm a_1$.

We can resume these possibilities in two cases:

$$\begin{array}{lll} q \in (a) & \text{if } \exists p \in \{1, \dots, n-1\} \text{ and, if } p > 1,\ \exists \epsilon = \pm 1 & \\ & \text{such that } \varphi_n \ne \pm a_1 \text{ and, if } p > 1, & \\ & \varphi_{n-p+1} = \epsilon a_1, \dots, \varphi_{n-1} = \epsilon a_{p-1}, \varphi_n \ne \epsilon a_p & (4.3) \\ q \in (b) & \text{if } \exists p,\ 1 \le p \le n-1, \text{ and } \epsilon = \pm 1 \text{ such that} & \\ & \varphi_{n-p+1} = \epsilon a_1, \dots, \varphi_n = \epsilon a_p. & \end{array}$$

For instance the generic case is $q \in (a)$ and $p = 1$. In this case, $\varphi_n \ne \pm\frac{\pi}{2}$ (since $a_1 = \frac{\pi}{2}$ from (3.1)) and there is no sequence $\varphi_{n-p+1}, \dots, \varphi_{n-1}$ equal to $\epsilon a_1, \ldots, \epsilon a_{p-1}$ for some $\epsilon = \pm 1$. Notice also that, in both cases (a) and (b) the ϵ must be the same for $\varphi_{n-p+1}, \ldots, \varphi_n$.

REMARK 4.4. If $q \in (b)$, then the functions $c_{n-p+1}, t_{n-p+2}, \ldots, t_n$ (defined by (4.1)) are all zero. If $q \in (a)$, then $c_{n-p+1} = t_{n-p+2} = \cdots = t_{n-1} = 0$ but $t_n \neq 0$ and $c_n \neq 0$.

4.2.2. RECALL OF SOME DEFINITIONS. Let us recall here some definitions used in the proof of Proposition 4.3. For each definition we indicate the original reference number.

- Set $\mathcal{A}_s$ (Formula (2.2)):

$$\mathcal{A}_1 = \{(1), (2)\},$$
$$\mathcal{A}_s = \{\underline{i} = (1, i_2, \ldots, i_s) \mid i_j = 1 \text{ or } 2\} \text{ if } s > 1$$

- Sequence a_p (Formula (3.1)):

$$\begin{cases} a_1 = \frac{\pi}{2}, \\ a_p = \arctan \sin a_{p-1} \end{cases}$$

- Brackets $[A^n(\beta, d)]$ (Formula (3.2)):

$$\begin{cases} [A^n(1,0)] & = & X_1^n \\ [A^n(1,1)] & = & X_2^n \\ [A^n(\beta,d)] & = & [X_1^n, \underbrace{X_2^n, \ldots, X_2^n}_{d}, \underbrace{X_1^n, \ldots, X_1^n}_{\beta-d-1}] \quad \textit{for } \beta > 2 \end{cases}$$

- The basis $\mathcal{B}^n = \{B_i^n, i = 1, \ldots, n+3\}$ of $T_q(\mathbf{R}^2 \times (\mathbf{S}^1)^{n+1})$ (Formula (3.3)):

$$B_i^n = [A^n(\beta_i^n, d_i^n)]_q$$

- Functions of φ_m, for $m \leq n$ (Formula (4.1)):

$$\begin{aligned} \varphi_m &= \theta_m - \theta_{m-1} \\ c_m &= \cos \varphi_m \\ s_m &= \sin \varphi_m \\ t_m &= \sin \varphi_m - \cos \varphi_m \sin \varphi_{m-1}. \end{aligned}$$

- Sets $I_{\underline{l}}$ and $I_{\underline{l}}^+$ (Lemma 4.1):

$$I_{\underline{l}} \subset \{\underline{b} = (b_0, \ldots, b_p) \mid b_0 = d(\underline{l}) \geq b_1 \geq \cdots \geq b_p \geq 0, \ \sum_{j=1}^{p} b_j \leq |\underline{i}| - |\underline{l}|\} \tag{4.4}$$

$$I_{\underline{l}}^+ = \{\underline{b} \in I_{\underline{l}} \mid \sum_{j=1}^{p} b_j = |\underline{i}| - |\underline{l}|\}. \tag{4.5}$$

4.2.3. INEQUALITIES. We are going now to prove some inequalities for d_i^n and β_i^n, and also some inclusion relationships between linear spaces. We begin with d_i^n, for which the inequality is the same in all of the cases.

LEMMA 4.5. *Let $n \geq 2$. Then, for $i = 2, \ldots, n+2$, we have:*

$$d_i^n \geq \beta_{i-1}^{n-1}$$

PROOF. Lemma 4.1, with $p=1$, implies that a bracket $[X^n_{\underline{j}}]$ ($\underline{j} \neq (1)$) can be written:

$$[X^n_{\underline{j}}] = \sum_{s=1}^{d(\underline{j})} \sum_{\underline{l} \in \mathcal{A}_s} f_{\underline{l}}(\varphi_n)\,[X^{n-1}_{\underline{l}}]. \tag{4.6}$$

Let $d < \beta^{n-1}_{i-1}$. We have:

$$\begin{aligned} \operatorname{span}\langle\ [X^n_{\underline{j}}]_q,\ |\underline{j}| \leq \beta^n_i,\ d(\underline{j}) \leq d\ \rangle\ &\subseteq\ \operatorname{span}\langle\ [X^n_{\underline{j}}]_q,\ d(\underline{j}) \leq d\ \rangle \\ &\subseteq\ \operatorname{span}\langle\ X^n_1, [X^{n-1}_{\underline{l}}]_{q^1},\ |\underline{l}| \leq d\ \rangle \\ &\subseteq\ \operatorname{span}\langle\ X^n_1, [X^{n-1}_{\underline{l}}]_{q^1},\ |\underline{l}| < \beta^{n-1}_{i-1}\ \rangle \end{aligned}$$

From the definition (2.4) of β^{n-1}_{i-1}, the dimension of the last linear space is strictly inferior to i Then we have $d^n_i > d$, which implies $d^n_i \geq \beta^{n-1}_{i-1}$.

■

REMARK 4.6. If we suppose that Proposition 4.3 is true for every $m < n$, that implies that d^{n-m}_i increases with respect to i. Then, by using the same kind of proof as in Lemma 4.5, we can see that:

$$\operatorname{span}\langle\ [X^{n-m}_{\underline{j}}]_{q^m},\ d(\underline{j}) < d^{n-m}_i\ \rangle \subseteq \operatorname{span}\langle\ B^{n-m}_1, \ldots, B^{n-m}_{i-1}\ \rangle \tag{4.7}$$

The relationships for β^n_i are more complicated because there are different cases according to the values of q and i.

LEMMA 4.7. *Let $n \geq 2$. Let us assume that Proposition 4.3 is true for every $m < n$. Then:*

1. *if $i \in [3, \ldots, n+3]$,*

$$|\underline{j}| \leq \beta^{n-1}_{i-1} \text{ and } \underline{j} \neq (1)\ \Rightarrow\ [X^n_{\underline{j}}]_q \in \mathit{span}\langle\ B^{n-1}_1, \ldots, B^{n-1}_{i-2}\ \rangle$$

2. *if $q \in (b)$ and $i \in [p+2, \ldots, n+3]$,*

$$|\underline{j}| < \beta^{n-p}_{i-p} + p\, d^{n-p}_{i-p} \text{ and } \underline{j} \neq (1)$$
$$\Rightarrow\ [X^n_{\underline{j}}]_q \in \mathit{span}\langle\ X^{n-1}_1, \ldots, X^{n-p+1}_1, B^{n-p}_1, \ldots, B^{n-p}_{i-p-1}\ \rangle$$

We can give a weaker formulation of this lemma:

COROLLARY 4.8. *Let $n \geq 2$. Let us assume that Proposition 4.3 is true for every $m < n$. Then:*

1. *if $i \in [3, \ldots, n+3]$,*

$$\beta^n_i \geq \beta^{n-1}_{i-1} + 1,$$

2. *if $q \in (b)$ and $i \in [p+2, \ldots, n+3]$,*

$$\beta^n_i \geq \beta^{n-p}_{i-p} + p\, d^{n-p}_{i-p}.$$

PROOF. The corollary is a direct consequence of the lemma. Moreover, if $|\underline{j}| \leq \beta^{n-1}_{i-1}$, then $d(\underline{j}) < \beta^{n-1}_{i-1}$. According to Formula (4.6), the first part of the lemma is obvious. Thus we have just to prove the second part of the lemma.

Let $i \in [p+2, \dots, n+3]$ and $\underline{j}$ ($\underline{j} \neq (1)$) such that $|\underline{j}| < \beta_{i-p}^{n-p} + p\, d_{i-p}^{n-p}$. According to Lemma 4.1 the bracket $[X_{\underline{j}}^n]$ can be written:

$$[X_{\underline{j}}^n] = \sum_{k=n-p+1}^{n-1} h_k\, X_1^k + \sum_{s=1}^{d} \sum_{\underline{l} \in \mathcal{A}_s} f_{\underline{l}}\, [X_{\underline{l}}^{n-p}]. \tag{4.8}$$

where $d = \max\{1, d(\underline{j}) - p + 1\}$.

If $|\underline{j}| < p+2$, then $d = 1$ and $[X_{\underline{j}}^n]_q$ is a linear combination of $X_1^{n-1}, \dots,$ X_1^{n-p+1}, B_1^{n-p} and B_2^{n-p} (recall (3.3) that $B_1^{n-p} = X_1^{n-p}(q^p)$ and $B_2^{n-p} = X_2^{n-p}(q^p)$).

If $|\underline{j}| \geq p+2$, we are going to prove that, in the sum (4.8), either $[X_{\underline{l}}^{n-p}]_{q^p}$ belongs to span$\langle\, B_1^{n-p}, \dots, B_{i-p-1}^{n-p} \,\rangle$, either $f_{\underline{l}} = 0$. First Lemma 4.1 gives the form of the functions $f_{\underline{l}}(\varphi_{n-p+1}, \dots, \varphi_n)$:

$$f_{\underline{l}} = \sum_{\underline{b} \in I_{\underline{l}}} (c_{n-p+1})^{b_0 - b_1} (t_{n-p+2})^{b_1 - b_2} \cdots (t_n)^{b_{p-1} - b_p} g_{\underline{b},\underline{l}}.$$

Since q belongs to case (b), the functions $c_{n-p+1}, \dots, t_n$ are equal to 0 (see Remark 4.4). Thus, if the function $f_{\underline{l}}$ is non zero, we have, from Remark 4.2:

$$b_0 = b_1 = \cdots = b_p = d(\underline{l})$$

and, by using the definition of the set $I_{\underline{l}}$ (see (4.4)):

$$pd(\underline{l}) \leq |\underline{j}| - |\underline{l}|.$$

Therefore, if $|\underline{l}| + pd(\underline{l}) > |\underline{j}|$, the function $f_{\underline{l}}$ is zero at q.

On the other hand, if a vector $[X_{\underline{l}}^{n-p}]_{q^p}$ doesn't belong to the space span$\langle\, B_1^{n-p}, \dots, B_{i-p-1}^{n-p} \,\rangle$, we have necessarily $|\underline{l}| \geq \beta_{i-p}^{n-p}$ and $d(\underline{l}) \geq d_{i-p}^{n-p}$ (see (4.7)). Thus, we have:

$$\begin{array}{rcl} |\underline{l}| + pd(\underline{l}) & \geq & \beta_{i-p}^{n-p} + p\, d_{i-p}^{n-p} \\ & > & |\underline{j}| \end{array}$$

Hence the vector $[X_{\underline{j}}^n]_q$ is a linear combination of $X_1^{n-1}, \dots, X_1^{n-p+1}$ and $B_1^{n-p}, \dots, B_{i-p-1}^{n-p}$.

■

4.2.4. Particular vector fields. Let us assume that Proposition 4.3 is true for a given m such that $m < n$. Then it is clear that the vectors $\{X_1^n, \dots, X_1^{m+1}, B_1^m, \dots, B_{m+3}^m\}$ form a basis of $T_q(\mathbf{R}^2 \times (\mathbf{S}^1)^{n+1})$. On this basis, the coordinates of a vector $X \in T_q(\mathbf{R}^2 \times (\mathbf{S}^1)^{n+1})$ are denoted by:

$$(\gamma_1^n(X), \dots, \gamma_1^{m+1}(X), \gamma_1^m(X), \dots, \gamma_{m+3}^m(X)).$$

Lemma 4.9. *Let $n \geq 2$. Let us assume that Proposition 4.3 is true for every $m < n$. Then:*

1. *if* $q \in (b)$ *or* (a) *and* $i \in [3, \ldots, p+2]$, *and if the vector* $\tilde{B}_i^n$ *is defined by:*
$$\tilde{B}_i^n = [A^n(\beta_{i-1}^{n-1} + 1, \beta_{i-1}^{n-1})]_q,$$
then $\gamma_{i-1}^{n-1}(\tilde{B}_i^n)$ *is non zero.*
2. *if* $q \in (b)$ *and* $i \in [p+2, \ldots, n+3]$, *and if the vector* $\tilde{B}_i^n$ *is defined by:*
$$\tilde{B}_i^n = [A^n(\beta_{i-p}^{n-p} + p\, d_{i-p}^{n-p}, \beta_{i-1}^{n-1})]_q,$$
then $\gamma_{i-p}^{n-p}(\tilde{B}_i^n)$ *is non zero.*
3. *if* $q \in (a)$ *and* $i \in [p+2, \ldots, n+3]$, *and if the vector* $\tilde{B}_i^n$ *is defined by:*
$$\tilde{B}_i^n = [A^n(\beta_{i-1}^{n-1} + 1, \beta_{i-1}^{n-1})]_q,$$
then $\gamma_{i-p}^{n-p}(\tilde{B}_i^n)$ *is non zero.*

LEMMA 4.10. *Let* $n \geq 2$. *Let us assume that Proposition 4.3 is true for every* $m < n$. *Then, with the notation of Lemma 4.9:*

1. *if* $q \in (b)$ *or* (a) *and* $i \in [3, \ldots, p+2]$, *every vector* $[X_{\underline{j}}^n]_q$ *such that* $|\underline{j}| = \beta_{i-1}^{n-1} + 1$ *and* $d(\underline{j}) = \beta_{i-1}^{n-1}$ *(and* $j_1 = 1$ *if* $|\underline{j}| > 1$*) satisfies:*
$$\frac{\gamma_{i-1}^{n-1}([X_{\underline{j}}^n]_q)}{\gamma_{i-1}^{n-1}(\tilde{B}_i^n)} \geq 0$$
2. *if* $q \in (b)$ *and* $i \in [p+2, \ldots, n+3]$, *every vector* $[X_{\underline{j}}^n]_q$ *such that* $|\underline{j}| = \beta_{i-p}^{n-p} + p\, d_{i-p}^{n-p}$ *and* $d(\underline{j}) = \beta_{i-1}^{n-1}$ *(and* $j_1 = 1$*) satisfies:*
$$\frac{\gamma_{i-p}^{n-p}([X_{\underline{j}}^n]_q)}{\gamma_{i-p}^{n-p}(\tilde{B}_i^n)} \geq 0$$
3. *if* $q \in (a)$ *and* $i \in [p+2, \ldots, n+3]$, *every vector* $[X_{\underline{j}}^n]_q$ *such that* $|\underline{j}| = \beta_{i-1}^{n-1} + 1$ *and* $d(\underline{j}) = \beta_{i-1}^{n-1}$ *(and* $j_1 = 1$*) satisfies:*
$$\frac{\gamma_{i-p}^{n-p}([X_{\underline{j}}^n]_q)}{\gamma_{i-p}^{n-p}(\tilde{B}_i^n)} \geq 0$$

REMARK 4.11. When $q \in (b)$, the vector $\tilde{B}_{p+2}^n$ has two different definitions, namely $[A^n(\beta_{p+1}^{n-1}+1, \beta_{p+1}^{n-1})]$ and $[A^n(\beta_2^{n-p}+pd_2^{n-p}, \beta_{p+1}^{n-1})]$. These definitions are compatible if $\beta_{p+1}^{n-1}+1$ and $\beta_2^{n-p}+pd_2^{n-p}$ are equal. Let us calculate these quantities (for $q \in (b)$):

- we know that $\beta_2^{n-p} = d_2^{n-p} = 1$ (see (2.7)), then:
$$\beta_2^{n-p} + p\, d_2^{n-p} = p + 1,$$
- by hypothesis Proposition 4.3 is true for $m < n$, then we have:
$$\beta_{p+1}^{n-1} + 1 = \beta_p^{n-2} + 2 = \cdots = \beta_2^{n-p} + p = p + 1.$$

Thus the two definitions of $\tilde{B}_{p+2}^n$ are compatible.

PROOF OF LEMMAE 4.9 AND 4.10.

We are going to prove both lemmas together, for instance in the case where $q \in (a)$ (recall that the cases (a) and (b) are given by (4.3)). In this case, for $i \leq n+3$, the only vector $[X_{\underline{j}}^n]_q$ such that $\underline{j} \in \mathcal{A}_{\beta_{i-1}^{n-1}+1}$ and $d(\underline{j}) = \beta_{i-1}^{n-1}$ is $\tilde{B}_i^n$. Therefore, if Lemma 4.9 is true, Lemma 4.10 in the case $q \in (a)$ states only that $1 \geq 0$!

By hypothesis, we can apply Proposition 4.3 to q^1 and we obtain:

$$\begin{cases} \beta_{i-1}^{n-1} = d_{i-1}^{n-1} + 1 & for\ i = 3, \dots, p+2 \\ \beta_{i-1}^{n-1} = \beta_{i-p}^{n-p} + (p-1)\, d_{i-p}^{n-p} & for\ i = p+2, \dots, n+3 \end{cases}$$

If $i \leq p+2$, the required property is given directly by the part 5 of Lemma 4.1 (recall that, when $q \in (a)$, $c_n \neq 0$ from Remark 4.4).

If $i \geq p+2$, we proceed as in the proof of Lemma 4.7 and write $\tilde{B}_i^n$ as:

$$\tilde{B}_i^n = \sum_{k=n-p+1}^{n-1} h_k\, X_1^k + \sum_{s=1}^{\beta_{i-1}^{n-1}} \sum_{\underline{l} \in \mathcal{A}_s} f_{\underline{l}}\, [X_{\underline{l}}^{n-p}].$$

The coefficient of $\tilde{B}_i^n$ on B_{i-p}^{n-p} is equal to:

$$\gamma_{i-p}^{n-p}(\tilde{B}_i^n) = \sum_{s=1}^{\beta_{i-1}^{n-1}} \sum_{\underline{l} \in \mathcal{A}_s} f_{\underline{l}}\, \gamma_{i-p}^{n-p}([X_{\underline{l}}^{n-p}]). \tag{4.9}$$

Claim.

- *If* $|\underline{l}| < \beta_{i-p}^{n-p}$ *or* $d(\underline{l}) < d_{i-p}^{n-p}$, *then* $\gamma_{i-p}^{n-p}([X_{\underline{l}}^{n-p}]) = 0$,
- *if* $|\underline{l}| + (p-1)d(\underline{l}) > \beta_{i-1}^{n-1} = \beta_{i-p}^{n-p} + (p-1)\, d_{i-p}^{n-p}$, *then* $f_{\underline{l}} = 0$.

Proof of the claim. The first point is a direct consequence of the definition of β_{i-p}^{n-p} (2.4) and of Formulas (4.7).

For the second point, we use the part 1 of Lemma 4.1:

$$f_{\underline{l}} = \sum_{\underline{b} \in I_{\underline{l}}} (c_{n-p+1})^{b_0 - b_1} (t_{n-p+2})^{b_1 - b_2} \cdots (t_n)^{b_{p-1} - b_p}\ g_{\underline{b},\underline{l}}. \tag{4.10}$$

Since q belongs to case (a), the functions $c_{n-p+1}, \dots, t_{n-1}$ are equal to 0 (Remark 4.4). Thus, if the function $f_{\underline{l}}$ is non zero, we have:

$$b_0 = b_1 = \cdots = b_{p-1} = d(\underline{l}). \tag{4.11}$$

Let us assume that $|\underline{l}| + (p-1)d(\underline{l}) > \beta_{i-1}^{n-1}$, that is $\beta_{i-1}^{n-1} - |\underline{l}| < (p-1)d(\underline{l})$. The definition of $I_{\underline{l}}$ (see (4.4)) and the part 2 of Lemma 4.1 implie that $\sum_{i=1}^{p-1} b_i \leq |\underline{i}| - |\underline{l}| - 1$, and so that $\sum_{i=1}^{p-1} b_i < (p-1)d(\underline{l})$ (recall that here $\beta_{i-1}^{n-1} = |\underline{i}| - 1$). Then, in this case, we can not have the relation (4.11). The claim is proved.

This claim allows us to reduce the sum (4.9) to:

$$\gamma_{i-p}^{n-p}(\tilde{B}_i^n) = \sum_{\substack{\underline{l} \in \mathcal{A}_{\beta_{i-p}^{n-p}} \\ d(\underline{l}) = d_{i-p}^{n-p}}} f_{\underline{l}}(\varphi_{n-p+1}, \dots, \varphi_n)\ \gamma_{i-p}^{n-p}([X_{\underline{l}}^{n-p}]) \tag{4.12}$$

Claim.

- *If* $\underline{l} \in \mathcal{A}_{\beta_{i-p}^{n-p}}$ *and* $d(\underline{l}) = d_{i-p}^{n-p}$, *then* $\gamma_{i-p}^{n-p}([X_{\underline{l}}^{n-p}])$ *is non negative,*

- *the sign of* $f_{\underline{l}}$ *is the same for all* $\underline{l} \in \mathcal{A}_{\beta_{i-p}^{n-p}}$ *such that* $d(\underline{l}) = d_{i-p}^{n-p}$ *(i.e. there exists* $\sigma = \pm 1$ *such that, if* $\underline{l} \in \mathcal{A}_{\beta_{i-p}^{n-p}}$ *and* $d(\underline{l}) = d_{i-p}^{n-p}$*, then* $f_{\underline{l}} = \sigma |f_{\underline{l}}|$*),*
- *if* $[X_{\underline{l}}^{n-p}] = B_{i-p}^{n-p}$*, then* $f_{\underline{l}}(\varphi_{n-p+1}, \dots, \varphi_n)\ \gamma_{i-p}^{n-p}([X_{\underline{l}}^{n-p}])$ *is non zero.*

Proof of the claim. The first point is given by Proposition 4.3, which is true for $n-p$ by induction hypothesis.

A function $f_{\underline{l}}$ is given by Formula (4.10) and is non zero if and only if there exists $\underline{b} \in I_{\underline{l}}$ such that $b_0 = b_1 = \cdots = b_{p-1} = d_{i-p}^{n-p}$ and $b_p > 0$, *i.e.* if $(d_{i-p}^{n-p}, \dots, d_{i-p}^{n-p}, 1)$ belongs to $I_{\underline{l}}^+$ (defined by (4.5)). Then, according to Lemma 4.1, part 3, if $f_{\underline{l}}$ is non zero, we have:

$$f_{\underline{l}} = (t_n)^{d_{i-p}^{n-p}-1} \alpha_{\underline{b}}\ G_{(d_{i-p}^{n-p}, \dots, d_{i-p}^{n-p}, 1), |\underline{l}|, d(\underline{l})}(\varphi_{n-p+1}, \dots, \varphi_n)$$

and the sign of this quantity depends only on $|\underline{l}|$ ($|\underline{l}| = \beta_{i-p}^{n-p}$) and on $d(\underline{l})$ ($d(\underline{l}) = d_{i-p}^{n-p}$). Therefore, the second point of the claim is proved.

Finally, by definition of the coordinates, $\gamma_{i-p}^{n-p}(B_{i-p}^{n-p}) = 1$ and the part 4 of Lemma 4.1 implies that $(d_{i-p}^{n-p}, \dots, d_{i-p}^{n-p}, 1)$ belongs to $I_{\underline{l}}^+$ if $[X_{\underline{l}}^{n-p}] = B_{i-p}^{n-p}$. Then the claim is proved.

According to this claim, the terms of the sum (4.12) are of same sign and non all zero, then $\gamma_{i-p}^{n-p}(\tilde{B}_i^n)$ is non nul.

■

4.2.5. Proof of Proposition 4.3. The base case $n = 0$ is obvious. Let us suppose that the proposition is true for every $m < n$ and consider for instance that $q \in (b)$.

Lemma 4.7 and Lemma 4.9 imply that, for $i \in [3, \dots, p+2]$, there exists $\gamma_{i-1}^{n-1} \neq 0$ and W_{i-1} in span$\langle\ X_1^n, B_1^{n-1}, \dots, B_{i-2}^{n-1}\ \rangle$ such that:

$$\tilde{B}_i^n = \gamma_{i-1}^{n-1} B_{i-1}^{n-1} + W_{i-1}. \tag{4.13}$$

where $\tilde{B}_i^n = [A^n(\beta_{i-1}^{n-1}+1, \beta_{i-1}^{n-1})]$.

Moreover it is easy to see that:

$$\mathrm{span}\langle\ X_1^n, X_2^n, [X_1^n, X_2^n]\ \rangle = \mathrm{span}\langle\ X_1^n, X_1^{n-1}, X_2^{n-1}\ \rangle$$

Therefore, for $i \in [2, \dots, p+2]$, we have:

$$\mathrm{span}\langle\ X_1^n, X_2^n, \tilde{B}_3^n, \dots, \tilde{B}_i^n \rangle = \mathrm{span}\langle\ X_1^n, B_1^{n-1}, \dots, B_{i-1}^{n-1}\ \rangle. \tag{4.14}$$

Let us consider the linear space spanned by $X_1^n, X_2^n, \tilde{B}_3^n, \dots, \tilde{B}_{p+2}^n$. According to the case 2 of Lemma 4.7, it is included in the one spanned by $X_1^n, \dots, X_1^{n-p+1}$, B_1^{n-p}, B_2^{n-p}, and its dimension is $p+2$ (from equality (4.14)). Then:

$$\mathrm{span}\langle\ X_1^n, X_2^n, \tilde{B}_3^n, \dots, \tilde{B}_{p+2}^n \rangle = \mathrm{span}\langle\ X_1^n, \dots, X_1^{n-p+1}, B_1^{n-p}, B_2^{n-p}\ \rangle. \tag{4.15}$$

For $i \in [p+2, \ldots, n+3]$, we use the same reasoning as for $i \in [3, \ldots, p+2]$. We set $\tilde{B}_i^n = [A^n(\beta_{i-p}^{n-p} + p\, d_{i-p}^{n-p}, \beta_{i-1}^{n-1})]$, and we obtain that the linear space:

$$\mathrm{span}\langle\ X_1^n, \ldots, X_1^{n-p+1}, \tilde{B}_{p+2}^n, \ldots, \tilde{B}_i^n\ \rangle$$

is equal to $\mathrm{span}\langle\ X_1^n, \ldots, X_1^{n-p+1}, B_1^{n-p}, \ldots, B_{i-p-1}^{n-p}\ \rangle$. Hence, by using the equality (4.15), we have, for $i \in [p+2, \ldots, n+3]$:

$$\mathrm{span}\langle\ X_1^n, X_2^n, \tilde{B}_3^n, \ldots, \tilde{B}_i^n\ \rangle = \mathrm{span}\langle\ X_1^n, \ldots, X_1^{n-p+1}, B_1^{n-p}, \ldots, B_{i-p-1}^{n-p}\ \rangle. \tag{4.16}$$

Finally, from equalities (4.14) and (4.16) we conclude:

$$\begin{array}{lcll} d_i^n & \leq & \beta_{i-1}^{n-1} & \text{for } i \in [3, \ldots, n+3] \\ \beta_i^n & \leq & \beta_{i-1}^{n-1} + 1 & \text{for } i \in [2, \ldots, p+2] \\ \beta_i^n & \leq & \beta_{i-p}^{n-p} + p\, d_{i-p}^{n-p} & \text{for } i \in [p+2, \ldots, n+3] \end{array}$$

By using Corollary 4.8, we have the values of β_i^n and d_i^n and the basis required. Thus we have proved the first four points of the proposition.

For the last point, let $i \in [3, \ldots, n+3]$ and $[X_{\underline{j}}^n]_q$ such that $\underline{j} \in \mathcal{A}_{\beta_i^n}$ and $d(\underline{j}) = d_i^n$. Suppose, for instance, that $q \in (b)$ and $i \in [3, \ldots, p+2]$. Formula (4.13) implies that:

$$\tilde{B}_i^n = \gamma_{i-1}^{n-1}(\tilde{B}_i^n)\ B_{i-1}^{n-1} + W_{i-1}$$

and Lemma 4.7 implies that there exists W'_{i-1} in $\mathrm{span}\langle\ X_1^n, B_1^{n-1}, \ldots, B_{i-2}^{n-1}\ \rangle$ such that:

$$[X_{\underline{j}}^n]_q = \gamma_{i-1}^{n-1}([X_{\underline{j}}^n]_q)\ B_{i-1}^{n-1} + W'_{i-1}.$$

Therefore, the coordinate $\gamma_i^n([X_{\underline{j}}^n]_q)$ of $[X_{\underline{j}}^n]_q$ on the basis vector $\tilde{B}_i^n$ is equal to:

$$\frac{\gamma_{i-1}^{n-1}([X_{\underline{j}}^n]_q)}{\gamma_{i-1}^{n-1}(\tilde{B}_i^n)}$$

which is ≥ 0, according to Lemma 4.10.

For $q \in (b)$ and $i \in [p+2, \ldots, n+3]$, the coordinate $\gamma_i^n([X_{\underline{j}}^n]_q)$ of $[X_{\underline{j}}^n]_q$ on the basis vector $\tilde{B}_i^n$ is equal to:

$$\frac{\gamma_{i-p}^{n-p}([X_{\underline{j}}^n]_q)}{\gamma_{i-p}^{n-p}(\tilde{B}_i^n)}$$

which is ≥ 0, according to Lemma 4.10.

The proof for $q \in (a)$ can be done in the same way. Then Proposition 4.3 is proved.

5. Appendix: proof of Lemma 4.1

To prove Lemma 4.1, we divide it in three different part, the lemmas 5.1, 5.2, 5.3 and 5.4 and the goal of this appendix is to prove these four results. We begin by Lemma 5.1, which establish some relationships between the Lie Algebras $\mathcal{L}(X_1^n, X_2^n)$ and $\mathcal{L}(X_1^{n-1}, X_2^{n-1})$. This lemma gives the part 5 of Lemma 4.1, and also provide a base case for the next lemma. Such a result has already been proved in [10].

Let us recall that many notations are gathered in Subsection 4.2.2. In particular the functions c_n and s_n used in the following lemma are defined by (4.1).

Lemma 5.1. *Let $n \geq 1$ and $\underline{i} \neq (1)$. Then there exists some functions $F_{\underline{l}}(\varphi_n) \in C^\infty(\mathbf{S}^1)$ such that:*

$$[X_{\underline{i}}^n] = \sum_{s=1}^{d(\underline{i})} \sum_{\underline{l} \in \mathcal{A}_s} F_{\underline{l}}(\varphi_n)\, [X_{\underline{l}}^{n-1}].$$

Moreover, for $|\underline{i}| \geq 3$, if $d(\underline{i}) = |\underline{i}| - 1$, then the coefficient of $[X_{\underline{l}}^{n-1}]$ such that $\underline{l} \in \mathcal{A}_{d(\underline{i})}$ and $d(\underline{l}) = d(\underline{i}) - 1$ is:

$$F_{\underline{l}}(\varphi_n) = (c_n)^{d(\underline{i})-2}.$$

Proof. We proceed by induction on $|\underline{i}|$. For $|\underline{i}| \leq 2$, the result is a consequence of the following formulas (see (2.9):

$$\begin{cases} X_2^n & = \quad s_n X_1^{n-1} + c_n X_2^{n-1} \\ [X_1^n, X_2^n] & = \quad c_n X_1^{n-1} - s_n X_2^{n-1} \end{cases} \tag{5.1}$$

With this formulas we can calculate the bracket $[X_{\underline{i}}^n]$ such that $\underline{i} \in \mathcal{A}_3$ and $d(\underline{i}) = |\underline{i}| - 1$:

$$[X_1^n, X_2^n, X_2^n] \quad = \quad -X_1^{n-1} + [X_1^{n-1}, X_2^{n-1}]$$

and verify that the coefficient of $[X_1^{n-1}, X_2^{n-1}]$ has the required form.

Let us assume now that the lemma is true for $|\underline{i}| = k$ and consider a bracket $[X_{\underline{j}}^n]$ such that $\underline{j} \in \mathcal{A}_{k+1}$. The sequence $\underline{j}$ can be written either $(\underline{i}, 1)$ either $(\underline{i}, 2)$ and we can apply the induction hypothesis to $[X_{\underline{i}}^n]$, that is:

$$[X_{\underline{i}}^n] = \sum_{s=1}^{d(\underline{i})} \sum_{\underline{l} \in \mathcal{A}_s} F_{\underline{l}}(\varphi_n)\, [X_{\underline{l}}^{n-1}].$$

Thus, if $\underline{j} = (\underline{i}, 1)$, we have $d(\underline{i}) = d(\underline{j})$ and:

$$[X_{\underline{j}}^n] = \sum_{s=1}^{d(\underline{j})} \sum_{\underline{l} \in \mathcal{A}_s} (X_1^n . F_{\underline{l}})\, [X_{\underline{l}}^{n-1}].$$

If $\underline{j} = (\underline{i}, 2)$, by using the expression (5.1) for X_2^n, $[X_{\underline{j}}^n]$ is equal to:

$$\sum_{s=1}^{d(\underline{i})} \sum_{\underline{l} \in \mathcal{A}_s} -s_n (X_1^{n-1} . F_{\underline{l}})\, [X_{\underline{l}}^{n-1}] + \sum_{s=1}^{d(\underline{i})} \sum_{\underline{l} \in \mathcal{A}_s} (s_n F_{\underline{l}}\, [X_{(\underline{l},1)}^{n-1}] + c_n F_{\underline{l}}\, [X_{(\underline{l},2)}^{n-1}]).$$

Hence $[X^n_{\underline{j}}]$ is of the required form. Moreover, if $d(\underline{j}) = |\underline{j}| - 1$, then $\underline{j} = (\underline{i}, 2)$ and the sequence $\underline{i}$ is such that $d(\underline{i}) = |\underline{i}| - 1$. A sequence $\underline{l'}$ such that $|\underline{l'}| = d(\underline{j})$ and $d(\underline{l'}) = d(\underline{j}) - 1$ is equal to $(\underline{l}, 2)$ and the coefficient of $[X^{n-1}_{(\underline{l},2)}]$ is:

$$c_n F_{\underline{l}} = c_n (c_n)^{d(\underline{i})-2} = (c_n)^{d(\underline{j})-2}$$

∎

We are going now to prove the part of Lemma 4.1 which concern the bracket of length inferior than $p+1$. This result gives also the base case for the proof of Lemma 5.3.

LEMMA 5.2. *Let p, $1 \leq p \leq n-1$ and $\underline{i} \neq (1)$ such that $|\underline{i}| \leq p+1$. Then there exist some functions $h_k(\varphi_{n-p+1}, \ldots, \varphi_n)$, $n-p \leq k \leq n-1$ and $f_2(\varphi_{n-p+1}, \ldots, \varphi_n)$ in $C^\infty(\mathbf{S}^p)$ such that:*

$$[X^n_{\underline{i}}] = \sum_{k=n-p}^{n-1} h_k\, X^k_1 + f_2\, X^{n-p}_2.$$

Moreover, there exists $g(\varphi_{n-p+1}, \ldots, \varphi_n)$ in $C^\infty(\mathbf{S}^p)$ such that:

- *if $|\underline{i}| = p+1$ and $d(\underline{i}) = p$, then:*

$$f_2 = -s_{n-p+1} c_{n-p+3} \ldots (c_n)^{p-2} + c_{n-p+1}\, g,$$

- *otherwise:*

$$f_2 = c_{n-p+1}\, g.$$

PROOF. We make the proof by induction on p. For $p = 1$, the result is a consequence of the formula (5.1). Let us assume now that the lemma is true for an integer $p-1$, and consider $\underline{i} \neq (1)$ such that $|\underline{i}| \leq p+1$. Lemma 5.1 implies that:

$$[X^n_{\underline{i}}] = \sum_{s=1}^{d(\underline{i})} \sum_{\underline{l} \in \mathcal{A}_s} F_{\underline{l}}\, [X^{n-1}_{\underline{l}}]$$

where $F_{\underline{l}} = (c_n)^{d(\underline{i})-2}$ if $|\underline{l}| = d(\underline{i})$ and $d(\underline{l}) = d(\underline{i}) - 1$.

By applying the induction hypothesis (with $p-1$) to each $[X^{n-1}_{\underline{l}}]$, we have:

$$[X^{n-1}_{\underline{l}}] = \sum_{k=n-p}^{n-2} h_{k,\underline{l}}\, X^k_1 + f_{2,\underline{l}}\, X^{n-p}_2,$$

thus $[X^n_{\underline{i}}]$ can be written:

$$[X^n_{\underline{i}}] = \sum_{k=n-p}^{n-1} h_k(\varphi_{n-p+1}, \ldots, \varphi_n)\, X^k_1 + f_2(\varphi_{n-p+1}, \ldots, \varphi_n)\, X^{n-p}_2$$

where

$$f_2 = \sum_{s=1}^{d(\underline{i})} \sum_{\underline{l} \in \mathcal{A}_s} F_{\underline{l}} f_{2,\underline{l}}.$$

If $d(\underline{i}) < p$, every $f_{2,\underline{l}}$ is equal to $c_{n-p+1} g_{\underline{l}}$ and then f_2 can be written $c_{n-p+1} g$.

If $d(\underline{i}) = p$ (and so $|\underline{i}| = p+1$), then there is a bracket $[X_{\underline{l}}^{n-1}]$ such that $\underline{l} \in \mathcal{A}_p$ and $d(\underline{l}) = p-1$ and for this bracket:

$$f_{2,\underline{l}}\, F_{\underline{l}} = \left(-s_{n-p+1} c_{n-p+3} \cdots (c_{n-1})^{p-3} + c_{n-p+1} g_{\underline{l}}\right)\ (c_n)^{p-2}.$$

Therefore we have the required properties.

■

We are going now to prove the parts 1, 2 and 3 of Lemma 4.1.

LEMMA 5.3. *Let p, $1 \leq p \leq n-1$ and $\underline{i} \in \mathcal{A}_{|\underline{i}|}$ such that $|\underline{i}| \geq p+1$. Then there exists some functions $h_k(\varphi_{n-p+1}, \ldots, \varphi_n)$, $n-p+1 \leq k \leq n-1$ and $f_{\underline{l}}(\varphi_{n-p+1}, \ldots, \varphi_n)$ in $C^\infty(\mathbf{S}^p)$ such that:*

$$[X_{\underline{i}}^n] = \sum_{k=n-p+1}^{n-1} h_k\, X_1^k + \sum_{s=1}^{d} \sum_{\underline{l} \in \mathcal{A}_s} f_{\underline{l}}\, [X_{\underline{l}}^{n-p}] \tag{5.2}$$

where $d = \max\{1, d(\underline{i}) - p + 1\}$.

Moreover,

1. *we have:*

$$f_{\underline{l}} = \sum_{\underline{b} \in I_{\underline{l}}} (c_{n-p+1})^{b_0 - b_1} (t_{n-p+2})^{b_1 - b_2} \cdots (t_n)^{b_{p-1} - b_p} g_{\underline{b},\underline{l}} \tag{5.3}$$

where $g_{\underline{b},\underline{l}}(\varphi_{n-p+1}, \ldots, \varphi_n)$ are in $C^\infty(\mathbf{S}^p)$ and $I_{\underline{l}} \subset \mathbf{Z}^{p+1}$ satisfies:

$$I_{\underline{l}} \subset \{\underline{b} = (b_0, \ldots, b_p) \mid d(\underline{l}) = b_0 \geq \cdots \geq b_p \geq 0,\ \sum_{i=1}^{p} b_i \leq |\underline{i}| - |\underline{l}|\};$$

2. *if $\underline{b} \in I_{\underline{l}}$ is such that $b_p = 0$, then $\sum_{i=1}^{p-1} b_i < d(\underline{i}) - |\underline{l}|$;*
3. *if we denote by $I_{\underline{l}}^+$ the following subset of $I_{\underline{l}}$:*

$$I_{\underline{l}}^+ = \{\underline{b} \in I_{\underline{l}} \mid \sum_{i=1}^{p} b_i = |\underline{i}| - |\underline{l}|\},$$

then, for every $\underline{b} \in I_{\underline{l}}^+$, there exist an integer $\alpha_{\underline{b}} > 0$ such that the function $g_{\underline{b},\underline{l}}(\varphi_{n-p+1}, \ldots, \varphi_n)$ is equal to:

$$\begin{aligned} g_{\underline{b},\underline{l}} = \alpha_{\underline{b}}\, (-1)^{l_1 - 1} s_{n-p+1} c_{n-p+3} \cdots (c_n)^{p-2} & \\ \times\, (c_n \cdots c_{n-p+2})^{|\underline{l}|-1} (s_{n-p+1})^{|\underline{l}| - d(\underline{l})} & \\ \times\, (c_n \cdots c_{n-p+3} s_{n-p+1})^{b_1 - 1} \cdots (-t'_{n-1})^{b_{p-1}-1} & \\ \times\, (s_n t'_n)^{b_p - |\underline{i}| + d(\underline{i})} (-t'_n)^{|\underline{i}| - d(\underline{i}) - 1} & \end{aligned} \tag{5.4}$$

where $t'_m = \frac{\partial t_m}{\partial \varphi_m}$ and t_m is defined by (4.1).

PROOF. We proceed by induction on $|\underline{i}|$.

Base case: $|\underline{i}| = p+1$.

Lemma 5.2 implies that a bracket $[X_{\underline{i}}^n]$ such that $\underline{i} \in \mathcal{A}_{p+1}$ is of the form (5.2) and that the coefficient f_2 of X_2^{n-p} satisfies Condition 1 with:

- $I_2 = \{(1,\dots,1),(1,0,\dots,0)\}$ if $|\underline{i}| = p+1$ and $d(\underline{i}) = p$,
- $I_2 = \{(1,0,\dots,0)\}$ otherwise.

These sets verify obviously Condition 2. We can see also that $I_2^+ = \emptyset$ in the second case, $I_2^+ = \{(1,\dots,1)\}$ in the first case and, according to Lemma 5.2, we have:

$$g_{(1,\dots,1)} = -s_{n-p+1}c_{n-p+3}\cdots(c_n)^{p-2}.$$

Thus the point 3 is also true.

Induction step: let us assume that the lemma is true for $|\underline{i}| \le m$ ($m \ge p+1$) and consider a bracket $[X^n_{\underline{j}}]$ such that $\underline{j} \in \mathcal{A}_{m+1}$.

- If $j_{m+1} = 1$, that is $\underline{j} = (\underline{i}, 1)$ with $|\underline{i}| = m$.
By applying the induction hypothesis to $[X^n_{\underline{i}}]$ and using the fact that $[X^n_{\underline{j}}] = -X^n_1.[X^n_{\underline{i}}]$, we obtain that $[X^n_{\underline{j}}]$ is equal to:

$$\sum_{k=n-p+1}^{n-1} (-X^n_1.h_k)\, X^k_1 + \sum_{s=1}^{d}\sum_{\underline{l}\in\mathcal{A}_s} (-X^n_1.f_{\underline{l}})\,[X^{n-p}_{\underline{l}}].$$

Since $d = \max\{1, d(\underline{i}) - p + 1\} = \max\{1, d(\underline{j}) - p + 1\}$, $[X^n_{\underline{j}}]$ is of the form (5.2). The coefficients of the brackets $[X^{n-p}_{\underline{l}}]$ are equal to $-X^n_1.f_{\underline{l}}$. According to the induction hypothesis, we have:

$$f_{\underline{l}} = \sum_{\underline{b}\in I_{\underline{l}}} (c_{n-p+1})^{b_0-b_1}(t_{n-p+2})^{b_1-b_2}\cdots(t_n)^{b_{p-1}-b_p}\, g_{\underline{b},\underline{l}}$$

and then, we can write $-X^n_1.f_{\underline{l}}$ as:

$$\sum_{\underline{b}\in I_{\underline{l}}} (c_{n-p+1})^{b_0-b_1}(t_{n-p+2})^{b_1-b_2}\dots(t_n)^{b_{p-1}-b_p-1}(b_{p-1}-b_p)(-t'_n)g_{\underline{b},\underline{l}} + \\ + \sum_{\underline{b}\in I_{\underline{l}}} (c_{n-p+1})^{b_0-b_1}(t_{n-p+2})^{b_1-b_2}\dots(t_n)^{b_{p-1}-b_p}(-X^n_1.g_{\underline{b},\underline{l}}). \tag{5.5}$$

This coefficient satisfies Condition 1 since:

$$b_1 + \dots + b_{p-1} + (b_p + 1) \le |\underline{i}| - |\underline{l}| + 1 = |\underline{j}| - |\underline{l}|,$$

and $b_p + 1 \le b_{p-1}$ (if $b_{p-1} = b_p$, the corresponding term in (5.5) is zero).

Condition 2 is also satisfied since, from the induction hypothesis, $b_p = 0$ implies that $\sum_{i=1}^{p-1} b_i$ is strictly inferior than $d(\underline{i}) - |\underline{l}| = d(\underline{j}) - |\underline{l}|$. Moreover, to have $b_1 + \dots + b_{p-1} + (b_p + 1) = |\underline{j}| - |\underline{l}|$, it is necessary that $(b_0,\dots,b_p)$ belongs to $I^+_{\underline{l}}$, and then $(b_{p-1} - b_p)(-t'_n)g_{\underline{b},\underline{l}}$ is in the form (5.4) and so property 3 is satisfied.

- If $j_{m+1} = 2$, that is $\underline{j} = (\underline{i}, 2)$ with $|\underline{i}| = m$.
We proceed in the same way as above: we apply the induction hypothesis to $[X^n_{\underline{i}}]$, which expression is then given by Formula (5.2). Since $[X^n_{\underline{j}}]$ is equal to $[X^n_{\underline{i}}, X^n_2]$, we can write it as $\Sigma' + \Sigma''$ where:

- Σ' is the sum, for $k = n-p+1, \dots, n-1$, of the terms:
$$\Sigma'_k = [h_k\, X_1^k, X_2^n],$$
- Σ'' is the sum, for $s = 1, \dots, d$ and $\underline{l} \in \mathcal{A}_s$ of the terms:
$$\Sigma''_{s,\underline{l}} = [f_{\underline{l}}\, [X_{\underline{l}}^{n-p}], X_2^n].$$

For completing the induction step, we have to prove that all the terms Σ'_k and $\Sigma''_{s,\underline{l}}$ are of the form (5.2) (for a length $|\underline{j}|$ and a "number of 2" $d(\underline{j})$) and satisfy the conditions 1, 2, 3 of the lemma.

Recall Formula (2.8), which states that X_2^n is equal to:
$$s_n X_1^{n-1} + c_n s_{n-1} X_1^{n-2} + \cdots + c_n \cdots c_{n-p+2} s_{n-p+1} X_1^{n-p} + \\ + c_n \cdots c_{n-p+2} c_{n-p+1} X_2^{n-p}.$$

By using this formula, we can expand a term Σ'_k in:
$$\Sigma'_k = \sum_{t=n-p+1}^{n-1} h'_t\, X_1^t + g'_1 X_1^{n-p} + g'_2 X_2^{n-p}$$

where $h'_t(\varphi_{n-p+1}, \dots, \varphi_n)$, $g'_1(\varphi_{n-p+1}, \dots, \varphi_n)$ and $g'_2(\varphi_{n-p+1}, \dots, \varphi_n)$ belong to $C^\infty(\mathbf{S}^p)$.

This expansion is in the form (5.2). Moreover, the functions g'_1 and g'_2 are in the form (5.3) with the sets $I_1 = \{(0, \dots, 0)\}$ and $I_2 = \{(1, \dots, 1)\}$, so Conditions 2 and 3 are trivially satisfied. Therefore the terms Σ'_k are of the required form.

By using again Formula (2.8), we can now expand $\Sigma''_{s,\underline{l}}$ in:
$$(\Delta . f_{\underline{l}})\, [X_{\underline{l}}^{n-p}] + c_n \cdots c_{n-p+2} s_{n-p+1} f_{\underline{l}}\, [X_{(\underline{l},1)}^{n-p}] + c_n \cdots c_{n-p+2} c_{n-p+1} f_{\underline{l}}\, [X_{(\underline{l},2)}^{n-p}] \tag{5.6}$$
where $\Delta = -\left(s_n X_1^{n-1} + c_n s_{n-1} X_1^{n-2} + \cdots + c_n \cdots c_{n-p+2} s_{n-p+1} X_1^{n-p} \right)$.

Thus $\Sigma''_{s,\underline{l}}$ can be write in the form (5.2). We denote by $f'_{\underline{l'}}(\varphi_{n-p+1}, \dots, \varphi_n)$ the coefficient of $[X_{\underline{l'}}^{n-p}]$ in this expression. The only non zero coefficients are obtained for $\underline{l'} = \underline{l}$, $(\underline{l}, 1)$ and $(\underline{l}, 2)$. Let us prove that these coefficients have the required properties.

We are going now to prove that we can associate to every $f'_{\underline{l'}}$ a set $\hat{I}_{\underline{l'}}$ of p-tuples which satisfy the properties required by the lemma. Recall that $I_{\underline{l}}$ denotes the set associated to the function $f_{\underline{l}}$ in the decomposition of the vector field $[X_{\underline{i}}^n]$.

Case $\underline{l'} = (\underline{l}, 1)$.
According to Formula (5.6), we have: $f'_{\underline{l'}} = c_n \cdots c_{n-p+2} s_{n-p+1} f_{\underline{l}}$. Since $|\underline{j}| - |\underline{l'}| = |\underline{i}| - |\underline{l}|$ and $d(\underline{j}) - |\underline{l'}| = d(\underline{i}) - |\underline{l}|$, we set $\hat{I}_{\underline{l'}} = I_{\underline{l}}$ and we can check that Conditions 1, 2, 3 are satisfied.

Case $\underline{l}' = (\underline{l}, 2)$.

According to Formula (5.6), we have: $f'_{\underline{l}'} = c_n \cdots c_{n-p+1} f_{\underline{l}}$. In this case $d(\underline{l}') = d(\underline{l}) + 1$, so we set:

$$\hat{I}_{\underline{l}'} = \{\underline{b} = (b_0 + 1, b_1, \ldots, b_p) \mid (b_0, \ldots, b_p) \in I_{\underline{l}}\}.$$

It is again easy to check that Conditions 1, 2, 3 are satisfied (for Condition 3, notice that $\hat{I}^+_{\underline{l}'}$ consists in the $(b_0 + 1, \ldots, b_p)$ such that $(b_0, \ldots, b_p)$ belongs to $I^+_{\underline{l}}$).

Case $\underline{l}' = \underline{l}$.

In this case $f'_{\underline{l}'} = \Delta.f_{\underline{l}}$. Since $X_1^m = \frac{\partial}{\partial \theta_m} = -\frac{\partial}{\partial \varphi_m} + \frac{\partial}{\partial \varphi_{m-1}}$, the vector field Δ (defined by (5.6)) can be written:

$$s_n \frac{\partial}{\partial \varphi_n} - t_n \frac{\partial}{\partial \varphi_{n-1}} - c_n t_{n-1} \frac{\partial}{\partial \varphi_{n-2}} - \cdots \\ - c_n \cdots c_{n-p+3} t_{n-p+2} \frac{\partial}{\partial \varphi_{n-p+1}} + K \frac{\partial}{\partial \varphi_{n-p}}.$$

Since $f_{\underline{l}}(\varphi_{n-p+1}, \ldots, \varphi_n)$ doesn't depend on φ_{n-p}, we doesn't need the expression of the function K. The form of the function $f_{\underline{l}}(\varphi_{n-p+1}, \ldots, \varphi_n)$ is given by the expression (5.3), therefore $f'_{\underline{l}'} = \Delta.f_{\underline{l}}$ is the sum, for $\underline{b} \in I_{\underline{l}}$, of:

$$\begin{aligned}(c_{n-p+1})^{b_0-b_1-1} &(t_{n-p+2})^{b_1+1-b_2} \cdots (t_n)^{b_{p-1}-b_p} \\ &\times (b_0 - b_1) c_n \cdots c_{n-p+3} s_{n-p+1} g_{\underline{b},\underline{l}} \\ &+ (c_{n-p+1})^{b_0-b_1} (t_{n-p+2})^{b_1-b_2} \cdots (t_n)^{b_{p-1}-b_p-1} (b_{p-1} - b_p) s_n t'_n g_{\underline{b},\underline{l}} \\ &+ (c_{n-p+1})^{b_0-b_1} (t_{n-p+2})^{b_1-b_2} \cdots (t_n)^{b_{p-1}-b_p} G_{\underline{b},\underline{l}}, \end{aligned} \tag{5.7}$$

where $G_{\underline{b},\underline{l}}(\varphi_{n-p+1}, \ldots, \varphi_n)$ belongs to $C^\infty(\mathbf{S}^p)$.

According to this expression, $f'_{\underline{l}'}$ can be written in the form (5.3) with the set:

$$\hat{I}_{\underline{l}'} = I_{\underline{l}} \cup \bigcup_{i=1}^{p} \{(b_0, \ldots, b_i + 1, \ldots, b_p) \mid (b_0, \ldots, b_p) \in I_{\underline{l}} \text{ and } b_{i-1} > b_i\}.$$

This set $\hat{I}_{\underline{l}'}$ has the form required in the part 1 of the lemma. Moreover, if $\underline{b} \in \hat{I}_{\underline{l}'}$ is such that $b_p = 0$, it is obtained from a $\underline{b}' \in I_{\underline{l}}$ such that $b'_p = 0$, and then $\sum_{i=1}^{p-1} b'_i < d(\underline{i}) - |\underline{l}|$. Since $\sum_{i=1}^{p-1} b_i \leq \sum_{i=1}^{p-1} b'_i + 1$, and $d(\underline{j}) = d(\underline{i}) + 1$, Condition 2 is satisfied.

At last, $\hat{I}^+_{\underline{l}'}$ consists only of elements $(b_0, \ldots, b_i + 1, \ldots, b_p)$, $1 \leq i \leq p$, such that $(b_0, \ldots, b_p)$ belongs to $I^+_{\underline{l}}$. Then the expression (5.4) for $(b_0, \ldots, b_p)$ and Formula (5.7) imply that Condition 3 is satisfied.

Finally we have proved that all the terms Σ'_k and $\Sigma''_{s,\underline{l}}$ can be written in the form (5.2) with the properties 1, 2, 3 of the lemma. Therefore the sum of these terms, that is $[X^n_{\underline{j}}]$, have the required form. That completes the proof of the induction step, and then the one of the lemma.

■

LEMMA 5.4. *Let β, δ and r such that $\beta > \delta \geq r \geq 1$, and $[X_{\underline{l}}^{n}] = [A^n(\beta + (p-1)\delta + r, \beta + (p-1)\delta)]$ (see (3.2). Then, if $[X_{\underline{l}}^{n-p}]$ is such that $|\underline{l}| = \beta$ and $d(\underline{l}) = \delta$, the sequence $(\delta, \ldots, \delta, r)$ belongs to $I_{\underline{l}}^{+}$.*

PROOF. Lemma 5.2 implies that, for $[A^n(p+1, p)]$, $(1, \ldots, 1)$ belongs to I_2^+. For $[A^n(p+\beta, p+\beta-1)]$, by using formula (5.6), we can see that, if $|\underline{l}| = \beta$ and $d(\underline{l}) = \delta$, $(\delta, 1, \ldots, 1)$ belongs to $I_{\underline{l}}^{+}$. By using now formula (5.7), we see that, for $[A^n(p+\beta+r, p+\beta+r-1)]$, $r \leq \delta - 1$, $(\delta, r+1, 1, \ldots, 1)$ belongs to $I_{\underline{l}}^{+}$, and so on until $[A^n(\beta + (p-1)\delta + r, \beta + (p-1)\delta)]$.

■

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